Handling conditionals adequately in uncertain reasoning and belief revision

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ABSTRACT. *Conditionals (*"if-then-rules"*) are most important objects in knowledge representation, commonsense reasoning and belief revision. Due to their non-classical nature, however, they are not easily dealt with. This paper presents a new approach to conditionals, which is apt to capture their dynamic power particularly well. We show how this approach can be applied to represent conditional knowledge inductively, and to guide revisions of epistemic states by sets of (propositional or conditional) beliefs. In particular, we generalize system-Z^* as an appropriate counterpart to maximum entropy-representations in a semi-quantitative setting, and provide a theoretical justification to make its basic ideas usable also for belief revision.*

RÉSUMÉ. *Les conditionnels (" règles si-alors ") sont des objets très importants en représentation des connaissances et en révision. En raison de leur nature non classique, les traiter est chose délicate. Cet article présente une nouvelle approche des conditionnels, qui est particuli'erement apte à rendre compte de leur côté dynamique. Nous montrons comment appliquer cette approche pour représenter les connaissances conditionnelles de manière inductive et guider les révisions d'états épistémiques par des ensembles de croyances (propositionnelles ou conditionnelles). En particulier, nous généralisons le système Z comme le pendant de représentations du maximum d'entropie dans un cadre semi-quantitatif, et nous fournissons une justification théorique permettant aussi l'application de ces idées de base à la révision.*

KEYWORDS: *Conditionals, uncertain reasoning, belief revision, default reasoning, inductive knowledge representation, probabilistics*

MOTS-CLÉS : *Conditionnels, raisonnement incertain, révision, raisonnement par défaut, représentation de connaissance inductive, probabilités*

1. Introduction

Relationships amongst propositions and propositional variables are crucial pieces of knowledge. Usually, they express plausible connections, bring isolated facts together and help us obtain a coherent image of the world. Such relationships may be represented in a most general form by *if-then-conditionals*.

Conditionals are omnipresent, in everyday life as well as in scientific environments. We make use of conditional knowledge when we avoid puddles on sidewalks (being aware of "If you step into a puddle, then your feet might get wet"), and when we expect a student to be young (due to "If someone is a student then he is mostly young.") Conditionals represent generic knowledge, acquired inductively from experience or learned from books. They tie a flexible and highly interrelated network of connections along which reasoning is possible and which can be applied to different situations. Moreover, as providing plausible yet defeasible conclusions, conditionals are intimately related to *nonmonotonic reasoning*, or, in general, to *uncertain reasoning*. In belief revision, they take the role of *revision policies*, guiding changes of beliefs when new (propositional) evidence becomes apparent. So, in contrast to factual knowledge which is mostly static, conditionals bear a clearly dynamic flavor.

The key to get conditionals right is to accept their non-classical nature – conditionals are not simply "true" or "false". In a particular situation, a conditional is *applicable* (you actually step into a puddle) or not (you simply walk around), it can be found *confirmed* (you step into a puddle and indeed, your feet get wet) or *violated* (you step into a puddle, but your feet remain dry because you are wearing rain boots). So the central problem in dealing with conditional knowledge is to handle adequately, on the one hand, inactive (or neutral, respectively) behavior, and, on the other hand, active as well as polarizing behavior. The main object of this paper is

– to sketch a new theory for conditionals which captures their three-valued, dynamic nature particularly well,

– to show how it is realized in an ordinal as well as in a probabilistic framework,

– to apply this theory to the problem of representing conditional knowledge inductively in both frameworks,

– to make use of it when revising epistemic states by sets of conditional beliefs.

We will focus on the process of establishing conditional beliefs, considering conditionals as agents shifting possible worlds in order to establish relationships and beliefs. We will represent the effects (the learning of) conditionals have on worlds by *conditional structures*. Handling conditionals adequately then means to observe these structures, and to choose representations and revisions, respectively, which are balanced with respect to the structures of the conditionals under consideration. In this way, highly complex interactions between different conditionals can be taken into account and maintained. A well-known probabilistic method that follows this idea is the *principle of maximum entropy (ME-principle)* , respectively, *the principle of minimum cross-entropy* for probabilistic belief revision [PAR 94]. In a semi-quantitative setting, system-Z^* [GOL 93] also realizes this approach, though it seems to be of restricted applicability. In this paper, we will generalize system-Z^* to make it applicable for any consistent set of conditionals. The generalized system-Z^*-approach will be shown to be theoretically justified, and will prove to handle even problematic examples adequately.

Although this generalization of system-Z^* is important in itself, it should be taken as only one example of how the conditional approach presented in this paper can be used. By virtue of separating strictly between structural and numerical aspects of conditionals, the new conditional theory can be applied in any (semi-)quantitative framework that allows the representation of conditional beliefs. Therefore, it has also important consequences for possibility theory, which has seen a lot of important work on conditionals in the last decade (see e.g. [BEN 97, BEN 00]). The realization of our approach in a possibilistic framework has been presented in [KER 01b]. Here, we will deal with conditionals in probabilistic as well as in ordinal settings, illustrating the applicability of the theory to different domains and thus pointing out its fundamental meaning for conditional reasoning. Moreover, we will elaborate in an ordinal setting how it gives rise to a simple and constructive approach to revise epistemic states by sets of conditionals. Both prior knowledge and new conditional information are observed in a well defined way as far as possible. Similar results can be obtained in a probabilistic framework.

This paper is based on the methods presented in [KER 01a] which are worked out in an ordinal environment and directly related to other approaches like system-Z^* here. The rest of this paper is organized as follows: The following section summarizes fundamental facts about conditionals, ordinal conditional functions and probability distributions. Section 3 briefly sketches the ME-principle, as well as system-Z and system-Z^*. Section 4 presents the generalized system-Z^* approach. The new dynamic approach to conditionals is presented in Section 5, and is linked to (semi-) quantitative representation of knowledge in Section 6. Section 7 shows the application of the conditional theory to belief revision. An outlook on further results concludes this paper in Section 8.

2. Conditionals, plausibility and probability

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{V} = \{a, b, c \ldots\}$. Let Ω be the complete set of interpretations of $\mathcal{L}$, where each $\omega \in \Omega$ is taken to be a *possible world* for $\mathcal{L}$. To simplify notation, we will write $\overline{A}$ instead of $\neg A$, and AB instead of $A \wedge B$, for formulas $A, B \in \mathcal{L}$. Conditionals $(B|A)$ represent statements of the form "*If A then (plausibly) B*", expressing a relationship between two (propositional) formulas A, the *antecedent* or *premise*, and B, the *consequent*. $(\mathcal{L} \mid \mathcal{L})$ denotes the set of all conditionals $(B|A)$ with $A, B \in \mathcal{L}$. A conditional $(B|\top)$ with a tautological antecedent is taken to correspond to its (propositional) consequent, B. A world $\omega \in \Omega$ is said to *verify* a conditional $(B|A)$, if $\omega \models AB$; it *falsifies* $(B|A)$, if $\omega \models A\overline{B}$; if $\omega \models \overline{A}$, then $(B|A)$ is not applicable to ω.

Generally, a conditional $(B|A)$ is considered to be accepted if its confirmation, AB, is judged to be more plausible than its refutation, $A\overline{B}$. So, to validate conditionals, we need richer epistemic structures than plain propositional interpretations, at least to compare different worlds with regard to their relevance for a conditional (see, for example, [DAR 97]). *Epistemic states* as representations of cognitive states of intelligent agents provide an adequate framework for conditionals. Two widely used types of epistemic states are *probability distributions* and *ordinal conditional functions, OCF's* (or simply *ranking functions*), which are based on the notion of plausibility. In short, OCF's are functions $\kappa : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$[1] from the set of worlds to the natural numbers, extended by ∞. They specify non-negative integers as degrees of plausibility – or, more precisely, as degrees of *disbelief* – for worlds. The smaller $\kappa(\omega)$ is, the more plausible the world ω appears, so that worlds ω with $\kappa(\omega) = 0$ are considered most plausible, and what is believed (for certain) in the epistemic state represented by κ is described precisely by the set $\mathit{Mod}(\kappa) := \{\omega \in \Omega \mid \kappa(\omega) = 0\}$. For propositional formulas $A, B \in \mathcal{L}$, we set $\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$. Thus, the ranking of a disjunction is the least ranking of each of its disjuncts: $\kappa(A \vee B) = \min\{\kappa(A), \kappa(B)\}$. In particular, $0 = \min\{\kappa(A), \kappa(\overline{A})\}$, so that at least one of A or $\overline{A}$ is considered most plausible. A proposition A is believed iff $\kappa(\overline{A}) > 0$, which is denoted by $\kappa \models A$. This may also be specified by degrees of plausibility (or of disbelief, respectively) by saying that $\kappa \models A[n]$ iff $\kappa(\overline{A}) > n$ ($n \in \mathbb{N}$). A conditional $(B|A) \in (\mathcal{L} \mid \mathcal{L})$ with $\kappa(A) < \infty$ may be assigned a degree of plausibility via $\kappa(B|A) = \kappa(AB) - \kappa(A)$. κ *satisfies* $(B|A)$, $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$, i.e. iff AB is more plausible than $A\overline{B}$. We can also specify a numerical degree of plausibility of a conditional by defining $\kappa \models (B|A)\,[n]$ iff $\kappa(AB) + n < \kappa(A\overline{B})$ ($n \in \mathbb{N}$). OCF's are the qualitative counterpart of probability distributions; their plausibility degrees may be taken as order-of-magnitude abstractions of probabilities (cf. [GOL 93, GOL 96]). For a probability distribution P, we have $P(B|A) = \frac{P(AB)}{P(A)}$ for $P(A) > 0$, and $P \models (B|A)\,[x]$ iff $P(B|A) = x$ ($x \in [0,1]$).

Plausibility structures, on the one hand, are much weaker than probability structures, but, on the other hand, they seem to be closer to commonsense reasoning. They have a crucial meaning for belief revision, too: Plausibility pre-orderings are known to characterize AGM-revisions of belief sets and of epistemic states (cf. [DAR 97]). The fundamental connection between revision operators, $*$, epistemic states, Ψ, and conditionals, $(B|A)$, is formalized by the so-called *Ramsey test*: $\Psi \models (B|A)$ iff $\Psi * A \models B$. That is, a $(B|A)$ is accepted in the epistemic state Ψ, iff revising Ψ by A yields belief in B. For an ordinal conditional function κ, this is in accordance with the plausibility relation imposed by κ. Each OCF κ induces a (propositional) AGM-revision operator $*$ by setting $\mathit{Mod}(\kappa * A) = \min_\kappa(\mathit{Mod}(A))$ (see [DAR 97]). The Ramsey test then reads $\kappa \models (B|A)$ iff $\kappa * A \models B$. As can easily be proved, this is equivalent to $\kappa(AB) < \kappa(A\overline{B})$, i.e. to AB being more plausible than $A\overline{B}$. So the Ramsey test is in accordance with the definitions given above.

We will consider mostly measure-free conditionals, focusing on structural aspects, but also allow quantifications. If $\mathcal{R}^* = \{(B_1|A_1)\,[x_1], \ldots, (B_n|A_n)\,[x_n]\}$ is a set of (appropriately) quantified conditionals, then $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ denotes

1. As usual, $\mathbb{N}$ is assumed to include 0.

the set of unquantified conditionals.Throughout this paper, we will assume all OCF's to be finite (i.e. taking on only natural numbers), and all probability distributions to be positive. All results to be presented also hold in the general case, but then need some technical modifications (see [KER 01a]).

3. Maximum entropy and System-$Z^{(*)}$

In this section, we review briefly well-known model-based approaches to represent conditional knowledge inductively.

For a consistent set $\mathcal{R}^* = \{(B_1|A_1)\,[x_1], \ldots, (B_n|A_n)\,[x_n]\}$ of probabilistic conditionals, the *ME-representation of* $\mathcal{R}^*$, *ME*$(\mathcal{R}^*)$, is the unique distribution P that maximizes the entropy $H(P) = -\sum_\omega P(\omega) \log P(\omega)$ subject to $P \models \mathcal{R}^*$ (see e.g. [PAR 94, KER 98]). *ME*$(\mathcal{R}^*)$ can be written as

$$ME(\mathcal{R}^*)(\omega) = \alpha_0 \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \alpha_i^{1-x_i} \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \alpha_i^{-x_i}, \qquad [1]$$

with the α_i's being exponentials of the Lagrange multipliers, appropriately chosen so as to make sure that all conditionals in $\mathcal{R}^*$ are satisfied by *ME*$(\mathcal{R}^*)$.

In a semi-quantitative framework, a well-known method to represent a (finite) set $\mathcal{R} = \{r_i : (B_i|A_i) \mid 1 \leqslant i \leqslant n\}$ of conditionals by an OCF is to apply the *system-Z* of Goldszmidt and Pearl [GOL 93, GOL 96]. The basic idea of system-Z is to observe the (logical) interactions of the conditionals in $\mathcal{R}$ which are described by the notion of *tolerance*. A conditional $(B|A)$ is said to be *tolerated* by $\mathcal{R}$ iff there is a world ω such that ω verifies $(B|A)$ and ω does not falsify any of the conditionals in $\mathcal{R}$. $\mathcal{R}$ is consistent iff there is an ordered partition $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_k$ of $\mathcal{R}$ such that each conditional in $\mathcal{R}_m$ is tolerated by $\bigcup_{j=m}^k \mathcal{R}_j$, $0 \leqslant m \leqslant k$ (see the procedure `Consistency-Test` of [GOL 96]). The system-Z ranking function, κ^z, representing $\mathcal{R}$ is given by

$$\kappa^z(\omega) = \begin{cases} 0, & \text{if } \omega \text{ does not falsify any } r_i, \\ 1 + \max\limits_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} Z(r_i), & \text{otherwise} \end{cases}$$

where $Z(r_i) = j$ iff $r_i \in \mathcal{R}_j$. κ^z assigns to each world ω the lowest possible rank admissible with respect to the constraints in $\mathcal{R}$.

A more sophisticated representation is obtained by combining the system-Z approach with the probabilistic ME-principle, yielding system-Z^* [GOL 93]. The corresponding Z^*-rankings of the conditionals in $\mathcal{R}$ have to satisfy (see [GOL 93])

$$Z^*(r_i) + \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) = 1 + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j), \qquad [2]$$

and κ^* is then calculated by

$$\kappa^*(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} Z^*(r_i) \qquad [3]$$

In [GOL 93], a procedure is given to calculate Z^*-rankings for so-called *minimal-core sets*, i.e. sets $\mathcal{R}$ of conditionals such that for each conditional $r \in \mathcal{R}$, there is a world $\omega \in \Omega$ that falsifies r and no other conditional in $\mathcal{R}$. We will illustrate system-Z and system-Z^* by an example.

Example 1 Consider the well-known *Nixon diamond*, consisting of the conditionals $r_1 : (p|q)$, *quakers are pacifists*, and $r_2 : (\overline{p}|r)$, *republicans are not pacifists*. $\{r_1, r_2\}$ is a minimal-core set, since $\omega = \overline{p}qr$ falsifies r_1 but not r_2, and $\omega = p\overline{q}r$ falsifies r_2 but not r_1. Each of the conditionals r_1, r_2 is tolerated by the other one, so the Z-rankings of r_1, r_2 are simply given by $Z(r_1) = Z(r_2) = 0$, and the Z^*-rankings are set to $Z^*(r_1) = Z^*(r_2) = 1$. For this example, both approaches yield the same ranking $\kappa^z = \kappa^*$. Table 1 shows the corresponding rankings for all possible worlds. As expected, κ^z (κ^*) is indeterminate with respect to the conditional $(p|qr)$:

ω	$\kappa^z(\omega)$	ω	$\kappa^z(\omega)$	ω	$\kappa^z(\omega)$	ω	$\kappa^z(\omega)$
pqr	1	$p\overline{q}r$	1	$\overline{p}qr$	1	$\overline{p}\,\overline{q}r$	0
$pq\overline{r}$	0	$p\overline{q}\,\overline{r}$	0	$\overline{p}q\overline{r}$	1	$\overline{p}\,\overline{q}\,\overline{r}$	0

Table 1. *Z- and Z^*-rankings for Example 1*

$\kappa^z(pqr) = 1 = \kappa^z(\overline{p}qr)$. From the knowledge base $\{r_1, r_2\}$, the system cannot decide whether a person, being both a quaker and a republican, is a pacifist or not. ∎

Z^*-entailment $\mid\!\sim^*$ is defined by $\mathcal{R} \mid\!\sim^* (B|A)$ iff $\kappa^*(AB) < \kappa^*(A\overline{B})$. System-$Z^*$ handles conditional knowledge quite appropriately (cf. [GOL 93]). In particular, Z^*-entailment satisfies the crucial property of *irrelevance*: If d is an atomic proposition not appearing in any of the conditionals in $\mathcal{R}$, then

$$\mathcal{R} \mid\!\sim^* (B|A) \quad \text{iff} \quad \mathcal{R} \mid\!\sim^* (B|Ad) \qquad [4]$$

Bourne and Parsons [BOU 99] generalized the system-Z^* approach in that they presented an algorithm that computes the Z^*-ranking, whenever [2] possesses a unique solution (also for non-minimal-core sets). This algorithm is also able to take variable strength of conditionals into account, that is, to calculate solutions to

$$Z^*(r_i) + \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) = m_i + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j), \qquad [5]$$

so that $\kappa^* \models (B|A)[m_i]$. There are, however, sets of conditionals that do not specify unique solutions to [2], or to [5], respectively, as the following example shows.

Example 2 For the set $\mathcal{R} = \{r_1 : (b|a), r_2 : (c|a), r_3 : (c|ab)\}$, [2] admits multiple solutions: $(Z^*(r_1), Z^*(r_2), Z^*(r_3))$ may be any one of, for instance, $(1, 0, 1)$, $(1, 1, 0)$, or even $(2, -1, 2)$ (cf. [BOU 99]). ■

Bourne and Parsons argue that some examples may be too complex to be dealt with by a "flat" system-Z^* approach and advise to use variable strength conditionals to enforce unique solvability. This seems a bit strange – Example 2 does not look very complex, and the probabilistic ME-method yields a unique solution even if all three conditionals are assigned the same probability.

The problem with Goldszmidt & Pearl's and Bourne & Parsons' work is that, in order to use as much of the inferential power of the ME-method as possible, they cling too closely to a probabilistic view. If the interactions between conditionals are too complex, as in the example above, then the rankings $Z^*(r_i)$ can not be computed uniquely from the (unique) ME-distribution. Instead, one should focus on how ME-techniques deal with conditional interactions, in order to understand the rationale behind the ME-probabilities.

This is exactly the idea we will present and pursue here. We will reveal the pattern that makes ME-methods a most adequate tool for handling conditional information and then transfer this pattern into the framework of semi-quantitative knowledge representation. This will bring forth a more general approach of which system-Z^* will turn out to be a special instance.

4. Generalizing system-Z^*

System-Z^* suggests a straightforward generalization for representing a set $\mathcal{R} = \{r_i : (B_i|A_i) \mid 1 \leqslant i \leqslant n\}$ of conditionals by an OCF: Use approach [3] and determine the $Z^*(r_i)$ so that κ^* satisfies all of the conditionals r_i, i.e. such that

$$Z^*(r_i) > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} Z^*(r_j) \qquad [6]$$

for $1 \leqslant i \leqslant n$. That means that condition [2] is weakened to be satisfied as an inequality constraint. Variable strengths of OCF-conditionals can be taken easily into account by adding them on the right-hand side of [6], thus generalizing [5]. Of course, thereby we obtain a bunch of possible rankings, losing the convenient unique solution. Moreover, this generalization seems to be quite ad hoc, leaving the ME-track and thus without theoretical justification.

Quite to the contrary – in the sequel, we will show that actually, this generalized approach emerged from a formal *principle of conditional indifference* that was formalized in [KER 98] as one of four axioms apt to characterize the ME-techniques. This principle is based on observing *conditional structures* and provides a powerful methodology, not only to represent conditional knowledge appropriately, but even to guide the revision of epistemic states by conditional beliefs. In the next section, we will briefly develop the necessary theoretical background, which is purely algebraic and therefore can be applied in a probabilistic as well as in a semi-quantitative setting. Just as for the ME-methods, it will provide an intelligible and appealing scheme for the inductive representation of conditionals.

5. A dynamic approach to conditionals

By observing the attitude of worlds with respect to it, each conditional $(B|A)$ can be considered as a generalized (three-valued) indicator function on worlds:

$$(B|A)(\omega) = \begin{cases} 1 & : \quad \omega \models AB \\ 0 & : \quad \omega \models A\overline{B} \\ u & : \quad \omega \models \overline{A} \end{cases} \qquad [7]$$

where u stands for *unknown*, *indeterminate*, or simply *inapplicable* (cf. [DEF 74, CAL 91]). Intuitively, representing or incorporating a conditional as a plausible conclusion in an epistemic state means to make – at least some – worlds verifying the conditional more plausible than the worlds falsifying it. In this sense, conditionals to be learned have effects on possible worlds (more exactly, on their degrees of plausibility or probability), shifting them appropriately to establish the intended relationship. Which worlds will actually be shifted depends on the chosen inductive representation procedure – for the conditional $(B|A)$, all worlds in either of the partitioning sets $AB, A\overline{B}$ and $\overline{A}$ are indistinguishable.

When we consider (finite) sets of conditionals $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$, we have to modify the representation [7] appropriately to identify the effect of each conditional in $\mathcal{R}$ on worlds in Ω. To this end, we replace the numbers 0 and 1 by abstract symbols, $\mathbf{a}_i^+, \mathbf{a}_i^-$, that we associate to each conditional $(B_i|A_i)$ in $\mathcal{R}$. Moreover, we will make use of a group structure to represent the joint impact of conditionals on worlds.

So let $\mathcal{F}_\mathcal{R} = \langle \mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^- \rangle$ be the free abelian group with generators $\mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^-$, i.e. $\mathcal{F}_\mathcal{R}$ consists of all elements of the form $(\mathbf{a}_1^+)^{r_1}(\mathbf{a}_1^-)^{s_1} \ldots (\mathbf{a}_n^+)^{r_n}(\mathbf{a}_n^-)^{s_n}$ with integers $r_i, s_i \in \mathbb{Z}$ (the ring of integers). Each element of $\mathcal{F}_\mathcal{R}$ can be identified by its exponents, so that $\mathcal{F}_\mathcal{R}$ is isomorphic to $\mathbb{Z}^{2n}$. (cf. [LYN 77, FIN 99]). The commutativity of $\mathcal{F}_\mathcal{R}$ corresponds to the fact that the conditionals in $\mathcal{R}$ shall be effective simultaneously, without assuming any order of application. So our way of dealing with conditionals is a symmetric, homogeneous one – we do not need user-defined priorities among conditionals. Note that, although we will speak of *multiplication* and *products* in $\mathcal{F}_\mathcal{R}$, the generators of $\mathcal{F}_\mathcal{R}$ are merely juxtaposed, like words.

For each $i, 1 \leqslant i \leqslant n$, we define a function $\sigma_i = \sigma_{(B_i|A_i)} : \Omega \to \mathcal{F}_\mathcal{R}$ by setting

$$\sigma_i(\omega) := \begin{cases} \mathbf{a}_i^+ & \text{if} \quad (B_i|A_i)(\omega) = 1 \\ \mathbf{a}_i^- & \text{if} \quad (B_i|A_i)(\omega) = 0 \\ 1 & \text{if} \quad (B_i|A_i)(\omega) = u \end{cases}$$

$\sigma_i(\omega)$ represents the manner in which the conditional $(B_i|A_i)$ applies to possible worlds $\omega \in \Omega$. The neutral element 1 of $\mathcal{F}_{\mathcal{R}}$ represents the non-applicability of $(B_i|A_i)$ in case that the antecedent A_i is not satisfied, so the neutral group element corresponds to a neutral attitude with respect to the conditional. The function $\sigma_{\mathcal{R}} : \Omega \to \mathcal{F}_{\mathcal{R}}$,

$$\sigma_{\mathcal{R}}(\omega) := \prod_{1 \leqslant i \leqslant n} \sigma_i(\omega) = \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \mathbf{a}_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \mathbf{a}_i^- \qquad [8]$$

describes the all-over effect of $\mathcal{R}$ on ω. $\sigma_{\mathcal{R}}(\omega)$ is called the *conditional structure of ω with respect to $\mathcal{R}$*. We will illustrate this notion of conditional structures in the following example which extends the *Nixon diamond* (see Example 1):

Example 3 Let $\mathcal{R}$ consist of the following conditionals:

$r_1 : (p|q)$ *Quakers are pacifists.*
$r_2 : (\overline{p}|r)$ *Republicans are not pacifists.*
$r_3 : (a|q)$ *Quakers are Americans.*
$r_4 : \ (b|a)$ *Americans like baseball.*
$r_5 : \ (\overline{b}|q)$ *Quakers do not like baseball.*

The conditional structure of a possible world, say $pqrab$, is calculated in the following way: $pqrab$ verifies the first conditional r_1 ($pqrab \models pq$), so we have $\sigma_1(pqrab) = \mathbf{a}_1^+$. $pqrab$, however, falsifies the second conditional $r_2 = (\overline{p}|r)$, thus $\sigma_2(pqrab) = \mathbf{a}_2^-$. In the same way, $\sigma_3(pqrab) = \mathbf{a}_3^+, \sigma_4(pqrab) = \mathbf{a}_4^+, \sigma_5(pqrab) = \mathbf{a}_5^-$. We obtain $\sigma_{\mathcal{R}}(pqrab) = \mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$. In Table 2, we list the conditional structures of all possible worlds with respect to $\mathcal{R}$. Using this table, it is easy to see that $\mathcal{R}$ is consistent, with partition $\mathcal{R}_0 = \{r_2, r_4\}$ and $\mathcal{R}_1 = \{r_1, r_3, r_5\}$. $\mathcal{R}$, however, is not a minimal-core set, because each world falsifying r_1 (i.e. worlds with label $\mathbf{a}_1^-$ in the table) also falsifies at least one of r_3, r_4, r_5. In fact, Goldszmidt & Pearl's system-Z^* approach fails for $\mathcal{R}$. Bourne & Parsons' algorithm, however, computes $Z^*(r_1) = Z^*(r_2) = Z^*(r_4) = 1, Z^*(r_3) = Z^*(r_5) = 2$. ■

$\sigma_{\mathcal{R}}$ labels each world appropriately and makes conditional effects on worlds comparable and computable. For instance, from Table 2 we see that the possible worlds $p\overline{q}\,\overline{r}ab$ and $\overline{p}\,\overline{q}\,\overline{r}ab$ have the same conditional structure (namely $\mathbf{a}_4^+$), and that forming the quotient $\dfrac{\sigma_{\mathcal{R}}(\overline{p}\,\overline{q}rab)}{\sigma_{\mathcal{R}}(\overline{p}\,\overline{q}ra\overline{b})} = \dfrac{\mathbf{a}_2^+\mathbf{a}_4^+}{\mathbf{a}_2^+\mathbf{a}_4^-} = \dfrac{\mathbf{a}_4^+}{\mathbf{a}_4^-}$ isolates the (net) effect of the fourth conditional. The following example illustrates that also multiple copies of worlds may be necessary to relate conditional structures:

Example 4 Consider the set $\mathcal{R} = \{(d|a), (d|b), (d|c)\}$ of conditionals using the atoms a, b, c, d. Let $\mathbf{a}_1^\pm, \mathbf{a}_2^\pm, \mathbf{a}_3^\pm$ be the group generators associated with $(d|a), (d|b)$, $(d|c)$, respectively. Then we have $\sigma_{\mathcal{R}}(ab\overline{c}d)\sigma_{\mathcal{R}}(a\overline{b}cd)\sigma_{\mathcal{R}}(\overline{a}bcd) = (\mathbf{a}_1^+\mathbf{a}_2^+)(\mathbf{a}_1^+\mathbf{a}_3^+)(\mathbf{a}_2^+\mathbf{a}_3^+) = (\mathbf{a}_1^+)^2(\mathbf{a}_2^+)^2(\mathbf{a}_3^+)^2 = (\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^+)^2 = \sigma_{\mathcal{R}}(abcd)^2$. Here two copies of $abcd$, or of its structure, respectively, are necessary to match the product of the conditional structures of $ab\overline{c}d, a\overline{b}cd$ and $\overline{a}bcd$. ■

Making calculations of conditional structures more convenient and more elegant, we take the worlds $\omega \in \Omega$ as formal generators of the free abelian group $\widehat{\Omega} := \langle \omega \mid$

ω	$\sigma_{\mathcal{R}}(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$
$pqrab$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$p\overline{q}rab$	$\mathbf{a}_2^-\mathbf{a}_4^+$
$pqra\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$p\overline{q}ra\overline{b}$	$\mathbf{a}_2^-\mathbf{a}_4^-$
$pqr\overline{a}b$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^-\mathbf{a}_5^-$	$p\overline{q}r\overline{a}b$	$\mathbf{a}_2^-$
$pqr\overline{a}\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^-\mathbf{a}_5^+$	$p\overline{q}r\overline{a}\overline{b}$	$\mathbf{a}_2^-$
$pq\overline{r}ab$	$\mathbf{a}_1^+\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$p\overline{q}\,\overline{r}ab$	$\mathbf{a}_4^+$
$pq\overline{r}a\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$p\overline{q}\,\overline{r}a\overline{b}$	$\mathbf{a}_4^-$
$pq\overline{r}\,\overline{a}b$	$\mathbf{a}_1^+\mathbf{a}_3^-\mathbf{a}_5^-$	$p\overline{q}\,\overline{r}\,\overline{a}b$	1
$pq\overline{r}\,\overline{a}\overline{b}$	$\mathbf{a}_1^+\mathbf{a}_3^-\mathbf{a}_5^+$	$p\overline{q}\,\overline{r}\,\overline{a}\,\overline{b}$	1
$\overline{p}qrab$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}\,\overline{q}rab$	$\mathbf{a}_2^+\mathbf{a}_4^+$
$\overline{p}qra\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}ra\overline{b}$	$\mathbf{a}_2^+\mathbf{a}_4^-$
$\overline{p}qr\overline{a}b$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_5^-$	$\overline{p}\,\overline{q}r\overline{a}b$	$\mathbf{a}_2^+$
$\overline{p}qr\overline{a}\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}r\overline{a}\overline{b}$	$\mathbf{a}_2^+$
$\overline{p}q\overline{r}ab$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-$	$\overline{p}\,\overline{q}\,\overline{r}ab$	$\mathbf{a}_4^+$
$\overline{p}q\overline{r}a\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}\,\overline{r}a\overline{b}$	$\mathbf{a}_4^-$
$\overline{p}q\overline{r}\,\overline{a}b$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_5^-$	$\overline{p}\,\overline{q}\,\overline{r}\,\overline{a}b$	1
$\overline{p}q\overline{r}\,\overline{a}\overline{b}$	$\mathbf{a}_1^-\mathbf{a}_3^-\mathbf{a}_5^+$	$\overline{p}\,\overline{q}\,\overline{r}\,\overline{a}\,\overline{b}$	1

Table 2. *Conditional structures for Example 3*

$\omega \in \Omega\rangle$. $\widehat{\Omega}$ consists of all products $\widehat{\omega} = {\omega_1}^{r_1} \ldots {\omega_m}^{r_m}$, with $\omega_1, \ldots, \omega_m \in \Omega$, and integers $r_1, \ldots r_m$. Introducing such a "multiplication between worlds" is nothing but a technical means to comply with the multiplicative structure the effects of conditionals impose on worlds. As in $\mathcal{F}_{\mathcal{R}}$, multiplication in $\widehat{\Omega}$ actually means juxtaposition. Now $\sigma_{\mathcal{R}}$ may be extended to $\widehat{\Omega}$ in a straightforward manner by setting $\sigma_{\mathcal{R}}({\omega_1}^{r_1} \ldots {\omega_m}^{r_m}) = \sigma_{\mathcal{R}}(\omega_1)^{r_1} \ldots \sigma_{\mathcal{R}}(\omega_m)^{r_m}$, yielding a *homomorphism of groups* $\sigma_{\mathcal{R}} : \widehat{\Omega} \to \mathcal{F}_{\mathcal{R}}$.

Having the same conditional structure defines an equivalence relation $\equiv_{\mathcal{R}}$ on $\widehat{\Omega}$: $\widehat{\omega}_1 \equiv_{\mathcal{R}} \widehat{\omega}_2$ iff $\sigma_{\mathcal{R}}(\widehat{\omega}_1) = \sigma_{\mathcal{R}}(\widehat{\omega}_2)$. Those elements of $\widehat{\Omega}$ that are balanced with respect to the effects of conditionals in $\mathcal{R}$ are contained in the *kernel of* $\sigma_{\mathcal{R}}$, *ker* $\sigma_{\mathcal{R}} = \{\widehat{\omega} \in \widehat{\Omega} \mid \sigma_{\mathcal{R}}(\widehat{\omega}) = 1\}$. *ker* $\sigma_{\mathcal{R}}$ does not depend on the chosen representation of conditional structures by symbols in $\mathcal{F}_{\mathcal{R}}$ and thus, it is an invariant of $\mathcal{R}$ [KER 01a]. In a semi-quantitative as well as in a probabilistic environment, implicit normalizing constraints have to be taken into account, namely, $\kappa(\top) = 0$ for OCF's, and $P(\top) = 1$ for probability distributions. This can be achieved by focusing on equivalence with respect to $\sigma_{(\top|\top)}$. Since $\sigma_{(\top|\top)}$ simply counts the worlds occurring in $\widehat{\omega}$, two elements $\widehat{\omega}_1 = \omega_1^{r_1} \ldots \omega_m^{r_m}$, $\widehat{\omega}_2 = \nu_1^{s_1} \ldots \nu_p^{s_p} \in \widehat{\Omega}$ are $\sigma_{(\top|\top)}$-equivalent, $\widehat{\omega}_1 \equiv_{\top} \widehat{\omega}_2$, iff $\sum_{1 \leqslant j \leqslant m} r_j = \sum_{1 \leqslant k \leqslant p} s_k$. This means, $\widehat{\omega}_1 \equiv_{\top} \widehat{\omega}_2$ iff they both are a (cancelled) product of the same number of generating worlds, each generator being counted with its corresponding exponent.

The conditional structures of generalized worlds, $\sigma_{\mathcal{R}}(\widehat{\omega})$, can be considered as generalized, algebraic versions of the *interaction quotients* investigated by Good (cf. [GOO 63]). $\sigma_{\mathcal{R}}(\widehat{\omega}) = 1$ (i.e. $\widehat{\omega} \in$ *ker* $\sigma_{\mathcal{R}}$) indicates that the interactions between the conditionals in $\mathcal{R}$ are balanced in $\widehat{\omega}$. We will elaborate the consequences of this idea for representing conditionals adequately in the next section.

6. Conditional indifference

To study conditional interactions, we now focus on the behavior of OCF's $\kappa : \Omega \to \mathbb{N} \cup \{0, \infty\}$ and probability functions $P : \Omega \to [0, 1]$ with respect to the elements in $\widehat{\Omega}$. We extend each such function to a homomorphism, $\kappa : \widehat{\Omega} \to (\mathbb{Z}, +)$, or $P : \widehat{\Omega} \to \mathbb{R}^+$, respectively, by setting

$$\begin{array}{rrcl} & \kappa({\omega_1}^{r_1} \cdot \ldots \cdot {\omega_m}^{r_m}) & = & r_1 \kappa(\omega_1) + \ldots + r_m \kappa(\omega_m) \\ \text{and} & P({\omega_1}^{r_1} \cdot \ldots \cdot {\omega_m}^{r_m}) & = & P(\omega_1)^{r_1} \cdot \ldots \cdot P(\omega_m)^{r_m} \end{array}$$

This allows us to analyze numerical relationships holding between different $\kappa(\omega)$, respectively $P(\omega)$, and to elaborate the conditionals whose structures these functions follow. In the sequel, let V denote an OCF or a probability function, i.e. $V = \kappa$ or $V = P$.

Definition 5 Assume $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ to be a set of conditionals. V is *indifferent with respect to* $\mathcal{R}$ iff the following condition is satisfied:

$$V(\widehat{\omega}_1) = V(\widehat{\omega}_2) \text{ whenever } \sigma_{\mathcal{R}}(\widehat{\omega}_1) = \sigma_{\mathcal{R}}(\widehat{\omega}_2) \text{ for } \widehat{\omega}_1 \equiv_{\top} \widehat{\omega}_2 \in \widehat{\Omega}, \qquad [9]$$

i.e. iff *ker* $\sigma_{\mathcal{R}} \cap$ *ker* $\sigma_{(\top|\top)} \subseteq$ *ker* V.

Note that we presupposed all OCF's to be finite and all probability functions to be positive in order to make this definition a very concise one. For the general case, cf. [KER 01a].

V being indifferent with respect to $\mathcal{R}$ means that it does not distinguish between different elements $\widehat{\omega}_1, \widehat{\omega}_2$ with the same conditional structure with respect to $\mathcal{R}$. Normalizing constraints are taken into account by observing $\equiv_{\top}$-equivalence. Conversely, any deviation $\kappa(\widehat{\omega}) \neq 0$, respectively $P(\widehat{\omega}) \neq 1$, can be explained by the conditionals in $\mathcal{R}$ acting on $\widehat{\omega}$ in a non-balanced way. Conditional indifference captures interactions of conditionals of arbitrary depth by taking into regard arbitrarily complex products, making use of the group homomorphism induced by V for their evaluation. It also respects, however, indifference on the superficial level of the function V itself:

Lemma 6 *If V is indifferent with respect to $\mathcal{R}$, then*

$$\sigma_{\mathcal{R}}(\omega_1) = \sigma_{\mathcal{R}}(\omega_2) \quad \textit{implies} \quad V(\omega_1) = V(\omega_2) \qquad [10]$$

for all worlds $\omega_1, \omega_2 \in \Omega$.

This is an easy consequence of conditional indifference, since each two worlds ω_1, ω_2 are $\sigma_{(\top|\top)}$-equivalent: $\omega_1 \equiv_\top \omega_2$ for all $\omega_1, \omega_2 \in \Omega$. Note that [10] only postulates that worlds showing the same conditional structures should be assigned the same numerical values by V. No information, however, is given as to how the numerical values of worlds with different conditional structures should be related. So only quite trivial interactions (namely those between two worlds having the same conditional structures) are dealt with by [10]. For more complex interactions, we need [9] to guide inductive representations of conditionals. We will discuss this in more detail when continuing Example 3 in Example 9 below.

But first, we will give a simple criterion to check conditional indifference in the next theorem (for a proof, cf. [KER 01a]):

Theorem 7 *Let* $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ *be a set of conditionals.*

A (finite) OCF κ *is indifferent with respect to* $\mathcal{R}$ *iff there are rational numbers* $\kappa_0, \kappa_i^+, \kappa_i^- \in \mathbb{Q}$, $1 \leqslant i \leqslant n$, *such that for all* $\omega \in \Omega$,

$$\kappa(\omega) = \kappa_0 + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \kappa_i^+ + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \kappa_i^- \qquad [11]$$

A (positive) probability function P *is indifferent with respect to* $\mathcal{R}$ *iff there are positive real numbers* $\alpha_0, \alpha_1^+, \alpha_1^-, \ldots, \alpha_n^+, \alpha_n^- \in \mathbb{R}^+$ *such that for all* $\omega \in \Omega$,

$$P(\omega) = \alpha_0 \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \alpha_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \alpha_i^- \qquad [12]$$

Although we admit rational *default rankings* κ_i^+, κ_i^-, we still assume the *world rankings* $\kappa(\omega)$ to be natural numbers. However, a generalization to rational world rankings, as is done in the work of Weydert (see e.g. [WEY 96, WEY 98, WEY 01, WEY 02]), also fits easily into our framework. For instance, Weydert's *constructible ranking measures* are conditionally indifferent ranking functions of type [11] with $\kappa_i^+ = 0$ and $\kappa_i^- \in [0, \infty]$.

Note that conditional indifference is a measure-free notion. In particular, it does not require the conditionals in $\mathcal{R}$ to be represented by κ or P, respectively. This qualitative or quantitative information is taken into account in a second step:

Definition 8 Ordinal conditional functions, κ, and probability functions, P, representing a set $\mathcal{R}$ of (quantified) conditionals and being indifferent with respect to it, are called *c-representations*.

Theorem 7 provides an intelligible schema to construct c-representations. For instance, for an OCF-c-representation, we simply set up κ according to [11] and choose κ_i^+, κ_i^- appropriately to ensure that $\kappa \models \mathcal{R}^{(*)}$ holds, i.e. such that $\kappa(A_i\overline{B}_i) - \kappa(A_iB_i) > 0$, respectively, $\kappa(A_i\overline{B}_i) - \kappa(A_iB_i) > m_i$, if the conditionals are quantified, for all $i, 1 \leqslant i \leqslant n$. This is equivalent to postulating

$$\begin{aligned}\kappa_i^- - \kappa_i^+ \quad > \quad (m_i +) \quad & \min_{\omega \models A_iB_i} \left(\sum_{\substack{j \neq i \\ \omega \models A_jB_j}} \kappa_j^+ + \sum_{\substack{j \neq i \\ \omega \models A_j\overline{B}_j}} \kappa_j^- \right) \\ & - \min_{\omega \models A_i\overline{B}_i} \left(\sum_{\substack{j \neq i \\ \omega \models A_jB_j}} \kappa_j^+ + \sum_{\substack{j \neq i \\ \omega \models A_j\overline{B}_j}} \kappa_j^- \right)\end{aligned} \qquad [13]$$

where the optional addition of m_i on the right hand side allows for variable strength conditionals. Furthermore, the normalizing constant κ_0 has to be chosen appropriately to ensure that actually an OCF is obtained. Once suitable numbers $\kappa_0, \kappa_i^+, \kappa_i^-$ are fixed, $\kappa(\omega)$ can easily be computed from the conditional structure $\sigma_{\mathcal{R}}(\omega)$ of ω by replacing each symbol $\mathbf{a}_i^+, \mathbf{a}_i^-$ by its numerical counterpart κ_i^+, κ_i^- (note the similarity of [8] with [11] and [12]).

By setting $\kappa_i^+ := 0$ for each conditional $r_i \in \mathcal{R}$, and determining $\kappa_i^- \geqslant 0$ according to [13], we recover the generalized system-Z^* (see [6]):

$$\kappa_i^- > (m_i +) \min_{\omega \models A_iB_i} \sum_{\substack{j \neq i \\ \omega \models A_j\overline{B}_j}} \kappa_j^- - \min_{\omega \models A_i\overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j\overline{B}_j}} \kappa_j^- \qquad [14]$$

Note that, due to the consistency of $\mathcal{R}$, also $\kappa_0 = 0$ in this case. By choosing $\kappa_i^- \geqslant 0$ minimally, we obtain system-Z^*. As long as we do not have multiple minimal solutions κ_i^-, the algorithm of Bourne & Parsons can be used to calculate these numbers. We also recover the ME-distribution from [12] (see [1], with $\alpha_i^+ = \alpha^{1-x_i}$ and $\alpha_i^- = \alpha^{-x_i}$ for $1 \leqslant i \leqslant n$). So both ME-methods and system-Z^* obey the principle of conditional indifference.

In [BEN 00], infinitesimal *lcd-plausibility functions* are introduced which are based (twofold) on the principle of least commitment (*lc*) and on Dempster's rule of combination (*d*) which motivates the specific multiplicative structure of lcd-functions. These lcd-functions can be constructed directly from our approach: Instead of *proto-infinitesimals* introduced in [BEN 00] to represent infinitesimals symbolically, we use the formal symbols $\mathbf{a}_1^-, \ldots, \mathbf{a}_n^-$ which can be assigned infinitesimals (or non-infinitesimal positive real numbers, in a non-infinitesimal framework) to give rise to plausibility functions of lcd-type. The specific form of lcd-functions is found to match the conditional structures of worlds – lcd-functions are indifferent with respect to $\mathcal{R}$. For a more detailed comparison in the framework of possibility theory, cf. [KER 01b].

Example 9 We continue Example 3, the extended Nixon diamond. r_2 and r_4 are tolerated by the whole set $\mathcal{R}$, so [14] yields $\kappa_2^-, \kappa_4^- > 0$. Therefore, we set $\kappa_2^- := \kappa_4^- := 1$. For determining $\kappa_1^-, \kappa_3^-, \kappa_5^-$, Table 2 proves to be helpful. We calculate

$$\begin{aligned}\kappa_1^- \quad &> \quad \min\{\kappa_3^-, \kappa_4^-, \kappa_5^-\} - \min\{\kappa_3^-, \kappa_4^-, \kappa_5^-\} = 0, \\ \kappa_3^- \quad &> \quad \min\{\kappa_4^-, \kappa_5^-\}, \\ \kappa_5^- \quad &> \quad \min\{\kappa_4^-, \kappa_3^-\}\end{aligned}$$

So we set $\kappa_1^- := 1, \kappa_3^- := \kappa_5^- := 2$. Because these numbers are minimal, they can also be obtained by applying the Bourne & Parsons-algorithm. Now, the appertaining c-representation $\kappa_c(\omega) = \sum_{\substack{1 \leqslant i \leqslant 5 \\ \omega \models A_i \overline{B_i}}} \kappa_i^-$ can be computed easily from Table 2. Table 3 shows κ_c-rankings for some worlds ω and compares them to system-Z-rankings. It is clearly seen that a c-representation imposes a more finely grained structure on

ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$	ω	$\kappa_c(\omega)$	$\kappa^z(\omega)$
$pqrab$	3	2	$\overline{p}qrab$	3	2
$pqra\overline{b}$	2	1	$\overline{p}qra\overline{b}$	2	2
$pqr\overline{a}b$	5	2	$\overline{p}qr\overline{a}b$	5	2
$pqr\overline{a}\overline{b}$	3	2	$\overline{p}qr\overline{a}\overline{b}$	3	2

Table 3. *Some conditional structures for Example 9*

the possible worlds and thus establishes more conditionals. For instance, we have $\kappa^z \not\models (\overline{b}|pqr\overline{a})$, but $\kappa_c \models (\overline{b}|pqr\overline{a})$, that is, a republican, who is a pacifist and a quaker, but not an American, is supposed not to like baseball. To show that this is more than pure speculation, compare the conditional structures of the two worlds involved, $pqr\overline{a}b$ and $pqr\overline{a}\overline{b}$: They are the same, except for conditional r_5, and due to r_5, the person is supposed not to like baseball.

In general, system-Z appears to be too cautious to handle conditional interactions adequately. But the extended Nixon diamond also shows it to be too bold sometimes: Now we have $\kappa^z \models (p|qr)$, which seems to be quite unintuitive. This problem also holds for other well-known approaches to conditional reasoning, cf. [DUB 96]. Again, the above c-representation κ_c proves to be more adequate: We have $\kappa_c(pqr) = 2 = \kappa_c(\overline{p}qr)$, so κ_c is completely indeterminate with respect to $(p|qr)$ – it preserves *ambiguity*, i.e. it does not give unjustified priority to arguments (cf. [BEN 00]).

We can now also illustrate the use of basing the concept of conditional indifference on considering generalized worlds $\widehat{\omega} \in \widehat{\Omega}$: The conditional structures of the worlds $pqrab, \overline{p}qrab, pqra\overline{b}, \overline{p}qra\overline{b}$ are pairwise different (see Table 2), so condition [10] would tell us nothing about the corresponding κ-values. By observing, however, that

$$\sigma_{\mathcal{R}}\left(\frac{pqrab \cdot \overline{p}qra\overline{b}}{pqra\overline{b} \cdot \overline{p}qrab}\right) = \frac{\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^- \cdot \mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+}{\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+\mathbf{a}_4^-\mathbf{a}_5^+ \cdot \mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^+\mathbf{a}_5^-} = 1,$$

[9] now yields

$$\kappa_c(pqrab) - \kappa_c(pqra\overline{b}) = \kappa_c(\overline{p}qrab) - \kappa_c(\overline{p}qra\overline{b})$$

for *any* c-representation κ_c. So balanced conditional interactions result in balanced numerical relationships. ∎

In contrast to the limited applicability of system-Z^*, an OCF-c-representation can be calculated for any consistent set of conditionals:

Proposition 10 *$\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ is consistent iff a c-representation of $\mathcal{R}$ exists.*

Proof. A set $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ is consistent iff an ordered partition $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_k$ of $\mathcal{R}$ exists such that each conditional in $\mathcal{R}_m$ is tolerated by $\bigcup_{j=m}^{k} \mathcal{R}_j$, $0 \leqslant m \leqslant k$. Set $\kappa_0 := 0$ and $\kappa_i^+ := 0$ for each conditional $r_i \in \mathcal{R}$. Furthermore, choose successively for each conditional $r_i \in \mathcal{R}_m$, $0 \leqslant m \leqslant k$,

$$\kappa_i^- > \min\{ \sum_{\substack{r_j \in \cup_{l=0}^{m-1} \mathcal{R}_l \\ \omega \models A_j \overline{B}_j}} \kappa_j^- \mid \omega \models A_i B_i \text{ and } r(\omega) \neq 0 \ \forall r \in \cup_{l=m}^{k} \mathcal{R}_l \}$$

□

It is already the approach [11] (or [3]) that guarantees a well-behavedness with respect to conditional interactions. In particular, all OCF-c-representations satisfy the property of irrelevance in the following sense (cf. [4]):

Lemma 11 *Suppose $\mathcal{R}$ is a set of conditionals, and d is an atomic proposition not appearing in any of the conditionals in $\mathcal{R}$. Then, for any OCF-c-representation κ_c of $\mathcal{R}$, and for any conditional $(B|A) \in (\mathcal{L} \mid \mathcal{L})$, $\kappa_c \models (B|A)$ iff $\kappa_c \models (B|Ad)$.*

This is easily proved by observing that for any propositional interpretation ω' over $\mathcal{V} - \{d\}$, we have $\sigma_{\mathcal{R}}(\omega' d) = \sigma_{\mathcal{R}}(\omega' \overline{d})$, and therefore, $\kappa_c(A\dot{B}) = \kappa_c(A\dot{B}d)$ for $\dot{B} \in \{B, \overline{B}\}$.

An analogous quantified version of this lemma holds for probabilistic c-representations, too.

Furthermore, let $\mathrel{|\!\sim}^c$ be the (nonmonotonic) inference relation based on considering all c-representations:

$$\mathcal{R} \mathrel{|\!\sim}^c (B|A) \quad \text{iff} \quad \kappa_c \models (B|A) \text{ for all c-representations } \kappa_c \text{ of } \mathcal{R} \qquad [15]$$

where $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L}), (B|A) \in (\mathcal{L} \mid \mathcal{L})$. Then Lemma 11 yields

Corollary 12 *Let $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L}), (B|A) \in (\mathcal{L} \mid \mathcal{L})$, let d be an atomic proposition not appearing in any of the conditionals in $\mathcal{R}$. Then $\mathcal{R} \mathrel{|\!\sim}^c (B|A)$ iff $\mathcal{R} \mathrel{|\!\sim}^c (B|Ad)$.*

Moreover, it is easy to prove that $\mathrel{|\!\sim}^c$ also fulfills the basic inferential properties of system P (see, e.g., [BEN 97]). But it does not suffer from the *drowning problem*, or, more specifically, from the *blocking of property inheritance problem* (see, e.g., [BEN 97]), as the following example shows:

Example 13 Let $\mathcal{R}$ consist of the rules $(f|b)$ (*birds fly*), $(b|p)$ (*penguins are birds*), $(\overline{f}|p)$ (*penguins do not fly*) and $(w|b)$ (*birds have wings*). We will show $\mathcal{R} \mathrel{|\!\sim}^c (w|pb)$,

ω	$\sigma_{\mathcal{R}}(\omega)$	$\kappa^*(\omega)$	ω	$\sigma_{\mathcal{R}}(\omega)$	$\kappa^*(\omega)$
$pbfw$	$\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_4^+$	2	$\overline{p}bfw$	$\mathbf{a}_1^+\mathbf{a}_4^+$	0
$pbf\overline{w}$	$\mathbf{a}_1^+\mathbf{a}_2^+\mathbf{a}_3^-\mathbf{a}_4^-$	3	$\overline{p}bf\overline{w}$	$\mathbf{a}_1^+\mathbf{a}_4^-$	1
$pb\overline{f}w$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^+$	1	$\overline{p}b\overline{f}w$	$\mathbf{a}_1^-\mathbf{a}_4^+$	1
$pb\overline{f}\overline{w}$	$\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{a}_3^+\mathbf{a}_4^-$	2	$\overline{p}b\overline{f}\overline{w}$	$\mathbf{a}_1^-\mathbf{a}_4^-$	2
$p\overline{b}fw$	$\mathbf{a}_2^-\mathbf{a}_3^-$	4	$\overline{p}\overline{b}fw$	1	0
$p\overline{b}f\overline{w}$	$\mathbf{a}_2^-\mathbf{a}_3^-$	4	$\overline{p}\overline{b}f\overline{w}$	1	0
$p\overline{b}\overline{f}w$	$\mathbf{a}_2^-\mathbf{a}_3^+$	2	$\overline{p}\overline{b}\overline{f}w$	1	0
$p\overline{b}\overline{f}\overline{w}$	$\mathbf{a}_2^-\mathbf{a}_3^+$	2	$\overline{p}\overline{b}\overline{f}\overline{w}$	1	0

Table 4. *Conditional structures and system-Z^* representation κ^* for Example 13*

that is, penguins inherit the property of having wings from their superclass *birds*, although they are non-typical birds.

Let κ_c be a c-representation of $\mathcal{R}$, with $\kappa_1^+, \kappa_1^-, \kappa_2^+, \kappa_2^-, \kappa_3^+, \kappa_3^-, \kappa_4^+, \kappa_4^-$ specifying the numerical effects of the conditionals $(f|b), (b|p), (\overline{f}|p), (w|b)$ (in that order), and with normalizing constant κ_0. For the fourth conditional $(w|b)$, [13] yields $\kappa_4^- - \kappa_4^+ > 0$ (this can be seen quickly by listing the conditional structures of all worlds involved, as is done in Table 4), so that $\kappa_4^- > \kappa_4^+$. By checking the conditional structures of the worlds verifying, or falsifying, respectively, $(w|pb)$, we find that

$$\begin{array}{rrcl} & \kappa_c(pbw) & = & \min\{\kappa_0 + \kappa_1^+ + \kappa_2^+ + \kappa_3^- + \kappa_4^+, \kappa_0 + \kappa_1^- + \kappa_2^+ + \kappa_3^+ + \kappa_4^+\} \\ \text{and} & \kappa_c(pb\overline{w}) & = & \min\{\kappa_0 + \kappa_1^+ + \kappa_2^+ + \kappa_3^- + \kappa_4^-, \kappa_0 + \kappa_1^- + \kappa_2^+ + \kappa_3^+ + \kappa_4^-\} \end{array}$$

Due to $\kappa_4^- > \kappa_4^+$, we conclude $\kappa_c(pbw) < \kappa_c(pb\overline{w})$, that is, $\kappa_c \models (w|pb)$ for all c-representations κ_c of $\mathcal{R}$. Without further assumptions concerning the κ_i^+, κ_i^-, however, it is not possible to show that also $(w|p)$ is valid in κ_c.

As should have been expected, the system-Z^* representation κ^* yields further inferences. It can be obtained from [14] by setting $\kappa_i^+ = 0$ and choosing $\kappa_1^- > 0, \kappa_2^- > \min\{\kappa_1^-, \kappa_3^-\}, \kappa_3^- > \min\{\kappa_1^-, \kappa_2^-\}, \kappa_4^- > 0$ in a minimal way, i.e. $\kappa_1^- = \kappa_4^- := 1, \kappa_2^- = \kappa_3^- := 2$ (see Table 4). κ^* also satisfies $(w|p)$: $\kappa^*(pw) = 1 < 2 = \kappa^*(p\overline{w})$.

Suppose now that we also take the conditionals $(e|b)$ (*birds lay eggs*), $(\overline{w}|k)$ (*kiwis do not have wings*) and $(b|k)$ (*kiwis are birds*) into account.[2] Then it can be shown that any c-representation κ_c of this extended set of rules treats both penguins and kiwis as normal birds with respect to laying eggs: $\kappa_c \models (e|pb), (e|kb)$. If we further restrict our attention to c-representations with $\kappa_i^+ = 0$ (that is, the generalized system-Z^* approach), then moreover, any such c-representation κ_c will also satisfy $(w|pb)$ and

2. I am grateful to an anonymous referee to raise this problem.

$(\overline{w}|kb)$. This shows, how most complicated interdependencies between conditionals are treated properly in our framework. ■

We described this example in detail to show that inference based on c-representations is largely qualitative and non-numerical. In fact, our argumentation used mainly structural information, provided by conditional structures.

The results presented so far reduce the necessity of having a *unique* conditionally indifferent OCF that represents conditional knowledge. Nevertheless, just as for system-Z and ME-distributions, the inferential power can be increased substantially by having such a unique, distinguished OCF-c-representation to base inferences on. We plead here for a way of solving this problem, which should be genuinely semi-quantitative, but also close to ME-methods: Conditional indifference, or [11], respectively, can be used as a fundamental schema of how an appropriate representation should be built. Then, further postulates have to make explicit the assumptions what actually makes the unique solution a most adequate one. This was exactly the way we took in [KER 98] to characterize the ME-approach. For instance, combining the postulate that only the plausibility of falsifying worlds should be changed (in a negative way, if at all) (this means postulating $\kappa_i^+ := 0$ and $\kappa_i^- \geqslant 0$ for each conditional $r_i \in \mathcal{R}$) with a minimality assumption for κ_i^-, we arrive at Goldszmidt & Pearl's system-Z^*, if a unique solution of [2] exists. (Bourne & Parsons obviously also allow κ_i^- to be negative.) But the failure of getting a unique OCF-c-representation for any consistent set of conditionals shows that these postulates are not sufficient.

We will discuss the problem of how to find a "best" c-representation in continuing the multiple solutions-example 2:

Example 14 Let $\mathcal{R} = \{r_1 : (b|a), r_2 : (c|a), r_3 : (c|ab)\}$. We assume $\kappa_i^+ = 0$ and $\kappa_i^- \geqslant 0$ for each of the three conditionals in $\mathcal{R}$, so also $\kappa_0 = 0$. Then the inequality constraints [14] yield $\kappa_1^- > 0$, $\kappa_2^- > 0 - \min\{\kappa_1^-, \kappa_3^-\}$, and $\kappa_3^- > 0 - \kappa_2^-$. So both $\kappa_1^- = 1, \kappa_2^- = 1, \kappa_3^- = 0$ and $\kappa_1^- = 1, \kappa_2^- = 0, \kappa_3^- = 1$ are minimal, non-negative solutions to [14]. In the table below, we list the resulting c-representations κ_1 (corresponding to the first solution) and κ_2 (corresponding to the second solution) (note, that for all worlds not mentioned in the table, $\kappa_1(\omega) = \kappa_2(\omega) = 0$).

ω	$\kappa_1(\omega)$	$\kappa_2(\omega)$	ω	$\kappa_1(\omega)$	$\kappa_2(\omega)$
abc	0	0	$a\overline{b}c$	1	1
$ab\overline{c}$	1	1	$a\overline{b}\overline{c}$	2	1

So κ_2 also satisfies the principle of minimum specificity – it yields a minimal c-representation (under the assumptions made). On the other hand, since the third conditional $(c|ab)$ represents quite specific knowledge, we might want to assign the more general conditional $(c|a)$ a greater strength, thus preferring the first solution with $\kappa_2^- = 1, \kappa_3^- = 0$. Note that κ_1 also accepts $(c|a\overline{b})$, establishing thoroughly the conditional belief $(c|a)$. ■

The exploration of further axioms apt to characterize a most adequate OCF-c-representation is a topic of our ongoing research.

7. Conditional belief revision – the principle of conditional preservation

In the last section, we described how observing conditional structures gives rise to most adequate representations of (quantified) conditional information. In this section, we show that the theory of conditional structures also provides a solid foundation to deal with conditionals appropriately when revising epistemic states by (propositional and conditional) beliefs.

Minimality of change is a crucial paradigm for belief revision, and a "principle of conditional preservation" is to realize this idea of minimality when conditionals are involved in change. Minimizing absolutely the changes in conditional beliefs, as in [BOU 93], is an important proposal to this aim, but it does not always lead to intuitive results (see [DAR 97]). The idea we will develop here rather aims at *preserving the conditional structure of knowledge* within an epistemic state which, in this section, we assume to be represented by an ordinal conditional function κ, since OCFs take quantitative as well as qualitative aspects into account. Similar results can be stated for probability functions, too.

We just explained what it means for an OCF, κ, to follow the structure imposed by a set of conditionals $\mathcal{R}$ on the possible worlds by introducing the notion of conditional indifference. Pursuing this approach further in the framework of belief revision, a revision of κ by simultaneously incorporating the conditionals in $\mathcal{R}$, $\kappa^* = \kappa * \mathcal{R}$, can be said to preserve the conditional structure of κ with respect to $\mathcal{R}$ if the *relative change function*[3] $\kappa^* - \kappa$ is indifferent with respect to $\mathcal{R}$.

Definition 15 Let κ be an ordinal conditional function, and let $\mathcal{R}^{(*)}$ be a (finite) set of (quantified) conditionals. Let $\kappa^* = \kappa * \mathcal{R}^{(*)}$ denote the result of revising κ by $\mathcal{R}^{(*)}$. κ^* is *indifferent with respect to* $\mathcal{R}$ *and* κ iff $(\kappa^* - \kappa)(\widehat{\omega}_1) = (\kappa^* - \kappa)(\widehat{\omega}_2)$ whenever $\sigma_{\mathcal{R}}(\widehat{\omega}_1) = \sigma_{\mathcal{R}}(\widehat{\omega}_2)$ and $\widehat{\omega}_1 \equiv_{\top} \widehat{\omega}_2$, for $\widehat{\omega}_1, \widehat{\omega}_2 \in \widehat{\Omega}$.

So for belief revision, this extended notion of conditional indifference is adequate to take prior knowledge as well as new (conditional) information into regard, and to preserve both as far as possible. Note that, once again, conditional indifference focusses on structural relationships; in particular, quantifications of the revising conditionals are irrelevant for this definition.

The principle of conditional preservation can now be realized as an indifference property:

Definition 16 Let κ be an ordinal conditional function, and let $\mathcal{R}^{(*)}$ be a finite set of (quantified) conditionals. A revision $\kappa^* = \kappa * \mathcal{R}^{(*)}$ satisfies the *principle of conditional preservation* iff κ^* is indifferent with respect to $\mathcal{R}$ and κ.

3. Note that we here make use of the conditional operation "$-$", considering revision as a generalized conditional operation.

So $\kappa * \mathcal{R}^{(*)}$ satisfies the principle of conditional preservation if any change in plausibility is clearly and unambigously induced by $\mathcal{R}$. The next theorem characterizes revisions of ordinal conditional functions that satisfy the principle of conditional preservation. The theorem is obvious by observing Theorem 7.

Theorem 17 *Let κ, κ^* be ordinal conditional functions, and let $\mathcal{R}^{(*)} = \{(B_1|A_1)([m_1]), \ldots, (B_n|A_n)([m_n])\}$ be a (finite) set of (quantified) conditionals in $(\mathcal{L} \mid \mathcal{L})$. A revision $\kappa^* = \kappa * \mathcal{R}^{(*)}$ satisfies the principle of conditional preservation iff there are numbers $\kappa_0, \kappa_i^+, \kappa_i^- \in \mathbb{Q}, 1 \leqslant i \leqslant n$, such that for all $\omega \in \Omega$,*

$$\kappa^*(\omega) = \kappa(\omega) + \kappa_0 + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \kappa_i^+ + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \kappa_i^- \qquad [16]$$

An analogous theorem for probabilistic revisions shows that also the minimum cross-entropy method satisfies the principle of conditional preservation (cf. [KER 01a]).

Revisions of ordinal conditional functions κ by sets $\mathcal{R}^{(*)}$ of (quantified) conditionals that also fulfill the so-called *success postulate* $\kappa * \mathcal{R}^{(*)} \models \mathcal{R}^{(*)}$ are termed *c-revisions*:

Definition 18 A revision $\kappa^* = \kappa * \mathcal{R}^{(*)}$ of an ordinary conditional function κ by a set $\mathcal{R}^{(*)}$ of (quantified) conditionals is called a *c-revision of κ by $\mathcal{R}^{(*)}$* iff κ^* satisfies the principle of conditional preservation with respect to κ and $\mathcal{R}$, and $\kappa^* \models \mathcal{R}^{(*)}$.

C-revisions in the framework of ordinal conditional functions can easily be obtained by using the schema provided by Theorem 17 and choosing the constants $\kappa_0, \kappa_i^+, \kappa_i^-$ appropriately so as to establish the necessary numerical relationships. So, a c-revision $\kappa^* = \kappa * \mathcal{R}^{(*)}$ of an OCF κ by $\mathcal{R}^{(*)} = \{(B_1|A_1)([m_1]), \ldots, (B_n|A_n)([m_n])\}$ has the form [16], and the postulate $\kappa^* \models \mathcal{R}^{(*)}$ yields the following conditions for κ_i^+, κ_i^- in a straightforward way:

$$\begin{aligned} \kappa_i^- - \kappa_i^+ \quad > \quad & (m_i+) \min_{\omega \models A_i B_i} (\kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \models A_j B_j}} \kappa_j^+ + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \kappa_j^-) \qquad [17] \\ & - \min_{\omega \models A_i \overline{B}_i} (\kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \models A_j B_j}} \kappa_j^+ + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \kappa_j^-) \end{aligned}$$

We will illustrate this by continuing Example 13.

Example 19 We take the system-Z^* representation κ^* of $\mathcal{R} = \{(f|b), (b|p), (\overline{f}|p), (w|b)\}$ as our current state of belief (see Example 13 and Table 4). Now we come to know that there are also *kiwis*, which are *birds* and *do not have wings*, this new knowledge being expressed by $\mathcal{R}_1 = \{(b|k), (\overline{w}|k)\}$. Since kiwis have not been considered yet, we first have to extend our beliefs κ^* so as to also take kiwis into account (without changing relevant relationships, of course). This can be done in a straightforward way by setting $\kappa^{**}(\nu k) = \kappa^{**}(\nu \overline{k}) := \kappa^*(\nu)$, where ν is a possible world over the variables $\{p, b, f, w\}$. This is not only intuitive, but can be justified by the concept of indifference, too: For any conceivable rule base $\mathcal{R}'$ underlying κ^*, we have that $\sigma_{\mathcal{R}'}(\nu k) = \sigma_{\mathcal{R}'}(\nu \overline{k})$, since the new variable k does not appear in any of the rules of $\mathcal{R}'$. So, whatever rule base has been used to build up κ^*, taking kiwis into account right from the beginning would result in κ^{**} instead of κ^*, since the principle of conditional indifference claims $\kappa^{**}(\nu k) = \kappa^{**}(\nu \overline{k})$, and as an extension of κ^*, κ^{**} should satisfy $\kappa^{**}(\nu) = \kappa^*(\nu)$ for all possible worlds ν over p, b, f, w.

Therefore, we now have to revise κ^{**} by $\mathcal{R}_1 = \{(b|k), (\overline{w}|k)\}$ according to the principle of conditional preservation. Each c-revision $\kappa_c^{**} = \kappa^{**} * \mathcal{R}_1$ has the form

$$\kappa_c^{**}(\omega) = \kappa^{**}(\omega) + \lambda_0 + \sum_{\substack{1 \leqslant i \leqslant 2 \\ \omega \models A_i B_i}} \lambda_i^+ + \sum_{\substack{1 \leqslant i \leqslant 2 \\ \omega \models A_i \overline{B_i}}} \lambda_i^-$$

where $(B_1|A_1) = (b|k)$ and $(B_2|A_2) = (\overline{w}|k)$ (see Table 5). According to [17], the λ_i^+, λ_i^- have to fulfill $\lambda_1^- - \lambda_1^+ > \min\{2+\lambda_2^-, 3+\lambda_2^+, 1+\lambda_2^-, 2+\lambda_2^+, \lambda_2^-, 1+\lambda_2^+\} - \min\{4+\lambda_2^-, 4+\lambda_2^+, 2+\lambda_2^-, 2+\lambda_2^+, \lambda_2^-, \lambda_2^+\}$, and $\lambda_2^- - \lambda_2^+ > \min\{3+\lambda_1^+, 2+\lambda_1^+, 4+\lambda_1^-, 2+\lambda_1^-, 1+\lambda_1^+, 2+\lambda_1^+, \lambda_1^-\} - \min\{2+\lambda_1^+, 1+\lambda_1^+, 4+\lambda_1^-, 2+\lambda_1^-, \lambda_1^+, \lambda_1^-\}$, respectively. Applying the same ideas as for system-Z^* means to set $\lambda_1^+ = \lambda_2^+ = 0$ and to choose $\lambda_1^-, \lambda_2^- \geqslant 0$ in a minimal way. The inequalities above then shrink to $\lambda_1^- > \min\{1, \lambda_2^-\}$ and $\lambda_2^- > \min\{1, \lambda_1^-\}$, and assuming minimality leads to $\lambda_1^- = \lambda_2^- = 2$. The resulting OCF κ_1^{**} is shown in Table 5, too.

It is interesting to note (and easily checked) that all conditionals from $\mathcal{R}$ – which actually gave rise to κ^* – still hold in κ_1^{**}. Moreover, *any* conditional relationship which holds in κ^* between formulas only involving the variables p, b, f, w is still satisfied by κ_1^{**} (even when quantifications are taken into account); this is due to the fact that for any formula A involving only p, b, f, w, we have $\kappa_1^{**}(A) = \min\{\kappa_1^{**}(Ak), \kappa_1^{**}(A\overline{k})\}$, and $\kappa_1^{**}(A\overline{k}) = \kappa^{**}(A\overline{k}) = \kappa^*(A)$, while $\kappa_1^{**}(Ak) \geqslant \kappa^{**}(Ak) = \kappa^*(A)$ (since all $\lambda_0, \lambda_i^+, \lambda_i^- \geqslant 0$). This is exactly what one would expect from a reasonable revision method – kiwis are not relevant to any of the formulas holding in κ^*, and so the new information concerning kiwis should not change this prior knowledge. ■

This example shows impressingly how the problem of irrelevant information can be solved by c-revisions. But the principle of conditional preservation means more than simply respecting irrelevance: By balancing numerical values according to structural information, highly complex interactions of conditional knowledge can be taken into regard and preserved under revision.

Comparing Theorems 7 and 17 with one another, we see that an ordinal conditional function κ is indifferent with respect to a finite set of conditionals $\mathcal{R}$ iff it can be taken as a revision $\kappa_0 * \mathcal{R}$ satisfying the principle of conditional preservation, where $\kappa_0(\omega) = 0$ for all $\omega \in \Omega$ is the *uniform ordinal conditional function*. So, (inductive) knowledge representation and belief revision can be dealt with in a common framework, based on conditionals as crucial pieces of knowledge. At least for this reason,

ω	$\kappa_c^{**}(\omega)$	$\kappa_1^{**}(\omega)$	ω	$\kappa_c^{**}(\omega)$	$\kappa_1^{**}(\omega)$
$pbfwk$	$2+\lambda_0+\lambda_1^{+}+\lambda_2^{-}$	4	$\overline{p}bfwk$	$0+\lambda_0+\lambda_1^{+}+\lambda_2^{-}$	2
$pbfw\overline{k}$	$2+\lambda_0$	2	$\overline{p}bfw\overline{k}$	$0+\lambda_0$	0
$pbf\overline{w}k$	$3+\lambda_0+\lambda_1^{+}+\lambda_2^{+}$	3	$\overline{p}bf\overline{w}k$	$1+\lambda_0+\lambda_1^{+}+\lambda_2^{+}$	1
$pbf\overline{w}\overline{k}$	$3+\lambda_0$	3	$\overline{p}bf\overline{w}\overline{k}$	$1+\lambda_0$	1
$pb\overline{f}wk$	$1+\lambda_0+\lambda_1^{+}+\lambda_2^{-}$	3	$\overline{p}b\overline{f}wk$	$1+\lambda_0+\lambda_1^{+}+\lambda_2^{-}$	3
$pb\overline{f}w\overline{k}$	$1+\lambda_0$	1	$\overline{p}b\overline{f}w\overline{k}$	$1+\lambda_0$	1
$pb\overline{f}\overline{w}k$	$2+\lambda_0+\lambda_1^{+}+\lambda_2^{+}$	2	$\overline{p}b\overline{f}\overline{w}k$	$2+\lambda_0+\lambda_1^{+}+\lambda_2^{+}$	2
$pb\overline{f}\overline{w}\overline{k}$	$2+\lambda_0$	2	$\overline{p}b\overline{f}\overline{w}\overline{k}$	$2+\lambda_0$	2
$p\overline{b}fwk$	$4+\lambda_0+\lambda_1^{-}+\lambda_2^{-}$	8	$\overline{p}\overline{b}fwk$	$0+\lambda_0+\lambda_1^{-}+\lambda_2^{-}$	4
$p\overline{b}fw\overline{k}$	$4+\lambda_0$	4	$\overline{p}\overline{b}fw\overline{k}$	$0+\lambda_0$	0
$p\overline{b}f\overline{w}k$	$4+\lambda_0+\lambda_1^{-}+\lambda_2^{+}$	6	$\overline{p}\overline{b}f\overline{w}k$	$0+\lambda_0+\lambda_1^{-}+\lambda_2^{+}$	2
$p\overline{b}f\overline{w}\overline{k}$	$4+\lambda_0$	4	$\overline{p}\overline{b}f\overline{w}\overline{k}$	$0+\lambda_0$	0
$p\overline{b}\,\overline{f}wk$	$2+\lambda_0+\lambda_1^{-}+\lambda_2^{-}$	6	$\overline{p}\overline{b}\,\overline{f}wk$	$0+\lambda_0+\lambda_1^{-}+\lambda_2^{-}$	4
$p\overline{b}\,\overline{f}w\overline{k}$	$2+\lambda_0$	2	$\overline{p}\overline{b}\,\overline{f}w\overline{k}$	$0+\lambda_0$	0
$p\overline{b}\,\overline{f}\overline{w}k$	$2+\lambda_0+\lambda_1^{-}+\lambda_2^{+}$	4	$\overline{p}\overline{b}\,\overline{f}\overline{w}k$	$0+\lambda_0+\lambda_1^{-}+\lambda_2^{+}$	2
$p\overline{b}\,\overline{f}\overline{w}\overline{k}$	$2+\lambda_0$	2	$\overline{p}\overline{b}\,\overline{f}\overline{w}\overline{k}$	$0+\lambda_0$	0

Table 5. *C-revision in Example 19*

the topic of revising epistemic states by conditional beliefs should not be looked upon as a superfluous artifact, but should be appreciated as providing a most general and enhanced framework for representing and revising knowledge. Actually, inductive representation of knowledge seems to be better understood in the context of conditional belief revision, taking it to be a special kind of revision. Due to the fact, that propositions, A, can be considered as degenerated conditionals, $(A|T)$, our approach can also be applied to propositional belief revision. In fact, the principle of conditional preservation, as defined here, can be shown to cover the postulates for iterative belief revision of [DAR 97] (cf. [KER 01a]).

8. Outlook

In this paper, we presented a new theory to handle conditionals most adequately in inductive knowledge representation, uncertain reasoning and belief revision. This new theory is based on the algebraic notion of *conditional structures* of worlds. Its realization via group theory does not only provide an elegant methodological framework for representing conditional knowledge, but also may guide the revision of epistemic states by sets of conditionals in a way that is compatible with the postulates for iterative belief revision of [DAR 97], and with their generalizations presented in [KER 99] (cf. [KER 01a]). We described how our approach can be applied both in a probabilistic and in an ordinal framework to illustrate its universality. Actually, in [KER 02a], we present it in the even more general environment of *conditional valuation functions* of which probability functions and ranking functions are examples. In [KER 02b], we use conditional structures to base qualitative default inference relations upon even the simplest one of which proves to be quite well-behaved.

Moreover, the new conditional theory sketched here can also be used for *knowledge discovery* (see [KER 00, KER 01a, BEI 02b]). In [KER 00, KER 01a, BEI 02a], an approach is presented to solve the "inverse representation problem": Which conditionals are most appropriate to represent an epistemic state? How to extract most relevant knowledge from, e.g., probability distributions? It is there where the techniques introduced in this paper reveal their full power by elaborating conditional structures from numerical relationships by group theoretical means.

So indeed, the conditional theory presented in this paper has important applications in the whole area of formal knowledge management – uncertain reasoning, belief revision and knowledge discovery. A comparison with conditional event theories can be found in [KER 01b].

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