Improvements of Basic TTE

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More than ten years have passed since my book Computable Analysis [3] has appeared in print. Meanwhile the representation approach, TTE, has been used to study computability in many applications. It has turned out that in the book some basic concepts are not quite adequate, that many theorems are not sufficiently general and that some tools are not sufficiently powerful for rigorous proofs. The three (free open access) articles [4, 5, 2] may remedy some of these deficiencies.

Multi-functions and multi-representations:
Not only multi-functions on represented sets but also multi-representations are useful. In [4, Section 3] two kinds of composition for multi-functions are discussed the "relational" composition "$\circ$" and the "function" composition "$\odot$". Realization of a multi-function w.r.t. multi-representations is defined in [4, Definition 17]. For the different use of the two kinds of composition see [4, Lemma 20 and Lemma 22]. (Because of their completely different nature the two structures "binary relations with $\circ$" and "binary relations with $\odot$" should have different names.)

Examples of useful multi-representations:

1. The multi-representation $\overline{\psi} : \Sigma^{\mathbb{N}} \rightrightarrows 2^{\mathbb{R}}$ of the power set of $\mathbb{R}$, $B \in \overline{\psi}(p)$ if p is a list of all rational intervals I such that $B \cap I \neq \emptyset$. See [5, Definition 5 and Theorem 38.4 and 38.5]. Also, for compact sets, $\kappa_{mincover} \equiv \kappa \wedge \overline{\psi}$. The restriction of $\overline{\psi}$ to the closed sets is the "inner" representation.

2. The multi-representation $[\gamma \rightarrow_p \delta]$ of the set of *all* partial (γ, δ)-continuous functions; allows to handle only the evaluation information without information about the domains of partial functions. See [5, Definition 25, Theorem 38].

3. The multi-representation μ of all effective metric spaces [3, Section 8.1]; $\mathcal{M} = (M, d, A, \alpha) \in \mu(p)$ iff p is a list of all (a, u, v, b) such that $a, b \in \mathbb{Q}$, $u, v \in \mathrm{dom}(\alpha)$, $a < d(\alpha(u), \alpha(v)) < b$. Then, for example, the distance on effective metric spaces is uniformly computable: from a μ-name of a space and two Cauchy-names of points we can compute their distance.

Type conversion for multi-representations and multi-functions
The important type conversion theorem [3, Theorem 3.3.15] (here only for single-valued representations and total functions) holds in full generality for multi-representations and multi-functions [4, Definition 32, Theorem 33]. As a special case also for partial functions.

Computable topological spaces
In the various publications considering computable topology as a foundation of computable analysis the basic definitions as well as the terminology are partly inconsistent so that the comparison of results is difficult. Furthermore, some definitions are unwieldy or inappropriate. Repeatedly facts from computable topology have been used in applications although they have never been proved or have not been proved in sufficient generality. In [5] a core of computable topology is developed in a more uniform and general manner. It can be considered as a careful revision of the corresponding parts in [3]. "Effective topological spaces", "computable topological spaces" and "predicate spaces" are defined (re-defined). Representations of points, open sets, closed sets, all subsets, compact sets and sets of partial continuous functions are studied for general computable topological spaces (and not only for Euclidean space or computable metric spaces).

Proving computability of functions on represented sets
Presently, many informal proofs of computability in analysis appeal to the intuition of the reader but cannot be refined to rigorous proofs by the available tools. In [2] we introduce a new type of generalized Turing machines (GTMs), which are intended as a tool for the mathematician who studies computability in Analysis. In a single tape cell a GTM can store a symbol, a real number, a continuous real function or a probability measure, for example. The model is based on TTE, the representation approach for computable analysis. As a main result we prove that the functions that are computable via given representations are closed under GTM programming. This generalizes the well known fact that these functions are closed under composition and primitive recursion [3, Theorem 3.1.7]. The theorem allows to speak about objects themselves instead of names in algorithms and proofs. By using GTMs for specifying algorithms, usually proofs become more rigorous and also simpler and more transparent since the GTM model is very simple and allows to apply well-known techniques from Turing machine theory.

Remark: In [4], corresponding theorems are proved for "flowcharts with indirect addressing". Since the GTM model is much simpler, the specific sections 4,5,7 and 8 (from Thm 30) can be ignored.

Uniform results (not yet satisfactorily handled)
Most theorems in the foundations developed so far have the form "Let $\mathcal{X}$ be a computable space ($\mathbb{R}^n$ for fixed n, Baire space, Banach space, topological space, ...). Then ...". But occasionally more uniform results are needed (ex-

ample: computable compactness results applied to finite dimensional subspaces (not uniform in n in [3, Section 5.2]) of a computable Hilbert space $\mathcal{X}$). For this purpose a multi-representation of the class of spaces under consideration should be defined (see the example for metric spaces above). Many non-uniform computability results on single computable spaces hold also uniformly. Whether a uniform version is true can be checked easily during the proof is written but it may be difficult to decide afterwards. (Example [1]: For computable Hilbert spaces a computable version of the Fréchet-Riesz representation theorem is true. Presumably it is true uniformly on all "effective Hilbert spaces", but the proof would require additional work.) Therefore, I suggest that in future authors add a remark on uniformity whenever possible.

References

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