Composition algebras over commutative rings

Holger P. Petersson
Fakultät für Mathematik und Informatik
FernUniversität in Hagen
D-58084 Hagen
Germany
Email: Holger.Petersson@FernUni-Hagen.de

Conference on linear algebraic groups and cohomology
Emory University, Atlanta, GA, USA, May 16-20, 2007

0. Introduction.

First of all, I would like to thank the organizers, most notably my good friend Skip Garibaldi, for the invitation and for the opportunity to give a talk at this wonderful conference.

My purpose in this lecture will be to give a survey of what is presently known about composition algebras over arbitrary commutative rings. While composition algebras over fields, thanks to their intimate connection with quadratic forms, are reasonably well understood, precious little is known about the subject in full generality. In fact, one could argue with some justification that the only things really new in this lecture are related in one way or another to the three open problems that I will discuss at the end. But rather than beginning at the end, I prefer to begin with the

1. Definition and elementary properties.

Even when it comes to the definition of composition algebras over fields, the literature offers a rather confusing picture, with many different concepts floating around under the same name. In this lecture, I will follow a suggestion of Ottmar Loos to define the concept of a composition algebra as follows.

Given an arbitrary commutative ring $k$, let us first fix some notation. I will always write $k$-alg for the category of commutative $k$-algebras (with 1). For a $k$-module $M$ and a $R \in k$-alg, I abbreviate $M_R = M \otimes R$ as $R$-modules.

1.1. Definition. By a composition algebra over $k$, I mean a non-associative $k$-algebra $C$ (so $C$ is a $k$-module together with a bilinear multiplication, subject to no further restrictions) satisfying the following conditions.

- $C$ has a unit.
- $C$ is faithful and finitely generated projective as a $k$-module.
- $C$ has a norm, i.e., there exists a quadratic form $n : C \to k$ such that

(i) $n$ permits composition: $n(xy) = n(x)n(y)$.

---

(ii) \(n\) is separable: Writing
\[ n(x, y) := n(x + y) - n(x) - n(y) \]
for the bilinearization of \(n\), then for all fields \(K \in k\text{-alg}\), the extended quadratic form \(n = n_K : C_K \to K\) is non-degenerate, so for all \(x \in C_K\),
\[ n(x) = n(x, y) = 0 \quad (y \in C_K) \implies x = 0. \]

Standard arguments as given in Knus [2, II §7], for example, then lead to what I call the

1.2. Basic facts. Composition algebras of rank \(r\) over \(k\)
- are invariant under base change,
- have a unique norm \(n = n_C\), so we are allowed to call \(C\) non-singular if the bilinearization of \(n_C\) is non-singular in the sense that it determines a linear isomorphism from the \(k\)-module \(C\) onto its dual \(C^*\),
- are quadratic: \(x^2 - t_C(x)x + n_C(x)1 = 0\), where \(t_C := n_C(1, -)\) is the trace of \(C\),
- exist only in ranks 1, 2, 4, 8,
- are non-singular unless \(C \cong k\) and \(2 \notin k^\times\), in which case they are not,
- are alternative: The associator
\[ C \times C \times C \to C, \quad (x, y, z) \mapsto [x, y, z] := (xy)z - x(yz) \]
is alternating,
- are associative iff \(r \leq 4\),
- are commutative associative iff \(r \leq 2\),
- have a canonical involution
\[ \iota_C : C \to C, \quad x \mapsto \overline{x} := t_C(x)1 - x. \]

Before proceeding, let’s give some

1.3. Examples of composition algebras. Let \(C\) be a composition algebra of rank \(r\) over \(k\). For
- \(r = 1, C \cong k, n_C(\alpha) = \alpha^2\),
- \(r = 2, C\) is a quadratic étale algebra, \(n_C(x) = \det L_x\),
- \(r = 4, C\) is a quaternion algebra, i.e., an Azumaya algebra of degree 2, \(n_C = \text{Nrd}_C\),
- \(r = 8, C\) by definition is an octonion algebra.
2. Construction methods.

Actually, Example 1.3 for \( r = 8 \) doesn't give us more than the definition of an octonion algebra. We are therefore in desperate need of construction methods. We begin with the

2.1. Cayley-Dickson construction. (Petersson [5], Pumplün [7]) The

**Input** consists of

- a non-singular associative composition algebra \( B \) over \( k \),
- a right \( B \)-module \( P \) that is *locally free of rank 1*, so locally looks just like \( B \) as a right \( B \)-module,
- a non-singular hermitian form \( h : P \times P \rightarrow B \) that is *diagonal* in the sense that \( h(x, x) \in k = k1_B \) for all \( x \in P \).

Then the

**Output** is

- a composition algebra \( C = \text{Cay}(B, P, h) \) that lives on the \( k \)-module \( B \oplus P \) under the multiplication

\[
(u \oplus x)(v \oplus y) := (uv + h(y, x)) \oplus (xv + yu).
\]

We then have:

- \( B \) embeds into \( C \) as a subalgebra through the first factor,
- \( \text{rk}(B) = r \implies \text{rk}(C) = 2r \),
- \( n_C(u \oplus x) = n_B(u) - h(x, x) \).

Conversely, one can prove

2.2. Theorem. **Let** \( C \) **be a composition algebra of rank** \( 2r \) **over** \( k \) **and** \( B \subseteq C \) **a non-singular composition subalgebra of rank** \( r \). **Then** there exist \( P, h \) **as above such that the inclusion** \( B \hookrightarrow C \) **extends to an isomorphism** \( \text{Cay}(B, P, h) \cong C \). **□**

2.3. Examples. If \( B \) is a quaternion algebra over \( k \), the Cayley-Dickson construction produces an octonion algebra \( C = \text{Cay}(B, P, h) \); conversely, every octonion algebra over \( k \) containing a quaternion subalgebra arises in this way. While this extra condition is always fulfilled if we are working over a field, it doesn’t hold in general. In fact, there are famous examples due to Knus-Parimala-Sridharan [3] of octonion algebras \( C \) over the polynomial ring \( k = K[X_1, \ldots, X_n] \), \( K \) a field of characteristic not 2, \( n \geq 2 \), having the trace zero elements

\[
C_0 := \{ x \in C \mid t_C(x) = 0 \}
\]

as an indecomposable quadratic subspace of rank 7.

The second construction method we would like to discuss is the
2.4. Zorn construction. (Thakur [8] for $\frac{1}{2} \in k$, Petersson [6]) Here the Input consists of

- a non-singular commutative associative composition algebra $D$ over $k$,
- a ternary hermitian space $(V, h)$ over $D$ with trivial discriminant,
- a trivialization of $\wedge^3(V, h)$, i.e., an isometry

$$\Delta: \wedge^3(V, h) \sim (D, \langle 1 \rangle),$$

giving rise to the induced hermitian vector product

$$V \times V \to V, \quad (x, y) \mapsto x \times_{h, \Delta} y,$$

via

$$h(x \times_{h, \Delta} y, z) = \Delta(x \wedge y \wedge z).$$

Then the Output is

- a composition algebra $C = Zor(D, V, h, \Delta)$ that lives on the $k$-module $D \oplus V$ under the multiplication

$$(a \oplus x)(b \oplus y) := (ab - h(x, y)) \oplus (xb + y\overline{x}).$$

We have

- $D$ embeds into $C$ as a subalgebra through the first factor,
- $\text{rk}(D) = r \implies \text{rk}(C) = 4r$,
- $n_C(a \oplus x) = n_D(a) + h(x, x)$.

Conversely, one can prove

2.5. Theorem. Let $C$ be a composition algebra of rank $4r$ over $k$ and $D \subseteq C$ a non-singular composition subalgebra of rank $r$. Then there exist $V, h, \Delta$ as above such that the inclusion $D \hookrightarrow C$ extends to an isomorphism $\text{Zor}(D, V, h, \Delta) \sim C$. \hfill $\square$

2.6. Remark. Again the Zorn construction doesn’t apply to the examples of Knus-Parimala-
Sridharan. However, it

- yields all quaternion algebras for $r = 1$ if 2 is a unit and, at the other extreme,
- it does apply if 2 is sufficiently far away from being a unit in the following sense.

2.7. Proposition. Let $C$ be a composition algebra of rank $> 1$ over $k$ and suppose $2 \in \text{Jac}(k)$, the Jacobson radical of $k$. Then $C$ contains a quadratic étale subalgebra. \hfill $\square$
2.8. The case $D$ split quadratic étale. (Petersson [5]) If $D = k \oplus k$, the Zorn construction boils down to the following. The
Input consists of

- a finitely generated projective $k$-module $M$ of rank 3 with trivial determinant,
- a trivialization of $\bigwedge^3(M)$, i.e., an isomorphism

$$\theta : \bigwedge^3(M) \xrightarrow{\sim} k,$$

giving rise to vector products

$$\times_\theta : M \times M \longrightarrow M^*, \quad \times_\theta : M^* \times M^* \longrightarrow M$$

defined by

$$\langle w, u \times_\theta v \rangle = \theta(u \wedge v \wedge w), \quad \langle u^* \times v^*, w^* \rangle = \theta^{-1}(u^* \wedge v^* \wedge w^*),$$

Then the
Output is

- an octonion algebra $C = \text{Zor}(M, \theta)$ living on the $k$-module

$$\text{Zor}(M, \theta) = \begin{pmatrix} k & M^* \\ M & k \end{pmatrix}$$

under the multiplication

$$\begin{pmatrix} \alpha_1 & v^* \\ v & \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 & w^* \\ w & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \beta_1 + \langle w, v^* \rangle & \alpha_1 w^* + \beta_2 v^* + v \times_\theta w \\ \beta_1 v + \alpha_2 w - v^* \times_\theta w & \langle v, w^* \rangle + \alpha_2 \beta_2 \end{pmatrix}.$$
3.2. The notion of isotopy. (McCrimmon [4]) Let $C$ be a composition algebra over $k$.

- $a \in C$ is defined to be invertible iff $n_C(a) \in k^\times$. In this case, $a^{-1} := n_C(a)^{-1}1$ is an honest-to-goodness inverse: $aa^{-1} = a^{-1}a = 1$. We put
  \[ C^\times := \{ a \in C \mid a \text{ is invertible} \}. \]

- Let $a, b \in C^\times$ and $C^{(a,b)}$ be the $k$-algebra that lives on the $k$-module $C$ under the multiplication
  \[ x \cdot a, b y := (xa)(by). \]
  Then one checks easily that
  - $C^{(a,b)}$ is a composition algebra with unit $1^{(a,b)} = (ab)^{-1}$ and norm $n_{C^{(a,b)}} = \langle n_C(ab) \rangle \cdot n_C$.

We can now phrase

3.3. The isotopy problem. Does isotopy of composition algebras reduce to isomorphism, i.e., given a composition algebra $C$ over $k$ and $a, b \in C^\times$, does it follow that
  \[ C^{(a,b)} \cong C? \]

Again I have a few comments.

- For $\text{rk}(C) \leq 4$, the answer is yes since $C$ is associative and
  \[ L_{ab} : C^{(a,b)} \xrightarrow{\sim} C \]
  is easily seen to be an isomorphism.

- For $\text{rk}(C) = 8$ ???

- A positive answer to the norm equivalence problem yields a positive answer to the isotopy problem since
  \[ L_{ab} : n_{C^{(a,b)}} \xrightarrow{\sim} n_C \]
  is an isometry, so a positive solution to the norm equivalence problem would imply $C^{(a,b)} \cong C$.
  In particular:
  - The answer is yes if $k$ is a field.
  - We may always assume $b = a^{-1}$ since one checks that
    \[ L_{ab} : C^{(a,b)} \xrightarrow{\sim} C^{(a^2b,b^{-1}a^{-2})} \]
    is an isomorphism. Setting
    \[ C^a := C^{(a^{-1},a)}, \]
    we have
    - $(C^a)^b = C^{ab}$,
    - $C^b \cong C^{ba^3} \cong C^{a^2ba} \cong C^{ab}$, in particular
    - $C^a \cong C$ if $a$ is a third power, so $3$ is indeed a bad prime for the group $G_2$,
    - $C^a \cong C$ if $k[a]^{-1} \cap C^\times \neq \emptyset$,
    - $C^a \cong C$ if $a$ has trace zero: $t_C(a) = 0$. 

3.4. The isotopy problem and the Cayley-Dickson construction. Let

- $B$ be a non-singular associative composition algebra over $k$,
- $P$ a locally free right $B$-module of rank 1,
- $h : P \times P \to B$ a non-singular hermitian form,
- $C = \text{Cay}(B, P, h)$,
- $a \in B^\times$

Then

- $C^a = \text{Cay}(B, P^a, h^a)$, where
- $P^a = P$ as $k$-modules with the twisted $B$-action
  $$P^a \times B \to P^a, \quad (x, u) \mapsto x(aua^{-1}),$$
- $h^a : P^a \times P^a \to B, h^a(x, y) = a^{-1}h(x, ya),$
- $C^a \cong C$.

3.5. The isotopy problem and the Zorn construction. Let

- $D$ be a non-singular commutative associative composition algebra over $k$,
- $(V, h)$ a ternary hermitian space over $D$ with trivial discriminant,
- $\Delta : \Lambda^3(V, h) \cong (D, \langle 1 \rangle)$ a trivialization of $\Lambda^3(V, h)$,
- $C = \text{Zor}(D, V, h, \Delta)$,
- $a \in D^\times$.

Then

- $s := a\bar{a}^{-1} \in D$ has norm 1, so $s\Delta$ is another trivialization of $\Lambda^3(V, h)$ and
- $C^a = \text{Zor}(D, V, h, s\Delta)$,
- $C^a \cong C$.

Since this is a conference on algebraic groups and cohomology, I would like to draw a connection between
3.6. The isotopy problem and invariants of $F_4$. If $k$ is a field, the 3-invariant mod 2 of a group of type $F_4$ may be described as follows: The group is completely determined by an Albert algebra $A$ over $k$, which in turn can be co-ordinatized by 3-by-3 hermitian matrices having entries in some octonion algebra $C$ over $k$ and the Arason invariant of $n_C$, the corresponding 3-fold Pfister form, is the group invariant we are looking for. The whole point of this is, of course, that $A$ determines $C$ uniquely.

It doesn’t seem at all clear whether this uniqueness continues to hold if $k$ is a ring. In fact, the best one can possibly hope for is uniqueness up to isotopy since it is easy to see that if $C$ co-ordinatizes $A$ in the manner described above, so does every isotope. But then it could very well be that passing to isotopes of $C$ is just about the only degree of freedom one is allowed here, in which case at least the norm of $C$ would be an invariant. So passing to the corresponding Arason invariant (assuming it exists) would yield a group invariant over rings that generalizes the 3-invariant mod 2 over fields.

But I have promised you three open problems. Here is the third one.

3.7. The co-ordinatization problem. For $i = 1, 2$, let $M_i$ be a finitely generated projective $k$-module of rank 3 and $\theta_i : \bigwedge^3(M_i) \cong k$ a trivialization of $\bigwedge^3(M)$. Find conditions in terms of $(M_1, \theta_1)$ and $(M_2, \theta_2)$ that are necessary and sufficient for the octonion algebras $\text{Zor}(M_1, \theta_1)$ and $\text{Zor}(M_2, \theta_2)$ to be isomorphic.

References


