Proceedings Sixth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, ECSQARU-2001

Handling conditionals adequately in uncertain reasoning

Gabriele Kern-Isberner

FernUniversität Hagen, Department of Computer Science,
P.O. Box 940, D-58084 Hagen, Germany
gabriele.kern-isberner@fernuni-hagen.de

Abstract. Conditionals ("*if-then-rules*") are most important objects in knowledge representation, commonsense reasoning and belief revision. Due to their non-classical nature, however, they are not easily dealt with. This paper presents a new approach to conditionals, which is apt to capture their dynamic power peculiarly well. We show how this approach can be applied to represent conditional knowledge inductively. In particular, we generalize system-Z^* as an appropriate counterpart to maximum entropy-representations in a semi-quantitative setting.

1 Introduction

Relationships amongst propositions are crucial pieces of knowledge. Usually, they express plausible connections, bring isolated facts together and help us obtain a coherent image of the world. Such relationships may be represented in a most general form by *if-then-conditionals*. Conditionals are omnipresent, in everyday life as well as in scientific environments. We make use of conditional knowledge when we avoid puddles on sidewalks (being aware of "If you step into a puddle, then your feet might get wet"), and when we expect high wheat prices from observing cold and rainy weather in spring and summer (due to "If the growing weather is poor then there will be an increase in price of wheat"). Conditionals represent generic knowledge, acquired inductively from experience or learned from books. They tie a flexible and highly interrelated network of connections along which reasoning is possible and which can be applied to different situations. Moreover, as plausible yet defeasible conclusions, conditionals are intimately related to *nonmonotonic reasoning*, or, in general, to *uncertain reasoning*. In belief revision, they take the role of *revision policies*, guiding changes of beliefs when new (propositional) evidence becomes apparent. So, in contrast to factual knowledge which is mostly static, conditionals bear a clearly dynamic flavor.

The key to get conditionals right is to accept their non-classical nature – conditionals are not simply "true" or "false". In a particular situation, a conditional is *applicable* (you actually step into a puddle) or not (you simply walk around), it can be found *confirmed* (you step into a puddle and indeed, your feet get wet) or *violated* (you step into a puddle, but your feet remain dry because you are wearing rain boots). So the central problem in dealing with conditional knowledge is to handle adequately, on the one hand, inactive (or neutral, respectively) behavior, and, on the other hand, active as well as polarizing behavior.

The main object of this paper is

- to sketch a new theory for conditionals which captures their three-valued, dynamic nature peculiarly well,
- to show how it is realized in a semi-quantitative as well as in a probabilistic framework,
- to apply this theory to the problem of representing conditional knowledge inductively in both frameworks.

We will focus on the process of establishing conditional beliefs, considering conditionals as agents shifting possible worlds in order to establish relationships and beliefs. We will represent the effects (the learning of) conditionals have on worlds by *conditional structures*. Handling conditionals adequately then means to choose representations which are balanced with respect to the structures of the conditionals under consideration. In this way, highly complex interactions between different conditionals can be taken into account and maintained.

A well-known probabilistic method that follows this idea is the *principle of maximum entropy (ME-principle)* [Par94]. In a semi-quantitative setting, system-Z^* [GMP93] also realizes this approach, though it seems to be of restricted applicability. In this paper, we will generalize system-Z^* to make it applicable for any consistent set of conditionals. The generalized system-Z^*-approach will be shown to be theoretically justified, and will prove to handle even problematic examples adequately. Although this generalization of system-Z^* is important in itself, it should be taken as only one example of how the conditional approach presented in this paper can be used. Due to separating strictly between structural and numerical aspects of conditionals, the new conditional theory can be applied in any (semi-)quantitative framework that allows the representation of conditional beliefs. Therefore, it has also important consequences for possibility theory, which has seen a lot of important work on conditionals in the last decade (see e.g. [BDP97,BSS00]). This connection to possibility theory is elaborated in [KI01b].

The following section summarizes fundamental facts about conditionals, ordinal conditional functions and probability distributions. Section 3 briefly sketches the ME-principle, as well as system-Z and system-Z^*. Section 4 presents the generalized system-Z^* approach. The new dynamic approach to conditionals is presented in Section 5, and is linked to (semi-)quantitative representation of knowledge in Section 6. An outlook on further results concludes this paper in Section 7.

2 Conditionals, plausibility and probability

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{V} = \{a, b, c \ldots\}$. Let Ω be the complete set of interpretations of $\mathcal{L}$, where each $\omega \in \Omega$ is taken to be a *possible world* for $\mathcal{L}$. To simplify notation, we will write $\overline{A}$ instead of $\neg A$, and AB instead of $A \wedge B$, for formulas $A, B \in \mathcal{L}$. Conditionals $(B|A)$ represent statements of the form "*If A then B*", expressing a relationship between two (propositional) formulas A, the *antecedent* or *premise*, and B, the *consequent*. $(\mathcal{L} \mid \mathcal{L})$ denotes the set of all conditionals $(B|A)$ with $A, B \in \mathcal{L}$. A world $\omega \in \Omega$ is said to *verify* a conditional $(B|A)$, if $\omega \models AB$; it *falsifies* $(B|A)$, if $\omega \models A\overline{B}$; if $\omega \models \overline{A}$, then $(B|A)$ is not applicable to ω.

Epistemic states as representations of cognitive states of intelligent agents provide an adequate framework for conditionals. Two widely used types of epistemic states are *probability distributions* and *ordinal conditional functions, OCF's*, which are based on the notion of plausibility. In short, OCF's are functions $\kappa : \Omega \rightarrow \mathbb{N} \cup \{0, \infty\}$ from the set of worlds to the natural numbers, extended by 0 and ∞. They specify non-negative integers as degrees of plausibility – or, more precisely, as degrees of *disbelief* – for worlds. For propositional formulas $A, B \in \mathcal{L}$, we set $\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$. A proposition A is believed iff $\kappa(\overline{A}) > 0$, which is denoted by $\kappa \models A$. This may also be specified by degrees of plausibility (or of disbelief, respectively) by saying that $\kappa \models A[n]$ iff $\kappa(\overline{A}) > n$ $(n \in \mathbb{N} \cup \{0\})$. A conditional $(B|A) \in (\mathcal{L} \mid \mathcal{L})$ may be assigned a degree of plausibility via $\kappa(B|A) = \kappa(AB) - \kappa(A)$. κ *satisfies* $(B|A)$, $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$, i.e. iff AB is more plausible than $A\overline{B}$. We can also specify a numerical degree of plausibility of a conditional by defining $\kappa \models (B|A)\,[n]$ iff $\kappa(AB) + n < \kappa(A\overline{B})$ $(n \in \mathbb{N} \cup \{0\})$. OCF's are the qualitative counterpart of probability distributions (cf. [GMP93,GP96]). For a probability distribution P, we have $P(B|A) = \frac{P(AB)}{P(A)}$ for $P(A) > 0$, and $P \models (B|A)\,[x]$ iff $P(B|A) = x$ $(x \in [0, 1])$.

We will consider mostly measure-free conditionals, focusing on structural aspects, but also allow quantifications. If $\mathcal{R}^* = \{(B_1|A_1)\,[x_1], \ldots, (B_n|A_n)\,[x_n]\}$ is a set of (appropriately) quantified conditionals, then $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ denotes the set of unquantified conditionals, and vice versa. Throughout this paper, we will assume all OCF's to be finite, and all probability distributions to be positive. All results to be presented also hold in the general case, but then need some technical modifications (see [KI01a]).

3 Maximum entropy and System-$Z^{(*)}$

In this section, we review briefly well-known model-based approaches to represent conditional knowledge inductively.

For a consistent set $\mathcal{R}^* = \{(B_1|A_1)\,[x_1], \ldots,\ (B_n|A_n)\,[x_n]\}$ of probabilistic conditionals, the *ME-representation of* $\mathcal{R}^*$, $\mathit{ME}(\mathcal{R}^*)$, is the unique distribution P that maximizes the entropy $H(P) = -\sum_\omega P(\omega) \log P(\omega)$ subject to $P \models \mathcal{R}^*$ (see e.g. [Par94,KI98]). $\mathit{ME}(\mathcal{R}^*)$ can be written as

$$\mathit{ME}(\mathcal{R}^*)(\omega) = \alpha_0 \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \alpha_i^{1-x_i} \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \alpha_i^{-x_i}, \tag{1}$$

with the α_i's being appropriately chosen so as to satisfy all conditionals in $\mathcal{R}^*$.

In a semi-quantitative framework, a well-known method to represent a (finite) set $\mathcal{R} = \{r_i = (B_i|A_i) \mid 1 \leqslant i \leqslant n\}$ of conditionals by an OCF is to apply the *system-Z* of Goldszmidt and Pearl [GMP93,GP96]. A conditional $(B|A)$ is said to be *tolerated* by $\mathcal{R}$ iff there is a world ω such that ω verifies $(B|A)$ and ω does not falsify any of the conditionals in $\mathcal{R}$. $\mathcal{R}$ is consistent iff there is an ordered partition $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_k$ of $\mathcal{R}$ such that each conditional in $\mathcal{R}_m$ is tolerated by $\bigcup_{j=m}^{k} \mathcal{R}_j$, $0 \leqslant m \leqslant k$. The system-Z ranking function, κ^z, representing $\mathcal{R}$ is given by

$$\kappa^z(\omega) = \begin{cases} 0, \text{ if } \omega \text{ does not falsify any } r_i, \\ 1 + \max\limits_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} Z(r_i), \text{otherwise} \end{cases}$$

where $Z(r_i) = j$ iff $r_i \in \mathcal{R}_j$. κ^z assigns to each world ω the lowest possible rank admissible with respect to the constraints in $\mathcal{R}$.

A more sophisticated representation is obtained by combining the system-Z approach with the probabilistic ME-principle, yielding system-Z^* [GMP93]. The corresponding Z^*-rankings of the conditionals in $\mathcal{R}$ have to satisfy (see [GMP93])

$$Z^*(r_i) + \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) = 1 + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j), \tag{2}$$

and κ^* is then calculated by

$$\kappa^*(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} Z^*(r_i) \tag{3}$$

Z^*-entailment $\mid\!\sim^*$ is defined by $\mathcal{R} \mid\!\sim^* (B|A)$ iff $\kappa^*(AB) < \kappa^*(A\overline{B})$. System-$Z^*$ handles conditional knowledge quite appropriately (cf. [GMP93]). In particular, Z^*-entailment satisfies the crucial property of *irrelevance*: If d is an atomic proposition not appearing in any of the conditionals in $\mathcal{R}$, then

$$\mathcal{R} \mid\!\sim^* (B|A) \quad \text{iff} \quad \mathcal{R} \mid\!\sim^* (B|Ad) \tag{4}$$

In [GMP93], a procedure is given to calculate Z^*-rankings for so-called *minimal-core sets*, i.e. sets $\mathcal{R}$ of conditionals such that for each conditional $r \in \mathcal{R}$, there is a world $\omega \in \Omega$ that falsifies r and no other conditional in $\mathcal{R}$. Bourne and Parsons [BP99] presented an algorithm that computes the Z^*-ranking, whenever (2) possesses a unique solution. This algorithm is also able to take variable strength of conditionals into account, that is, to calculate solutions to

$$Z^*(r_i) + \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) = n_i + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j), \tag{5}$$

so that $\kappa^* \models (B|A)[n_i]$. There are, however, sets of conditionals that do not specify unique solutions to (2), as the following example shows.

Example 1. For the set $\mathcal{R} = \{r_1 : (b|a), r_2 : (c|a), r_3 : (c|ab)\}$, (2) admits multiple solutions: $(Z^*(r_1), Z^*(r_2), Z^*(r_3))$ may be any one of, for instance, $(1, 0, 1), (1, 1, 0)$, or even $(2, -1, 2)$ (cf. [BP99]).

Bourne and Parsons argue that some examples may be too complex to be dealt with by a "flat" system-Z^* approach and advise to use variable strength conditionals to enforce unique solvability. This seems a bit strange – Example 1 does not look very complex, and the probabilistic ME-method yields a unique solution even if all three conditionals are assigned the same probability.

The problem with Goldszmidt & Pearl's and Bourne & Parsons' work is that, in order to use as much of the inferential power of the ME-method as possible, they cling too closely to *probabilistic* ME-techniques. The idea we will present and pursue here is to reveal the pattern that makes ME-methods a most adequate tool for handling conditional information and to transfer this pattern into the framework of semi-quantitative knowledge representation. This will bring forth a more general approach of which system-Z^* turns out to be a special instance.

4 Generalizing system-Z^*

System-Z^* suggests a straightforward generalization for representing a set $\mathcal{R} = \{r_i = (B_i|A_i) \mid 1 \leqslant i \leqslant n\}$ of conditionals by an OCF: Use approach (3) and determine the $Z^*(r_i)$ so that κ^* satisfies all of the conditionals r_i, i.e. such that

$$Z^*(r_i) > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) \tag{6}$$

for $1 \leqslant i \leqslant n$. That means that condition (2) is weakened to be satisfied as an inequality constraint. Variable strengths of OCF-conditionals can be taken easily into account by adding them on the right-hand side of (6), thus generalizing (5). This generalization, however, seems to be quite ad hoc, leaving the ME-track and thus without theoretical justification.

Quite to the contrary – in the sequel, we will show that actually, this generalized approach emerged from a formal *principle of conditional indifference* that was formalized in [KI98] as one of four axioms apt to characterize the ME-techniques. This principle is based on observing *conditional structures* and provides a powerful methodology, not only to represent conditional knowledge appropriately, but even to guide the revision of epistemic states by conditional beliefs. In the next section, we will briefly develop the necessary theoretical background, which is purely algebraic and therefore can be applied in a probabilistic as well as in a semi-quantitative setting. Just as for the ME-methods, it will provide an intelligible and appealing scheme for the inductive representation of conditionals.

5 A dynamic approach to conditionals

By observing the attitude of worlds with respect to it, each conditional $(B|A)$ can be considered as a generalized (three-valued) indicator function on worlds:

$$(B|A)(\omega) = \begin{cases} 1 & : \quad \omega \models AB \\ 0 & : \quad \omega \models A\overline{B} \\ u & : \quad \omega \models \overline{A} \end{cases} \tag{7}$$

where u stands for *unknown* or *indeterminate* (see, e.g., [Cal91]). Intuitively, representing or incorporating a conditional as a plausible conclusion in an epistemic state means to make – at least some – worlds verifying the conditional more plausible than the worlds falsifying it. In this sense, conditionals to be learned have effects on possible worlds, shifting them appropriately to establish the intended relationship. Which worlds will actually be shifted depends on the chosen inductive representation procedure – for the conditional $(B|A)$, all worlds in either of the partitioning sets $AB, A\overline{B}$ and $\overline{A}$ are indistinguishable.

When we consider (finite) sets of conditionals $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$, we have to modify the representation (7) appropriately to identify the effect of each conditional in $\mathcal{R}$ on worlds in Ω. To this end, we replace the numbers 0 and 1 by abstract symbols, $\mathbf{a}_i^+, \mathbf{a}_i^-$, that we associate to each conditional $(B_i|A_i)$ in $\mathcal{R}$. Moreover, we will make use of a group structure to represent the joint impact of conditionals on worlds.

So let $\mathcal{F}_\mathcal{R} = \langle \mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^- \rangle$ be the free abelian group with generators $\mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^-$, i.e. $\mathcal{F}_\mathcal{R}$ consists of all elements of the form $(\mathbf{a}_1^+)^{r_1}(\mathbf{a}_1^-)^{s_1} \ldots (\mathbf{a}_n^+)^{r_n}(\mathbf{a}_n^-)^{s_n}$ with integers $r_i, s_i \in \mathbb{Z}$ (the ring of integers). Each element of $\mathcal{F}_\mathcal{R}$ can be identified by its exponents, so that $\mathcal{F}_\mathcal{R}$ is isomorphic to $\mathbb{Z}^{2n}$. The commutativity of $\mathcal{F}_\mathcal{R}$ corresponds to the fact that the conditionals in $\mathcal{R}$ shall be effective simultaneously, without assuming any order of application. Note that, although we will speak of *multiplication* and *products* in $\mathcal{F}_\mathcal{R}$, the generators of $\mathcal{F}_\mathcal{R}$ are merely juxtaposed, like words.

For each $i, 1 \leqslant i \leqslant n$, we define a function $\sigma_i = \sigma_{(B_i|A_i)} : \Omega \to \mathcal{F}_\mathcal{R}$ by setting

$$\sigma_i(\omega) := \begin{cases} \mathbf{a}_i^+ & \text{if} \quad (B_i|A_i)(\omega) = 1 \\ \mathbf{a}_i^- & \text{if} \quad (B_i|A_i)(\omega) = 0 \\ 1 & \text{if} \quad (B_i|A_i)(\omega) = u \end{cases}$$

$\sigma_i(\omega)$ represents the manner in which the conditional $(B_i|A_i)$ applies to the possible world ω. The neutral element 1 of $\mathcal{F}_\mathcal{R}$ represents the non-applicability of $(B_i|A_i)$ in case that the antecedent A_i is not satisfied, so the neutral group element corresponds to a neutral attitude with respect to the conditional. The function $\sigma_\mathcal{R} : \Omega \to \mathcal{F}_\mathcal{R}$,

$$\sigma_\mathcal{R}(\omega) := \prod_{1 \leqslant i \leqslant n} \sigma_i(\omega) = \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \mathbf{a}_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \mathbf{a}_i^- \tag{8}$$

describes the all-over effect of $\mathcal{R}$ on ω. $\sigma_\mathcal{R}(\omega)$ is called the *conditional structure of ω with respect to $\mathcal{R}$*. We will illustrate this notion of conditional structures in the following example which extends the well-known *Nixon diamond*:

Example 2. Let $\mathcal{R}$ consist of the following conditionals:

$r_1 : (p|q)$ *Quakers are pacifists.*

$r_2 : (\overline{p}|r)$ *Republicans are not pacifists.*

$r_3 : (a|q)$ *Quakers are Americans.*

$r_4 : (b|a)$ *Americans like baseball.*

$r_5 : (\overline{b}|q)$ *Quakers do not like baseball.*

The conditional structure of a possible world, say $pqrab$, is calculated in the following way: $pqrab$ verifies the first conditional r_1 ($pqrab \models pq$), so we have $\sigma_1(pqrab) = \mathbf{a}_1^+$. $pqrab$, however, falsifies the second conditional $r_2 = (\overline{p}|r)$, thus

$\sigma_2(pqrab) = \mathbf{a}_2^-$. In the same way, $\sigma_3(pqrab) = \mathbf{a}_3^+$, $\sigma_4(pqrab) = \mathbf{a}_4^+$, $\sigma_5(pqrab) = \mathbf{a}_5^-$. We obtain $\sigma_{\mathcal{R}}(pqrab) = \mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$. In the table below, we list the conditional structures of all possible worlds with respect to $\mathcal{R}$:

ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$
$pqrab$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$p\overline{q}rab$	$\mathbf{a}_2^-\mathbf{a}_4^+$	$\overline{p}qrab$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}\,\overline{q}rab$	$\mathbf{a}_2^+\mathbf{a}_4^+$
$pqra\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$p\overline{q}ra\overline{b}$	$\mathbf{a}_2^-\mathbf{a}_4^-$	$\overline{p}qra\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}ra\overline{b}$	$\mathbf{a}_2^+\mathbf{a}_4^-$
$pqr\overline{a}b$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^-\mathbf{a}_5^-$	$p\overline{q}r\overline{a}b$	$\mathbf{a}_2^-$	$\overline{p}qr\overline{a}b$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_5^-$	$\overline{p}\,\overline{q}r\overline{a}b$	$\mathbf{a}_2^+$
$pqr\overline{a}\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^-\mathbf{a}_5^+$	$p\overline{q}r\overline{a}\overline{b}$	$\mathbf{a}_2^-$	$\overline{p}qr\overline{a}\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}r\overline{a}\overline{b}$	$\mathbf{a}_2^+$
$pq\overline{r}ab$	$\mathbf{a}_1^+\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$p\overline{q}\,\overline{r}ab$	$\mathbf{a}_4^+$	$\overline{p}q\overline{r}ab$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}\,\overline{q}\,\overline{r}ab$	$\mathbf{a}_4^+$
$pq\overline{r}a\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$p\overline{q}\,\overline{r}a\overline{b}$	$\mathbf{a}_4^-$	$\overline{p}q\overline{r}a\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}\,\overline{r}a\overline{b}$	$\mathbf{a}_4^-$
$pq\overline{r}\,\overline{a}b$	$\mathbf{a}_1^+\mathbf{a}_3^-\mathbf{a}_5^-$	$p\overline{q}\,\overline{r}\,\overline{a}b$	1	$\overline{p}q\overline{r}\,\overline{a}b$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_5^-$	$\overline{p}\,\overline{q}\,\overline{r}\,\overline{a}b$	1
$pq\overline{r}\,\overline{a}\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_3^-\mathbf{a}_5^+$	$p\overline{q}\,\overline{r}\,\overline{a}\,\overline{b}$	1	$\overline{p}q\overline{r}\,\overline{a}\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}\,\overline{r}\,\overline{a}\,\overline{b}$	1

Using this table, it is easy to see that $\mathcal{R}$ is consistent, with partition $\mathcal{R}_0 = \{r_2, r_4\}$ and $\mathcal{R}_1 = \{r_1, r_3, r_5\}$. $\mathcal{R}$, however, is not a minimal-core set, because each world falsifying r_1 (i.e. worlds with label $\mathbf{a}_1^-$ in the table) also falsifies at least one of r_3, r_4, r_5. In fact, Goldszmidt & Pearl's system-Z^* approach fails for $\mathcal{R}$. Bourne & Parsons' algorithm, however, computes $Z^*(r_1) = Z^*(r_2) = Z^*(r_4) = 1$, $Z^*(r_3) = Z^*(r_5) = 2$.

$\sigma_{\mathcal{R}}$ labels each world appropriately and makes conditional effects on worlds comparable and computable. For instance, from the table of Example 2 we see that $p\overline{q}\,\overline{r}ab$ and $\overline{p}\,\overline{q}\,\overline{r}ab$ have the same conditional structure (namely $\mathbf{a}_4^+$), and that forming the quotient $\dfrac{\sigma_{\mathcal{R}}(\overline{p}\,\overline{q}rab)}{\sigma_{\mathcal{R}}(\overline{p}\,\overline{q}ra\overline{b})} = \dfrac{\mathbf{a}_2^+\mathbf{a}_4^+}{\mathbf{a}_2^+\mathbf{a}_4^-} = \dfrac{\mathbf{a}_4^+}{\mathbf{a}_4^-}$ isolates the effect of the fourth conditional. Making calculations of conditional structures more convenient and more elegant, we take the worlds $\omega \in \Omega$ as formal generators of the free abelian group $\widehat{\Omega} := \langle \omega \mid \omega \in \Omega \rangle$. $\widehat{\Omega}$ consists of all products $\widehat{\omega} = {\omega_1}^{r_1} \ldots {\omega_m}^{r_m}$, with $\omega_1, \ldots, \omega_m \in \Omega$, and integers $r_1, \ldots r_m$. Introducing such a "multiplication between worlds" is nothing but a technical means to comply with the multiplicative structure the effects of conditionals impose on worlds. As in $\mathcal{F}_{\mathcal{R}}$, multiplication in $\widehat{\Omega}$ actually means juxtaposition. Now $\sigma_{\mathcal{R}}$ may be extended to $\widehat{\Omega}$ in a straightforward manner by setting $\sigma_{\mathcal{R}}({\omega_1}^{r_1} \ldots {\omega_m}^{r_m}) = \sigma_{\mathcal{R}}(\omega_1)^{r_1} \ldots \sigma_{\mathcal{R}}(\omega_m)^{r_m}$, yielding a *homomorphism of groups* $\sigma_{\mathcal{R}} : \widehat{\Omega} \to \mathcal{F}_{\mathcal{R}}$.

Having the same conditional structure defines an equivalence relation $\equiv_{\mathcal{R}}$ on $\widehat{\Omega}$: $\widehat{\omega}_1 \equiv_{\mathcal{R}} \widehat{\omega}_2$ iff $\sigma_{\mathcal{R}}(\widehat{\omega}_1) = \sigma_{\mathcal{R}}(\widehat{\omega}_2)$. Those elements of $\widehat{\Omega}$ that are balanced with respect to the effects of conditionals in $\mathcal{R}$ are contained in the *kernel of* $\sigma_{\mathcal{R}}$, $ker\ \sigma_{\mathcal{R}} = \{\widehat{\omega} \in \widehat{\Omega} \mid \sigma_{\mathcal{R}}(\widehat{\omega}) = 1\}$. $ker\ \sigma_{\mathcal{R}}$ does not depend on the chosen representation of conditional structures by symbols in $\mathcal{F}_{\mathcal{R}}$ and thus, it is an invariant of $\mathcal{R}$ [KI01a]. In a semi-quantitative as well as in a probabilistic environment, implicit normalizing constraints have to be taken into account, namely, $\kappa(\top) = 0$ for OCF's, and $P(\top) = 1$ for probability distributions. This can be achieved by focusing on equivalence with respect to $\sigma_{(\top|\top)}$. Since $\sigma_{(\top|\top)}$ simply counts the worlds occurring in $\widehat{\omega}$, two elements $\widehat{\omega}_1 = \omega_1^{r_1} \ldots \omega_m^{r_m}$, $\widehat{\omega}_2 = \nu_1^{s_1} \ldots \nu_p^{s_p} \in \widehat{\Omega}$ are

$\sigma_{(\top|\top)}$-equivalent, $\widehat{\omega}_1 \equiv_\top \widehat{\omega}_2$, iff $\sum_{1 \leqslant j \leqslant m} r_j = \sum_{1 \leqslant k \leqslant p} s_k$. This means, $\widehat{\omega}_1 \equiv_\top \widehat{\omega}_2$ iff they both are a (cancelled) product of the same number of generating worlds, each generator being counted with its corresponding exponent.

The conditional structures of generalized worlds, $\sigma_{\mathcal{R}}(\widehat{\omega})$, can be considered as generalized, algebraic versions of the *interaction quotients* investigated by Good [Goo63]. $\sigma_{\mathcal{R}}(\widehat{\omega}) = 1$ (i.e. $\widehat{\omega} \in \mathit{ker}\, \sigma_{\mathcal{R}}$) indicates that the interactions between the conditionals in $\mathcal{R}$ are balanced in $\widehat{\omega}$. We will elaborate the consequences of this idea for representing conditionals adequately in the next section.

6 Conditional indifference

To study conditional interactions, we now focus on the behavior of OCF's $\kappa : \Omega \to \mathbb{N} \cup \{0, \infty\}$ and probability functions $P : \Omega \to [0,1]$ with respect to the elements in $\widehat{\Omega}$. We extend each such function to a homomorphism, $\kappa : \widehat{\Omega} \to (\mathbb{Z}, +)$, or $P : \widehat{\Omega} \to \mathbb{R}^+$, respectively, by setting $\kappa({\omega_1}^{r_1} \cdot \ldots \cdot {\omega_m}^{r_m}) = r_1\kappa(\omega_1) + \ldots + r_m\kappa(\omega_m)$ and $P({\omega_1}^{r_1} \cdot \ldots \cdot {\omega_m}^{r_m}) = P(\omega_1)^{r_1} \cdot \ldots \cdot P(\omega_m)^{r_m}$. This allows us to analyze numerical relationships holding between different $\kappa(\omega)$, respectively $P(\omega)$, and to elaborate the conditionals whose structures these functions follow. In the sequel, let V denote an OCF or a probability function, i.e. $V = \kappa$ or $V = P$.

Definition 1. *Assume $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ to be a set of conditionals. V is* indifferent with respect to $\mathcal{R}$ *iff $V(\widehat{\omega}_1) = V(\widehat{\omega}_2)$ whenever $\sigma_{\mathcal{R}}(\widehat{\omega}_1) = \sigma_{\mathcal{R}}(\widehat{\omega}_2)$ for $\widehat{\omega}_1 \equiv_\top \widehat{\omega}_2 \in \widehat{\Omega}$, i.e. iff $\mathit{ker}\, \sigma_{\mathcal{R}} \cap \mathit{ker}\, \sigma_{(\top|\top)} \subseteq \mathit{ker}\, V$.*

Note that we presupposed all OCF's to be finite and all probability functions to be positive in order to make this definition a very concise one. For the general case, cf. [KI01a].

V being indifferent with respect to $\mathcal{R}$ means that it does not distinguish between different elements $\widehat{\omega}_1, \widehat{\omega}_2$ with the same conditional structure with respect to $\mathcal{R}$. Normalizing constraints are taken into account by observing $\equiv_\top$-equivalence. Conversely, any deviation $\kappa(\widehat{\omega}) \neq 0$, respectively $P(\widehat{\omega}) \neq 1$, can be explained by the conditionals in $\mathcal{R}$ acting on $\widehat{\omega}$ in a non-balanced way. Conditional indifference captures interactions of conditionals of arbitrary depth by making use of the group homomorphism induced by V. The next theorem gives a simple criterion to check conditional indifference (for a proof, cf. [KI01a]):

Theorem 1. *Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a set of conditionals.*

A (finite) OCF κ is indifferent with respect to $\mathcal{R}$ iff there are rational numbers $\kappa_0, \kappa_i^+, \kappa_i^- \in \mathbb{Q}$, $1 \leqslant i \leqslant n$, such that for all $\omega \in \Omega$,

$$\kappa(\omega) = \kappa_0 + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \kappa_i^+ + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \kappa_i^- \tag{9}$$

A (positive) probability function P is indifferent with respect to $\mathcal{R}$ iff there are positive real numbers $\alpha_0, \alpha_1^+, \alpha_1^-, \ldots, \alpha_n^+, \alpha_n^- \in \mathbb{R}^+$ such that for all $\omega \in \Omega$,

$$P(\omega) = \alpha_0 \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \alpha_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \alpha_i^- \tag{10}$$

Note that conditional indifference is a measure-free notion. In particular, it does not require the conditionals in $\mathcal{R}$ to be represented by κ or P, respectively. This qualitative or quantitative information is taken into account in a second step:

Definition 2. *Ordinal conditional functions, κ, and probability functions, P, representing a set $\mathcal{R}$ of (quantified) conditionals and being indifferent with respect to it, are called* c-representations.

Theorem 1 provides an intelligible schema to construct c-representations. For instance, for an OCF-c-representation, we simply set up κ according to (9) and choose the κ_i^+, κ_i^- appropriately to ensure that $\kappa \models \mathcal{R}^{(*)}$ holds, i.e. such that

$$\kappa_i^- - \kappa_i^+ > (n_i +) \min_{\omega \models A_i B_i} \Big(\sum_{\substack{j \neq i \\ \omega \models A_j B_j}} \kappa_j^+ + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \kappa_j^- \Big) - \min_{\omega \models A_i \overline{B}_i} \Big(\sum_{\substack{j \neq i \\ \omega \models A_j B_j}} \kappa_j^+ + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \kappa_j^- \Big) \quad (11)$$

where the optional addition of n_i on the right-hand side allows for variable strength conditionals. Furthermore, the normalizing constant κ_0 has to be chosen appropriately to ensure that actually an OCF is obtained. Once suitable numbers $\kappa_0, \kappa_i^+, \kappa_i^-$ are fixed, $\kappa(\omega)$ can easily be computed from the conditional structure $\sigma_{\mathcal{R}}(\omega)$ of ω by replacing each symbol $\mathbf{a}_i^+, \mathbf{a}_i^-$ by its numerical counterpart κ_i^+, κ_i^- (note the similarity of (8) with (9) and (10)).

By setting $\kappa_i^+ := 0$ for each conditional $r_i \in \mathcal{R}$, and determining $\kappa_i^- \geqslant 0$ according to (11), we recover the generalized system-Z^* (see (6)):

$$\kappa_i^- > (n_i +) \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \kappa_j^- - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \kappa_j^- \quad (12)$$

Note that, due to the consistency of $\mathcal{R}$, also $\kappa_0 = 0$ in this case. By choosing $\kappa_i^- \geqslant 0$ minimally, we obtain system-Z^*. As long as we do not have multiple minimal solutions κ_i^-, the algorithm of Bourne & Parsons can be used to calculate these numbers. We also recover the ME-distribution from (10) (see (1), with $\alpha_i^+ = \alpha^{1-x_i}$ and $\alpha_i^- = \alpha^{-x_i}$ for $1 \leqslant i \leqslant n$). So both ME-methods and system-Z^* obey the principle of conditional indifference.

In [BSS00], infinitesimal *lcd-plausibility functions* are introduced which are based (twofold) on the principle of least commitment (*lc*) and on the principle of auto-deduction (*d*) which simply states that the plausibility function should satisfy the conditionals in $\mathcal{R}$. The specific multiplicative structure of lcd-functions is motivated by using Dempster's rule of combination. These lcd-functions can be constructed directly from our approach: Instead of *proto-infinitesimals* introduced in [BSS00] to represent infinitesimals symbolically, we use the formal symbols $\mathbf{a}_1^-, \ldots, \mathbf{a}_n^-$ which can be assigned infinitesimals (or non-infinitesimal positive real numbers, in a non-infinitesimal framework) to give rise to plausibility functions of lcd-type. The specific form of lcd-functions is found to match the conditional structures of worlds – lcd-functions are indifferent with respect to $\mathcal{R}$. For a more detailed comparison in the framework of possibility theory, cf. [KI01b].

Example 3. We continue Example 2, the extended Nixon diamond. (12) yields $\kappa_2^-, \kappa_4^- > 0$. Therefore, we set $\kappa_2^- := \kappa_4^- := 1$. For determining $\kappa_1^-, \kappa_3^-, \kappa_5^-$, the table from Example 2 proves to be helpful. We calculate $\kappa_1^- > \min\{\kappa_3^-, \kappa_4^-, \kappa_5^-\} - \min\{\kappa_3^-, \kappa_4^-, \kappa_5^-\} = 0, \kappa_3^- > \min\{\kappa_4^-, \kappa_5^-\}, \kappa_5^- > \min\{\kappa_4^-, \kappa_3^-\}$. So we set $\kappa_1^- := 1, \kappa_3^- := \kappa_5^- := 2$. Because these numbers are minimal, they can also be obtained by applying the Bourne & Parsons-algorithm. Now, $\kappa_c(\omega) = \sum_{\substack{1 \leqslant i \leqslant 5 \\ \omega \models A_i \overline{B}_i}} \kappa_i^-$ can be computed easily. The table below shows κ_c-rankings for some worlds ω and compares them to system-Z-rankings.

ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$	ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$	ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$	ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$
$pqrab$	3	2	$pqr\overline{a}b$	5	2	$\overline{p}qrab$	3	2	$\overline{p}qr\overline{a}b$	5	2
$pqra\overline{b}$	2	1	$pqr\overline{a}\overline{b}$	3	2	$\overline{p}qra\overline{b}$	2	2	$\overline{p}qr\overline{a}\overline{b}$	3	2

It is clearly seen that a c-representation imposes a more finely grained structure on the possible worlds and thus establishes more conditionals. For instance, we have $\kappa^z \not\models (\overline{b}|pqr\overline{a})$, but $\kappa_c \models (\overline{b}|pqr\overline{a})$, that is, a republican, who is a pacifist and a quaker, but not an American, is supposed not to like baseball. To show that this is more than pure speculation, compare the conditional structures of the two worlds involved, $pqr\overline{a}b$ and $pqr\overline{a}\,\overline{b}$: They are the same, except for conditional r_5, and due to r_5, the person is supposed not to like baseball.

In general, system-Z appears to be too cautious to handle conditional interactions adequately. But the extended Nixon diamond also shows it to be too bold sometimes: Now we have $\kappa^z \models (p|qr)$, which seems to be quite unintuitive. This problem also holds for other well-known approaches to conditional reasoning, cf. [DP96]. Again, the c-representation κ_c proves to be more adequate: We have $\kappa_c(pqr) = 2 = \kappa_c(\overline{p}qr)$, so κ_c is completely indeterminate with respect to $(p|qr)$ – it preserves *ambiguity* (cf. [BSS00]).

In contrast to the limited applicability of system-Z^*, an OCF-c-representation can be calculated for any consistent set of conditionals:

Proposition 1. *$\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ is consistent iff a c-representation of $\mathcal{R}$ exists.*

It is already the approach (9) (or (3)) that guarantees a well-behavedness with respect to conditional interactions. In particular, all OCF-c-representations satisfy the property of irrelevance in the following sense (cf. (4)):

Lemma 1. *Suppose $\mathcal{R}$ is a set of conditionals, and d is an atomic proposition not appearing in any of the conditionals in $\mathcal{R}$. Then, for any OCF-c-representation κ_c of $\mathcal{R}$, and for any conditional $(B|A) \in (\mathcal{L} \mid \mathcal{L})$, $\kappa_c \models (B|A)$ iff $\kappa_c \models (B|Ad)$.*

An analogous quantified version of this lemma holds for probability functions, too. Furthermore, let $\mid\!\sim^c$ be the (nonmonotonic) inference relation based on considering all c-representations:

$$\mathcal{R} \mid\!\sim^c (B|A) \quad \text{iff} \quad \kappa_c \models (B|A) \text{ for all c-representations of } \mathcal{R} \tag{13}$$

where $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L}), (B|A) \in (\mathcal{L} \mid \mathcal{L})$. Then Lemma 1 yields

Corollary 1. *Let $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L}), (B|A) \in (\mathcal{L} \mid \mathcal{L})$, let d be an atomic proposition not appearing in any of the conditionals in $\mathcal{R}$. Then $\mathcal{R} \mid\!\sim^c (B|A)$ iff $\mathcal{R} \mid\!\sim^c (B|Ad)$.*

Moreover, it is easy to prove that $\mid\!\sim^c$ also fulfills the basic inferential properties of system P (see, e.g., [BDP97]). But it does not suffer from the *drowning problem*, or, more specifically, from the *blocking of property inheritance problem* (see, e.g., [BDP97]), as the following example shows:

Example 4. Let $\mathcal{R}$ consist of the rules $(f|b)$ (*birds fly*), $(b|p)$ (*penguins are birds*), $(\overline{f}|p)$ (*penguins do not fly*) and $(w|b)$ (*birds have wings*). We will show $\mathcal{R} \mid\!\sim^c (w|pb)$, that is, penguins inherit the property of having wings from their superclass *birds*, although they are non-typical birds.

Let κ_c be a c-representation of $\mathcal{R}$, with $\kappa_1^+, \kappa_1^-, \kappa_2^+, \kappa_2^-, \kappa_3^+, \kappa_3^-, \kappa_4^+, \kappa_4^-$ specifying the numerical effects of the conditionals, and with normalizing constant κ_0. For the fourth conditional $(w|b)$, (11) yields $\kappa_4^- - \kappa_4^+ > 0$ (this can be seen quickly by listing the conditional structures of all worlds involved), so that $\kappa_4^- > \kappa_4^+$. By checking the conditional structures of the worlds verifying, or falsifying, respectively, $(w|pb)$ we find that $\kappa_c(pbw) = \min\{\kappa_0 + \kappa_1^+ + \kappa_2^+ + \kappa_3^- + \kappa_4^+, \kappa_0 + \kappa_1^- + \kappa_2^+ + \kappa_3^+ + \kappa_4^+\}$ and $\kappa_c(pb\overline{w}) = \min\{\kappa_0 + \kappa_1^+ + \kappa_2^+ + \kappa_3^- + \kappa_4^-, \kappa_0 + \kappa_1^- + \kappa_2^+ + \kappa_3^+ + \kappa_4^-\}$. Due to $\kappa_4^- > \kappa_4^+$, we conclude $\kappa_c(pbw) < \kappa_c(pb\overline{w})$, that is, $\kappa_c \models (w|pb)$.

Suppose now that we also take the conditionals $(e|b)$ (*birds lay eggs*), $(\overline{w}|k)$ (*kiwis do not have wings*) and $(b|k)$ (*kiwis are birds*) into account.[1] Then it can be shown that any c-representation κ_c of this extended set of rules treats both penguins and kiwis as normal birds with respect to laying eggs: $\kappa_c \models (e|pb), (e|kb)$. If we further restrict our attention to c-representations with $\kappa_i^+ = 0$ (that is, the extended system-Z^* approach), then moreover, any such c-representation κ_c will also satisfy $(w|pb)$ and $(\overline{w}|kb)$. This shows, how most complicated interdependencies between conditionals are treated properly in our framework.

We described this example in detail to show that inference based on c-representations is largely qualitative and non-numerical. In fact, our argumentation used mainly structural information, provided by conditional structures.

7 Outlook

In this paper, we presented a new theory to handle conditionals most adequately in inductive knowledge representation and uncertain reasoning. This new theory is based on the algebraic notion of *conditional structures* of worlds. Its realization via group theory does not only provide an elegant methodological framework for representing conditional knowledge. The same machinery can be applied to guide the revision of epistemic states by sets of conditionals in a way that is compatible with the postulates for iterative belief revision of [DP97], and with their generalizations presented in [KI99] (cf. [KI01a]). Moreover, the new conditional theory sketched here can also be used for *knowledge discovery* (see [KI00,KI01a]). It is there where the techniques introduced in this paper reveal their full power by

[1] I am grateful to an anonymous referee to raise this problem.

elaborating conditional structures from numerical relationships by group theoretical means.

So indeed, the conditional theory presented in this paper has important applications in the whole area of formal knowledge management – uncertain reasoning, belief revision and knowledge discovery. A comparison with conditional event theories can be found in [KI01b].

References

[BDP97] S. Benferhat, D. Dubois, and H. Prade. Nonmonotonic reasoning, conditional objects and possibility theory. *Artificial Intelligence*, 92:259–276, 1997.

[BP99] R.A. Bourne and S. Parsons. Maximum entropy and variable strength defaults. In *Proceedings Sixteenth International Joint Conference on Artificial Intelligence, IJCAI'99*, pages 50–55, 1999.

[BSS00] S. Benferhat, A. Saffiotti, and P. Smets. Belief functions and default reasoning. *Artificial Intelligence*, 122:1–69, 2000.

[Cal91] P.G. Calabrese. Deduction and inference using conditional logic and probability. In I.R. Goodman, M.M. Gupta, H.T. Nguyen, and G.S. Rogers, editors, *Conditional Logic in Expert Systems*, pages 71–100. Elsevier, North Holland, 1991.

[DP96] D. Dubois and H. Prade. Non-standard theories of uncertainty in plausible reasoning. In G. Brewka, editor, *Principles of Knowledge Representation*. CSLI Publications, 1996.

[DP97] A. Darwiche and J. Pearl. On the logic of iterated belief revision. *Artificial Intelligence*, 89:1–29, 1997.

[GMP93] M. Goldszmidt, P. Morris, and J. Pearl. A maximum entropy approach to nonmonotonic reasoning. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 15(3):220–232, 1993.

[Goo63] I.J. Good. Maximum entropy for hypothesis formulation, especially for multidimensional contingency tables. *Ann. Math. Statist.*, 34:911–934, 1963.

[GP96] M. Goldszmidt and J. Pearl. Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artificial Intelligence*, 84:57–112, 1996.

[KI98] G. Kern-Isberner. Characterizing the principle of minimum cross-entropy within a conditional-logical framework. *Artificial Intelligence*, 98:169–208, 1998.

[KI99] G. Kern-Isberner. Postulates for conditional belief revision. In *Proceedings Sixteenth International Joint Conference on Artificial Intelligence, IJCAI-99*, pages 186–191. Morgan Kaufmann, 1999.

[KI00] G. Kern-Isberner. Solving the inverse representation problem. In *Proceedings 14th European Conference on Artificial Intelligence, ECAI'2000*, pages 581–585, Berlin, 2000. IOS Press.

[KI01a] G. Kern-Isberner. *Conditionals in nonmonotonic reasoning and belief revision*. Springer, Lecture Notes in Artificial Intelligence, 2001. (to appear).

[KI01b] G. Kern-Isberner. Representing and learning conditional information in possibility theory. In *Proceedings 7th Fuzzy Days, Dortmund, Germany*. Springer LNCS-Series, 2001. (to appear).

[Par94] J.B. Paris. *The uncertain reasoner's companion – A mathematical perspective*. Cambridge University Press, 1994.