Conditionals in knowledge representation and belief revision

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Abstract

Conditionals ("*if-then-rules*") are most important objects in knowledge representation, commonsense reasoning and belief revision. Due to their non-classical nature, however, they are not easily dealt with. This paper presents a new approach to conditionals, which is apt to capture their dynamic power peculiarly well. We show how this approach can be applied to represent conditional knowledge inductively and to guide revisions of epistemic states by (propositional or conditional) beliefs. In particular, we generalize system-Z^* as an appropriate counterpart to maximum entropy-representations in a semi-quantitative setting.

1 Introduction

Relationships amongst propositions are crucial pieces of knowledge. Usually, they express plausible connections, bring isolated facts together and help us obtain a coherent image of the world. Such relationships may be represented in a most general form by *if-then-conditionals*.

Conditionals are omnipresent, in everyday life as well as in scientific environments. We make use of conditional knowledge when we avoid puddles on sidewalks (being aware of "If you step into a puddle, then your feet might get wet"), and when we expect high wheat prices from observing cold and rainy wheather in spring and summer (due to "If the growing wheather is poor then there will be an increase in price of wheat"). Conditionals represent generic knowledge, acquired inductively from experience or learned from books. They tie a flexible and highly interrelated network of connections along which reasoning is possible and which can be applied to different situations. Moreover, as plausible yet defeasible conclusions, conditionals are intimately related to *nonmonotonic reasoning*. In belief revision, they take the role of *revision policies*, guiding changes of beliefs when new (propositional) evidence becomes apparent.

The key to get conditionals right is to accept their non-classical nature – conditionals are not simply "true" or "false". In a particular situation, a conditional is *applicable* (you actually step into a puddle) or not (you simply walk around), it can be found *confirmed* (you step into a puddle and indeed, your feet get wet) or *violated* (you step into a puddle, but your feet remain dry because you are wearing rain boots). So the central problem in dealing with conditional knowledge is to handle adequately, on the one hand, inactive (or neutral, respectively) behavior, and, on the other hand, active as well as polarizing behavior.

The main object of this paper is

- to sketch a new theory for conditionals which captures their three-valued, dynamic nature peculiarly well,
- to show how it is realized in a semi-quantitative as well as in a probabilistic framework,
- to apply this theory to the problem of representing conditional knowledge inductively in both frameworks,
- to make use of it when revising epistemic states by conditional beliefs.

The central idea in that new approach to be presented is to consider conditionals as agents shifting possible worlds in order to establish relationships and beliefs, and to represent the effects conditionals have on worlds by *conditional structures.* Handling conditionals adequately then means to choose representations and revisions, respectively, which are balanced with respect to the structures of the conditionals under consideration. In this way, highly complex interactions between different conditionals can be taken into account and maintained.

A well-known probabilistic method that follows this idea is the *principle of maximum entropy (ME-principle)*, respectively, *the principle of minimum cross-entropy* for probabilistic belief revision [PV90, Par94, PV96]. In a semi-quantitative setting, system-Z^* [GMP93] also realizes this approach, though it seems to be of restricted applicability. In this paper, we will generalize system-Z^* to make it applicable for any consistent set of conditionals. The generalized Z^*-approach will be shown to be theoretically justified, and will prove to handle even problematic examples adequately.

The following section summarizes fundamental facts about conditionals, ordinal conditional functions and probability distributions. Section 3 briefly sketches the ME-principle, as well as system-Z and system-Z^*. The new dynamic approach to conditionals is presented in Section 4, and is linked to (semi-)quantitative representation of knowledge in Section 5. Section 6 shows the application of the conditional theory to belief revision. Section 7 compares our approach to related work. An outlook on further results concludes this paper in Section 8.

2 Conditionals, plausibility and probability

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{V} = \{a, b, c \ldots\}$. Let Ω be a complete set of interpretations of $\mathcal{L}$, where each $\omega \in \Omega$ is taken to be a *possible world* for $\mathcal{L}$. To simplify notation, we will write $\overline{A}$ instead of $\neg A$, and AB instead of $A \wedge B$, for formulas $A, B \in \mathcal{L}$.

Conditionals $(B|A)$ represent statements of the form "*If A then B*", expressing a relationship between two (propositional) formulas A, the *antecedent* or *premise*, and B, the *consequent.* $(\mathcal{L} \mid \mathcal{L})$ denotes the set of all conditionals $(B|A)$ with $A, B \in \mathcal{L}$. A conditional $(B|\top)$ with a tautological antecedent is taken to correspond to its (propositional) consequent, B.

Generally, a conditional $(B|A)$ is considered to be accepted if its confirmation, AB, is judged to be more plausible than its refutation, $A\overline{B}$. So, to validate conditionals, we need richer epistemic structures than plain propositional interpretations, at least to compare different worlds with regard to their relevance for a conditional (see, for example, [Nut80, Bou94, DP97]). A world $\omega \in \Omega$ is said to *verify* a conditional $(B|A)$, if $\omega \models AB$; it *falsifies* $(B|A)$, if $\omega \models A\overline{B}$; if $\omega \models \overline{A}$, then $(B|A)$ is not applicable to ω.

Epistemic states as representations of cognitive states of intelligent agents provide an adequate framework for conditionals. Two widely used types of epistemic states are *probability distributions* and *ordinal conditional functions, OCF's*, which are based on the notion of plausibility. In short, OCF's are functions $\kappa : \Omega \to \mathbb{N} \cup \{0, \infty\}$ from the set of worlds to the natural numbers, extended by 0 and ∞. They specify non-negative integers as degrees of plausibility – or, more precisely, as degrees of *disbelief* – for worlds. For propositional formulas $A, B \in \mathcal{L}$, we set $\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$. In particular, $0 = \min\{\kappa(A), \kappa(\overline{A})\}$, so that at least one of A or $\overline{A}$ is considered most plausible. A proposition A is believed iff $\kappa(\overline{A}) > 0$, which is denoted by $\kappa \models A$. This may also be specified by degrees of plausibility (or of disbelief, respectively) by saying that $\kappa \models A[n]$ iff $\kappa(\overline{A}) > n$ $(n \in \mathbb{N} \cup \{0\})$. A conditional $(B|A) \in (\mathcal{L} \mid \mathcal{L})$ may be assigned a degree of plausibility via $\kappa(B|A) = \kappa(AB) - \kappa(A)$. This is quite analogous as for probability distributions P, where we have $P(B|A) = \frac{P(AB)}{P(A)}$. Indeed, plausibility degrees via OCF's may be taken as order-of-magnitude abstractions of probabilities (cf. [GMP93, GP96]). For a probability distribution P, we have $P \models (B|A)\,[x]$ iff $P(B|A) = x$ $(x \in [0, 1])$.

Plausibility structures, on the one hand, are much weaker than probability structures, but, on the other hand, they seem to be closer to commonsense reasoning. They have a crucial meaning for belief revision, too: Plausibility pre-orderings are known to characterize AGM-revisions of belief sets and of epistemic states (cf. [DP97]). The fundamental connection between revision operators, $*$, epistemic states, Ψ, and conditionals, $(B|A)$, is formalized by the so-called *Ramsey test*: $\Psi \models (B|A)$ iff $\Psi * A \models B$. That is, a $(B|A)$ is accepted in the epistemic state Ψ, iff revising Ψ by A yields belief in B. For an OCF κ, this is in accordance with the plausibility relation imposed by κ. Each

OCF κ induces a (propositional) AGM-revision operator $*$ by setting $Mod(\kappa * A) = \min_\kappa(Mod(A))$ (see [DP97]). The Ramsey test then reads $\kappa \models (B|A)$ iff $\kappa * A \models B$. As can easily be proved, this is equivalent to $\kappa(AB) < \kappa(A\overline{B})$, i.e. to AB being more plausible than $A\overline{B}$. The Ramsey test can be generalized to handle quantified conditionals: $\Psi \models (B|A)\,[x]$ iff $\Psi * A \models B\,[x]$, where x is an appropriate quantification of $(B|A)$. For an OCF κ, we thus obtain $\kappa \models (B|A)\,[n]$ iff $\kappa(AB) + n < \kappa(A\overline{B})$ ($n \in \mathbb{N} \cup \{0\}$). We will consider mostly measure-free conditionals, focusing on structural aspects, but also allow quantifications. If $\mathcal{R}^* = \{(B_1|A_1)\,[x_1], \ldots, (B_n|A_n)\,[x_n]\}$ is a set of (appropriately) quantified conditionals, then $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ denotes the set of unquantified conditionals, and vice versa.

Throughout this paper, we will assume all OCF's to be finite, and all probability distributions to be positive. All results to be presented also hold in the general case, but then need some technical modifications (see [KI01]).

3 Maximum entropy and System-$Z^{(*)}$

In this section, we review well-known model-based approaches to represent conditional knowledge inductively.

For a consistent set $\mathcal{R}^* = \{(B_1|A_1)\,[x_1], \ldots,\ (B_n|A_n)\,[x_n]\}$ of probabilistic conditionals, the *ME-representation of* $\mathcal{R}^*$, $ME(\mathcal{R}^*)$, is the unique distribution P that maximizes the entropy $H(P) = -\sum_\omega P(\omega) \log P(\omega)$ subject to $P \models \mathcal{R}^*$ (see e.g. [Par94]). $ME(\mathcal{R}^*)$ can be written as

$$ME(\mathcal{R}^*)(\omega) = \alpha_0 \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \alpha_i^{1-x_i} \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \alpha_i^{-x_i}, \tag{1}$$

with the α_i's being appropriately chosen so as to satisfy all conditionals in $\mathcal{R}^*$.

In a semi-quantitative framework, a well-known method to represent a (finite) set $\mathcal{R} = \{r_i = (B_i|A_i) \mid 1 \leqslant i \leqslant n\}$ of conditionals by an OCF is to apply the *system-Z* of Goldszmidt and Pearl [GMP93, GP96]. The basic idea of system-Z is to observe the (logical) interactions of the conditionals in $\mathcal{R}$ which are described by the notion of *tolerance*. A conditional $(B|A)$ is *tolerated* by a set of conditionals $\mathcal{R}$ iff there is a world ω such that ω verifies $(B|A)$ and ω does not falsify any of the conditionals in $\mathcal{R}$. $\mathcal{R}$ is consistent iff there is an ordered partition $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_k$ of $\mathcal{R}$ such that each conditional in $\mathcal{R}_m$ is tolerated by $\bigcup_{j=m}^k \mathcal{R}_j$, $0 \leqslant m \leqslant k$.

The system-Z ranking function, κ^z, representing $\mathcal{R}$ is given by

$$\kappa^z(\omega) = \begin{cases} 0, \text{ if } \omega \text{ does not falsify any } r_i, \\ 1 + \max\limits_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} Z(r_i), \text{otherwise} \end{cases}$$

where $Z(r_i) = j$ iff $r_i \in \mathcal{R}_j$. κ^z assigns to each world ω the lowest possible rank admissible with respect to the constraints in $\mathcal{R}$.

An even more sophisticated representation is obtained by combining the system-Z approach with the probabilistic ME-principle, yielding system-Z^* [GMP93]. The corresponding Z^*-rankings of the conditionals in $\mathcal{R}$ have to satisfy the following equation (see [GMP93])

$$Z^*(r_i) + \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) = 1 + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j), \tag{2}$$

and κ^* is then calculated by

$$\kappa^*(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} Z^*(r_i) \tag{3}$$

Z^*-entailment $\mid\!\sim^*$ is defined by $\mathcal{R} \mid\!\sim^* (B|A)$ iff $\kappa^*(AB) < \kappa^*(A\overline{B})$.

A set $\mathcal{R}$ of conditionals is called a *minimal-core set* [GMP93] if for each conditional $r \in \mathcal{R}$, there is a world $\omega \in \Omega$ that falsifies r and no other conditional in $\mathcal{R}$. For minimal-core sets, a procedure is given to calculate Z^*-rankings in [GMP93]. If $\mathcal{R}$ is not a minimal-core set, however, the system-Z^* approach may fail.

Example 1 Consider the well-known *Nixon diamond*, consisting of the conditionals $r_1 : (p|q)$, *quakers are pacifists*, and $r_2 : (\overline{p}|r)$, *republicans are not pacifists*. $\{r_1, r_2\}$ is a minimal-core set, since $\omega = \overline{p}qr$ falsifies r_1 but not r_2, and $\omega = p\overline{q}r$ falsifies r_2 but not r_1. Each of the conditionals r_1, r_2 is tolerated by the other one, so the Z-rankings of r_1, r_2 are simply given by $Z(r_1) = Z(r_2) = 0$, and the Z^*-rankings are set to $Z^*(r_1) = Z^*(r_2) = 1$. For this example, both approaches yield the same ranking $\kappa^z = \kappa^*$. The table below shows the corresponding rankings for all possible worlds. As expected, κ^z (κ^*) is indeterminate with respect to the conditional $(p|qr)$:

ω	$\kappa^z(\omega) = \kappa^*(\omega)$	ω	$\kappa^z(\omega) = \kappa^*(\omega)$	ω	$\kappa^z(\omega) = \kappa^*(\omega)$	ω	$\kappa^z(\omega) = \kappa^*(\omega)$
pqr	1	$p\overline{q}r$	1	$\overline{p}qr$	1	$\overline{p}\,\overline{q}r$	0
$pq\overline{r}$	0	$p\overline{q}\,\overline{r}$	0	$\overline{p}q\overline{r}$	1	$\overline{p}\,\overline{q}\,\overline{r}$	0

Table 1: Z- and Z^*-rankings for Example 1

$\kappa^z(pqr) = 1 = \kappa^z(\overline{p}qr)$. From the knowledge base $\{r_1, r_2\}$, the system cannot decide whether a person, being both a quaker and a republican, is a pacifist or not. ■

System-Z^* handles conditional knowledge quite appropriately (cf. [GMP93]). In particular, Z^*-entailment satisfies the crucial property of *irrelevance*: If d is an atomic proposition not appearing in any of the conditionals in $\mathcal{R}$, then

$$\mathcal{R} \mid\!\sim^* (B|A) \quad \text{iff} \quad \mathcal{R} \mid\!\sim^* (B|Ad) \tag{4}$$

System-Z^* suggests a straightforward generalization for representing a set $\mathcal{R} = \{r_i = (B_i|A_i) \mid 1 \leqslant i \leqslant n\}$ of conditionals by an OCF: Use approach (3) and determine the $Z^*(r_i)$ so that κ^* satisfies all of the conditionals r_i, i.e. such that

$$Z^*(r_i) > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) \tag{5}$$

for $1 \leqslant i \leqslant n$. That means, that condition (2) is weakened to be satisfied as an inequality constraint. This generalization, however, seems to be quite ad hoc, leaving the ME-track and thus without theoretical justification.

Quite to the contrary – in the sequel, we will show that actually, this generalized approach emerged from a formal *principle of conditional preservation* that underlies the ME-techniques and can also be applied directly in the framework of ordinal conditional functions. This principle is based on observing *conditional structures* and provides a powerful methodology, not only to represent conditional knowledge appropriately, but even to guide the revision of epistemic states by conditional beliefs. In the next section, we will briefly develope the necessary theoretical background. For a more comprehensive introduction, cf. [KI01, KI00a].

4 A dynamic approach to conditionals

By observing the behavior of worlds with respect to it, each conditional $(B|A)$ can be considered as a generalized (namely three-valued) indicator function on worlds:

$$(B|A)(\omega) = \begin{cases} 1 & : \quad \omega \models AB \\ 0 & : \quad \omega \models A\overline{B} \\ u & : \quad \omega \models \overline{A} \end{cases} \tag{6}$$

where u stands for *unknown* or *indeterminate* [DeF74, Cal91]). Intuitively, representing or incorporating a conditional as a plausible conclusion in an epistemic state means to make – at least some – worlds confirming the conditional more plausible than the worlds refuting it. In this sense, conditionals to be learned have effects on possible worlds (more exactly, on their degrees of plausibility or probability), shifting them appropriately to establish the intended relationship. (6) then provides a classification of worlds for achieving this: On confirming worlds $\omega \models AB$, i.e. $(B|A)(\omega) = 1$, $(B|A)$ possibly has a positive effect, while on refuting worlds $\omega \models A\overline{B}$, $(B|A)$ possibly has a negative effect; the effects on worlds ω with $(B|A)(\omega) = u$ is unclear. Which worlds will actually be shifted depends on the chosen revision procedure – for the conditional, all worlds in either of the partitioning sets are indistinguishable.

When we consider (finite) sets of conditionals $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$, we have to modify the representation (6) appropriately to identify the effect of each conditional in $\mathcal{R}$ on worlds in Ω. To this end, we replace the numbers 0 and 1 by abstract symbols, $\mathbf{a}_i^+, \mathbf{a}_i^-$, that we associate to each conditional $(B_i|A_i)$ in $\mathcal{R}$. Moreover, we will make use of a group structure to represent the joint impact of conditionals on worlds.

So let $\mathcal{F}_\mathcal{R} = \langle \mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^- \rangle$ be the free abelian group with generators $\mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^-$, i.e. $\mathcal{F}_\mathcal{R}$ consists of all elements of the form $(\mathbf{a}_1^+)^{r_1}(\mathbf{a}_1^-)^{s_1} \ldots (\mathbf{a}_n^+)^{r_n}(\mathbf{a}_n^-)^{s_n}$ with integers $r_i, s_i \in \mathbb{Z}$ (the ring of integers). Each element of $\mathcal{F}_\mathcal{R}$ can be identified by its exponents, so that $\mathcal{F}_\mathcal{R}$ is isomorphic to $\mathbb{Z}^{2n}$ (cf. [LS77]). The commutativity of $\mathcal{F}_\mathcal{R}$ corresponds to the fact that the conditionals in $\mathcal{R}$ shall be effective simultaneously, without assuming any order of application. So our way of dealing with conditionals is a symmetric, homogeneous one – we do not need (user-defined) priorities among conditionals. Note that, although we will speak of *multiplication* and *products* in $\mathcal{F}_\mathcal{R}$, the generators of $\mathcal{F}_\mathcal{R}$ are merely juxtaposed, like words.

For each $i, 1 \leqslant i \leqslant n$, we define a function $\sigma_i : \Omega \to \mathcal{F}_\mathcal{R}$ by setting

$$\sigma_i(\omega) = \begin{cases} \mathbf{a}_i^+ & \text{if} \quad (B_i|A_i)(\omega) = 1 \\ \mathbf{a}_i^- & \text{if} \quad (B_i|A_i)(\omega) = 0 \\ 1 & \text{if} \quad (B_i|A_i)(\omega) = u \end{cases}$$

$\sigma_i(\omega)$ represents the manner in which the conditional $(B_i|A_i)$ applies to the possible world ω. The neutral element 1 of $\mathcal{F}_\mathcal{R}$ corresponds to the non-applicability of $(B_i|A_i)$ in case that the antecedent A_i is not satisfied. The function $\sigma_\mathcal{R} : \Omega \to \mathcal{F}_\mathcal{R}$,

$$\sigma_\mathcal{R}(\omega) = \prod_{1 \leqslant i \leqslant n} \sigma_i(\omega) = \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \mathbf{a}_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \mathbf{a}_i^-$$

describes the all-over effect of $\mathcal{R}$ on ω. $\sigma_\mathcal{R}(\omega)$ is called the *conditional structure of ω with respect to $\mathcal{R}$*. $\sigma_\mathcal{R}$ labels each world appropriately and allows us to compare different worlds with respect to the impact the conditionals in $\mathcal{R}$ exert on them. Making calculations of conditional structures more convenient and more elegant, we take the worlds $\omega \in \Omega$ as formal generators of the free abelian group $\widehat{\Omega} := \langle \omega \mid \omega \in \Omega \rangle$. $\widehat{\Omega}$ consists of all products $\widehat{\omega} = {\omega_1}^{r_1} \ldots {\omega_m}^{r_m}$, with $\omega_1, \ldots, \omega_m \in \Omega$, and integers $r_1, \ldots r_m$. Introducing such a "multiplication between worlds" is nothing but a technical means to comply with the multiplicative structure the effects of conditionals impose on worlds. As in $\mathcal{F}_\mathcal{R}$, multiplication in $\widehat{\Omega}$ actually means juxtaposition. Now $\sigma_\mathcal{R}$ may be extended to $\widehat{\Omega}$ in a straightforward manner by setting $\sigma_\mathcal{R}({\omega_1}^{r_1} \ldots {\omega_m}^{r_m}) = \sigma_\mathcal{R}(\omega_1)^{r_1} \ldots \sigma_\mathcal{R}(\omega_m)^{r_m}$, yielding a *homomorphism of groups* $\sigma_\mathcal{R} : \widehat{\Omega} \to \mathcal{F}_\mathcal{R}$. The following example extends the Nixon diamond (Example 1):

Example 2 Let $\mathcal{R}$ consist of the following conditionals:

$r_1 : (p\|q)$	*Quakers are pacifists.*	r_4 :	$(b\|a)$	*Americans like baseball.*
$r_2 : (\overline{p}\|r)$	*Republicans are not pacifists.*	r_5 :	$(\overline{b}\|q)$	*Quakers do not like baseball.*
$r_3 : (a\|q)$	*Quakers are Americans.*			

In Table 2, we list the conditional structures of all possible worlds with respect to $\mathcal{R}$. Using this table, it is easy to see that $\mathcal{R}$ is consistent, with partition $\mathcal{R}_0 = \{r_2, r_4\}$ and $\mathcal{R}_1 = \{r_1, r_3, r_5\}$. $\mathcal{R}$, however, is not a minimal-core set, because each world falsifying r_1 (i.e. worlds with label $\mathbf{a}_1^-$ in the table) also falsifies at least one of r_3, r_4, r_5. In fact, the system-Z^* approach fails for $\mathcal{R}$. ■

Having the same conditional structure defines an equivalence relation $\equiv_\mathcal{R}$ on $\widehat{\Omega}$: $\widehat{\omega}_1 \equiv_\mathcal{R} \widehat{\omega}_2$ iff $\sigma_\mathcal{R}(\widehat{\omega}_1) = \sigma_\mathcal{R}(\widehat{\omega}_2)$. Those elements of $\widehat{\Omega}$ that are balanced with respect to the effects of conditionals in $\mathcal{R}$ are contained in the *kernel of* $\sigma_\mathcal{R}$, *ker* $\sigma_\mathcal{R} = \{\widehat{\omega} \in \widehat{\Omega} \mid \sigma_\mathcal{R}(\widehat{\omega}) = 1\}$. *ker* $\sigma_\mathcal{R}$ does not depend on the chosen representation of conditional structures by symbols in $\mathcal{F}_\mathcal{R}$ and thus, it

ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$
$pqrab$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$p\overline{q}rab$	$\mathbf{a}_2^-\mathbf{a}_4^+$	$\overline{p}qrab$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}\,\overline{q}rab$	$\mathbf{a}_2^+\mathbf{a}_4^+$
$pqra\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$p\overline{q}ra\overline{b}$	$\mathbf{a}_2^-\mathbf{a}_4^-$	$\overline{p}qra\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}ra\overline{b}$	$\mathbf{a}_2^+\mathbf{a}_4^-$
$pqr\overline{a}b$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^-\mathbf{a}_5^-$	$p\overline{q}r\overline{a}b$	$\mathbf{a}_2^-$	$\overline{p}qr\overline{a}b$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_5^-$	$\overline{p}\,\overline{q}r\overline{a}b$	$\mathbf{a}_2^+$
$pqr\overline{a}\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^-\mathbf{a}_5^+$	$p\overline{q}r\overline{a}\overline{b}$	$\mathbf{a}_2^-$	$\overline{p}qr\overline{a}\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}r\overline{a}\overline{b}$	$\mathbf{a}_2^+$
$pq\overline{r}ab$	$\mathbf{a}_1^+\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$p\overline{q}\,\overline{r}ab$	$\mathbf{a}_4^+$	$\overline{p}q\overline{r}ab$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}\,\overline{q}\,\overline{r}ab$	$\mathbf{a}_4^+$
$pq\overline{r}a\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$p\overline{q}\,\overline{r}a\overline{b}$	$\mathbf{a}_4^-$	$\overline{p}q\overline{r}a\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}\,\overline{r}a\overline{b}$	$\mathbf{a}_4^-$
$pq\overline{r}\,\overline{a}b$	$\mathbf{a}_1^+\mathbf{a}_3^-\mathbf{a}_5^-$	$p\overline{q}\,\overline{r}\,\overline{a}b$	1	$\overline{p}q\overline{r}\,\overline{a}b$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_5^-$	$\overline{p}\,\overline{q}\,\overline{r}\,\overline{a}b$	1
$pq\overline{r}\,\overline{a}\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_3^-\mathbf{a}_5^+$	$p\overline{q}\,\overline{r}\,\overline{a}\,\overline{b}$	1	$\overline{p}q\overline{r}\,\overline{a}\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}\,\overline{r}\,\overline{a}\,\overline{b}$	1

Table 2: Conditional structures for Example 2

is an invariant of $\mathcal{R}$ [KI01]. In a semi-quantitative as well as in a probabilistic environment, implicit normalizing constraints have to be taken into account, namely, $\kappa(\top) = 0$ for OCF's, and $P(\top) = 1$ for probability distributions. This can be achieved by focusing on equivalence with repect to $\sigma_{(\top|\top)}$. Since $\sigma_{(\top|\top)}$ simply counts the generators occurring in $\widehat{\omega}$, two elements $\widehat{\omega}_1 = \omega_1^{r_1} \ldots \omega_m^{r_m}, \widehat{\omega}_2 = \nu_1^{s_1} \ldots \nu_p^{s_p} \in \widehat{\Omega}$ are $\sigma_{(\top|\top)}$-equivalent, $\widehat{\omega}_1 \equiv_\top \widehat{\omega}_2$, iff $\sum_{1 \leqslant j \leqslant m} r_j = \sum_{1 \leqslant k \leqslant p} s_k$. This means, $\widehat{\omega}_1 \equiv_\top \widehat{\omega}_2$ iff they both are a (cancelled) product of the same number of generators, each generator being counted with its corresponding exponent.

5 Conditional indifference

To study conditional interactions, we now focus on the behavior of OCF's $\kappa : \Omega \to \mathbb{N} \cup \{0, \infty\}$ and probability functions $P : \Omega \to [0, 1]$ with respect to the elements in $\widehat{\Omega}$.

What is common to both ordinal conditional functions and probability functions is that they make use of *two different operations* to distinguish between purely propositional information and conditional knowledge (minimization and summation, and summation and multiplication, respectively). Such functions where introduced and investigated as *conditional valuation functions* in [KI01]. It is only this property that is needed to apply the theory of conditional structures, where a crucial demand for the conditional operation (summation, respectively multiplication) is its invertibility. Therefore, it is possible to extend each such function to a homomorphism, $\kappa : \widehat{\Omega} \to (\mathbb{Z}, +)$, or $P : \widehat{\Omega} \to \mathbb{R}^+$, respectively, by setting

$$
\begin{aligned}
\kappa({\omega_1}^{r_1} \cdot \ldots \cdot {\omega_m}^{r_m}) &= r_1\kappa(\omega_1) + \ldots + r_m\kappa(\omega_m), \\
P({\omega_1}^{r_1} \cdot \ldots \cdot {\omega_m}^{r_m}) &= P(\omega_1)^{r_1} \cdot \ldots \cdot P(\omega_m)^{r_m}
\end{aligned}
$$

This allows us to analyze numerical relationships holding between different $\kappa(\omega)$, respectively $P(\omega)$, and to elaborate the conditionals whose structures these functions follow. In the sequel, let V denote an OCF or a probability function.

Definition 3 Assume $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ to be a set of conditonals. V is *indifferent with respect to* $\mathcal{R}$ iff $V(\widehat{\omega}_1) = V(\widehat{\omega}_2)$ whenever $\sigma_{\mathcal{R}}(\widehat{\omega}_1) = \sigma_{\mathcal{R}}(\widehat{\omega}_2)$ for $\widehat{\omega}_1 \equiv_\top \widehat{\omega}_2 \in \widehat{\Omega}$.

Note that we presupposed all OCF's to be finite and all probability functions to be positive in order to make this definition a very concise one. For the general case, cf. [KI01].

V being indifferent with respect to $\mathcal{R}$ means that it does not distinguish between different elements $\widehat{\omega}_1, \widehat{\omega}_2$ with the same conditional structure with respect to $\mathcal{R}$. Normalizing constraints are taken into account by observing $\equiv_\top$-equivalence. Conversely, any deviation $\kappa(\widehat{\omega}) \neq 0$, respectively $P(\widehat{\omega}) \neq 1$, can be explained by the conditionals in $\mathcal{R}$ acting on $\widehat{\omega}$ in a non-balanced way. Conditional indifference captures interactions of conditionals of arbitrary depth by making use of the group homomorphism induced by V.

The next theorem gives a simple criterion to check conditional indifference (cf. [KI01]).

Theorem 4 *Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a set of conditionals.*

An OCF κ is indifferent with respect to $\mathcal{R}$ iff there are rational numbers $\kappa_0, \kappa_i^+, \kappa_i^- \in \mathbb{Q}$, $1 \leqslant i \leqslant n$, such that for all $\omega \in \Omega$,

$$\kappa(\omega) = \kappa_0 + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \kappa_i^+ + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \kappa_i^- \tag{7}$$

A probability function P is indifferent with respect to $\mathcal{R}$ iff there are positive real numbers $\alpha_0, \alpha_1^+, \alpha_1^-, \ldots, \alpha_n^+, \alpha_n^- \in \mathbb{R}^+$ such that for all $\omega \in \Omega$,

$$P(\omega) = \alpha_0 \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \alpha_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \alpha_i^- \tag{8}$$

Ordinal conditional functions, κ, and probability functions, P, representing a set $\mathcal{R}$ of (quantified) conditionals and being indifferent with respect to it, are called *c-representations*. Theorem 4 provides an intelligible schema to construct conditionally indifferent functions. For instance, for an OCF-c-representation, we simply set up κ according to (7) and choose the κ_i^+, κ_i^- appropriately to ensure that $\kappa \models \mathcal{R}$ holds, i.e. such that

$$\kappa_i^- - \kappa_i^+ > \min_{\omega \models A_i B_i} \left(\sum_{\substack{j \neq i \\ \omega \models A_j B_j}} \kappa_j^+ + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- \right) - \min_{\omega \models A_i \overline{B_i}} \left(\sum_{\substack{j \neq i \\ \omega \models A_j B_j}} \kappa_j^+ + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- \right) \tag{9}$$

Furthermore, the normalizing constant κ_0 has to be chosen appropriately to ensure that actually an OCF is obtained.

Example 5 We continue Example 2, the extended Nixon diamond. We set $\kappa_i^+ := 0$ for each conditional $r_i \in \mathcal{R}$, and determine $\kappa_i^- \geqslant 0$ according to (9); actually, this amounts to applying the generalized system-Z^*. Note that, due to the consistency of $\mathcal{R}$, also $\kappa_0 = 0$. r_2 and r_4 are tolerated by the whole set $\mathcal{R}$, so (9) yields $\kappa_2^-, \kappa_4^- > 0$. Therefore, we set $\kappa_2^- := \kappa_4^- := 1$. For determining $\kappa_1^-, \kappa_3^-, \kappa_5^-$, Table 2 proves to be helpful. We calculate $\kappa_1^- > \min\{\kappa_3^-, \kappa_4^-, \kappa_5^-\} - \min\{\kappa_3^-, \kappa_4^-, \kappa_5^-\} = 0, \kappa_3^- > \min\{\kappa_4^-, \kappa_5^-\}, \kappa_5^- > \min\{\kappa_4^-, \kappa_3^-\}$. So we set $\kappa_1^- := 1, \kappa_3^- := \kappa_5^- := 2$. Now, $\kappa_c(\omega) = \sum_{\substack{1 \leqslant i \leqslant 5 \\ \omega \models A_i \overline{B_i}}} \kappa_i^-$ can be computed easily. Table 3 shows κ_c-rankings for some worlds ω and compares them to system-Z-rankings. It is clearly seen that a c-representation imposes a

ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$	ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$	ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$	ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$
$pqrab$	3	2	$pqr\overline{a}b$	5	2	$\overline{p}qrab$	3	2	$\overline{p}qr\overline{a}b$	5	2
$pqra\overline{b}$	2	1	$pqr\overline{a}\overline{b}$	3	2	$\overline{p}qra\overline{b}$	2	2	$\overline{p}qr\overline{a}\overline{b}$	3	2

Table 3: Some conditional structures for Example 5

more finely grained structure on the possible worlds and thus establishes more conditionals. For instance, we have $\kappa^z \not\models (\overline{b}|pqr\overline{a})$, but $\kappa_c \models (\overline{b}|pqr\overline{a})$, that is, a republican, who is a pacifist and a quaker, but not an American, is supposed not to like baseball. To show that this is more than pure speculation, compare the conditional structures of the two worlds involved, $pqr\overline{a}b$ and $pqr\overline{a}\,\overline{b}$: They are the same, except for conditional r_5, and due to r_5, the person is supposed not to like baseball.

In general, system-Z appears to be too cautious to handle conditional interactions adequately. But the extended Nixon diamond also shows it to be too bold sometimes: Now we have $\kappa^z \models (p|qr)$, which seems to be quite unintuitive. This problem also holds for other well-known approaches to conditional reasoning, cf. [DP96]. Again, the c-representation κ_c proves to be more adequate: We have $\kappa_c(pqr) = 2 = \kappa_c(\overline{p}qr)$, so κ_c is completely indeterminate with respect to $(p|qr)$. ■

Comparing (7) to (3) and (8) to (1), respectively, we see that both (the generalized) system-Z^* (with $\kappa_0 = \kappa_i^+ = 0$ and $\kappa_i^- = Z^*(r_i)$) and the ME-techniques (with $\alpha_i^+ = \alpha^{1-x_i}$ and $\alpha_i^- = \alpha^{-x_i}$)

determine c-representations. In contrast to the limited applicability of system-Z^*, an OCF-c-representation can be calculated for any consistent set of conditionals, not only for minimal-core sets:

Proposition 6 *$\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ is consistent iff a c-representation of $\mathcal{R}$ exists.*

It is already the approach (7) (or (3)) that guarantees a well-behavedness with respect to conditional interactions. In particular, all OCF-c-representations satisfy the property of irrelevance in the following sense (cf. (4)):

Lemma 7 *Suppose $\mathcal{R}$ is a set of conditionals, and d is an atomic proposition not appearing in any of the conditionals in $\mathcal{R}$. Then, for any OCF-c-representation κ_c of $\mathcal{R}$, and for any conditional $(B|A) \in (\mathcal{L} \mid \mathcal{L})$, $\kappa_c \models (B|A)$ iff $\kappa_c \models (B|Ad)$.*

This is easily proved by observing that for any propositional interpretation ω' over $\mathcal{V} - \{d\}$, we have $\sigma_{\mathcal{R}}(\omega' d) = \sigma_{\mathcal{R}}(\omega' \overline{d})$, and therefore, $\kappa_c(A\dot{B}) = \kappa_c(A\dot{B}d)$ for $\dot{B} \in \{B, \overline{B}\}$. An analogous quantified version of this lemma holds for probability functions, too.

Furthermore, let $\mid\!\sim^c$ be the (nonmonotonic) inference relation based on considering all c-representations:

$$\mathcal{R} \mid\!\sim^c (B|A) \quad \text{iff} \quad \kappa_c \models (B|A) \text{ for all c-representations of } \mathcal{R}, \tag{10}$$

where $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L}), (B|A) \in (\mathcal{L} \mid \mathcal{L})$. Then Lemma 7 yields

Corollary 8 *Let $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L}), (B|A) \in (\mathcal{L} \mid \mathcal{L})$, let d be an atomic proposition not appearing in any of the conditionals in $\mathcal{R}$. Then $\mathcal{R} \mid\!\sim^c (B|A)$ iff $\mathcal{R} \mid\!\sim^c (B|Ad)$.*

In particular, the generalized system-Z^* approach – combining (3) with (5) – is thus justified as a theoretically profound method for representing conditionals via OCF's, following the same idea as the probabilistic ME-approach.

6 The principle of conditional preservation in belief revision

The theory of conditional structures also provides a solid foundation to deal with conditionals appropriately when revising epistemic states by (propositional and conditional) beliefs. Minimality of change is a crucial paradigm for belief revision, and a "principle of conditional preservation" is to realize this idea of minimality when conditionals are involved in change. Minimizing absolutely the changes in conditional beliefs, as in [BG93], is an important proposal to this aim, but it does not always lead to intuitive results (see [DP97]). The idea we will develop here rather aims at *preserving the conditional structure of knowledge* within an epistemic state which, in this section, we assume to be represented by an OCF κ. Similar results can be stated for probability functions, too.

We just explained what it means for an OCF, κ, to follow the structure imposed by a set of conditionals $\mathcal{R}$ on the possible worlds by introducing the notion of conditional indifference. Pursuing this approach further in the framework of belief revision, a revision of κ by simultaneously incorporating the conditionals in $\mathcal{R}$, $\kappa^* = \kappa * \mathcal{R}$, can be said to preserve the conditional structure of κ with respect to $\mathcal{R}$ if the *relative change function* $\kappa^* - \kappa$ is indifferent with respect to $\mathcal{R}$.

Definition 9 Let κ be an OCF, and let $\mathcal{R}$ be a (finite) set of conditionals. Let $\kappa^* = \kappa * \mathcal{R}$ denote the result of revising κ by $\mathcal{R}$. κ^* is *indifferent with respect to $\mathcal{R}$ and κ* iff $(\kappa^* - \kappa)(\widehat{\omega}_1) = (\kappa^* - \kappa)(\widehat{\omega}_2)$ whenever $\sigma_{\mathcal{R}}(\widehat{\omega}_1) = \sigma_{\mathcal{R}}(\widehat{\omega}_2)$ and $\widehat{\omega}_1 \equiv_{\top} \widehat{\omega}_2$, for $\widehat{\omega}_1, \widehat{\omega}_2 \in \widehat{\Omega}$.

The principle of conditional preservation is now realized as an indifference property:

Definition 10 Let κ be an OCF, and let $\mathcal{R}$ be a finite set of conditionals. A revision $\kappa^* = \kappa * \mathcal{R}$ satisfies the *principle of conditional preservation* iff κ^* is indifferent with respect to $\mathcal{R}$ and κ.

So $\kappa * \mathcal{R}$ satisfies the principle of conditional preservation if any change in plausibility is clearly and unambigously induced by $\mathcal{R}$. The next theorem characterizes revisions of ordinal conditional functions that satisfy the principle of conditional preservation. The theorem is obvious by observing Theorem 4.

Theorem 11 *Let κ, κ^* be OCF's, and let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a (finite) set of conditionals in $(\mathcal{L} \mid \mathcal{L})$. A revision $\kappa^* = \kappa * \mathcal{R}$ satisfies the principle of conditional preservation iff there are numbers $\kappa_0, \kappa_i^+, \kappa_i^- \in \mathbb{Q}, 1 \leqslant i \leqslant n$, such that for all $\omega \in \Omega$,*

$$\kappa^*(\omega) = \kappa(\omega) + \kappa_0 + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \kappa_i^+ + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \kappa_i^- \tag{11}$$

An analogous theorem for probabilistic revisions shows that also the minimum cross-entropy method satisfies the principle of conditional preservation (cf. [KI98, KI01]).

Comparing Theorems 4 and 11 with one another, we see that an OCF κ is indifferent with respect to a finite set of conditionals $\mathcal{R}$ iff it can be taken as a revision $\kappa_0 * \mathcal{R}$ satisfying the principle of conditional preservation, where $\kappa_0(\omega) = 0$ for all $\omega \in \Omega$ is the *uniform ordinal conditional function*. So knowledge representation and belief revision can be dealt with in a common framework, based on conditionals as crucial pieces of knowledge. At least for this reason, the topic of revising epistemic states by conditional beliefs should not be looked upon as a superfluous artifact, but should be appreciated as providing a most general and enhanced framework for representing and revising knowledge. Due to the fact, that propositions, A, can be considered as degenerated conditionals, $(A|T)$, the theory may also be applied to propositional belief revision. In fact, the principle of conditional preservation, as defined here, can be shown to cover the postulates for iterative belief revision of [DP97] (cf. [KI01]).

7 Conditional structures and related work

We announced our approach to conditionals as a substantially new one, capturing the nature of conditionals peculiarly well. The results and examples presented above illustrate this, but what is actually new and different, compared with other, established conditional theories?

First formal work on conditionals dates back to the days of Boole. *Conditional events* were investigated particularly intensively in the nineties of the past century, when their crucial meaning for knowledge representation in AI had become apparent (for a historical review, cf. [NW94]; for a survey of conditional logics, cf. [Nut80]). Lewis [Lew76] made clear that conditionals cannot be handled as boolean entities, so new ways of representing and dealing with conditionals had to be explored. A very popular approach, e.g., is the following one (for an overview of important approaches to conditionals, see [DGC94]):

It is generally agreed that two conditionals $(B|A), (D|C)$ are equivalent, $(B|A) \equiv (D|C)$, iff $A \equiv C$ and $AB \equiv CD$. So it seems adequate to define a conditional $(B|A)$ by a set of boolean propositions X all fulfilling $XB = AB$: $(B|A) := \{X \in \mathcal{L} \mid XB = AB\} = \{X \in \mathcal{L} \mid AB \leqslant X \leqslant \overline{B} \vee A\}$, where $\leqslant$ is the usual entailment relation between boolean propositions (see [GN88]). That is, a conditional is identified with an interval of boolean propositions. This gives rise to a partial ordering $\leqslant$ between conditionals which is also frequently understood as an entailment relation between conditionals (cf. [DP91, BDP97]):

$$(D|C) \leqslant (B|A) \quad \text{iff} \quad CD \leqslant AB \text{ and } A\overline{B} \leqslant C\overline{D} \tag{12}$$

Walker lists in [Wal94] several possible representations of conditional events and investigates different ways of defining logical connectives between conditionals. The representation that seems most adequate to compare our approach with others is the one that identifies a conditional $(B|A)$ with an ordered pair of its crucial components $AB, A\overline{B}$:

$$(B|A) \sim (AB, A\overline{B}), \tag{13}$$

so that $(\mathcal{L} \mid \mathcal{L}) \sim \{(X, Y) \mid X, Y \in \mathcal{L}, XY \equiv \perp\}$. Moreover, the appropriateness of the three-valued semantics of conditionals (6) which we base our theory upon is accepted by most other researchers in the field of conditionals. So the essential ingredients to our theory are established and nearly undebatable. The difference lies in the way we make use of them. Our focusing not on logical properties but on the effects conditionals exert on possible worlds suggests another way of relating conditionals to one another. First of all, both positive and negative effects are considered to be equally important, no priority is given to the confirming component AB (as is done in most approaches). So, a conditional $(D|C)$ is called a *subconditional* of $(B|A)$, iff its effect on worlds is in line with $(B|A)$, but possibly $(D|C)$ applies to fewer worlds:

$$(D|C) \sqsubseteq (B|A) \quad \text{iff for all } \omega \in \Omega, \begin{cases} \sigma_{(D|C)}(\omega) = 1 \Rightarrow \sigma_{(B|A)}(\omega) = 1 \\ \sigma_{(D|C)}(\omega) = 0 \Rightarrow \sigma_{(B|A)}(\omega) = 0 \end{cases} \tag{14}$$

which amounts to saying that $(D|C) \sqsubseteq (B|A)$ iff $CD \leqslant AB$ and $C\overline{D} \leqslant A\overline{B}$. Note the essential difference with (12), but we still have $(B|A) \equiv (D|C)$ iff $(B|A) \sqsubseteq (D|C)$ and $(D|C) \sqsubseteq (B|A)$, due to $(B|A) \equiv (D|C)$ iff $AB \equiv CD$ and $A\overline{B} \equiv C\overline{D}$.

Further, we can define connectives $\sqcap, \sqcup$ by setting $(B|A) \sqcap (D|C) := \inf\{(B|A), (D|C)\}$ and $(B|A) \sqcup (D|C) := \sup\{(B|A), (D|C)\}$, where the supremum is defined only if $ABC\overline{D} \equiv A\overline{B}CD \equiv \bot$. It is easy to check that

$$\begin{aligned} (B|A) \sqcap (D|C) &= (BD \mid AC(BD \vee \overline{B}\,\overline{D})) &(15)\\ (B|A) \sqcup (D|C) &= (AB \vee CD \mid A \vee C) &(16) \end{aligned}$$

Although $(B|A) \sqcup (D|C)$ in its original sense as a supremum is defined only for $ABC\overline{D} \equiv A\overline{B}CD \equiv \bot$, (16) may be used for defining a connective for arbitrary $(B|A), (D|C)$. If the condition $ABC\overline{D} \equiv A\overline{B}CD \equiv \bot$ is not fulfilled, however, then $(B|A) \sqcup (D|C)$ is not the supremum of $(B|A)$ and $(D|C)$ (as can easily be seen by considering $(B|A)$ and $(\overline{B}|A)$). $\sqcap$ and $\sqcup$ are associative and distributive operations on $(\mathcal{L} \mid \mathcal{L})$, if all required suprema exist (cf. [KI01]). Their connections to the boolean operations $\wedge$ and $\vee$ can be seen more directly by using the representation (13)

$$\begin{aligned} (X,Y) \sqcap (W,Z) &= (X \wedge W, Y \wedge Z), \\ (X,Y) \sqcup (W,Z) &= (X \vee W, Y\overline{W} \vee Z\overline{X}), \\ &= (X \vee W, Y \vee Z) \quad \text{if } XZ \equiv YW \equiv \bot, \end{aligned}$$

that is, the $\wedge$- and $\vee$-operator, respectively, are applied to the confirming and the refuting components separately, respectively. Both $\sqcap$ and $\sqcup$ are truth-functional, with $((B|A) \sqcap (D|C))(\omega) = 1$ iff $(B|A)(\omega) = 1$ and $(D|C)(\omega) = 1$, and $((B|A) \sqcap (D|C))(\omega) = 0$ iff $(B|A)(\omega) = 0$ and $(D|C)(\omega) = 0$, whereas $((B|A) \sqcup (D|C))(\omega) = 1$ iff $(B|A)(\omega) = 1$ or $(D|C)(\omega) = 1$, and $((B|A) \sqcup (D|C))(\omega) = 0$ iff $(B|A)(\omega) = 0$ and $(D|C)(\omega) \neq 1$, or $(D|C)(\omega) = 0$ and $(B|A)(\omega) \neq 1$. Therefore, we obtain the following truth tables for combining conditionals by $\sqcap$ and $\sqcup$, respectively (note that for the supremum, $0 \sqcup 1$ and $1 \sqcup 0$ do not exist):

$\sqcap$	0	1	u
0	0	u	u
1	u	1	u
u	u	u	u

$\sqcup$	0	1	u
0	0	1/*undef.*	0
1	1/*undef.*	1	1
u	0	1	u

These truth-tables reveal some incompatibilities with boolean logic: We have $0 \sqcap 1 = 1 \sqcap 0 = u$, and $0 \sqcup 1$ and $1 \sqcup 0$ are not defined, if $\sqcup$ is taken as a supremum. If one conditional is confirmed and the other refuted, nothing is said how they combine. So no boolean rules are forced upon the interactions of conditionals. Therefore, the difference between our theory and most other approaches is that we do not expect the connectives $\sqcup$ and $\sqcap$ to observe boolean laws; in particular, though $\sqcup$ and $\sqcap$ are based upon $\vee$ and $\wedge$, respectively, they cannot be taken as extensions of those boolean operators (note, however, that the generalized version of $\sqcup$ does extend $\vee$ and is identical to Calabrese's conditional $\vee$-operator; see [Cal91]). Moreover, $\sqcap$ and $\sqcup$ are not linked by DeMorgan's laws; rather, the negation $(B|A)' := (\overline{B}|A)$ acts as a homomorphism with respect to both connectives: $((B|A) \sqcap (D|C))' = (B|A)' \sqcap (D|C)'$ and $((B|A) \sqcup (D|C))' = (B|A)' \sqcup (D|C)'$.

This breaking with boolean conditional traditions also has consequences for considering facts (i.e., boolean propositions) as conditionals: Usually, a proposition A is identified with the degenerate (two-valued) conditional $(A|\top)$. This appears to be justified by considering (6). Note, however, that actually, stating A means "*A is true*", whereas $(A|\top)$ means "*A is plausible*". This distinction is respected in our framework, in that $A \wedge B$ does not correspond to $(A|\top) \sqcap (B|\top)$

$(= (AB|AB \vee \overline{AB}) \neq (AB|\top)$; the conjunction of two plausible propositions need not be plausible), nor does $(A|\top) \sqsubseteq (B|\top)$ mean $A \subseteq B$; rather, we have $(A|\top) \sqsubseteq (B|\top)$ if and only if $A \equiv B$.

In summary, it can be said that, although the roots of our approach are firmly established in the foundations of conditional theories, the thorough application of the new concept of conditional structures brings forth quite a new theory of conditionals which is non-boolean to a very large extent. We think, however, that the loss of boolean conveniences is more than compensated by gaining a rich methodological framework for handling the dynamic power of conditionals more adequately, in particular in the areas of inductive knowledge representation and belief revision. To exemplify what it means that c-representations impose conditional structures upon worlds, we state the following proposition which relates $\sqsubseteq$ to $\mid\!\sim^c$-inference (cf. (10)) and extends Corollary 8:

Proposition 12 *$(B|A) \mid\!\sim^c (D|C)$ iff $(D|C) \sqsubseteq (B|A)$, that is, if $(D|C)$ is a subconditional of $(B|A)$.*

Proof. Any (OCF-)c-representation of $(B|A)$ has the form

$$\kappa(\omega) = \begin{cases} \kappa_0 & \text{if} \quad \omega \models \overline{A}, \\ \kappa_0 + \kappa^+ & \text{if} \quad \omega \models AB, \\ \kappa_0 - \kappa^- & \text{if} \quad \omega \models A\overline{B} \end{cases}$$

with rational numbers $\kappa_0, \kappa^+, \kappa^-$ such that $\kappa^- - \kappa^+ > 0$ (cf. (11)). If $(D|C) \sqsubseteq (B|A)$, then also $\kappa \models (D|C)$, and conversely, if $(D|C)$ is not a subconditional of $(B|A)$, then $\kappa_0, \kappa^+, \kappa^-$ can be chosen appropriately so that $\kappa \not\models (D|C)$. □

So, in the simple case where only one conditional is considered, inferring by observing conditional structures means accepting all subconditionals, too. If the inference is to be based on a set of conditionals, interactions between these conditionals have to be taken into account, and c-entailment should not be expected to be this simple. A thorough investigation of $\mid\!\sim^c$ is a topic of our current research; results will soon be published in a further paper.

8 Outlook

In this paper, the algebraic notion of *conditional structures* of worlds plays a crucial part for developing a new approach to handle conditionals more adequately in inductive knowledge representation and belief revision. Its realization via group theory does not only provide an elegant methodological framework for dealing with conditional information. In [KI00b, KI01], an approach is presented to solve the "inverse representation problem" – which conditionals are most appropriate to represent an epistemic state? – by making extensive use of group theoretical means to elaborate conditional structures from, e.g., probability distributions.

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