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Postulates for conditional belief revision

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Abstract

In this paper, we present a scheme of postulates for revising epistemic states by conditional beliefs. These postulates are supported mainly by following the specific, non-classical nature of conditionals, and the aim of preserving conditional beliefs is achieved by studying specific interactions between conditionals, represented properly by two relations. Because one of the postulates claims propositional belief revision to be a special case of conditional belief revision, our framework also covers the work of Darwiche and Pearl [Darwiche and Pearl, 1997], and we show that all postulates presented there may be derived from our postulates. We state representation theorems for the principal postulates, and finally, we present a conditional belief operator obeying all of the postulates by using ordinal conditional functions as representations of epistemic states.

1 Introduction

Belief revision deals with the *dynamics of belief* – how should currently held beliefs be modified in the light of new information? Results in this area are mainly influenced by the so-called *AGM theory*, named after Alchourron, Gärdenfors and Makinson who set up a framework of postulates for a reasonable change of beliefs (cf. [Alchourrón *et al.*, 1985], [Gärdenfors, 1988]). Usually, the *belief sets* in AGM theory are assumed to be deductively closed sets of propositional formulas, or to be represented by one single propositional formula, respectively, and the revising beliefs are taken to be propositional formulas. So the AGM postulates constrain revisions of the form $\psi \star A$, the revision operator $\star$ connecting two propositional formulas ψ and A, where ψ represents the initial state of belief and A stands for the new information. A representation theorem (cf. [Katsuno and Mendelzon, 1991]) establishes a relationship between AGM revision operators and total pre-orders $\leq_\psi$ on the set of possible worlds, proving the revised belief set $\psi \star A$ to be satisfied precisely by all minimal A-worlds.

Though belief sets representing what is known for certain are of specific interest they are only poor reflections of the complex attitudes an individual may hold. The limitation to propositional beliefs severely restricts the frame of AGM theory, in particular, when iterated revision has to be performed. So belief revision should not only be concerned with the revision of propositional beliefs but also with the modification of *revision strategies* when new information arrives (cf. [Darwiche and Pearl, 1997], [Boutilier, 1993], [Boutilier and Goldszmidt, 1993]). These revision strategies may be taken as conditional beliefs, therefore revision should be concerned with changes in conditional beliefs and, the other way around, with the preservation of conditional beliefs.

Darwiche and Pearl [Darwiche and Pearl, 1997] explicitly took conditional beliefs into account by considering epistemic states instead of belief sets, and they advanced four postulates in addition to the AGM axioms to model what may be called *conditional preservation* under revision by propositional beliefs.

In the present paper, we broaden the framework for revising epistemic states presented in [Darwiche and Pearl, 1997] so as to include also the *revision by conditional beliefs.* Thus belief revision is considered here in quite a general framework, exceeding the AGM-theory in two respects:

- We revise epistemic states; this makes it necessary to allow for the changes in conditional beliefs caused by new information.
- The new belief A may be of a conditional nature, thus reflecting a changed or newly acquired revision policy that has to be incorporated adequately.

We present a scheme of eight postulates appropriate to guide the revision of epistemic states by conditional beliefs. These postulates are supported mainly by following the specific, non-classical nature of conditionals, and the aim of preserving conditional beliefs is achieved by studying specific interactions between conditionals, represented properly by two relations. Because one of the postulates claims propositional belief revision to be a special case of conditional belief revision, our framework also covers the topic of Darwiche and Pearl's work [Darwiche and Pearl, 1997], and we show that all four postulates presented there may be derived from our postulates. We state representation theorems for the principal postulates, and finally, we present a conditional belief operator obeying all of the postulates by using ordinal conditional functions as representations of epistemic states.

The organization of this paper is as follows: In section 2, we briefly summarize the results of Darwiche and Pearl concerning the revision of epistemic states and lay down some foundations for this paper. In section 3, we describe conditionals as objects of a three-valued nature and introduce the relations $\perp$ and $\sqsubseteq$ between conditionals which play an important part for studying interactions between conditionals. Section 4 presents and explains the eight postulates for conditional revision and shows correspondences to the axioms of [Darwiche and Pearl, 1997]. Section 5 contains representation theorems and some consequences of the postulates. In section 6, we introduce a conditional revision operator for ordinal conditional functions that realizes the ideas of this paper, and section 7 concludes this paper with a short summary and an outlook.

2 Revising epistemic states

An epistemic state Ψ represents the cognitive state of some individual at a given time. In particular, beside the set of beliefs $Bel(\Psi)$ the individual accepts for certain, Ψ contains the revision policies the individual entertains at that time. These revision policies reflect the beliefs (B) the individual is inclined to hold if new information (A) becomes obvious, and are adequately represented by conditionals $(B \mid A)$, i.e. expressions of the form "*If A then B*", conjoining two propositional formulas A and B. So the conditional $(B \mid A)$ is accepted in the epistemic state Ψ iff revising Ψ by A yields belief in B. This defines a fundamental relationship between conditionals and the process of revision, known as the *Ramsey test* (cf. e.g. [Boutilier and Goldszmidt, 1993], [Gärdenfors, 1988]):

(RT) $\Psi \models (B \mid A) \quad \text{iff} \quad Bel(\Psi \star A) \models B$

where $\star$ is a revision operator, taking an epistemic state Ψ and some new belief A as inputs and yielding a revised epistemic state $\Psi \star A$ as output.

Each epistemic state Ψ is associated with its belief set $Bel(\Psi)$ which is supposed to be a deductively closed set of formulas of a propositional language $\mathcal{L}$. The revision of Ψ by $A \in \mathcal{L}$ also yields a revised belief set $Bel(\Psi \star A) \subseteq \mathcal{L}$, and of course, this revision should obey the standards of the AGM theory. But the revision of epistemic states cannot be reduced to propositional revision because two *different* epistemic states Ψ_1, Ψ_2 may have *equivalent* belief sets $Bel(\Psi_1) \equiv Bel(\Psi_2)$. Thus an epistemic state is not described uniquely by its belief set, and revising Ψ_1 and Ψ_2 by new information A may result in different revised belief sets $Bel(\Psi_1 \star A) \not\equiv Bel(\Psi_2 \star A)$.

Example. Two physicians have to make a diagnosis when confronted with a patient showing certain symptoms. They both agree that disease A is by far the most adequate diagnosis, so they both hold belief in A. Moreover, as the physicians know, diseases B and C might also cause the symptoms, but here the experts disagree: One physician regards B to be a possible diagnosis, too, but excludes C, whereas the other physician is inclined to take C into consideration, but not B.

Suppose now that a specific blood test definitely proves that the patient is not suffering from disease A. So both experts have to change their beliefs, the first physician now takes B to be the correct diagnosis, the second one takes C for granted. Though initially the physicians' opinions may be described by the same belief set, they end up with different belief sets after revision.

It is important to note that Gärdenfors' famous triviality result [Gärdenfors, 1988] complaining the incompatibility of the Ramsey test with some of the AGM-postulates does not hold if conditional beliefs are considered essentially different from propositional beliefs, as is emphasized here and elsewhere (cf. e.g. [Darwiche and Pearl, 1997]). Therefore obeying the difference between $Bel(\Psi_1) \equiv Bel(\Psi_2)$ and $\Psi_1 = \Psi_2$ makes the Ramsey test compatible with the AGM-theory for propositional belief revision: Whereas $Bel(\Psi_1) \equiv Bel(\Psi_2)$ only means that both epistemic states have equivalent belief sets, $\Psi_1 = \Psi_2$ requires the two epistemic states to be identical, i.e. to incorporate in particular the same propositional beliefs as well as the same conditional beliefs.

Darwiche and Pearl [Darwiche and Pearl, 1997] consider the revision of epistemic states with propositional beliefs, mainly concerned with handling iterated revisions. They generalize the AGM-postulates for belief revision to the framework of revising epistemic states (cf. [Darwiche and Pearl, 1997]):

Suppose Ψ, Ψ_1, Ψ_2 to be epistemic states and $A, A_1, A_2, B \in \mathcal{L}$;

(R*1) A is believed in $\Psi \star A$: $Bel(\Psi \star A) \models A$.

(R*2) If $Bel(\Psi) \wedge A$ is satisfiable, then $Bel(\Psi \star A) \equiv Bel(\Psi) \wedge A$.

(R*3) If A is satisfiable, then $Bel(\Psi \star A)$ is also satisfiable.

(R*4) If $\Psi_1 = \Psi_2$ and $A_1 \equiv A_2$, then $Bel(\Psi_1 \star A_1) \equiv Bel(\Psi_2 \star A_2)$.

(R*5) $Bel(\Psi \star A) \wedge B$ implies $Bel(\Psi \star (A \wedge B))$.

(R*6) If $Bel(\Psi \star A) \wedge B$ is satisfiable then $Bel(\Psi \star (A \wedge B))$ implies $Bel(\Psi \star A) \wedge B$.

Considered superficially, these postulates are exact reformulations of the AGM postulates, as stated in [Katsuno and Mendelzon, 1991], with belief sets replaced throughout by belief sets of epistemic states. So the postulates above ensure that the revision of epistemic states is in line with the AGM theory as long as the revision of the corresponding belief sets is considered. The most important new aspect by contrast with propositional belief revision is given by postulate (R*4): Only identical epistemic states are supposed to yield equivalent revised belief sets. This is a clear but adequate weakening of the corresponding AGM-postulate

(R4) If $Bel(\Psi_1) \equiv Bel(\Psi_2)$ and $A_1 \equiv A_2$, then $Bel(\Psi_1 \star A_1) \equiv Bel(\Psi_2 \star A_2)$

which amounts to reducing the revision of epistemic states to propositional belief revision. As we explained above, such a reduction is inappropriate.

Darwiche and Pearl [Darwiche and Pearl, 1997] proved a representation theorem for their postulates which parallels the corresponding theorem in AGM theory (cf. [Katsuno and Mendelzon, 1991]), using the notion of *faithful assignments*:

Definition 1 ([Darwiche and Pearl, 1997]) Let W be the set of all worlds (interpretations) of the propositional language $\mathcal{L}$ and consider epistemic states Ψ the belief sets of which belong to $\mathcal{L}$.

A *faithful assignment* is a function that maps each such epistemic state Ψ to a total pre-order $\leq_\Psi$ on the worlds W satisfying the following conditions:

(1) $\omega_1, \omega_2 \models \Psi$ only if $\omega_1 =_\Psi \omega_2$;

(2) $\omega_1 \models \Psi$ and $\omega_2 \not\models \Psi$ only if $\omega_1 <_\Psi \omega_2$.

for worlds $\omega_1, \omega_2 \in W$ and epistemic states Ψ, Φ.

As usual, $\omega_1 <_\Psi \omega_2$ means $\omega_1 \leq_\Psi \omega_2$ and $\omega_2 \not\leq_\Psi \omega_1$; $\omega_1 =_\Psi \omega_2$ iff $\omega_1 \leq_\Psi \omega_2$ and $\omega_2 \leq_\Psi \omega_1$.

Given the set W of all worlds of the language $\mathcal{L}$ and a propositional formula $A \in \mathcal{L}$, we denote by $Mod(A)$ the set of all A-worlds, $Mod(A) = \{\omega \in W \mid \omega \models A\}$. If Ψ is an epistemic state, we set $Mod(\Psi) = Mod(Bel(\Psi))$.

Theorem 2 ([Darwiche and Pearl, 1997]) *A revision operator $\star$ satisfies postulates* (R*1)-(R*6) *precisely when there exists a faithful assignment that maps each epistemic state Ψ to a total pre-order $\leq_\Psi$ such that*

$$Mod(\Psi \star A) = \min(A; \Psi) := \min(Mod(A); \leq_\Psi)$$

i.e. the worlds satisfying $Bel(\Psi \star A)$ are precisely those worlds satisfying A that are minimal with respect to $\leq_\Psi$.

This theorem shows an important connection between the ordering $\leq_\Psi$ associated with an epistemic state Ψ and the process of revising Ψ by propositional beliefs. Therefore, at least in the context of revision, epistemic states are properly represented as pairs $(\Psi, \leq_\Psi)$ with a total pre-order $\leq_\Psi$ satisfying conditions (1)-(2) of definition 1 and the so-called *smoothness condition* $min(A; \Psi) \neq \emptyset$ for any satisfiable $A \in \mathcal{L}$ (cf. e.g. [Boutilier and Goldszmidt, 1993]), and such that $Mod(\Psi) = \min(W; \leq_\Psi)$. Using the relationship (RT) between revision and conditionals, theorem 2 immediately yields

Lemma 3 *A conditional $(B \mid A)$ is accepted in an epistemic state $(\Psi, \leq_\Psi)$ iff all minimal A-worlds satisfy B, i.e. $\Psi \models (B \mid A)$ iff $\min(A; \Psi) \subseteq Mod(B)$.*

Thus the pre-order $\leq_\Psi$ encodes the conditional beliefs held in Ψ.

For two propositional formulas A, B, we define $A \leq_\Psi B$ iff for all $\omega \in \min(A; \Psi)$, $\omega' \in \min(B; \Psi)$, we have $\omega \leq_\Psi \omega'$, i.e. iff the minimal A-worlds are at least as plausible as the minimal B-worlds. To simplify notations, we will replace a conjunction by juxtaposition and indicate the negation of a proposition by barring, i.e. $AB = A \wedge B$ and $\overline{B} = \neg B$. Using this, the lemma above may be reformulated as

Lemma 4 *A conditional $(B \mid A)$ is accepted in an epistemic state $(\Psi, \leq_\Psi)$ iff $AB <_\Psi A\overline{B}$.*

Boutilier (cf. e.g. [Boutilier, 1994]) also took conditional beliefs into account. He presented in [Boutilier, 1993] his *natural revision* that preserves as many conditional beliefs as possible, in accordance with the AGM postulates, and he generalized this approach to deal with the revision by conditional beliefs [Boutilier and Goldszmidt, 1993]. As Darwiche and Pearl emphasized, however, Boutilier's natural revision seems to be too restrictive in that it preserves conditional beliefs at the cost of compromising propositional beliefs (cf. [Darwiche and Pearl, 1997]). Thus the question which conditional beliefs should be kept under revision turns out to be a crucial problem when revising epistemic states. In the framework of iterated revision, Darwiche and Pearl [Darwiche and Pearl, 1997] proposed four postulates concerning the preservation of conditional beliefs under propositional revision:

(C1) If $C \models B$ then $\Psi \models (D \mid C)$ iff $\Psi \star B \models (D \mid C)$.

(C2) If $C \models \overline{B}$ then $\Psi \models (D \mid C)$ iff $\Psi \star B \models (D \mid C)$.

(C3) If $\Psi \models (B \mid A)$ then $\Psi \star B \models (B \mid A)$.

(C4) If $\Psi \star B \models (\overline{B} \mid A)$ then $\Psi \models (\overline{B} \mid A)$.

For discussion of these postulates, cf. the original paper [Darwiche and Pearl, 1997].

In this paper, we present postulates for the revision of epistemic states by *conditional beliefs* which generalize the postulates of Darwiche and Pearl and support them with new conditional arguments. The rationale behind these postulates is not to minimize conditional change, as in Boutilier's work, but to preserve the *conditional structure* of the knowledge, as far as possible, which is made obvious by studying interactions between conditionals.

3 Conditionals

Conditionals may be given a lot of different interpretations, e.g. as counterfactuals, as indicative, subjunctive or normative conditionals etc. (cf. e.g. [Nute, 1980], [Boutilier, 1994]). In the context of revision, a subjunctive meaning fits particularly well, in accordance with the Ramsey test (RT): *If A were true, B would hold*, implicitly referring to a revision of the actual epistemic state by A.

Independently of its given meaning, a conditional $(B \mid A)$ is an object of a three-valued nature, partitioning the set of worlds W in three parts: those worlds satisfying $A \wedge B$ and thus confirming the conditional, those worlds satisfying $A \wedge \neg B$, thus contradicting the conditional, and those worlds not fulfilling the premise A and so which the conditional may not be applied to at all. Therefore Calabrese represents a conditional as a *generalized indicator function* (cf. [Calabrese, 1991])

$$(B \mid A)(\omega) = \begin{cases} 1 & \text{if} \quad \omega \models AB \\ 0 & \text{if} \quad \omega \models A\overline{B} \\ u & \text{if} \quad \omega \models \overline{A} \end{cases}$$

where u means *undefined*. Two conditionals are considered to be equivalent iff they are identical as indicator functions, i.e. $(B \mid A) \equiv (D \mid C)$ iff $A \equiv C$ and $AB \equiv CD$ (cf. [Calabrese, 1991]). Usually, a propositional fact $A \in \mathcal{L}$ is identified with the conditional $(A \mid \top)$, where $\top$ is tautological.

For a conditional $(B \mid A)$, we define the *affirmative set* $(B \mid A)^+$ and the *contradictory set* $(B \mid A)^-$ of worlds as

$$\begin{aligned} (B \mid A)^+ &= \{\omega \in W \mid \omega \models AB\} = Mod(AB) \\ (B \mid A)^- &= \{\omega \in W \mid \omega \models A\overline{B}\} = Mod(A\overline{B}) \end{aligned}$$

Lemma 5 *Two conditionals $(B \mid A), (D \mid C)$ are equivalent iff their corresponding affirmative and contradictory sets are equal, i.e. $(B \mid A) \equiv (D \mid C)$ iff $(B \mid A)^+ = (D \mid C)^+$ and $(B \mid A)^- = (D \mid C)^-$.*

It is difficult to capture interactions between conditionals. In [Calabrese, 1991], logical connectives and implications between conditionals are defined and investigated. Here we will pursue a different idea of interaction. Having the effects of conditionals on worlds in mind, we define two relations $\sqsubseteq$ and $\perp$ between conditionals by

$$\begin{aligned} (D \mid C) \sqsubseteq (B \mid A) \quad & \text{iff} \quad (D \mid C)^+ \subseteq (B \mid A)^+ \\ & \text{and} \quad (D \mid C)^- \subseteq (B \mid A)^- \end{aligned}$$

and

$$\begin{aligned} (D \mid C) \perp (B \mid A) \quad & \text{iff} \quad Mod(C) \subseteq (B \mid A)^+ \\ \text{or} \quad Mod(C) \subseteq (B \mid A)^- \quad & \text{or} \quad Mod(C) \subseteq Mod(\overline{A}). \end{aligned}$$

Thus $(D \mid C) \sqsubseteq (B \mid A)$ if the effect of the former conditional on worlds is in line with the latter one, but $(D \mid C)$ applies to fewer worlds. Thus $(D \mid C)$ may be called a *subconditional* of $(B \mid A)$ in this case. In contrast to this, the second relation $\perp$ symbolizes a kind of *independency between conditionals*. We have $(D \mid C) \perp (B \mid A)$ if $Mod(C)$, i.e. the range of application of the conditional $(D \mid C)$, is completely contained in one of the sets $(B \mid A)^+, (B \mid A)^-$ or $Mod(\overline{A})$. So for all worlds which $(D \mid C)$ may be applied to, $(B \mid A)$ has the same effect and yields no further partitioning. Note, however, that $\perp$ is not a symmetric independence relation; $(D \mid C) \perp (B \mid A)$ rather expresses that $(D \mid C)$ is *not affected* by $(B \mid A)$.

Both relations may be expressed using the standard ordering $\leq$ between propositional formulas: $A \leq B$ iff $A \models B$, i.e. iff $Mod(A) \subseteq Mod(B)$.

Lemma 6 *(i) $(D \mid C) \sqsubseteq (B \mid A)$ iff $CD \leq AB$ and $C\overline{D} \leq A\overline{B}$; in particular, if $(D \mid C) \sqsubseteq (B \mid A)$ then $C \leq A$.*

(ii) $(D \mid C) \sqsubseteq (B \mid A)$ and $(B \mid A) \sqsubseteq (D \mid C)$ iff $(D \mid C) \equiv (B \mid A)$.

(iii) $(D \mid C) \perp (B \mid A)$ iff $C \leq AB$ or $C \leq A\overline{B}$ or $C \leq \overline{A}$.

4 Revision by conditionals

Revising an epistemic state Ψ by a conditional $(B \mid A)$ becomes necessary if a new conditional belief resp. a new revision policy should be included in Ψ, yielding a changed epistemic state $\Psi' = \Psi \star (B \mid A)$ such that $\Psi' \models (B \mid A)$, i.e. $\Psi' \star A \models B$. We will use the same operator $\star$ for propositional as well as for conditional revision, thus expressing that conditional revision should extend propositional revision in accordance with the Ramsey test (RT).

Boutilier and Goldszmidt [Boutilier and Goldszmidt, 1993] presented a generalized version of the natural revision operator of Boutilier to perform such an adaptation to conditional beliefs; their method minimizes changes in conditional beliefs in accordance with the AGM theory.

Below, we propose several postulates a revision of an epistemic state by a conditional should satisfy. The key idea is to follow the conditionals in Ψ as long as there is no conflict between them and the new conditional belief, and we will use $\sqsubseteq$ and $\perp$ to relate conditionals appropriately.

Postulates for conditional revision:

Suppose $(\Psi, \leq_\Psi)$ is an epistemic state and $(B \mid A)$, $(D \mid C)$ are conditionals. Let $\Psi \star (B \mid A)$ denote the result of revising Ψ by $(B \mid A)$.

(CR0) $\Psi \star (B \mid A)$ is an epistemic state.

(CR1) $\Psi \star (B \mid A) \models (B \mid A)$.

(CR2) $\Psi \star (B \mid A) = \Psi$ iff $\Psi \models (B \mid A)$.

(CR3) $\Psi \star B = \Psi \star (B \mid \top)$ induces a propositional AGM-revision operator.

(CR4) $\Psi \star (B \mid A) = \Psi \star (D \mid C)$ whenever $(B \mid A) \equiv (D \mid C)$.

(CR5) If $(D \mid C) \perp (B \mid A)$ then $\Psi \models (D \mid C)$ iff $\Psi \star (B \mid A) \models (D \mid C)$.

(CR6) If $(D \mid C) \sqsubseteq (B \mid A)$ and $\Psi \models (D \mid C)$ then $\Psi \star (B \mid A) \models (D \mid C)$.

(CR7) If $(D \mid C) \sqsubseteq (\overline{B} \mid A)$ and $\Psi \star (B \mid A) \models (D \mid C)$ then $\Psi \models (D \mid C)$.

Postulates (CR0) and (CR1) are self-evident. (CR2) postulates that Ψ should be left unchanged precisely if it already entails the conditional. (CR3) says that the induced propositional revision operator should be in accordance with the AGM postulates. (CR4) requires the result of the revision process to be independent of the syntactical representation of conditionals.

The next three postulates aim at preserving the conditional structure of knowledge:

(CR5) claims that revising by a conditional should preserve all conditionals that are independent of that conditional, in the sense given by the relation $\perp$. The rationale behind this postulate is the following: The validity of a conditional $(B \mid A)$ in an epistemic state Ψ depends on the relation between (some) worlds in $Mod(AB)$ and (some) worlds in $Mod(A\overline{B})$ (cf. lemmata 3, 4). So incorporating $(B \mid A)$ to Ψ may require a shift between $Mod(AB)$ on one side and $Mod(A\overline{B})$ on the other side, but should leave intact any relations between worlds within $Mod(AB)$, $Mod(A\overline{B})$, or $Mod(\overline{A})$. These relations may be captured by conditionals not affected by $(B \mid A)$, i.e. by conditionals $(D \mid C) \perp (B \mid A)$.

(CR6) states that conditional revision should bring about no change for conditionals that are already in line with the revising conditional, and (CR7) guarantees that no conditional change contrary to the revising conditional is caused by conditional revision.

An idea of *conditional preservation* is also inherent to the postulates (C1)-(C4) of Darwiche and Pearl [Darwiche and Pearl, 1997] which we will show to be generalized by our postulates.

Theorem 7 *Suppose $\star$ is a conditional revision operator obeying the postulates* (CR0)-(CR7). *Then for the induced propositional revision operator, postulates* (C1)-(C4) *are satisfied, too.*

Proof: Let $A, B, C, D \in \mathcal{L}$.

Suppose $C \leq B$ or $C \leq \overline{B}$. Then, according to lemma 6, $(D \mid C) \perp (B \mid \top)$. (CR3) and (CR5) now imply (C1) and (C2).

(C3) and (C4) are direct consequences of (CR6) and (CR7) by using that $(B \mid A) \sqsubseteq (B \mid \top)$ and $(\overline{B} \mid A) \sqsubseteq (\overline{B} \mid \top)$, respectively, due to lemma 6. $\square$

This theorem provides further justifications for the postulates of Darwiche and Pearl from within the framework of conditionals.

5 Representation theorems

Postulates (CR5)-(CR7) claim specific connections to hold between Ψ and the revised $\Psi \star (B \mid A)$, thus relating $\leq_\Psi$ and $\leq_{\Psi\star(B|A)}$. We will elaborate this relationship in order to characterize those postulates by properties of the pre-orders associated with Ψ and $\Psi \star (B \mid A)$.

Postulate (CR5) proves to be of particular importance because it guarantees the ordering within $Mod(AB), Mod(A\overline{B}), Mod(\overline{A})$, respectively, to be preserved:

Theorem 8 *The conditional revision operator $\star$ satisfies* (CR5) *iff for each epistemic state $(\Psi, \leq_\Psi)$ and for each conditional $(B \mid A)$ it holds that:*

$$\omega \leq_\Psi \omega' \quad iff \quad \omega \leq_{\Psi\star(B|A)} \omega' \tag{1}$$

for all worlds $\omega, \omega' \in Mod(AB)$ $(Mod(A\overline{B}), Mod(\overline{A})$, respectively).

As an immediate consequence, (1) yields

Lemma 9 *Suppose (1) holds for all worlds $\omega, \omega' \in Mod(AB)$ $(Mod(A\overline{B}), Mod(\overline{A})$, respectively). Let proposition $E \leq AB$ $(A\overline{B}, \overline{A}$, respectively). Then*

$$min(E; \Psi) = min(E; \Psi \star (B \mid A))$$

Together with the Ramsey test (RT), (CR5) yields equalities of belief sets as stated in the following proposition:

Proposition 10 *If the conditional revision operator $\star$ satisfies postulate* (CR5)*, then*

$$\begin{aligned} Bel((\Psi \star (B \mid A)) \star AB) &= Bel(\Psi \star AB) \\ Bel((\Psi \star (B \mid A)) \star A\overline{B}) &= Bel(\Psi \star A\overline{B}) \\ Bel((\Psi \star (B \mid A)) \star \overline{A}) &= Bel(\Psi \star \overline{A}) \end{aligned}$$

For the representation theorems of postulates (C6) and (C7), we need postulate (CR5), respectively equation (1) and its consequence, lemma 9, to ensure that the property of being a minimal world in the affirmative or in the contradictory set associated with some conditionals is not touched under revision.

Theorem 11 *Suppose $\star$ is a conditional revision operator satisfying (CR5). Let Ψ be an epistemic state, and let $(B \mid A)$ be a conditional.*

1. *$\star$ satisfies* (CR6) *iff for all $\omega \in Mod(AB)$, $\omega' \in Mod(A\overline{B})$, $\omega <_\Psi \omega'$ implies $\omega <_{\Psi\star(B|A)} \omega'$.*
2. *$\star$ satisfies* (CR7) *iff for all $\omega \in Mod(AB)$, $\omega' \in Mod(A\overline{B})$, $\omega' <_{\Psi\star(B|A)} \omega$ implies $\omega' <_\Psi \omega$.*

6 Ordinal conditional functions

Ordinal conditional functions (*rankings*), as introduced by Spohn [Spohn, 1988], are functions κ from worlds to ordinals, i.e. to non-negative integers, such that some worlds are mapped to the minimal element 0. They are considered adequate representations of epistemic states (cf. e.g. [Spohn, 1988], [Darwiche and Pearl, 1997]), inducing a total pre-order on the set W of worlds by setting $\omega_1 \leq_\kappa \omega_2$ iff $\kappa(\omega_1) \leq \kappa(\omega_2)$. So the smaller $\kappa(\omega)$ is, the more plausible appears the world ω, and what is believed (for certain) in the epistemic state represented by κ is described precisely by the set $\{\omega \in W \mid \kappa(\omega) = 0\} =: Mod(Bel(\kappa))$. For a propositional formula $A \in \mathcal{L}$, we set $\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$, so that $\kappa(A \vee B) = \min\{\kappa(A), \kappa(B)\}$. In particular, $0 = \min\{\kappa(A), \kappa(\overline{A})\}$, so that at least one of A or $\overline{A}$ is considered mostly plausible. A proposition A is believed iff $\kappa(\overline{A}) > 0$ (which implies $\kappa(A) = 0$), so that A is believed iff $Bel(\kappa) \models A$. We abbreviate this again by $\kappa \models A$.

Let $\kappa \star A$ denote the revision of the ranking κ (of the corresponding epistemic state, respectively) by the proposition $A \in \mathcal{L}$ (for examples of such revision operators, cf. [Spohn, 1988], [Darwiche and Pearl, 1997]). For a conditional $(B \mid A)$, we set $\kappa \models (B \mid A)$ iff $\kappa \star A \models B$, that is iff $\kappa(AB) < \kappa(A\overline{B})$ (cf. lemma 4). Similar as in probability theory, we define $\kappa(B \mid A) = \kappa(AB) - \kappa(A)$

(for the connections between ordinal conditional functions and qualitative probabilistic reasoning, cf. e.g. [Spohn, 1988], [Darwiche and Pearl, 1997], [Goldszmidt and Pearl, 1996]).

We are now going to present a conditional revision operator for ordinal conditional functions that satisfy all of the postulates (CR0)-(CR7) and thus realizes the idea of conditional revision developed in this paper:

For an ordinal conditional function κ and a conditional $(B \mid A)$, we define $\kappa \star (B \mid A)$ by setting

$$\kappa \star (B \mid A)\,(\omega) = \begin{cases} \kappa(\omega) - \kappa(B \mid A) & : \omega \models AB \\ \kappa(\omega) + \alpha + 1 & : \omega \models A\overline{B} \\ \kappa(\omega) & : \omega \models \overline{A} \end{cases} \qquad (2)$$

where $\alpha = -1$, if $\kappa(AB) < \kappa(A\overline{B})$, and $= 0$, else.

The check of the postulates is straightforward, due to the representation theorems 8 and 11. So we have

Proposition 12 *The conditional revision operator defined by (2) satisfies all of the postulates (CR0) - (CR7).*

7 Concluding remarks

We presented a scheme of postulates (CR0)-(CR7) a revision of an epistemic state by a conditional should satisfy, with propositional revision and conditionals being connected via the Ramsey test (RT). These postulates are supported by arguments using the conditional structure of knowledge which can be made obvious by considering the relations $\perp$ and $\sqsubseteq$ between conditionals. We showed that our axioms cover the postulates of Darwiche and Pearl in [Darwiche and Pearl, 1997] and hence are of relevance for iterated belief revision, too. For the most crucial postulates (CR5)-(CR7), we formulated representation theorems, and we proved that our postulates are satisfiable by presenting a suitable conditional revision operator for ordinal conditional functions.

In addition to the postulates (CR0)-(CR7), another postulate may be worthwhile discussion:

(CR8) If $(D \mid C) \sqsubseteq (B \mid A)$ and $\Psi \not\models (B \mid A), (\overline{D} \mid C)$, then $\Psi \star (B \mid A) \models (D \mid C)$.

(CR8) clearly exceeds the paradigm of conditional preservation, in favor of imposing conditional structure as long as this does not contradict stated knowledge. The revision operator introduced in (2) satisfies (CR8), too.

The notion of conditional preservation is also mentioned in the area of quantified uncertain reasoning in [Kern-Isberner, 1998], within the framework of probabilistic reasoning at optimum entropy. Here we introduced so-called *c-adaptations* which adjust a prior probability distribution P to new quantified conditional information in a manner that preserves the conditional structure inherent to P "as far as possible". Though the axiomatization of conditional preservation given in [Kern-Isberner, 1998] is quite complex, it is easy to prove that the c-adaptations satisfy a probabilistic version of postulate (CR5):

$\mathbf{(CR5)_{prob}}$ If $(D \mid C) \perp (B \mid A)$ then $P \models (D \mid C)[y]$ iff $P \star (B \mid A)[x] \models (D \mid C)[y]$

where $x, y \in [0, 1]$ are probabilities and $P \models (D \mid C)[y]$ means that the conditional probability of D given C in P is y, i.e. $P(D \mid C) = y$.

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