A structural approach to default reasoning

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Abstract

In this paper, we present several approaches to default reasoning in a possibilistic framework which all are based on purely structural information, encoded by the *conditional structures* of worlds. Conditional structures symbolize the effects conditionals have on worlds and make conditional interactions transparent and computable, providing a sound algebraic framework for default inferences. Even the weakest of our inference relations, which merely demands a compatibility with conditional structures, proved to be quite well-behaved, according to widely accepted standards. The other two inference relations investigated here are based on faithful possibilistic realizations of conditional structures. Both these approaches allow to handle implicit priorities among the default rules under consideration and thereby combine structural with qualitative information.

1 Introduction

Default reasoning is based on using default rules as plausible, but uncertain rules for inference. It is only tentative, and its conclusions can be revised when new or more precise information becomes available. It is just this defeasibility, however, which makes default reasoning a major topic in Artificial Intelligence: Knowledge is seldom complete nor certain, and defeasible inference mechanisms provide a convenient framework to realize human reasoning. Within the last two decades, a great variety of different systems for default reasoning have been presented, in symbolic as well as in ordinal or numerical and even infinitesimal environments (see, e.g., [Adams, 1975; Reiter, 1980; Lehmann and Magidor, 1992; Goldszmidt *et al.*, 1993; Goldszmidt and Pearl, 1996; Benferhat *et al.*, 1992; 1999; 2000; Geffner, 1992; Lehmann, 1995]).

The crucial question in default reasoning is the following: Given a set $\mathcal{R}$ of default rules, which conclusions can be drawn from $\mathcal{R}$? Default reasoning is expected to be plausible, and it can even be bold sometimes, but it should never be arbitrary, so two further points have to be made clear: What are the mechanisms that give rise to these conclusions, and what are the properties the corresponding inference relation has? In [Benferhat *et al.*, 2000], the desirable properties a default inference relation $\mid\!\sim$ should have are summarized, as follows:

- *System P*: The inference relation should satisfy the following basic properties of system P (see, e.g., [Kraus *et al.*, 1990]):
  Reflexivity: $A \mid\!\sim A$ for all formulas A;
  Left Logical Equivalence:
  from $A \Leftrightarrow A' = \top$ and $A \mid\!\sim B$ deduce $A' \mid\!\sim B$;
  Right Weakening:
  from $B \models B'$ and $A \mid\!\sim B$ deduce $A \mid\!\sim B'$;
  OR: from $A \mid\!\sim C$ and $B \mid\!\sim C$ deduce $A \vee B \mid\!\sim C$;
  Cautious Monotony:
  from $A \mid\!\sim B$ and $A \mid\!\sim C$ deduce $A \wedge B \mid\!\sim C$;
  Cut: from $A \wedge B \mid\!\sim C$ and $A \mid\!\sim B$ deduce $A \mid\!\sim C$.
- *Rational Monotonicity*: If B can be (defeasibly) inferred from A, and $\neg C$ can *not* be inferred from A, then B can also be inferred from $A \wedge C$. Rational Monotonicity claims that information not in conflict with the knowledge given should not affect the possible conclusions.
- *Specificity*: More specific information should dominate general information. Knowing that Tweety is a bird, but also a member of the subclass "penguins", should yield the conclusion that Tweety cannot fly, in spite of the more general default rule that birds are normally able to fly.

- *Irrelevance*: Atoms not occurring in any of the defaults should be irrelevant to default conclusions.
- *Property inheritance*: Members of a subclass which are exceptional with respect to one property should be able to inherit other properties from the superclass. In our penguin example, we would expect Tweety to lay eggs, as all birds do, although he is an exceptional bird.
- *Ambiguity preservation*: Conflicts between defaults should not be resolved arbitrarily. In case that we have two equally strong arguments, one in favor of and one against a proposition, no conclusion should be drawn at all.
- *Duplication stability*[1]: Duplications of defaults in the rule base should not affect the conclusions.

In this paper, we will present methods for reasoning with defaults which are based on mainly structural arguments and which satisfy most of the properties listed above, with the only exception of *Rational Monotonicity*. The structures that are crucial for the inference process here are *conditional structures*, assigned to possible worlds and reflecting a world's attitude with respect to the set of default rules under consideration. The first inference relation, $\mid\!\sim^{s}_{\mathcal{R}}$, is actually quite a weak one and only needs a purely qualitative environment. Hence, it is based on comparative possibility distributions [Benferhat *et al.*, 1999] which are compatible with conditional structures. The second inference relation, $\mid\!\sim^{c}_{\mathcal{R}}$, is a more sophisticated one; it is realized by considering *c-representations*, i.e. possibility distributions which are built from conditional structures in a straightforward way. $\mid\!\sim^{c}_{\mathcal{R}}$ is also based on structural information, but, in using numbers, it allows of taking implicit information about priorities between defaults into account. Therefore, it is stronger than $\mid\!\sim^{s}_{\mathcal{R}}$ and provides a more convenient environment for inferences. The last inference relation, $\mid\!\sim^{lcc}_{\mathcal{R}}$, is obtained by focusing on least specific and least committed c-representations. It turns out to be a possibilistic realization of $\mid\!\sim_{lcd}$, the default inference relation of [Benferhat *et al.*, 2000] using lcd-belief functions.

This paper is organized as follows: The following section recalls the notion of (comparative) possibility distributions and how these can be used to validate default rules (or *conditionals*). Section 3 gives a brief overview on the theory of conditional structures which is basic to the methods of this paper. In particular, we define a partial ordering on conditional structures of possible worlds here. In Section 4, we introduce the notion of *structural compatibility* of comparative possibility distributions with sets of defaults, and investigate the corresponding structural inference relation. Section 5 deals with *c-representations* of default rules which can be regarded as numerical realizations of algebraic conditional structures. This section also introduces *lcc-representations* as a special type of c-representations. Section 6 concludes this paper by highlighting its main results, and by pointing out connections to other approaches to uncertain reasoning.

[1] This property is referred to as *syntax independence* or *redundancy* in [Benferhat *et al.*, 2000]. We prefer *duplication stability* here since the notions used in [Benferhat *et al.*, 2000] suggest quite stronger properties to hold.

2 Default rules in possibility theory

Default rules describe what is normally the case, or what is expected. In this paper, we will take quite a general view and understand default rules simply as plausible relationships "*If A then B*" between two formulas A, B of a (finitely generated) propositional language $\mathcal{L}$. We will use here the conditional notation $(B|A)$ for a default rule, with *antecedent* or *premise* A, and *consequent* B, and also speak of default rules simply as "conditionals". Let $(\mathcal{L} \mid \mathcal{L})$ denote the set of all default rules $(B|A)$ with $A, B \in \mathcal{L}$. We will only consider finite sets $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ of conditionals. To simplify notation, we will write $\overline{A}$ instead of $\neg A$, and AB instead of $A \wedge B$, for $A, B \in \mathcal{L}$. Let Ω be the complete set of interpretations of $\mathcal{L}$, where each $\omega \in \Omega$ is taken to be a *possible world* for $\mathcal{L}$.

A *possibility distribution* is a mapping $\pi : \Omega \to [0,1]$, where each $\pi(\omega)$ reflects the degree of possibility an intelligent agent assigns to ω. π induces a *possibility measure* $\Pi : \mathcal{L} \to [0,1], \Pi(A) = \max\{\pi(\omega) \mid \omega \models A\}$ and a dual *necessity measure* $N = N_\pi : \mathcal{L} \to [0,1], N(A) = 1 - \Pi(\overline{A})$ (cf., e.g., [Dubois *et al.*, 1994]). We will not distinguish between a possibility distribution, π, on Ω, and its induced possibility measure, Π, on $\mathcal{L}$, and will use the same symbol π for both.

A default rule $(B|A)$ *is represented* (or *accepted*) *by* π, $\pi \models (B|A)$, iff $\pi(AB) > \pi(A\overline{B})$. So, in accordance with intuition, a default rule $(B|A)$ is accepted in the epistemic state modeled by a possibility distribution, if its confirmation (AB) is considered to be more possible (or plausible) than its refutation $(A\overline{B})$. We assume all possibility distributions to be normalized, i.e. $\pi(\top) = 1$, and, only for conciseness of presentation, to be positive, i.e. $\pi(\omega) > 0$ for all $\omega \in \Omega$.

Each possibility distribution induces uniquely a complete pre-ordering on possible worlds by $\omega_1 \succeq_\pi \omega_2$ iff $\pi(\omega_1) \geqslant \pi(\omega_2)$. Different possibility distributions, however, may give rise to the same pre-ordering $\succeq_\pi$. If one is only concerned with qualitative relationships, one can abstract from numbers and focus on such complete pre-orderings $\succeq_\pi$ on Ω, which are called *comparative possibility distributions* (cf. also [Benferhat *et al.*, 1999]). The subscript π does not refer to a particular possibility distribution, but is to symbolize possibility or plausibility, in general. In the usual way, we define $\omega \succ_\pi \omega'$ by $\omega \succeq_\pi \omega'$ and not $(\omega' \succeq_\pi \omega)$, and $\omega \approx_\pi \omega'$ iff both $\omega \succeq_\pi \omega'$ and $\omega' \succeq_\pi \omega$.

From a comparative possibility distribution $\succeq_\pi$, a *comparative possibility measure* on $\mathcal{L}$, also denoted by $\succeq_\pi$, can be obtained by $A \succeq_\pi B$ ($A \succ_\pi B$) iff there exists $\omega \models A$ such that for all $\omega' \models B$, $\omega \succeq_\pi \omega'$ ($\omega \succ_\pi \omega'$). $\succeq_\pi$ defines a complete pre-ordering on the set of formulas $\mathcal{L}$, in particular, $A \succ_\pi \perp$ for any consistent formula $A \in \mathcal{L}$. Again we hope that no confusion may arise when using the same symbol $\succeq_\pi$ for comparative possibility distributions and comparative possibility measures.

Like possibility distributions, each comparative possibility distribution provides an epistemic structure to validate default rules: A comparative possibility distribution $\succ_\pi$ *accepts* (or *represents*) *a default rule* $(B|A)$ iff $AB \succ_\pi A\overline{B}$.

For (comparative) possibility distributions, the definition of representing default rules is easily extended to sets $\mathcal{R}$ of default rules by postulating that each rule in $\mathcal{R}$ must be represented. In particular, the set of all comparative possibility distributions which accept $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ is denoted by $\Pi^{\succeq}(\mathcal{R})$:

$$\Pi^{\succeq}(\mathcal{R}) = \{\succeq_\pi \mid A_iB_i \succ_\pi A_i\overline{B_i} \text{ for all } i, 1 \leqslant i \leqslant n\}$$

3 Conditional structures

The theory of conditional structures has been developed in detail in several publications (cf. [Kern-Isberner, 2000; 2001a; 2001c; 2001b]); so we only summarize the main ideas here.

From the theory of conditionals, the following distinction concerning the possible relationships between worlds and conditionals is well-known: A world $\omega \in \Omega$ is said to *verify* a conditional (default rule) $(B|A)$, if $\omega \models AB$; it *falsifies* $(B|A)$, if $\omega \models A\overline{B}$; if $\omega \models \overline{A}$, then $(B|A)$ is not applicable to ω (cf. e.g. [DeFinetti, 1974; Calabrese, 1991]).

Intuitively, representing a conditional as a plausible conclusion in an epistemic state means to make – at least some – worlds verifying the conditional more plausible than the worlds falsifying it. In this sense, conditionals have effects on possible worlds – more exactly, on their degrees of plausibility or possibility, shifting them appropriately to establish the intended relationship. When we consider (finite) sets of conditionals $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$, we have to identify the effect of each conditional in $\mathcal{R}$ on worlds in Ω. To this end, we use a pair of abstract symbols, $\mathbf{a}_i^+, \mathbf{a}_i^-$, that we associate to each conditional $(B_i|A_i)$ in $\mathcal{R}$. Moreover, we will make use of a group structure to represent the joint impact of conditionals on worlds as products.

So let $\mathcal{F}_\mathcal{R} = \langle \mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^- \rangle$ be the free abelian group with generators $\mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^-$, i.e. $\mathcal{F}_\mathcal{R}$ consists of all elements of the form $(\mathbf{a}_1^+)^{r_1}(\mathbf{a}_1^-)^{s_1} \ldots (\mathbf{a}_n^+)^{r_n}(\mathbf{a}_n^-)^{s_n}$ with integers $r_i, s_i \in \mathbb{Z}$ (the ring of integers) (cf. e.g. [Lyndon and Schupp, 1977; Fine and Rosenberger, 1999]). Note that, although we will speak of *multiplication* and *products* in $\mathcal{F}_\mathcal{R}$, the generators of $\mathcal{F}_\mathcal{R}$ are merely juxtaposed, like words. Moreover, it should be clear that it does not matter which symbols $\mathbf{a}_i^+, \mathbf{a}_i^-$ have actually been chosen – taking a set of different symbols will yield isomorphic structures.

For each $i, 1 \leqslant i \leqslant n$, we define a function $\sigma_i = \sigma_{(B_i|A_i)} : \Omega \to \mathcal{F}_\mathcal{R}$ by setting

$$\sigma_i(\omega) := \sigma_{(B_i|A_i)}(\omega) := \begin{cases} \mathbf{a}_i^+ & \text{if } \omega \models A_iB_i \\ \mathbf{a}_i^- & \text{if } \omega \models A_i\overline{B_i} \\ 1 & \text{if } \omega \models \overline{A_i} \end{cases}$$

$\sigma_i(\omega)$ represents the manner in which the conditional $(B_i|A_i)$ applies to the possible world ω. The neutral element 1 of $\mathcal{F}_\mathcal{R}$ represents the non-applicability of $(B_i|A_i)$ in case that the antecedent A_i is not satisfied, so the neutral group element corresponds to a neutral attitude with respect to the conditional. The function $\sigma_\mathcal{R} : \Omega \to \mathcal{F}_\mathcal{R}$,

$$\sigma_\mathcal{R}(\omega) := \prod_{1 \leqslant i \leqslant n} \sigma_i(\omega) = \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_iB_i}} \mathbf{a}_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i\overline{B_i}}} \mathbf{a}_i^-$$

describes the all-over effect of $\mathcal{R}$ on ω. $\sigma_\mathcal{R}(\omega)$ is called the *conditional structure of* ω *with respect to* $\mathcal{R}$. Since $\mathcal{F}_\mathcal{R}$ is a free (abelian) group, the conditional structures of worlds are uniquely determined by its σ_i-components: For any two worlds ω_1, ω_2, we have

$$\sigma_\mathcal{R}(\omega_1) = \sigma_\mathcal{R}(\omega_2) \quad \text{iff} \quad \sigma_i(\omega_1) = \sigma_i(\omega_2) \text{ for all } 1 \leqslant i \leqslant n \tag{1}$$

So, each conditional is taken to be an independent piece of information. To make notations more convenient in the following, we write $\sigma_\mathcal{R}(A) := \{\sigma_\mathcal{R}(\omega) \mid \omega \in \Omega, \omega \models A\}$ for the set of conditional structures appertaining to the models of $A \in \mathcal{L}$.

We will illustrate this notion of conditional structures in the following example which extends the well-known *Nixon diamond*:

Example 1 Let $\mathcal{R}$ consist of the following conditionals:

$r_1 : (p|q)$ *Quakers are pacifists.*
$r_2 : (\overline{p}|r)$ *Republicans are not pacifists.*
$r_3 : (a|q)$ *Quakers are Americans.*
$r_4 : (b|a)$ *Americans like baseball.*
$r_5 : (\overline{b}|q)$ *Quakers do not like baseball.*

The conditional structure of a possible world, say $pqrab$, is calculated in the following way: $pqrab$ verifies the first conditional r_1 ($pqrab \models pq$), so we have $\sigma_1(pqrab) = \mathbf{a}_1^+$. $pqrab$, however, falsifies the second conditional $r_2 = (\overline{p}|r)$, thus $\sigma_2(pqrab) = \mathbf{a}_2^-$. In the same way, $\sigma_3(pqrab) = \mathbf{a}_3^+, \sigma_4(pqrab) = \mathbf{a}_4^+, \sigma_5(pqrab) = \mathbf{a}_5^-$. We obtain $\sigma_{\mathcal{R}}(pqrab) = \mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$. ■

The conditional structures of worlds are very special elements of $\mathcal{F}_{\mathcal{R}}$: For each i, $1 \leqslant i \leqslant n$, at most one of $\mathbf{a}_i^+, \mathbf{a}_i^-$ occurs once in $\sigma_{\mathcal{R}}(\omega)$ since each conditional in $\mathcal{R}$ applies to ω in a well-defined way. Group elements of this kind will be called simple products:

Definition 2 An element $\mathbf{a} \in \mathcal{F}_{\mathcal{R}}$ is called a *simple product* iff for each i, $1 \leqslant i \leqslant n$, at most one of $\mathbf{a}_i^+, \mathbf{a}_i^-$ occurs in $\mathbf{a}$, with exponent 1.

The set of all simple products in $\mathcal{F}_{\mathcal{R}}$ will be denoted by $\mathcal{SP}_{\mathcal{R}}$.

So every conditional structure $\sigma_{\mathcal{R}}(\omega)$ of a possible world $\omega \in \Omega$ is a simple product, i.e. $\sigma_{\mathcal{R}}(\Omega) \subseteq \mathcal{SP}_{\mathcal{R}}$. In general, however, we will have $\sigma_{\mathcal{R}}(\Omega) \neq \mathcal{SP}_{\mathcal{R}}$.

Next, we define a partial ordering $>$ on simple products. To this end, we have to check which of the generators of $\mathcal{F}_{\mathcal{R}}$ actually occur in a simple product. This is done by the aid of *signature functions* $sign_i : \mathcal{SP}_{\mathcal{R}} \to \{+, -, 0\}$, which are defined for each conditional $(B_i|A_i) \in \mathcal{R}$, $1 \leqslant i \leqslant n$, by

$$sign_i(\mathbf{a}) := \begin{cases} +, & \text{if } \mathbf{a}_i^+ \text{ occurs in } \mathbf{a} \\ -, & \text{if } \mathbf{a}_i^- \text{ occurs in } \mathbf{a} \\ 0, & \text{otherwise} \end{cases}$$

Note that the signature functions $sign_i$ (defined on group elements) correspond to the structure functions σ_i (defined on possible worlds) in that $sign_i(\sigma_{\mathcal{R}}(\omega))$ reflects the behavior of ω with respect to $(B_i|A_i)$ independent of the symbols $\mathbf{a}_i^+, \mathbf{a}_i^-$. In particular, from (1), we have that $\sigma_{\mathcal{R}}(\omega_1) = \sigma_{\mathcal{R}}(\omega_2)$ iff $sign_i(\sigma_{\mathcal{R}}(\omega_1)) = sign_i(\sigma_{\mathcal{R}}(\omega_2))$ for all $1 \leqslant i \leqslant n$, for any two worlds $\omega_1, \omega_2 \in \Omega$.

Let $\mathbf{a}, \mathbf{b} \in \mathcal{SP}_{\mathcal{R}}$ be simple products. We define $\mathbf{a} > \mathbf{b}$ iff for each i, $1 \leqslant i \leqslant n$,

- $sign_i(\mathbf{a}) \neq 0$ iff $sign_i(\mathbf{b}) \neq 0$, and
- $sign_i(\mathbf{a}) = -$ implies $sign_i(\mathbf{b}) = -$, and
- there is at least one i, $1 \leqslant i \leqslant n$, such that $sign_i(\mathbf{a}) = +$ and $sign_i(\mathbf{b}) = -$.

So, for instance, $\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^+ > \mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^- > \mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^- > \mathbf{a}_1^-\mathbf{a}_2^-\mathbf{a}_3^-$, whereas $\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+$ and $\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^-$ are not comparable with respect to $>$.

This partial ordering $>$ can now be used to compare the conditional structures of possible worlds $\omega_1, \omega_2 \in \Omega$: We have $\sigma_{\mathcal{R}}(\omega_1) > \sigma_{\mathcal{R}}(\omega_2)$ iff the same conditionals in $\mathcal{R}$ are applicable to either of them, and each conditional, that is falsified by ω_1, is also falsified by ω_2, but there is at least one conditional in $\mathcal{R}$ which is verified by ω_1 and falsified by ω_2. Note that only the behavior of worlds with respect to the same conditionals is relevant to compare their conditional structures. $>$ provides no means to relate the conditional structures, for instance, of two worlds ω_1, ω_2 such that ω_1 verifies $(B_1|A_1)$ but falsifies $(B_2|A_2)$, say, and ω_2 verifies $(B_2|A_2)$ but falsifies $(B_1|A_1)$. No information about default priorities and the like is taken into account for the purely structural partial ordering $>$, as is e.g. done in the *prioritized preferential structures* of [Geffner and Pearl, 1992]. Note also that we do not focus here merely on which conditionals are falsified by the corresponding worlds, which is a most common approach in default reasoning to make conditionals fit a two-valued logic (see, e.g., [Geffner and Pearl, 1992; Goldszmidt and Pearl, 1996; Benferhat *et al.*, 2000]). We also distinguish between the case that a conditional is verified by a world, and the case that it is not applicable to that world, thus respecting the inherent three-valued nature of a conditional.

4 Default reasoning compatible with conditional structures

We will now consider comparative possibility distributions that not only represent a set of conditionals $\mathcal{R}$, but which also observe the information carried by the conditional structure of worlds. Such a comparative possibility distribution should not distinguish between worlds having the same conditional structures, and should give priority to those worlds that verify conditionals, in order to establish conditional beliefs:

Definition 3 A comparative possibility distribution $\succeq_\pi$ is said to be *structurally compatible* (*s-compatible*) *with* $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ iff it represents $\mathcal{R}$ (i.e. $\succeq_\pi \in \Pi^{\succeq}(\mathcal{R})$) and for all possible worlds $\omega, \omega' \in \Omega$, the following conditions are fulfilled:

- $\sigma_{\mathcal{R}}(\omega) = \sigma_{\mathcal{R}}(\omega')$ implies $\omega \approx_\pi \omega'$;
- $\sigma_{\mathcal{R}}(\omega) > \sigma_{\mathcal{R}}(\omega')$ implies $\omega \succ_\pi \omega'$.

The set of comparative possibility distributions which are s-compatible with $\mathcal{R}$ is denoted by $\Pi^s(\mathcal{R})$. Considering all comparative possibility distributions which are s-compatible with $\mathcal{R}$ gives rise to a (nonmonotonic) *structural inference relation* $|\!\!\sim^{s}_{\mathcal{R}}$ on $\mathcal{L}$:

$$\mathrm{A} \,|\!\!\sim^{s}_{\mathcal{R}} B \quad \text{iff} \quad AB \succ_\pi A\overline{B} \text{ for all } \succeq_\pi \in \Pi^s(\mathcal{R})$$

The following proposition shows how structural default reasoning can be done by considering conditional structures:

Proposition 4 *Let A, B be two propositional formulas such that for all $\omega \models A\overline{B}$, there is $\omega' \models AB$ such that $\sigma_{\mathcal{R}}(\omega') > \sigma_{\mathcal{R}}(\omega)$. Then $A \,|\!\!\sim^{s}_{\mathcal{R}} B$.*

Proof. We have to show that $AB \succ_\pi A\overline{B}$ for all s-compatible comparative possibility distributions $\succeq_\pi \in \Pi^s(\mathcal{R})$. So let $\succeq_\pi \in \Pi^s(\mathcal{R})$, and let $\omega_1 \models AB$ be maximal with respect to $\succeq_\pi$. Let $\omega \models A\overline{B}$. By presupposition, there is $\omega' \models AB$ with $\sigma_{\mathcal{R}}(\omega') > \sigma_{\mathcal{R}}(\omega)$. Due to s-compatibility, this implies $\omega' \succ_\pi \omega$, hence $\omega_1 \succeq_\pi \omega' \succ_\pi \omega$ for all $\omega \models A\overline{B}$. This proves $AB \succ_\pi A\overline{B}$. □

We will now check which of the properties of our catalogue (see Section 1) hold for $|\!\!\sim^{s}_{\mathcal{R}}$. It is straightforward to verify all properties of system P, so $|\!\!\sim^{s}_{\mathcal{R}}$ satisfies the very basic properties a nonmonotonic inference operation should obey. Furthermore, $|\!\!\sim^{s}_{\mathcal{R}}$ also respects *specificity*, as the simple penguin-example shows:

Example 5 Let $\mathcal{R}$ consist of the conditionals $(f|b)$ (*birds fly*), $(b|p)$ (*penguins are birds*), and $(\overline{f}|p)$ (*penguins do not fly*), with symbols $\mathbf{a}_1^+, \mathbf{a}_1^-, \mathbf{a}_2^+, \mathbf{a}_2^-, \mathbf{a}_3^+, \mathbf{a}_3^-$ associated to the conditionals (in that order). The conditional structures are listed in the following table:

ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$
pbf	$\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^-$	$\overline{p}bf$	$\mathbf{a}_1^+$
$pb\overline{f}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+$	$\overline{p}b\overline{f}$	$\mathbf{a}_1^-$
$p\overline{b}f$	$\mathbf{a}_2^-\mathbf{a}_3^-$	$\overline{p}\overline{b}f$	1
$p\overline{b}\,\overline{f}$	$\mathbf{a}_2^-\mathbf{a}_3^+$	$\overline{p}\overline{b}\,\overline{f}$	1

Specificity would postulate here that from knowing that an individual is both a penguin and a bird, we could derive that this individual can not fly – the more specific information is expected to override the general knowledge.

Let $\succeq_\pi \in \Pi^s(\mathcal{R})$. Then $\succeq_\pi$ represents $\mathcal{R}$, so, in particular, we have $pb \succ_\pi p\overline{b}$ and $p\overline{f} \succ_\pi pf$, which means $\max\{pbf, pb\overline{f}\} \succ_\pi \max\{p\overline{b}f, p\overline{b}\,\overline{f}\}$ and $\max\{pb\overline{f}, p\overline{b}\,\overline{f}\} \succ_\pi \max\{pbf, p\overline{b}f\}$. Furthermore, s-compatibility enforces $p\overline{b}\,\overline{f} \succ_\pi p\overline{b}f$ and $\overline{p}bf \succ_\pi \overline{p}b\overline{f}$. The first inequality, together with $pb \succ_\pi p\overline{b}$, implies $\max\{pbf, pb\overline{f}\} \succ_\pi p\overline{b}\,\overline{f}$. Now, if $pbf \succeq_\pi pb\overline{f}$, then $pbf \succ_\pi p\overline{b}\,\overline{f} \succ_\pi pbf$ would follow, a contradiction. Hence $pb\overline{f} \succ_\pi pbf$ for all $\succeq_\pi \in \Pi^s(\mathcal{R})$. Therefore, we have $pb \,|\!\!\sim^{s}_{\mathcal{R}} \overline{f}$. ■

The next proposition shows that $|\!\!\sim^{s}_{\mathcal{R}}$ also fulfills *irrelevance*:

Proposition 6 *Let $A, B, C \in \mathcal{L}$ be propositions such that C is composed of atomic propositions not occurring in any of the conditionals in $\mathcal{R}$, nor in $(B|A)$. Then $A \,|\!\!\sim^{s}_{\mathcal{R}} B$ iff $AC \,|\!\!\sim^{s}_{\mathcal{R}} BC$.*

Proof. Let $\mathcal{V}_1 \subset \mathcal{V}$ be the set of atomic propositions occurring in C, let $\mathcal{V}_2 := \mathcal{V} - \mathcal{V}_1$, $\mathcal{V}_1, \mathcal{V}_2 \neq \emptyset$. Then each possible world $\omega \in \Omega$ can be written as $\omega = \nu_1\nu_2$ with ν_1, ν_2 possible worlds (i.e. interpretations) over $\mathcal{V}_1, \mathcal{V}_2$, respectively. The conditional structure $\sigma_{\mathcal{R}}(\omega)$ is determined by the $\mathcal{V}_2$-part of ω, since no variable in $\mathcal{V}_1$ occurs in $\mathcal{R}$. Therefore, $\sigma_{\mathcal{R}}(\nu_1\nu_2) = \sigma_{\mathcal{R}}(\nu_1'\nu_2)$ for all ν_1, ν_1', and for a fixed (but arbitrary) ν_2.

Let $\succ_\pi \in \Pi^s(\mathcal{R})$, and let $\dot{B}$ be one of $B, \overline{B}$. In particular, for any $\succ_\pi$-maximal world $\omega = \nu_1\nu_2$ with $\omega \models A\dot{B}C$, ν_1' can be chosen such that $\nu_1'\nu_2 \models A\dot{B}\overline{C}$ and $\sigma_{\mathcal{R}}(\nu_1\nu_2) = \sigma_{\mathcal{R}}(\nu_1'\nu_2)$. Hence $\nu_1\nu_2 \approx_\pi \nu_1'\nu_2$. A similar argumentation for the $\succ_\pi$-maximal worlds of $A\dot{B}\overline{C}$ shows $A\dot{B}C \approx_\pi A\dot{B}\overline{C} \approx_\pi A\dot{B}$. Therefore, $AB \succ_\pi A\overline{B}$ iff $ABC \succ_\pi A\overline{B}C$ for each $\succ_\pi \in \Pi^s(\mathcal{R})$. □

The next two examples will illustrate that also exceptional subtypes can inherit properties from their superclass:

Example 7 Let $\mathcal{R}$ consist of the following conditionals:

r_1: $(f|b)$ *birds fly*
r_2: $(\overline{f}|p)$ *penguins do not fly*
r_3: $(b|p)$ *penguins are birds*
r_4: $(l|b)$ *birds have legs*

Let $\mathbf{a}_i^+, \mathbf{a}_i^-$ be associated with conditional r_i, $1 \leqslant i \leqslant 4$. Then $pb \,|\!\!\sim^{s}_{\mathcal{R}} l$ can be derived, i.e. penguin-birds are supposed to have legs: Due to Proposition 4, we only have to check the conditional structures in $\sigma_{\mathcal{R}}(pbl) = \{\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+, \mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^+\}$ and $\sigma_{\mathcal{R}}(pb\overline{l}) = \{\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^-, \mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^-\}$. Since $\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+ > \mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^-$ and $\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^+ > \mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^-$, we have $pb \,|\!\!\sim^{s}_{\mathcal{R}} l$. ■

Example 8 Here, the rule base $\mathcal{R}$ represents knowledge about *Swedes* having the properties *blond* and *tall*: $\mathcal{R} = \{(b|s), (t|s)\}$ (*Swedes are blond* and *Swedes are tall*). Then, short Swedes are also expected to be blond: Let $\mathbf{a}_1^+, \mathbf{a}_1^-$ be associated with the first conditional, and $\mathbf{a}_2^+, \mathbf{a}_2^-$ with the second one. Since $\sigma_{\mathcal{R}}(sb\overline{t}) = \mathbf{a}_1^+\mathbf{a}_2^- > \mathbf{a}_1^-\mathbf{a}_2^- = \sigma_{\mathcal{R}}(s\overline{b}\,\overline{t})$, Proposition 4 implies $s\overline{t} \mathrel{|\!\sim}^s_{\mathcal{R}} b$. ■

In continuing Example 1, we show that $\mathrel{|\!\sim}^s_{\mathcal{R}}$ preserves ambiguity.

Example 1 [(continued)] $\mathrel{|\!\sim}^s_{\mathcal{R}}$ yields no conclusion as to whether a person who is a quaker and a Republican is a pacifist or not. This is not surprising, since $\mathrel{|\!\sim}^s_{\mathcal{R}}$ is actually quite weak. It is noticeable, however, *how* ambiguity is preserved by $\mathrel{|\!\sim}^s_{\mathcal{R}}$ by considering the conditional structures of pqr and $\overline{p}qr$ which are listed below:

ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$
$pqrab$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}qrab$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$
$pqra\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$\overline{p}qra\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$
$pqr\overline{a}b$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^-\mathbf{a}_5^-$	$\overline{p}qr\overline{a}b$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_5^-$
$pqr\overline{a}\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^-\mathbf{a}_5^+$	$\overline{p}qr\overline{a}\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_5^+$

As to the third, fourth and fifth conditional in $\mathcal{R}$, there is a complete symmetry in the conditional structures of the worlds satisfying pqr and $\overline{p}qr$, respectively. More exactly, we have $\sigma_{\mathcal{R}}(pqr\dot{a}\dot{b}) = \sigma_{\mathcal{R}}(\overline{p}qr\dot{a}\dot{b})$ for $\dot{a} \in \{a, \overline{a}\}, \dot{b} \in \{b, \overline{b}\}$, and it is clearly seen that the ambiguity is only due to the conflict between the first and the second conditional. The other conditionals in $\mathcal{R}$ do not have any influence on the decision whether a person who is both a quaker and a Republican is a pacifist or not, as one should expect. Note that standard possibilistic inference does not preserve ambiguity in this example (cf. [Dubois and Prade, 1996]). ■

Finally, the following proposition proves that $\mathrel{|\!\sim}^s_{\mathcal{R}}$ is not altered by repeating rules in the knowledge base:

Proposition 9 *Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a finite set of conditionals, and let $\mathcal{R}' = \{(B_0|A_0), (B_1|A_1), \ldots, (B_n|A_n)\}$ contain the same conditionals $(B_1|A_1), \ldots, (B_n|A_n)$ with a duplicated first conditional $(B_0|A_0) \equiv (B_1|A_1)$. Then $A \mathrel{|\!\sim}^s_{\mathcal{R}} B$ iff $A \mathrel{|\!\sim}^s_{\mathcal{R}'} B$ for $A, B \in \mathcal{L}$.*

Proof. Let $\sigma_{\mathcal{R}}, \sigma_{\mathcal{R}'}$ be two conditional structure homomorphisms appertaining to $\mathcal{R}, \mathcal{R}'$, respectively, with signature functions $sign_i, sign'_j$, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n$. Then for all $\omega \in \Omega$ and for all i, $2 \leqslant i \leqslant n$, we have $sign_i(\sigma_{\mathcal{R}}(\omega)) = sign'_i(\sigma_{\mathcal{R}'}(\omega))$, and $sign_1(\sigma_{\mathcal{R}}(\omega)) = sign'_1(\sigma_{\mathcal{R}'}(\omega)) = sign'_0(\sigma_{\mathcal{R}'}(\omega))$. Therefore $\sigma_{\mathcal{R}}(\omega_1) = \sigma_{\mathcal{R}}(\omega_2)$ iff $\sigma_{\mathcal{R}'}(\omega_1) = \sigma_{\mathcal{R}'}(\omega_2)$, and $\sigma_{\mathcal{R}}(\omega_1) > \sigma_{\mathcal{R}}(\omega_2)$ iff $\sigma_{\mathcal{R}'}(\omega_1) > \sigma_{\mathcal{R}'}(\omega_2)$, for $\omega_1, \omega_2 \in \Omega$. Hence, $\Pi^s(\mathcal{R}) = \Pi^s(\mathcal{R}')$, and the statement follows trivially. □

So, nearly all of the postulates listed in Section 1 could be shown to be fulfilled by the structural inference relation $\mathrel{|\!\sim}^s_{\mathcal{R}}$. Only Rational Monotonicity has not yet been considered, and here we observe a failure – $\mathrel{|\!\sim}^s_{\mathcal{R}}$ does not satisfy Rational Monotonicity, as the following example shows.

Example 10 Let a, b, c be atoms of the language $\mathcal{L}$, and let $\mathcal{R} = \{(a\overline{b}c|ab\overline{c} \vee a\overline{b}c), (abc|abc \vee a\overline{b}\overline{c})\}$, with α_1^+, α_1^- be associated with the first conditional, and α_2^+, α_2^- with the second. Then $a \mathrel{|\!\sim}^s_{\mathcal{R}} c$, since $\sigma_{\mathcal{R}}(ac) = \{\alpha_1^+, \alpha_2^+\}$ and $\sigma_{\mathcal{R}}(a\overline{c}) = \{\alpha_1^-, \alpha_2^-\}$. On the other hand, $\sigma_{\mathcal{R}}(ab) = \{\alpha_1^-, \alpha_2^+\}$, and $\sigma_{\mathcal{R}}(a\overline{b}) = \{\alpha_1^+, \alpha_2^-\}$, so we can not derive that $a \mathrel{|\!\sim}^s_{\mathcal{R}} b$. But also $\alpha_2^+ = \sigma_{\mathcal{R}}(abc)$ and $\alpha_1^- = \sigma_{\mathcal{R}}(ab\overline{c})$ are incomparable with respect to $>$, so we do not have $ab \mathrel{|\!\sim}^s_{\mathcal{R}} c$. ■

This example is quite a formal one and gives little evidence whether this inference behavior is more reasonable than Rational Monotonicity, or the other way around. There are, however, other works that question Rational Monotonicity as a general inference rule (for a more intuitive counterexample, cf. [Benferhat *et al.*, 2000, p. 35]). The point with Rational Monotonicity is, that it considers only bilateral relationships (between A and B, respectively A and C), but is not apt to take complex interdependencies between all three formulas A, B, C, or between the corresponding conditionals, respectively, into account. A way to make interactions between conditionals explicit is to use the two relations $\sqsubseteq$ (*subconditionality*) and $\perp\!\!\!\perp$ (*perpendicularity*) which are defined on the set of conditionals, as follows (see also [Kern-Isberner, 2001b]):

$$\begin{array}{lll} (D|C) \sqsubseteq (B|A) & \text{iff} & CD \models AB \text{ and } C\overline{D} \models A\overline{B} \\ (D|C) \perp\!\!\!\perp (B|A) & \text{iff} & \text{either } C \models AB, \\ & & \text{or } C \models A\overline{B}, \text{ or } C \models \overline{A} \end{array}$$

If $(D|C) \sqsubseteq (B|A)$, then $(D|C)$ is a *subconditional* of $(B|A)$; if $(D|C) \perp\!\!\!\perp (B|A)$, then $(D|C)$ is called *perpendicular* to $(B|A)$. Both relations are non-symmetrical. Although they symbolize quite opposite ways of conditional interaction, they both aim at excluding conflicts between the involved conditionals. Subconditionals of a conditional are in complete accordance with that conditional, they only possibly apply to fewer worlds. On the other hand, perpendicularity expresses a kind of irrelevance of one conditional for another one. In our structural framework, it can be characterized, as follows:

$$(D|C) \perp\!\!\!\perp (B|A) \quad \text{iff} \quad \sigma_{(B|A)}(\omega) = \sigma_{(B|A)}(\omega') \text{ for all } \omega, \omega' \models C$$

The next proposition illustrates a connection between perpendicularity and an irrelevance property of the structural inference relation $\mid\!\sim^s_{\mathcal{R}}$.

Proposition 11 *Let C, D, E be propositional formulas such that $C \mid\!\sim^s_{\mathcal{R}} D$, and for all $(B|A) \in \mathcal{R}$, it holds that $(E|CD) \perp\!\!\!\perp (B|A)$. Then $CE \mid\!\sim^s_{\mathcal{R}} D$.*

Proof. Let $\succeq_\pi \in \Pi^s(\mathcal{R})$, arbitrary. Then $CD \succ_\pi C\overline{D}$, since $C \mid\!\sim^s_{\mathcal{R}} D$. In particular, for each $\omega \models CE\overline{D}$, there is $\omega_0 \models CD$ such that $\omega_0 \succ_\pi \omega$. By presupposition, $(E|CD)$ is perpendicular to any conditional in $\mathcal{R}$, so for all $\omega' \models CD$, $\sigma_{\mathcal{R}}(\omega') = \sigma_{\mathcal{R}}(\omega_0)$. From the s-compatibility of $\succeq_\pi$, we conclude $\omega' \approx_\pi \omega_0$, and hence $\omega' \succ_\pi \omega$, in particular, for all $\omega' \models CED$. Therefore, we have $CE \mid\!\sim^s_{\mathcal{R}} D$. $\square$

Moreover, based on the two relations $\sqsubseteq$ and $\perp\!\!\!\perp$, a sufficient condition for a defeasible consequence from $\mathcal{R}$ can be given:

Proposition 12 *Let $C, D \in \mathcal{L}$ be propositional formulas such that for all $(B|A) \in \mathcal{R}$, it holds that either $(D|C) \sqsubseteq (B|A)$, or $(D|C) \perp\!\!\!\perp (B|A)$, and there is at least one conditional $(B|A) \in \mathcal{R}$ such that $(D|C) \sqsubseteq (B|A)$. Then $C \mid\!\sim^s_{\mathcal{R}} D$.*

Proof. Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$. Without loss of generality, we assume $(D|C) \sqsubseteq (B_1|A_1), \ldots, (B_m|A_m)$, and $(D|C) \perp\!\!\!\perp (B_{m+1}|A_{m+1}), \ldots, (B_n|A_n)$, where $1 \leqslant m \leqslant n$. Then for any $\omega \models CD$, $\omega' \models C\overline{D}$, we have $sign_i(\sigma_{\mathcal{R}}(\omega)) = +, sign_i(\sigma_{\mathcal{R}}(\omega')) = -$, for $1 \leqslant i \leqslant m$, and $sign_i(\sigma_{\mathcal{R}}(\omega)) = sign_i(\sigma_{\mathcal{R}}(\omega'))$ for $m < i \leqslant n$. Hence $\sigma_{\mathcal{R}}(\omega) > \sigma_{\mathcal{R}}(\omega')$, and Proposition 4 yields $C \mid\!\sim^s_{\mathcal{R}} D$. $\square$

In summary, the structural inference relation $\mid\!\sim^s_{\mathcal{R}}$ satisfies all of the postulates listed in Section 1, except for Rational Monotonicity. It must be emphasized that, although s-compatible comparative possibility distributions are only superficially linked to conditional structures, the concept of conditional structures is strong enough to establish a surprising well-behavedness of the corresponding inference relation. There is, however, an even more powerful application of conditional structures for possibility distributions, which will be developed in the following section.

5 Default reasoning based on conditional structures

In this section, we will leave the purely qualitative framework and deal with possibility distributions instead of the induced orderings. Although we are still mainly interested in qualitative relationships, calculating with numbers will make it easier to deal with complex interactions of conditionals. It is straightforward to define s-compatibility with a set of conditionals $\mathcal{R}$ (see Definition 3) for possibility distributions, too, but we will now go one step further and focus on possibility distributions which can be considered as numerical realizations of conditional structures – each of the generators $\mathbf{a}_i^+, \mathbf{a}_i^-$ of $\mathcal{F}_{\mathcal{R}}$ is mapped to a positive real number α_i^+, α_i^-, respectively, in an appropriate way to build up a possitibility distribution representing $\mathcal{R}$. Such possibility distributions will be called *c-representations*.

Definition 13 A positive possibility distribution π is called a *c-representation* of $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ iff $\pi \models \mathcal{R}$ and there are positive real numbers $\alpha_0, \alpha_1^+, \alpha_1^-, \ldots, \alpha_n^+, \alpha_n^- \in \mathbb{R}^+$ such that for all $\omega \in \Omega$,

$$\pi(\omega) = \alpha_0 \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \alpha_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \alpha_i^- \tag{2}$$

The α_0 in (2) is only a normalization factor to ensure that $\pi(\top) = 1$. The representation property of a distribution of type (2) can be expressed directly by the factors $\alpha_1^+, \alpha_1^-, \ldots, \alpha_n^+, \alpha_n^-$:

Proposition 14 *A possibility distribution $\pi(\omega) = \alpha_0 \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \alpha_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \alpha_i^-$ with $\alpha_0, \alpha_1^+, \alpha_1^-, \ldots, \alpha_n^+, \alpha_n^- \in \mathbb{R}^+$ represents $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ iff for all $i, 1 \leqslant i \leqslant n$,*

$$\alpha_i^+ \cdot \max_{\omega \models A_i B_i} \prod_{\substack{j \neq i \\ \omega \models A_j B_j}} \alpha_j^+ \prod_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \alpha_j^- \;>\; \alpha_i^- \cdot \max_{\omega \models A_i \overline{B}_i} \prod_{\substack{j \neq i \\ \omega \models A_j B_j}} \alpha_j^+ \prod_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \alpha_j^- \tag{3}$$

Actually, (3) is equivalent to $\pi(A_i B_i) > \pi(A_i \overline{B}_i)$, $1 \leqslant i \leqslant n$. We will call a mapping $I : \{\mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^-\} \to \mathbb{R}^+, \mathbf{a}_i^+ \mapsto \alpha_i^+, \mathbf{a}_i^- \mapsto \alpha_i^-, 1 \leqslant i \leqslant n$, such that (3) holds, a *c-interpretation of $\mathcal{R}$*. From c-interpretations, c-representations arise in a straightforward way via the formal tool of conditional structures.

The structure imposed on possible worlds by a c-representation is as finely grained as can be justified by differences in conditional structures. More precisely, c-representations are *conditionally indifferent with respect to* $\mathcal{R}$ in the following sense: For any two multi-sets of worlds, $\{\omega_1 : r_1, \ldots, \omega_m : r_m\}$ and $\{\nu_1 : s_1, \ldots, \nu_p : s_p\}$[2] with the same number of elements $\sum_{j=1}^{m} r_j = \sum_{k=1}^{p} s_k$, it holds that $\pi(\omega_1)^{r_1} \ldots \pi(\omega_m)^{r_m} = \pi(\nu_1)^{s_1} \ldots \pi(\nu_p)^{s_p}$ whenever $\sigma_{\mathcal{R}}(\omega_1)^{r_1} \ldots \sigma_{\mathcal{R}}(\omega_m)^{r_m} = \sigma_{\mathcal{R}}(\nu_1)^{s_1} \ldots \sigma_{\mathcal{R}}(\nu_p)^{s_p}$. This means that, in following conditional structures, c-representations handle interactions of conditionals in a most adequate way. For a more thorough discussion of conditional indifference, cf. [Kern-Isberner, 2001a; 2001b; 2001d; 2001c].

Let $\mid\!\sim^{c}_{\mathcal{R}}$ be the (nonmonotonic) inference relation based on considering all c-representations:

$$A \mid\!\sim^{c}_{\mathcal{R}} B \quad \text{iff} \quad \pi_c(AB) > \pi_c(A\overline{B}) \quad \text{for all c-representations } \pi_c \text{ of } \mathcal{R},$$

where $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L}), A, B \in \mathcal{L}$.

It is easily proved that $\mid\!\sim^{c}_{\mathcal{R}}$ also satisfies the basic properties of system P, *irrelevance*, *specificity*, *property inheritance*, *ambiguity preservation* and *duplication stability*, just in the same way as $\mid\!\sim^{s}_{\mathcal{R}}$ does. In fact, Propositions 6 and 9 can be rephrased for $\mid\!\sim^{c}_{\mathcal{R}}$, and all examples of Section 4 work for $\mid\!\sim^{s}_{\mathcal{R}}$ as well. Furthermore, in the next example, also illustrating *property inheritance*, we will show how inference based on c-representations yields more plausible conclusions than that based on s-representations. The reason for this is that equations (3) often imply qualitative relations between the $\alpha_1^+, \alpha_1^-, \ldots, \alpha_n^+, \alpha_n^-$ which can be used for inferences.

Example 15 Let $\mathcal{R}$ consist of the following conditionals:

$r_1 : (w\|p)$	*Penguins have wings.*
$r_2 : (f\|w)$	*Winged animals fly.*
$r_3 : (\overline{f}\|p)$	*Penguins do not fly.*
$r_4 : (b\|p)$	*Penguins are birds.*
$r_5 : (e\|b)$	*Birds lay eggs.*

In Table 1, we list the conditional structures of all possible worlds with respect to $\mathcal{R}$. Possibility values of c-representations π_c can be calculated directly from that table. For instance, $\pi_c(pw\overline{f}be) = \alpha_0 \alpha_1^+ \alpha_2^- \alpha_3^+ \alpha_4^+ \alpha_5^+$, where the α's are suitably chosen so as to fulfill (3) for that special set of conditionals. For α_4^+, α_4^- and α_5^+, α_5^-, e.g., we obtain from (3)

$$\begin{aligned}
&\alpha_4^+ \cdot \max\{\alpha_1^+\alpha_2^+\alpha_3^-\alpha_5^+, \alpha_1^+\alpha_2^+\alpha_3^-\alpha_5^-, \alpha_1^+\alpha_2^-\alpha_3^+\alpha_5^+, \\
&\qquad \alpha_1^+\alpha_2^-\alpha_3^+\alpha_5^-, \alpha_1^-\alpha_3^-\alpha_5^+, \alpha_1^-\alpha_3^-\alpha_5^-, \\
&\qquad \alpha_1^-\alpha_3^+\alpha_5^+, \alpha_1^-\alpha_3^+\alpha_5^-\} \\
&> \alpha_4^- \cdot \max\{\alpha_1^+\alpha_2^+\alpha_3^-, \alpha_1^+\alpha_2^-\alpha_3^+, \alpha_1^-\alpha_3^-, \alpha_1^-\alpha_3^+\}
\end{aligned}$$

$$\begin{aligned}
&\alpha_5^+ \cdot \max\{\alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^+, \alpha_1^+\alpha_2^-\alpha_3^+\alpha_4^+, \alpha_1^-\alpha_3^-\alpha_4^+, \\
&\qquad \alpha_1^-\alpha_3^+\alpha_4^+, \alpha_2^+, \alpha_2^-, 1\} \\
&> \alpha_5^- \cdot \max\{\alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^+, \alpha_1^+\alpha_2^-\alpha_3^+\alpha_4^+, \alpha_1^-\alpha_3^-\alpha_4^+, \\
&\qquad \alpha_1^-\alpha_3^+\alpha_4^+, \alpha_2^+, \alpha_2^-, 1\}
\end{aligned}$$

So, $\alpha_5^+ > \alpha_5^-$, and therefore,

$$\begin{aligned}
&\alpha_4^+\alpha_5^+ \cdot \max\{\alpha_1^+\alpha_2^+\alpha_3^-, \alpha_1^+\alpha_2^-\alpha_3^+, \alpha_1^-\alpha_3^-, \alpha_1^-\alpha_3^+\} \\
&> \alpha_4^- \cdot \max\{\alpha_1^+\alpha_2^+\alpha_3^-, \alpha_1^+\alpha_2^-\alpha_3^+, \alpha_1^-\alpha_3^-, \alpha_1^-\alpha_3^+\},
\end{aligned}$$

which implies $\alpha_4^+\alpha_5^+ > \alpha_4^-$.

ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$
$pwfbe$	$\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_4^+\mathbf{a}_5^+$	$\overline{p}wfbe$	$\mathbf{a}_2^+\mathbf{a}_5^+$
$pwfb\overline{e}$	$\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}wfb\overline{e}$	$\mathbf{a}_2^+\mathbf{a}_5^-$
$pwf\overline{b}e$	$\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_4^-$	$\overline{p}wf\overline{b}e$	$\mathbf{a}_2^+$
$pwf\overline{b}\overline{e}$	$\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_4^-$	$\overline{p}wf\overline{b}\overline{e}$	$\mathbf{a}_2^+$
$pw\overline{f}be$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^+$	$\overline{p}w\overline{f}be$	$\mathbf{a}_2^-\mathbf{a}_5^+$
$pw\overline{f}b\overline{e}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}w\overline{f}b\overline{e}$	$\mathbf{a}_2^-\mathbf{a}_5^-$
$pw\overline{f}\,\overline{b}e$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^-$	$\overline{p}w\overline{f}\,\overline{b}e$	$\mathbf{a}_2^-$
$pw\overline{f}\,\overline{b}\overline{e}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^-$	$\overline{p}w\overline{f}\,\overline{b}\overline{e}$	$\mathbf{a}_2^-$
$p\overline{w}fbe$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_4^+\mathbf{a}_5^+$	$\overline{p}\,\overline{w}fbe$	$\mathbf{a}_5^+$
$p\overline{w}fb\overline{e}$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}\,\overline{w}fb\overline{e}$	$\mathbf{a}_5^-$
$p\overline{w}f\overline{b}e$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_4^-$	$\overline{p}\,\overline{w}f\overline{b}e$	1
$p\overline{w}f\overline{b}\overline{e}$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_4^-$	$\overline{p}\,\overline{w}f\overline{b}\overline{e}$	1
$p\overline{w}\overline{f}be$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^+$	$\overline{p}\,\overline{w}\overline{f}be$	$\mathbf{a}_5^+$
$p\overline{w}\overline{f}b\overline{e}$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}\,\overline{w}\overline{f}b\overline{e}$	$\mathbf{a}_5^-$
$p\overline{w}\overline{f}\,\overline{b}e$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^-$	$\overline{p}\,\overline{w}\overline{f}\,\overline{b}e$	1
$p\overline{w}\overline{f}\,\overline{b}\overline{e}$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^-$	$\overline{p}\,\overline{w}\overline{f}\,\overline{b}\overline{e}$	1

Table 1: Conditional structures for Example 15

Although penguins are exceptional birds in that they are not able to fly, they should still inherit the property of laying eggs from their superclass *birds*. This is not guaranteed in general possibilistic inference – the conditional $(e|b)$ is "drowned" by the conditionals $(w|p), (\overline{f}|p)$ which obtain a higher degree of possibility (see [Benferhat *et al.*, 1997, p. 273]). Note that also no application of Proposition 4 is possible to yield an easy $\mid\!\sim^{s}_{\mathcal{R}}$-inference, since $\sigma_{\mathcal{R}}(p\overline{e})$ contain some conditional structures which are not dominated by conditional structures in $\sigma_{\mathcal{R}}(pe)$. By following conditional

[2]The numbers r_j, s_k denote the number of copies of the corresponding world to be contained in the multi-set.

structures as in c-representations, however, the impact of $(e|b)$ can be preserved to establish the validity of $(e|p)$: For any c-representation π_c of $\mathcal{R}$, we compute

$$\begin{aligned}\pi_c(pe) &= \alpha_0 \max\{\alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^+\alpha_5^+, \alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^-, \\ &\quad \alpha_1^+\alpha_2^-\alpha_3^+\alpha_4^+\alpha_5^+, \alpha_1^+\alpha_2^-\alpha_3^+\alpha_4^-, \\ &\quad \alpha_1^-\alpha_3^-\alpha_4^+\alpha_5^+, \alpha_1^-\alpha_3^-\alpha_4^-, \\ &\quad \alpha_1^-\alpha_3^+\alpha_4^+\alpha_5^+, \alpha_1^-\alpha_3^+\alpha_4^-\} \\ \pi_c(p\overline{e}) &= \alpha_0 \max\{\alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^+\alpha_5^-, \alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^-, \\ &\quad \alpha_1^+\alpha_2^-\alpha_3^+\alpha_4^+\alpha_5^-, \alpha_1^+\alpha_2^-\alpha_3^+\alpha_4^-, \\ &\quad \alpha_1^-\alpha_3^-\alpha_4^+\alpha_5^-, \alpha_1^-\alpha_3^-\alpha_4^-, \\ &\quad \alpha_1^-\alpha_3^+\alpha_4^+\alpha_5^-, \alpha_1^-\alpha_3^+\alpha_4^-\}\end{aligned}$$

By comparing the sets which contribute to the maxima element by element, we see that $\pi_c(pe) > \pi_c(p\overline{e})$, due to $\alpha_5^+ > \alpha_5^-$ and $\alpha_4^+\alpha_5^+ > \alpha_4^-$ (e.g., $\alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^+\alpha_5^+ > \alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^-$ and $\alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^+\alpha_5^+ > \alpha_1^+\alpha_2^+\alpha_3^-\alpha_4^+\alpha_5^-$). This means, that $\pi_c \models (e|p)$, for all c-representations π_c of $\mathcal{R}$. Hence, $p \mid\!\sim^c_{\mathcal{R}} e$. ■

In this way, c-representations combine structural with qualitative information to handle conditional interactions.

Our general approach based on conditional structures and c-representations may be simplified when combined with the *principle of least commitment* and with the *principle of minimum specificity*, that is, *choose the α's in a least committed and least specific way*. The consequences of these principles are twofold:

1. To allow worlds verifying conditionals to be as normal and least specific as possible, set $\alpha_i^+ = 1$ for $1 \leqslant i \leqslant n$. Since $\mathcal{R}$ is consistent iff there are worlds not falsifying any of the conditionals in $\mathcal{R}$, this implies $\alpha_0 = 1$. Furthermore, to comply with intuition (falsifying conditionals should decrease the possibility of worlds), we choose $\alpha_i^- < 1$ for $1 \leqslant i \leqslant n$.
2. In order not to introduce any relations which are not explicitly sanctioned by the representation postulate $\pi \models \mathcal{R}$, choose $\alpha_1^-, \ldots, \alpha_n^-$ according to (3), but in a "least committed way". Since only qualitative relationships count, this comes down to the following: The set $\{\alpha_1^-, \ldots, \alpha_n^-\}$ is partitioned into sets $\mathcal{A}_1, \ldots, \mathcal{A}_m$ such that the number m of partitioning sets is minimal, each α_i^- is in a partitioning set $\mathcal{A}_j$ with an index as minimal as possible, and the partitioning is compatible with the ordering between $\alpha_1^-, \ldots, \alpha_n^-$, in that if $\alpha_{i_1}^- \in \mathcal{A}_j$ and $\alpha_{i_2}^- \in \mathcal{A}_k$ with $j < k$, then $\alpha_{i_1}^- > \alpha_{i_2}^-$.

An *lcc-representation* of $\mathcal{R}$ is a c-representation $\pi_{lcc}(\omega) = \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i\overline{B_i}}} \alpha_i^-$ of $\mathcal{R}$ whose factors α_i^- satisfy both requirements 1. and 2. above. In particular, $\alpha_i^- < 1$, and

$$\max_{\omega \models A_iB_i} \prod_{\substack{j \neq i \\ \omega \models A_j\overline{B}_j}} \alpha_j^- > \alpha_i^- \cdot \max_{\omega \models A_i\overline{B}_i} \prod_{\substack{j \neq i \\ \omega \models A_j\overline{B}_j}} \alpha_j^-,$$

$1 \leqslant i \leqslant n$, for lcc-representations π_{lcc}. Let $\mid\!\sim^{lcc}_{\mathcal{R}}$ be the inference relation based on considering all lcc-representations:

$$A \mid\!\sim^{lcc}_{\mathcal{R}} B \quad \text{iff} \quad \pi_{lcc}(AB) > \pi_{lcc}(A\overline{B}) \quad \text{for all lcc-representations } \pi_{lcc} \text{ of } \mathcal{R}.$$

The approach we thereby obtain is very similar to the *lcd-approach* of Benferhat, Saffiotti and Smets [Benferhat *et al.*, 2000]. Instead of *proto-infinitesimals* and *infinitesimals* which are used in that paper, we use formal symbols $\mathbf{a}_1^-, \ldots, \mathbf{a}_n^-$ and positive real numbers $\alpha_1^-, \ldots, \alpha_n^-$. So our c-interpretations are nearly the same as their interpretations of proto-infinitesimals. Their *auto-deduction principle* is the same as our (quite implicit) representation postulate $\pi \models \mathcal{R}$. They use belief functions, but actually base their considerations upon the corresponding plausibility functions; we use possibility distributions (which correspond to plausibility functions) directly. The basic idea to construct infinitesimal lcd-belief functions is to interpret proto-infinitesimals according to a *least committed ϵ-stratification*, where each ϵ-stratification corresponds to a partition of $\mathcal{R}$, and least committedness comes down to minimizing the number of classes, and to assign as many proto-infinitesimals (i.e. α_i^- in our framework) as possible to classes with low index. This amounts exactly to choose $\alpha_1^-, \ldots, \alpha_n^-$ according to (3) in a least committed way, as described above in 2. So our $\mid\!\sim^{lcc}_{\mathcal{R}}$ and $\mid\!\sim_{lcd}$ of [Benferhat *et al.*, 2000] coincide.

When comparing this present paper to [Benferhat *et al.*, 2000], it must be emphasized that both papers agree on the ideas of taking each default rule as an independent piece of information and of aggregating the effects of default rules in a mainly structural way. The crucial difference between our work and that of Benferhat, Saffiotti and Smets [Benferhat *et al.*, 2000] is how these ideas are realized, in particular, how the special product form of possibility distributions (see (2)), or belief functions in [Benferhat *et al.*, 2000], respectively, is motivated: Benferhat, Saffiotti and Smets derive the product form (of the corresponding plausibility functions) from Dempster's combination rule, whereas in our framework, (2) arises from an underlying conditional theory, which is much more general and applies not only in a possibilistic environment, but also in other quantitative (probabilistic) and semi-quantitative (ordinal conditional functions) environments and has also important consequences for handling conditionals adequately in a purely qualitative manner (cf. [Kern-Isberner, 2001b]). Actually, in the framework of ordinal conditional functions, the application of our conditional approach turns out to be a generalization of system-Z^* (cf. [Kern-Isberner, 2001c]).

6 Conclusion

In this paper, we presented several approaches to default reasoning in a possibilistic framework which all are based on considering *conditional structures* of worlds. Conditional structures encode the effects conditionals have on worlds and make conditional interactions transparent and computable, providing a sound algebraic framework for default inferences. Even the weakest of our inference relations, $\mid\!\sim^{s}_{\mathcal{R}}$, which merely demands a compatibility with conditional structures, proved to be quite well-behaved, according to widely accepted standards. The other two inference relations investigated here, $\mid\!\sim^{c}_{\mathcal{R}}$ and $\mid\!\sim^{lcc}_{\mathcal{R}}$, are based on faithful possibilistic realizations of conditional structures. Both these approaches allow to handle implicit priorities among the default rules under consideration and thereby combine structural with qualitative information. $\mid\!\sim^{c}_{\mathcal{R}}$ reflects conditional structures in full generality, whereas $\mid\!\sim^{lcc}_{\mathcal{R}}$ (which turns out to be equivalent to $\mid\!\sim_{lcd}$ of [Benferhat *et al.*, 2000]) is also guided by postulates of least specificity and least committedness.

The concept of conditional structures was initiated when investigating maximum entropy inference (cf. [Kern-Isberner, 1998]). The same crucial principle, namely that of observing conditional structures, underlies the possibilistic approaches to default reasoning presented in this paper, as well as the probabilistic maximum entropy approach, the generalized system-Z^* [Goldszmidt *et al.*, 1993; Kern-Isberner, 2001c] and the lcd-approach of Benferhat, Saffiotti and Smets [Benferhat *et al.*, 2000]. Independent of numerical realizations, however, this paper reveals clearly the relevance of structural arguments in default reasoning.

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