4. Computability on the Real Numbers

Real numbers are the basic objects in analysis. For most non-mathematicians a real number is an infinite decimal fraction, for example $\pi = \texttt{3.14159}\ldots$. Mathematicians prefer to define the real numbers *axiomatically* as follows: $(\mathbb{R}, +, \cdot, 0, 1, <)$ is, up to isomorphism, the only Archimedean ordered field satisfying the axiom of continuity [Die60]. The set of real numbers can also be *constructed* in various ways, for example by means of Dedekind cuts or by completion of the (metric space of) rational numbers. We will neglect all foundational problems and assume that the real numbers form a well-defined set $\mathbb{R}$ with all the properties which are proved in analysis. We will denote the real line topology, that is, the set of all open subsets of $\mathbb{R}$, by $\tau_{\mathbb{R}}$.

In Sect. 4.1 we introduce several representations of the real numbers, three of which (and the equivalent ones) will survive as useful. We introduce a representation ρ^n of $\mathbb{R}^n$ by generalizing the definition of the main representation ρ of the set $\mathbb{R}$ of real numbers. In Sect. 4.2 we discuss the computable real numbers. Sect. 4.3 is devoted to computable real functions. We show that many well known functions are computable, and we show that partial summation of sequences is computable and that limit operator on sequences of real numbers is computable, if a modulus of convergence is given. We also prove a computability theorem for power series.

Convention 4.0.1. We still assume that Σ is a fixed finite alphabet containing all the symbols we will need.

4.1 Various Representations of the Real Numbers

According to the principles of TTE we introduce computability on $\mathbb{R}$ by naming systems. Since the set $\mathbb{R}$ is not countable, it has no notation $\nu :\subseteq \Sigma^* \to \mathbb{R}$ (onto) but only representations. Most of its numerous representations have no applications. In this section we introduce three representations $\rho, \rho_<$ and $\rho_>$ of the set of real numbers (and some equivalent ones) which induce the most important computability concepts. We will also discuss some other representations which, for various reasons, are only of little interest in computable analysis.

Since the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$, every real number has arbitrarily tight lower and arbitrarily tight upper rational bounds. Every real number x can be identified by the set

$$\{(a;b) \mid a,b \in \mathbb{Q},\ a < x < b\}$$

of all open intervals with rational endpoints containing x, by the set

$$\{a \in \mathbb{Q} \mid a < x\}$$

of all rational numbers smaller than x or by the set

$$\{a \in \mathbb{Q} \mid a > x\}$$

of all rational numbers greater than x. According to the concept of standard admissible representations

(Definitions 3.2.1, 3.2.2) a name of x will be a list of all open intervals with rational endpoints containing x, a list of all rational lower bounds of x or a list of all rational upper bounds of x, respectively.

Convention 4.1.1. In the following we will abbreviate $\nu_{\mathbb{Q}}(w)$ by $\overline{w}$ where $\nu_{\mathbb{Q}}$ is our standard notation of the rational numbers (Definition 3.1.2).

First, we introduce a standard notation I^n of all rational n-dimensional cubes with edges parallel to the coordinate axes and rational vertices.

Definition 4.1.2 (notation of rational cubes). *Assume $n \geq 1$.*

1. *For $(a_1, \ldots, a_n) \in \mathbb{R}^n$ define the (maximum) norm*

$$||(a_1, \ldots, a_n)|| := \max |a_1|, \ldots, |a_n|$$

and for $x, y \in \mathbb{R}^n$ define the (maximum) distance by

$$d(x,y) := ||x - y|| \ .$$

2. *Let $\mathrm{Cb}^{(n)} := \{B(a,r) \mid a \in \mathbb{Q}^n, r \in \mathbb{Q}, r > 0\}$ be the set of open rational balls (or cubes), where $B(a,r) := \{x \in \mathbb{R}^n \mid d(x,a) < r\}$.*
3. *Define a notation I^n of the set $\mathrm{Cb}^{(n)}$ by*

$$\mathrm{I}^n(\iota(v_1)\ldots\iota(v_n)\iota(w)) := B((\overline{v_1}, \ldots, \overline{v_n}), \overline{w}) \ .$$

4. *By $\overline{\mathrm{I}}^n(w)$ we denote the closure of the cube $\mathrm{I}^n(w)$.*

In particular, $\mathrm{Cb}^{(1)}$ is the set of all open intervals with rational endpoints and $\mathrm{I}^1(\iota(v)\iota(w))$ is the open interval $(\overline{v} - \overline{w}; \overline{v} + \overline{w})$, $\mathrm{Cb}^{(2)}$ is the set of all open squares with rational vertices (edges parallel to the coordinate axes) and $\mathrm{I}^2(\iota(v_1)\iota(v_2)\iota(w))$ is the open square $(\overline{v_1} - \overline{w}; \overline{v_1} + \overline{w}) \times (\overline{v_2} - \overline{w}; \overline{v_2} + \overline{w})$, $\mathrm{Cb}^{(3)}$ is the set of all open cubes with rational vertices (edges parallel to the coordinate axes), etc. .

We introduce representations $\rho, \rho_<$ and $\rho_>$ as our standard representations of computable topological spaces as follows.

Definition 4.1.3 (the representations $\rho, \rho_<$ and $\rho_>$). *Define computable topological spaces*

- $\mathbf{S}_= := (\mathbb{R}, \mathrm{Cb}^{(1)}, \mathrm{I}^1)$,
- $\mathbf{S}_< := (\mathbb{R}, \sigma_<, \nu_<)$, $\nu_<(w) := (\overline{w}; \infty)$,
- $\mathbf{S}_> := (\mathbb{R}, \sigma_>, \nu_>)$, $\nu_>(w) := (-\infty; \overline{w})$,

and let $\rho := \delta_{\mathbf{S}_=}$, $\rho_< := \delta_{\mathbf{S}_<}$ *and* $\rho_> := \delta_{\mathbf{S}_>}$.

(The sets $\sigma_<$ and $\sigma_>$ are defined implicitly.) Notice that $\mathbf{S}_=$, $\mathbf{S}_<$ and $\mathbf{S}_>$ are computable topological spaces (Definition 3.2.1), since the properties $\nu_\mathbb{Q}(u) = \nu_\mathbb{Q}(v)$ and $\mathrm{I}^1(u) = \mathrm{I}^1(v)$ are decidable in (u, v). By Definition 3.2.2,

$$\rho(p) = x \iff \{J \in \mathrm{Cb}^{(1)} \mid x \in J\} = \{\mathrm{I}^1(w) \mid \iota(w) \lhd p\} ,$$

or roughly speaking, iff p is a list of all $J \in \mathrm{Cb}^{(1)}$ such that $x \in J$.

Similarly, $\rho_<(p) = x$, iff p is a list of all rational numbers a such that $a < x$, and $\rho_>(p) = x$, iff p is a list of all rational numbers a such that $a > x$. Fig. 4.1 shows some open intervals $J \in \mathrm{Cb}^{(1)}$ with $x \in J$ and some rational numbers a with $a < y$.

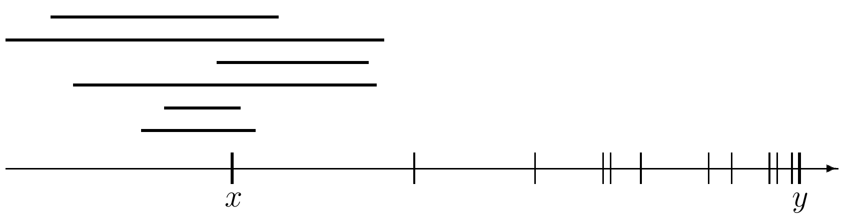

Fig. 4.1. Some open intervals $J \in \mathrm{Cb}^{(1)}$ with $x \in J$ and some rational numbers a with $a < y$

The final topologies of the above representations can be characterized easily (Definition 3.1.3.2, Lemma 3.2.5.3):

Lemma 4.1.4 (final topologies of $\rho, \rho_<$ and $\rho_>$).
1. The final topology of ρ is the real line topology $\tau_\mathbb{R}$.
2. The final topology of $\rho_<$ is $\tau_{\rho_<} := \{(x; \infty) \mid x \in \mathbb{R}\}$.
3. The final topology of $\rho_>$ is $\tau_{\rho_>} := \{(-\infty; x) \mid x \in \mathbb{R}\}$.

Proof: $\mathrm{Cb}^{(1)}$ generates $\tau_\mathbb{R}$, $\sigma_<$ generates $\tau_{\rho_<}$ and $\sigma_>$ generates $\tau_{\rho_>}$. □

Another important representation of the real numbers is the *Cauchy representation*. Since every real number is the limit of a Cauchy sequence of rational numbers, such sequences can be used as names of real numbers. However, the "naive Cauchy representation" which considers *all* converging sequences of rational numbers as names is not very useful (Example 4.1.14.1).

For the Cauchy representation we consider merely the "rapidly converging" sequences of rational numbers.

Definition 4.1.5 (Cauchy representation). *Define the Cauchy representation* $\rho_{\mathrm{C}} :\subseteq \Sigma^\omega \to \mathbb{R}$ *by*

$$\rho_{\mathrm{C}}(p) = x :\Longleftrightarrow \begin{cases} \textit{there are words } \ w_0, w_1 \ldots \in \mathrm{dom}(\nu_{\mathbb{Q}}) \\ \textit{such that } \ p = \iota(w_0)\iota(w_1)\iota(w_2)\ldots, \\ |\overline{w}_i - \overline{w}_k| \le 2^{-i} \ \textit{ for } \ i < k \ \textit{ and } \ x = \lim_{i\to\infty} \overline{w}_i \ . \end{cases}$$

The representation ρ, the Cauchy representation ρ_{C} and many variants of them are equivalent:

Lemma 4.1.6 (representations equivalent to ρ). The following representations of the real numbers are equivalent to the standard representation $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$:
1. the Cauchy representation ρ_{C},
2. the representations ρ'_{C}, ρ''_{C} and ρ'''_{C} obtained by substituting

$$|\overline{w}_i - \overline{w}_k| < 2^{-i}, \qquad |\overline{w}_i - x| < 2^{-i} \qquad \text{or} \qquad |\overline{w}_i - x| \le 2^{-i} \ ,$$

   respectively, for $|\overline{w}_i - \overline{w}_k| \le 2^{-i}$ in the definition of ρ_{C},
3.
$$\rho^a(p) = x :\Longleftrightarrow \begin{cases} \text{there are words } \ u_0, u_1 \ldots \in \mathrm{dom}(\mathrm{I}^1) \\ \text{such that } \ p = \iota(u_0)\iota(u_1)\ldots, \\ (\forall k)\left(\overline{\mathrm{I}}^1(u_{k+1}) \subseteq \mathrm{I}^1(u_k) \text{ and length}(\mathrm{I}^1(u_k)) < 2^{-k}\right) \\ \text{and } \ \{x\} = \mathrm{I}^1(u_0) \cap \mathrm{I}^1(u_1) \cap \ldots \ , \end{cases}$$
4.
$$\rho^b(p) = x \ :\Longleftrightarrow \ \ \{x\} = \bigcap \left\{ \overline{\mathrm{I}}^1(v) \mid \iota(v) \lhd p \right\} \ .$$

The representations ρ'_{C}, ρ''_{C} and ρ'''_{C} are inessential modifications of the Cauchy representation, and ρ^a is a representation by strongly nested, rapidly converging sequences of open intervals. A ρ^b-name of x is a list of closed rational intervals for which x is the only common point. While a ρ-name of x is a list of *all* open rational intervals containing x, these characterizations show that it suffices to list merely "sufficiently many" of them.

Proof:
$\rho^b \le \rho^a$: Define representations δ_1 and δ_2 of $\mathbb{R}$ by $\delta_1(p) = x$, iff

$$p = \iota(u_0)\iota(u_1)\ldots, \quad \overline{\mathrm{I}}^1(u_{k+1}) \subseteq \overline{\mathrm{I}}^1(u_k) \text{ and } \{x\} = \overline{\mathrm{I}}^1(u_0) \cap \overline{\mathrm{I}}^1(u_1) \cap \ldots \ ,$$

and $\delta_2(p) = x$, iff

$$p = \iota(u_0)\iota(u_1)\ldots, \quad \overline{\mathrm{I}}^1(u_{k+1}) \subseteq \mathrm{I}^1(u_k) \text{ and } \{x\} = \mathrm{I}^1(u_0) \cap \mathrm{I}^1(u_1) \cap \ldots .$$

There is a computable function $h :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that

$$h(p) = \iota(v_0)\iota(v_1)\ldots \text{ where } \overline{\mathrm{I}}^1(v_i) = \bigcap \left\{ \overline{\mathrm{I}}^1(v) \mid \iota(v) \lhd p_{<i+i_p} \right\} ,$$

where i_p is the smallest number k such that $\iota(v) \lhd p_{<k}$ for some $v \in \mathrm{dom}(\mathrm{I}^1)$. Then obviously, the function h translates ρ^b to δ_1.
There is a computable function $g : \mathbb{N} \times \Sigma^* \to \Sigma^*$ such that

$$\mathrm{I}^1(w) = B(a, r) \implies \mathrm{I}^1 \circ g(i, w) = B(a, (1 + 2^{-i}) \cdot r) .$$

There is a computable function $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that

$$f(\iota(w_0)\iota(w_1)\iota(w_2)\ldots) = \iota \circ g(0, w_0)\iota \circ g(1, w_1)\iota \circ g(2, w_2)\ldots .$$

Then f translates δ_1 to δ_2. It remains to select from any δ_2-name a rapidly converging subsequence of intervals. There is a Type-2 machine M which on input $p = \iota(u_0)\iota(u_1)\ldots \in \mathrm{dom}(\delta_2)$ computes a sequence $q = \iota(u_{m_0})\iota(u_{m_1})\ldots$ where m_k is the smallest number $i > m_{k-1}$ such that $\mathrm{length}(\mathrm{I}^1(u_i)) < 2^{-k}$. Then f_M, the function computed by the machine M, translates δ_2 to ρ^a. Therefore, $\rho^b \le \rho^a$.

$\rho^a \le \rho'_\mathrm{C}$ and $\rho^a \le \rho''_\mathrm{C}$: There is a computable function $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ mapping any $p = \iota(u_0)\iota(u_1)\iota(u_1)\ldots \in \mathrm{dom}(\rho^a)$ to $q = \iota(v_0)\iota(v_1)\iota(v_1)\ldots$ such that $\nu_\mathbb{Q}(v_i) = \inf(\mathrm{I}^1(u_i))$ for all i. If $\rho^a(p) = x$, then $\overline{v}_0 < \overline{v}_1 < \overline{v}_2 < \ldots < x$ and $|\overline{v}_i - x| < 2^{-i}$ for all i. Therefore, f translates ρ^a to ρ'_C and ρ''_C.

$\rho'_\mathrm{C} \le \rho_\mathrm{C}$ and $\rho''_\mathrm{C} \le \rho'''_\mathrm{C}$: The identity in Σ^ω translates ρ'_C into ρ_C and ρ''_C into ρ'''_C.

$\rho_\mathrm{C} \le \rho'''_\mathrm{C}$: Suppose $|\overline{w}_i - \overline{w}_k| \le 2^{-i}$ for $i < k$ and $x = \lim_{i\to\infty} \overline{w}_i$. Since

$$|\overline{w}_i - x| \le |\overline{w}_i - \overline{w}_k| + |\overline{w}_k - x| \le 2^{-i} + |\overline{w}_k - x|$$

for all $i < k$, $|\overline{w}_i - x| \le 2^{-i}$, and so the identity translates ρ_C to ρ'''_C.

$\rho'''_\mathrm{C} \le \rho$: If $\rho'''_\mathrm{C}(\iota(w_0)\iota(w_1)\iota(w_2)\ldots) = x$, then for any $v \in \mathrm{dom}(\mathrm{I}^1)$:

$$x \in \mathrm{I}^1(v) \iff (\exists\, i)[\overline{w}_i - 2^{-i}; \overline{w}_i + 2^{-i}] \subseteq \mathrm{I}^1(v) .$$

Let ν_Σ be the standard numbering of Σ^* (Sect. 1.4). There is a Type-2 machine M mapping every $p = \iota(w_0)\iota(w_1)\iota(w_2)\ldots \in \mathrm{dom}(\rho'''_\mathrm{C})$ to a sequence $q = \iota(v_0)\iota(v_1)\iota(v_2)\ldots$ such that

$$v_{\langle i,k\rangle} = \begin{cases} \nu_\Sigma(k) & \text{if } \nu_\Sigma(k) \in \mathrm{dom}(\mathrm{I}^1) \\ & \quad \text{and } [\overline{w}_i - 2^{-i}; \overline{w}_i + 2^{-i}] \subseteq \mathrm{I}^1(\nu_\Sigma(k)) \\ u & \text{otherwise} \end{cases}$$

where u is a word with $\mathrm{I}^1(u) = (\overline{w}_0 - 2; \overline{w}_0 + 2)$. If $\rho'''_\mathrm{C}(p) = x$, then q is a list of all words v such that $x \in \mathrm{I}^1(v)$. Therefore, the function f_M translates ρ'''_C to ρ.

$\rho \le \rho^b$: The identity on Σ^ω translates ρ to ρ^b.
Therefore, the given representations are equivalent. □

The representation δ informally introduced in Sect. 1.3.2 has the property $\rho^a \le \delta \le \rho^b$ (Lemma 4.1.6) and so is equivalent to ρ.

By definition, a ρ-name of x is a list of all intervals $(a;b)$ with rational endpoints such that $x \in (a;b)$. Every arbitrarily tight lower and every arbitrarily tight upper rational bound of x can be obtained from a finite prefix of p. This is the characteristic common property of all representations equivalent to ρ:

Lemma 4.1.7 (characterization of ρ). For every representation $\delta :\subseteq \Sigma^\omega \to \mathbb{R}$,

$$\delta \le \rho \iff \begin{cases} \{(x,a) \in \mathbb{R} \times \mathbb{Q} \mid a < x\} \text{ is } (\delta, \nu_\mathbb{Q})\text{-r.e. and} \\ \{(x,a) \in \mathbb{R} \times \mathbb{Q} \mid x < a\} \text{ is } (\delta, \nu_\mathbb{Q})\text{-r.e.} . \end{cases}$$

This is essentially a special case of Theorem 3.2.10. For a proof see Exercise 4.1.3. Therefore, ρ is up to equivalence the "poorest" representation δ of the real numbers such that the properties "$a < x$" and "$x < a$" are r.e.

Also the representations $\rho_<$ and $\rho_>$ can be simplified.

Lemma 4.1.8 (representations equivalent to $\rho_<$). The following representations of the real numbers are equivalent to the representation $\rho_< :\subseteq \Sigma^\omega \to \mathbb{R}$:

1. $$\rho^a_<(p) = x :\iff \begin{cases} \text{there are } u_0, u_1 \ldots \in \mathrm{dom}(\nu_\mathbb{Q}) \\ \text{such that } p = \iota(u_0)\iota(u_1)\ldots, \\ \overline{u_0} < \overline{u_1} < \ldots < x \text{ and } x = \lim_{i\to\infty} \overline{u_i} , \end{cases}$$
2. $$\rho^b_<(p) = x :\iff x = \sup\{\overline{v} \mid \iota(v) \lhd p\} .$$

The proof is left as Exercise 4.1.5. However, if we force rapid convergence, we obtain a representation equivalent to ρ (Exercise 4.1.6). Lemma 4.1.7 holds correspondingly for $\rho_<$ replacing ρ. Therefore, $\rho_<$ is the "poorest" representation δ of the real numbers such that the property "$a < x$" is r.e. The above considerations hold correspondingly for $\rho_>$ replacing $\rho_<$.

The following lemma is obvious already from our informal characterizations of ρ, $\rho_<$ and $\rho_>$.

Lemma 4.1.9.
1. $\rho \equiv \rho_< \wedge \rho_>$ (that is, ρ is the greatest lower bound of $\rho_<$ and $\rho_>$), in particular, $\rho \le \rho_<$ and $\rho \le \rho_>$.
2. $\rho_< \not\le_\mathrm{t} \rho$, $\rho_> \not\le_\mathrm{t} \rho$, $\rho_< \not\le_\mathrm{t} \rho_>$ and $\rho_> \not\le_\mathrm{t} \rho_<$.

Proof: 1. By Definition 3.3.7, $(\rho_< \wedge \rho_>)\langle p, q\rangle = x \iff \rho_<(p) = \rho_>(q) = x$. There is a Type-2 machine M which on input $p \in \mathrm{dom}(\rho)$ produces a list of all $\iota(w)$ for which there is some word v such that $\iota(v) \lhd p$ and $\overline{w}$ is the left endpoint of the interval $\mathrm{I}^1(v)$. Then the function f_M translates ρ to $\rho_<$. For a similar reason, $\rho \leq \rho_>$. By Lemma 3.3.8, $\rho \leq \rho_< \wedge \rho_>$.
For the other direction it suffices to prove $\rho^a_< \wedge \rho^a_> \leq \rho^b$. There is a Type-2 machine M which on input $\langle p, q\rangle \in \mathrm{dom}(\rho^a_< \wedge \rho^a_>)$, where $p = \iota(u_0)\iota(u_1)\ldots$ and $q = \iota(v_0)\iota(v_1)\ldots$, produces a sequence $\iota(w_0)\iota(w_1)\ldots$, such that $\mathrm{I}^1(w_i) = (\overline{u}_i; \overline{v}_i)$. The function f_M translates $\rho^a_< \wedge \rho^a_>$ to ρ^b.

2. This follows from the simple observation that a lower bound cannot be obtained from a finite set of upper bounds and vice versa. More formally we can use the fact $\gamma' \leq_{\mathrm{t}} \gamma \Longrightarrow \tau_\gamma \subseteq \tau_{\gamma'}$ from Theorem 3.1.8: $\rho_< \not\leq_{\mathrm{t}} \rho$ since $\tau_\rho \not\subseteq \tau_{\rho_<}$ etc.. □

To sum up, one can roughly say that for a real number x, a $\rho_<$-name consists of all rational lower bounds of x, a $\rho_>$-name consists of all rational upper bounds of x, and a ρ-name consists of all rational lower bounds and all rational upper bounds of x. Since by Corollary 3.2.12 for admissible

representations only continuous functions can be computable, Lemma 4.1.4 tells us which of the three computability concepts is adequate in a given topological setting.

Computability of elements and functions induced by the representations ρ, $\rho_<$ and $\rho_>$ will be discussed in the following sections. As an instructive example we discuss the problem of finding a rational upper bound for a real number.

Example 4.1.10. Consider the multi-valued function

$$F : \mathbb{R} \rightrightarrows \mathbb{Q}, \quad \mathrm{R}_F := \{(x, a) \in \mathbb{R} \times \mathbb{Q} \mid x < a\} .$$

We prove the following effectiveness properties (Definition 3.1.3):

1. F is not $(\rho_<, \nu_{\mathbb{Q}})$-continuous.
2. F is $(\rho_>, \nu_{\mathbb{Q}})$-computable (and therefore, $(\rho, \nu_{\mathbb{Q}})$-computable).
3. F has no $(\rho, \nu_{\mathbb{Q}})$-continuous choice function (and therefore, no $(\rho_>, \nu_{\mathbb{Q}})$-continuous choice function).

Remember, that $\nu_{\mathbb{Q}}$ is admissible with discrete final topology $\tau_{\nu_{\mathbb{Q}}}$ (Example 3.2.4.1) .

1. Suppose F is $(\rho_<, \nu_{\mathbb{Q}})$-continuous. Then F has a continuous $(\rho^b_<, \nu_{\mathbb{Q}})$-realization $g :\subseteq \Sigma^\omega \to \Sigma^*$ (Lemma 4.1.8). Consider $\rho^b_<(p) = x$. By assumption, $g(p) = w$ with $x < \nu_{\mathbb{Q}}(w) = \overline{w}$ for some $w \in \Sigma^*$. Since g is continuous, there is some number n with $g[p_{<n}\Sigma^\omega] = \{w\}$. For some $k \geq n$, `11` is the suffix of $p_{<k}$. Choose $u \in \mathrm{dom}(\nu_{\mathbb{Q}})$ with $\overline{w} < \overline{u}$ and define $q := p_{<k}\iota(u)\iota(u)\ldots$. Then $\rho^b_<(q) = \overline{u}$ and $\nu_{\mathbb{Q}} \circ g(q) = \overline{w}$, since $q \in p_{<n}\Sigma^\omega \cap \mathrm{dom}(g)$. However, by assumption on g we must have $\overline{u} = \rho^b_<(q) < \nu_{\mathbb{Q}} \circ g(q) = \overline{w}$. Contradiction!

2. Let M be a Type-2 machine which for any input $p = \iota(w_0)\iota(w_1)\ldots$ prints the word w_0 and halts. Then $\rho_>(p) < \overline{w}_0 = \nu_{\mathbb{Q}} \circ f_M(p)$ for all

$p = \iota(w_0)\iota(w_1)\ldots \in \mathrm{dom}(\rho_>)$. Therefore, f_M is a $(\rho_>, \nu_\mathbb{Q})$-realization of the relation R.

3. Since the final topology $\tau_\mathbb{R}$ of the admissible representation ρ is connected, every $(\rho, \nu_\mathbb{Q})$-continuous function is constant by Corollary 3.2.13. But a choice function of F cannot be constant. □

The definitions of ρ, ρ_C, $\rho_<$ and $\rho_>$ (and their variants) can be modified in various other ways without affecting the induced continuity or computability, respectively. So far we have used the set $\mathbb{Q}$ of the rational numbers as a "standard" dense countable subset of the real numbers and $\nu_\mathbb{Q}$ as its standard notation. Can we replace $\nu_\mathbb{Q}$ by some other notation ν_Q of a dense subset Q of $\mathbb{R}$ such that the resulting representations induce the same continuity or computability concepts on $\mathbb{R}$? The following "robustness" theorem gives an answer.

Theorem 4.1.11 (robustness). For any representation δ of the real numbers introduced in Definitions 4.1.3, 4.1.5, 4.1.6 and 4.1.8 consider δ as a function of $\nu_\mathbb{Q}$, that is, $\delta = D(\nu_\mathbb{Q})$. Then for every notation ν_Q of a dense subset $Q \subseteq \mathbb{R}$,

1. $\delta \equiv_\mathrm{t} D(\nu_Q)$,
2. $\delta \equiv D(\nu_Q)$, if $\nu_\mathbb{Q}$ and ν_Q are r.e.-related (that is, if $\{(v,w,i) \mid |\nu_\mathbb{Q}(v) - \nu_Q(w)| < 2^{-i}\}$ is r.e.).

The proof is left as Exercise 4.1.8. Examples for notations r.e.-related to $\nu_\mathbb{Q}$ are any notation of $\mathbb{Q}$ equivalent to $\nu_\mathbb{Q}$ and any standard notation of the binary rational numbers $\mathbb{Q}_2 := \{z/2^n \mid z \in \mathbb{Z}, n \in \mathbb{N}\}$.

Computability concepts introduced via robust definitions are not sensitive to "inessential" modifications. It can be expected that they occur in many applications. On the other hand, computability concepts introduced via non-robust definitions are not very relevant.

The *Turing machine* is another famous example of a robust definition. Numerous variants of the original definition are used in the literature, all of which define the same notion of computable functions. Usually the representation by infinite decimal fractions is considered to be the most natural representation of the real numbers.

Definition 4.1.12 (finite and infinite base-n fractions). *For $n \geq 2$ define the notation $\nu_{\mathrm{b},n}$ of the finite base-n fractions and the representation $\rho_{\mathrm{b},n} :\subseteq \Sigma^\omega \to \mathbb{R}$ of the real numbers by infinite -n fractions as follows:*

$$\mathrm{dom}(\nu_{\mathrm{b},n}) := \{\lambda, \texttt{-}\}(\Gamma^* \setminus \texttt{0}\Gamma^*)\bullet\Gamma^*,$$
$$\mathrm{dom}(\rho_{\mathrm{b},n}) := \{\lambda, \texttt{-}\}(\Gamma^* \setminus \texttt{0}\Gamma^*)\bullet\Gamma^\omega,$$
$$\nu_{\mathrm{b},n}(sa_k \ldots a_0 \bullet a_{-1}a_{-2}\ldots a_{-m}) := \overline{s} \cdot \sum_{k \geq i \geq -m} a_i \cdot n^i,$$
$$\rho_{\mathrm{b},n}(sa_k \ldots a_0 \bullet a_{-1}a_{-2}\ldots) := \overline{s} \cdot \sum_{i \leq k} a_i \cdot n^i,$$

where $a_i \in \Gamma_n := \{0, 1, \ldots, n-1\}$ *for all* $i \le k$, $\overline{s} := 1$, *if* $s = \lambda$, *and* $\overline{s} := -1$, *if* $s =$ -. *(We assume tacitly* $\Gamma_n \subseteq \Sigma$.*)*

(Remember that λ is the empty word.) The following theorem summarizes some interesting properties of the representations by infinite base-n fractions.

Theorem 4.1.13 (infinite base-n fractions). For any $m, n \ge 2$ (assuming $\Gamma_m, \Gamma_n \subseteq \Sigma$)

1. $\rho_{\mathrm{b},n}|^{\mathrm{Ir}} \equiv \rho|^{\mathrm{Ir}}$ (where $\mathrm{Ir} := \mathbb{R} \setminus \mathbb{Q}$ is the set of irrational real numbers),
2. x is $\rho_{\mathrm{b},n}$-computable, iff x is ρ-computable.
3. $\rho_{\mathrm{b},n} \le \rho$ and $\rho \not\le_{\mathrm{t}} \rho_{\mathrm{b},n}$,
4. $\rho_{\mathrm{b},m} \le \rho_{\mathrm{b},n}$, if $\mathrm{pd}(n) \subseteq \mathrm{pd}(m)$, $\rho_{\mathrm{b},m} \not\le_{\mathrm{t}} \rho_{\mathrm{b},n}$ otherwise (where $\mathrm{pd}(n)$ denotes the set of prime divisors of n),
5. $\rho_{\mathrm{b},n}$ has the final topology $\tau_{\mathbb{R}}$.
6. $\rho_{\mathrm{b},n}$ is not admissible,
7. $f :\subseteq \mathbb{R} \to \mathbb{R}$ is $(\rho_{\mathrm{b},n}, \rho)$-computable (-continuous), iff it is (ρ, ρ)-computable (-continuous).

Proof: 1. and 2. See Exercise 4.1.10

3. There is a Type-2 machine M which maps any sequence $p = sa_k \ldots a_0 \bullet a_{-1} a_{-2} \ldots \in \mathrm{dom}(\rho_{\mathrm{b},n})$ to a sequence $\iota(u_0)\iota(u_1)\ldots$ with $\overline{u_j} = \nu_{\mathrm{b},n}(sa_k \ldots a_0 \bullet a_{-1} a_{-2} \ldots a_{-j})$. Then M translates $\rho_{\mathrm{b},n}$ to ρ_{C}. The relation $\rho \not\le_{\mathrm{t}} \rho_{\mathrm{b},n}$ will be concluded from Property 4.

4. Suppose, $\mathrm{pd}(n) \subseteq \mathrm{pd}(m)$. Then for each $k \in \mathbb{N}$ numbers $l_k, b_k \in \mathbb{N}$ can be determined such that $m^{l_k} = b_k \cdot n^k$. For computing a $\rho_{\mathrm{b},n}$-name of $x = \rho_{\mathrm{b},m}(p)$, it suffices to compute integers $c_0, c_1, \ldots$ such that $c_k \le n^k \cdot x \le c_k + 1$ for all k. There is a Type-2 machine M, which on input (p, u) ($x := \rho_{\mathrm{b},m}(p)$, $k := \nu_{\mathbb{N}}(u)$) determines (names of) numbers $l_k, b_k \in \mathbb{N}$ such that $m^{l_k} = b_k \cdot n^k$ and then (a name of) a number c_k such that

$$c_k \cdot b_k \le m^{l_k} \cdot x \le (c_k + 1) \cdot b_k \ .$$

(For this purpose, M must read only the first l_k digits of p after the dot.) Since $m^{l_k} = b_k \cdot n^k$, we obtain immediately $c_k \le n^k \cdot x \le c_k + 1$, as required. Therefore, $\rho_{\mathrm{b},m}$ can be translated to $\rho_{\mathrm{b},n}$ by a Type-2 machine.

Now consider that the prime number $r \in \mathbb{N}$ divides n but not m. There is some $c \in \mathbb{N}$, $1 \le c < n$, with $n = c \cdot r$. The number $1/r$ has a $\rho_{\mathrm{b},m}$-name $p := \bullet a_{-1} a_{-2} \ldots$ which has neither the period 0 nor the period $(m-1)$. Suppose some continuous function $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ translates $\rho_{\mathrm{b},m}$ to $\rho_{\mathrm{b},n}$. Then $f(p) = \bullet c00\ldots$ or $f(p) = \bullet 0(c-1)(n-1)(n-1)\ldots$.
Consider the first case $f(p) = \bullet c00\ldots$. By continuity of f there is some l such that $f[\bullet a_{-1} a_{-2} \ldots a_{-l} \Sigma^\omega] \subseteq \bullet c\Sigma^\omega$. Choose $p' := \bullet a_{-1} a_{-2} \ldots a_{-l} 00\ldots$. Then $f(p') \in \bullet c\Sigma^\omega$. We have $\rho_{\mathrm{b},m}(p') < 1/r$ (since p does not have the period 0) and $\rho_{\mathrm{b},n}(f(p')) \in \rho_{\mathrm{b},n}[\bullet c\Sigma^\omega]$, hence $\rho_{\mathrm{b},n}(f(p')) \ge 1/r$ (contradiction). Therefore, $\rho_{\mathrm{b},m} \not\le_{\mathrm{t}} \rho_{\mathrm{b},n}$.
The second case can be treated similarly.

3. (continued) Suppose $\rho \leq_{\mathrm{t}} \rho_{\mathrm{b},n}$. Choose some prime number m which does not divide n. Then $\rho_{\mathrm{b},m} \leq_{\mathrm{t}} \rho \leq_{\mathrm{t}} \rho_{\mathrm{b},n}$ by Property 1, but $\rho_{\mathrm{b},m} \not\leq_{\mathrm{t}} \rho_{\mathrm{b},n}$ as proved above (contradiction). This shows $\rho \not\leq_{\mathrm{t}} \rho_{\mathrm{b},n}$.

5. If $X \subseteq \mathbb{R}$ is open, then it is ρ-open by Lemma 4.1.4. Since $\rho_{\mathrm{b},n} \leq \rho$ by Property 3, X is $\rho_{\mathrm{b},n}$-open by Theorem 3.1.8. Let $X \subseteq \mathbb{R}$ be $\rho_{\mathrm{b},n}$-open. Then $\rho_{\mathrm{b},n}^{-1}[X] = A\Sigma^\omega \cap \mathrm{dom}(\rho_{\mathrm{b},n})$ for some $A \subseteq \Sigma^*$. Consider $x \in X$, $x > 0$.
Case 1: $x = a/n^j$ for some integer a and some natural number j.
Then x has two $\rho_{\mathrm{b},n}$-names $p :=$ "$a_k \ldots a_0{\bullet}a_{-1} \ldots$" and $q :=$ "$b_k \ldots b_0{\bullet}b_{-1} \ldots$", such that there is some some $m \leq k$ with $a_m > 0$, $a_i = 0$ for $i < m$, $b_m = a_m - 1$ and $b_i = n - 1$ for $i < m$. Since $\rho_{\mathrm{b},n}^{-1}[X]$ is open in $\mathrm{dom}(\rho_{\mathrm{b},n})$, there is a number $l > m$ such that $\rho_{\mathrm{b},n}[$"$a_k \ldots a_0{\bullet}a_{-1} \ldots a_{-l}$"$\Sigma^\omega] \subseteq X$ and $\rho_{\mathrm{b},n}[$"$b_k \ldots b_0{\bullet}b_{-1} \ldots b_{-l}$"$\Sigma^\omega] \subseteq X$. We obtain $[x; x + n^{-l}) \subseteq X$ and $(x - n^{-l}; x] \subseteq X$, i,e, x has the open neighborhood $(x - n^{-l}; x + n^{-l}) \subseteq X$.
Case 2: Not Case 1.
Then x has a $\rho_{\mathrm{b},n}$-name $p :=$ "$a_k \ldots a_0{\bullet}a_{-1} \ldots$" which has neither the period 0 nor the period $(n-1)$. Since $\rho_{\mathrm{b},n}^{-1}[X]$ is open in $\mathrm{dom}(\rho_{\mathrm{b},n})$, there is a number l such that $\rho_{\mathrm{b},n}[$"$a_k \ldots a_0{\bullet}a_{-1} \ldots a_{-l}$"$\Sigma^\omega] \subseteq X$. Then $x \in (y; z) \subseteq X$, where $y := \rho_{\mathrm{b},n}($"$a_k \ldots a_0{\bullet}a_{-1} \ldots a_{-l}00 \ldots$"$)$ and
$z := \rho_{\mathrm{b},n}($"$a_k \ldots a_0{\bullet}a_{-1} \ldots a_{-l}(n-1)(n-1) \ldots$"$)$, hence x has an open neighborhood in X.
For the case $x < 0$ the proof is similar. If $x = 0$ consider the two names "$\bullet 00 \ldots$" and "$-\bullet 00 \ldots$".

6. By Lemma 4.1.4, the representation ρ is admissible with final topology $\tau_{\mathbb{R}}$. By Theorem 3.2.8.1 any two admissible representations with final topology $\tau_{\mathbb{R}}$ are continuously equivalent. Since the representation $\rho_{\mathrm{b},n}$ has the final topology $\tau_{\mathbb{R}}$ but is not continuously equivalent to ρ, it cannot be admissible.

7. See Exercise 4.1.12. □

Restricted to the irrational numbers, $\rho_{\mathrm{b},n}$ and ρ are equivalent. Therefore, a real number x is $\rho_{\mathrm{b},n}$-computable, iff it is ρ-computable (notice that the rational numbers are $\rho_{\mathrm{b},n}$-computable and ρ-computable).

Since $\rho_{\mathrm{b},n}$ is not admissible, a $\rho_{\mathrm{b},n}$-name cannot be interpreted as a (sufficiently rich) list of atomic properties from a subbase of its final topology $\tau_{\mathbb{R}}$. Although $\rho_{\mathrm{b},n}$-names are richer than ρ-names by Property 3, this additional information is useless for computing real functions by Property 7. From Property 7 we conclude also that every $(\rho_{\mathrm{b},n}, \rho_{\mathrm{b},m})$-computable function is (ρ, ρ)-computable. However, already the simple (ρ, ρ)-computable real function $x \mapsto 3 \cdot x$ is not $(\rho_{\mathrm{b},10}, \rho_{\mathrm{b},10})$-continuous (Example 2.1.4.7).

We discuss some further representations of the real numbers which, however, have only very few applications.

Example 4.1.14 (further representations of $\mathbb{R}$).

1. *Naive Cauchy representation:* Define the naive Cauchy representation ρ_{Cn} of the real numbers by

$$\rho_{\mathrm{Cn}}(p) = x :\iff p = \iota(w_0)\iota(w_1)\ldots \text{ and } \lim_{i\to\infty} \overline{w_i} = x \;.$$

Since ρ_{C}, $\rho_<$ and $\rho_>$ are restrictions of ρ_{Cn}, we have $\rho_{\mathrm{C}}, \rho_<, \rho_> \le \rho_{\mathrm{Cn}}$. ρ_{Cn} has the final topology $\{\emptyset, \mathbb{R}\}$ (no property of $x = \rho_{\mathrm{Cn}}(p)$ can be concluded from a finite prefix w of p). From this fact and Theorem 3.1.8 we conclude $\rho_{\mathrm{Cn}} \not\le_{\mathrm{t}} \rho_{\mathrm{C}}$, $\rho_{\mathrm{Cn}} \not\le_{\mathrm{t}} \rho_<$ and $\rho_{\mathrm{Cn}} \not\le_{\mathrm{t}} \rho_>$.

2. *Cut representations:* Define computable topological spaces
   - $\mathbf{S}_\le = (\mathbb{R}, \sigma_\le, \nu_\le)$ by $\nu_\le(w) := [\nu_{\mathbb{Q}}(w); \infty)$ and
   - $\mathbf{S}_\ge = (\mathbb{R}, \sigma_\ge, \nu_\ge)$ by $\nu_\ge(w) := (-\infty; \nu_{\mathbb{Q}}(w)]$

   (cf. Definition 4.1.3). Define the *left cut* representation and the *right cut* representation by $\rho_\le := \delta_{\mathbf{S}_\le}$ and $\rho_\ge := \delta_{\mathbf{S}_\ge}$, respectively.
   If $\rho_\le(p) = x$ ($\rho_\ge(p) = x$), then p is a list of all $a \in \mathbb{Q}$ with $a \le x$ ($a \ge x$). The final topology of $\rho_\le$ is $\tau_{\rho_<} \cup \sigma_\le$, that is, also intervals ("atomic properties") $[a; \infty)$ with $a \in \mathbb{Q}$ are called "open". We have $\rho_\le \le \rho_<$, but neither $\rho_<$, $\rho_>$, ρ_{C} nor $\rho_\ge$ are t-reducible to $\rho_\le$.
   As an example assume $\rho_{\mathrm{C}} \le_{\mathrm{t}} \rho_\le$. Then $\tau_{\rho_\le} \subseteq \tau_\rho$ by Theorem 3.1.8. But $[0; \infty) \in \tau_{\rho_\le} \setminus \tau_\rho$. Restricted to the irrational numbers $\rho_<$ and $\rho_\le$ are equivalent: $\rho_<|^{\mathrm{Ir}} \equiv \rho_\le|^{\mathrm{Ir}}$. (The properties hold accordingly for $\rho_\ge$.) The definitions of $\rho_\le$ and $\rho_\ge$ are not topologically robust, that is, replacement of $\mathbb{Q}$ in the definitions by another dense subset Q yields representations which are not continuously equivalent to $\rho_\le$ and $\rho_\ge$, respectively.
3. *Continued fraction representation:* For any real number $x \ge 0$ define its continued fraction $\mathrm{fr}(x) := [a_0, a_1, \ldots]$ ($a_i \in \mathbb{N}$) inductively by:

$$x_0 := x\,,$$

$$a_n := \lfloor x_n \rfloor\,,\ x_{n+1} := \begin{cases} \frac{1}{x_n - a_n} & \text{if } a_n \ne x_n \\ 0 & \text{otherwise.} \end{cases}$$

   Then, informally

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}}\,,$$

   where the fraction is finite ($a_n = 0$ for all $n \ge n_0$), iff the number x is rational. Define the continued fraction representation ρ_{cf} of the real numbers by $\rho_{\mathrm{cf}}(p) = x$, iff ($x \ge 0$ and $p = \mathtt{1}^{a_0}\mathtt{0}\mathtt{1}^{a_1}\mathtt{0}\ldots$ where $\mathrm{fr}(x) := [a_0, a_1, \ldots]$) or ($x < 0$ and $p = \mathtt{-1}^{a_0}\mathtt{0}\mathtt{1}^{a_1}\mathtt{0}\ldots$ where $\mathrm{fr}(-x) := [a_0, a_1, \ldots]$). One can show

$$\rho_{\mathrm{cf}} \equiv \rho_\le \wedge \rho_\ge\,,$$

   in particular, $\rho_{\mathrm{cf}} \le \rho_\le$ and $\rho_{\mathrm{cf}} \le \rho_\ge$. Furthermore, $\rho_{\mathrm{cf}} \le \rho_{\mathrm{b},n}$ for all $n \ge 2$. Neither $\rho_\le$, $\rho_\ge$ nor $\rho_{\mathrm{b},n}$ for $n \ge 2$ are t-reducible to ρ_{cf}. Restricted to the irrational numbers, ρ and ρ_{cf} are equivalent: $\rho|^{\mathrm{Ir}} \equiv \rho_{\mathrm{cf}}|^{\mathrm{Ir}}$. □

The definitions of the cut representations $\rho_\le$ and $\rho_\ge$ are not even topologically robust. Neither the representations by infinite base-n fractions and the naive Cauchy representation nor the cut representations $\rho_\le$ and $\rho_\ge$ and their greatest lower bound ρ_{cf} are of much interest in computable analysis. Fig. 4.2 shows the reducibility order of the representations of the real numbers we have introduced so far. Many other representations of the real numbers are introduced and compared in [Hau73, Dei84].

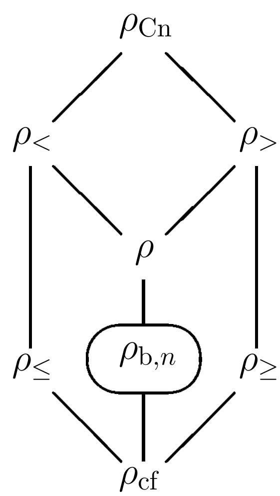

Fig. 4.2. Reducibility order of some representations of $\mathbb{R}$

Our main representation $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$ of the real numbers is not a total function and it is not injective. The same holds for all representations equivalent to it which we have discussed. It would be convenient to have an injective representation δ of the real numbers which is equivalent to ρ. Unfortunately this is not possible.

Theorem 4.1.15.
1. There is no total representation $\delta : \Sigma^\omega \to \mathbb{R}$ with $\delta \equiv \rho$.
2. There is no injective representation $\delta :\subseteq \Sigma^\omega \to \mathbb{R}$ with $\delta \equiv \rho$.

The two statements hold accordingly for $\rho_<$ and $\rho_>$ instead of ρ.

Proof: 1. Let $\delta : \Sigma^\omega \to \mathbb{R}$ be a total function with $\delta \equiv \rho$. δ is continuous, since ρ is continuous. By Lemma 2.2.5 the metric space (Σ^ω, d) with Cantor topology τ_{C} is compact. A continuous function maps compact sets to compact sets, therefore, $\mathrm{range}(\delta) = \delta[\Sigma^\omega]$ is compact. Since $\mathbb{R}$ is not compact (for example the open cover $\{(z; z+2) \mid z \in \mathbb{Z}\}$ has no finite subcover), $\mathbb{R} \neq \mathrm{range}(\delta)$.

2. Let $\delta :\subseteq \Sigma^\omega \to \mathbb{R}$ be an injective function with $\delta \equiv \rho$. By Lemma 3.2.5, $\tau_\mathbb{R}$ is the final topology of δ, that is, X is open, iff $\delta^{-1}[X]$ is open in $\mathrm{dom}(\delta)$. Since δ is injective, there is some $w \in \Sigma^*$ such that $0 \in \delta[w\Sigma^\omega]$ and $1 \notin \delta[w\Sigma^\omega]$. Since $w\Sigma^\omega$ and $\Sigma^\omega \setminus w\Sigma^\omega$ are open, $A := \delta[w\Sigma^\omega]$ and

$B := \delta[\Sigma^\omega \setminus w\Sigma^\omega]$ are open subsets of $\mathbb{R}$ such that $0 \in A$, $1 \in B$, $A \cup B = \mathbb{R}$ and $A \cap B = \emptyset$. This is impossible.

The proofs for $\rho_<$ and $\rho_>$ are left for Exercise 4.1.17. □

In the "real-RAM" model of computation which is used, for example, in computational geometry [PS85] and studied in detail by Blum et al. [BCSS98] one uses the test $x < y$ for real numbers x, y as a basic operation. We show that no representation makes this test decidable. The test $x < y$ is "absolutely non-decidable", at least in the framework of TTE.

Theorem 4.1.16 ($x \leq y$ is absolutely undecidable). For every representation $\delta :\subseteq \Sigma^\omega \to \mathbb{R}$ the relations "$x = y$" and "$x \leq y$" are not (δ, δ)-open and the relation "$x < y$" is not (δ, δ)-clopen.

Proof: Assume that the relation "$x = y$" is (δ, δ)-open. By Definitions 2.4.1 and 3.1.3.2 there is a continuous function $f :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ such that $f(p, q) = 0$, if $\delta(p) = \delta(q)$, and $f(p, q) = \mathrm{div}$ otherwise for all $p, q \in \mathrm{dom}(\delta)$. Consider z and p with $\delta(p) = z$. We obtain $f(p, p) = 0$. Since f is continuous, $f[w\Sigma^\omega \times w\Sigma^\omega] = \{0\}$ for some prefix $w \in \Sigma^*$ of p. We obtain $x = y$ for any $x, y \in \delta[w\Sigma^\omega]$, hence $\{z\} = \delta[w\Sigma^\omega]$. Therefore, for every $z \in \mathbb{R}$ there is some $w \in \Sigma^*$ with $\{z\} = \delta[w\Sigma^\omega]$. But this is impossible, since Σ^* is countable and $\mathbb{R}$ is uncountable. If "$x \leq y$" is (δ, δ)-open, then also "$x \geq y$" is (δ, δ)-open, hence "$x = y$" is (δ, δ)-open by Theorem 2.4.5. If "$x < y$" is (δ, δ)-clopen, then "$x \geq y$" is (δ, δ)-open. □

Among the representations of the real numbers discussed in this section, $\rho, \rho_<$ and $\rho_>$ (and equivalent ones) are the most natural ones. For representations of $\mathbb{R}^n$ ($n \geq 2$) we generalize Definition 4.1.3 straightforwardly.

Definition 4.1.17 (standard representation ρ^n of $\mathbb{R}^n$). *For $n \geq 1$ let ρ^n be the standard representation of $\mathbb{R}^n$ derived from the computable topological space $\mathbf{S}^n := (\mathbb{R}^n, \mathrm{Cb}^{(n)}, \mathrm{I}^n)$ (Definition 4.1.2).*

A sequence $p \in \Sigma^\omega$ is a ρ^n-name of $x \in \mathbb{R}^n$, iff it is a list of all n-dimensional open rational cubes $J \in \mathrm{Cb}^{(n)}$ such that $x \in J$. Fig. 4.3 shows some rational squares $J \in \mathrm{Cb}^{(2)}$ and a point $x \in \mathbb{R}^2$ such that $x \in J$.

The definitions of the other representations equivalent to ρ given above can be generalized straightforwardly to n dimensions by substituting the n-dimensional norm $|| \,.\, ||$ for the absolute value $| \,.\, |$ and the notation I^n for I^1.

Lemma 4.1.18 (representations equivalent to ρ^n). For $n \geq 2$ generalize the definitions of $\rho, \rho_\mathrm{C}, \rho'_\mathrm{C}, \rho''_\mathrm{C}, \rho'''_\mathrm{C}, \rho^a$ and ρ^b from Definitions 4.1.3, 4.1.5 and Lemma 4.1.6, respectively, from $\mathbb{R}$ to $\mathbb{R}^n$ by substituting the B-dimensional norm $|| \,.\, ||$ for the absolute value $| \,.\, |$ and the notation I^n for I^1. Then all the resulting representations are equivalent to ρ^n.

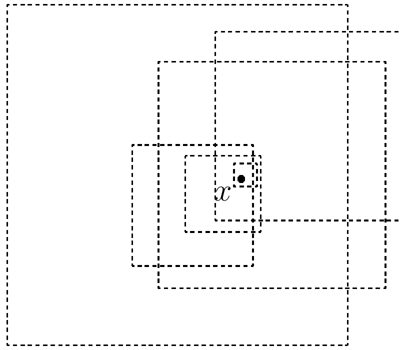

Fig. 4.3. Some open rational squares $J \in \mathrm{Cb}^{(2)}$ such that $x \in J$

Proof: The proof of Lemma 4.1.6 can be generalized straightforwardly. □

Instead of maximum norm, distance and balls, sometimes we will use Euclidean norm (absolute value), distance and balls:

- $|(a_1, \ldots, a_n)| = \sqrt{a_1^2 + \ldots + a_n^2}$,
- $d^{\mathrm{e}}(x, y) := |x - y| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$,
- $B^{\mathrm{e}}(a, r) := \{x \in \mathbb{R}^n \mid |x - a| < r\}$.

The two metrics are related by

$$d(x, y) \leq d^{\mathrm{e}}(x, y) \leq \sqrt{n}\; d(x, y)$$

and generate the same topology on the set $\mathbb{R}^n$.

If we replace the maximum metric by the Euclidean metric in the above definitions, we obtain representations of $\mathbb{R}^n$ which are equivalent to ρ^n. We leave the proofs to the reader. According to Definition 3.3.3, the product $[\rho]^n = [\rho, \ldots, \rho]$ is defined by

$$[\rho]^n \langle p_1, \ldots, p_n \rangle = (\rho(p_1), \ldots, \rho(p_n))\ .$$

The following lemma summarizes some useful properties.

Lemma 4.1.19.

1. $[\rho]^n \equiv \rho^n$, ρ^n is admissible with final topology $\tau_{\mathbb{R}^n}$.
2. A tuple $(x_1, \ldots, x_n)$ is $(\rho, \ldots, \rho)$-computable, iff it is ρ^n-computable.
3. A set $X \subseteq \mathbb{R}^n$ is $(\rho, \ldots, \rho)$-decidable (-r.e. , -clopen, -open), iff it is ρ^n-decidable (-r.e. , -clopen, -open).
4. A function or multi-valued function $f :\subseteq \mathbb{R}^n \rightrightarrows M$ is $(\rho, \ldots, \rho, \delta)$-computable (-continuous), iff it is (ρ^n, δ)-computable (-continuous).
5. A function $f :\subseteq M \to \mathbb{R}^k$ is (δ, ρ^k)-computable (-continuous), iff $\mathrm{pr}_i \circ f$ is (δ, ρ)-computable (-continuous) for $i = 1, \ldots, k$.

Proof: 1. Every ρ^n-name of $(x_1,\ldots,x_n)$ essentially consists of arbitrarily tight upper and arbitrarily tight lower bounds for every component x_i. The same is true for every $[\rho]^n$-name of $(x_1,\ldots,x_n)$. Translations from ρ^n to $[\rho]^n$ and from $[\rho]^n$ to ρ^n can be constructed straightforwardly.

2.-5. These properties follow from Property 1 and Lemma 3.3.6. □

Convention 4.1.20. From now on we will consider as standard the notations $\nu_\mathbb{N}, \nu_\mathbb{Z}, \nu_\mathbb{Q}$ and id_{Σ^*} of the natural numbers, the integers, rational numbers and the set Σ^*, respectively, and the representations $\mathrm{id}_{\Sigma^\omega} : \Sigma^\omega \to \Sigma^\omega$, ρ and ρ^n of Σ^ω, $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Occasionally, we will omit prefixes $\nu_\mathbb{N}$-, $\nu_\mathbb{Z}$-, $\nu_\mathbb{Q}$-, id_{Σ^*}-, $\mathrm{id}_{\Sigma^\omega}$- ρ- and ρ^n- and say *computable* instead of ρ-computable, *recursively enumerable (r.e.)* instead of $(\nu_\mathbb{N}, \rho)$-r.e., *computable* instead of $(\rho, \nu_\mathbb{Q}, \rho)$-computable etc. . Often we will use representations which we have proved to be equivalent to ρ or ρ^n.

The representations ρ, $\rho_<$ and $\rho_>$ can be extended to representations of $\overline{\mathbb{R}}$, the closure of $\mathbb{R}$ under supremum and infimum. We now modify Definition 4.1.3.

Definition 4.1.21 (representations of $\overline{\mathbb{R}}$). *For* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ *define computable topological spaces*

- $\mathbf{S}_< := (\overline{\mathbb{R}}, \sigma_<, \nu_<)$, $\quad \nu_<(w) := (\overline{w}; \infty]$,
- $\mathbf{S}_> := (\overline{\mathbb{R}}, \sigma_>, \nu_>)$, $\quad \nu_>(w) := [-\infty; \overline{w})$,

and let $\overline{\rho}_< := \delta_{\mathbf{S}_<}$, $\overline{\rho}_> := \delta_{\mathbf{S}_>}$ *and* $\overline{\rho} := \overline{\rho}_< \wedge \overline{\rho}_>$.

Exercises 4.1.

◊ 1. Show that the set of real numbers is not countable.
2. Show $\rho|^\mathbb{N} \equiv \nu_\mathbb{N}$, $\rho|^\mathbb{Z} \equiv \nu_\mathbb{Z}$, $\nu_\mathbb{Q} \le \rho$ and $\rho|^\mathbb{Q} \not\le_\mathrm{t} \nu_\mathbb{Q}$ (Sect. 1.4).
3. Prove Lemma 4.1.7.
4. Show that a representation δ of the real numbers is reducible to $\rho_<$, iff the set $\{(x,a) \in \mathbb{R} \times \mathbb{Q} \mid a < x\}$ is $(\delta, \nu_\mathbb{Q})$-r.e. (cf. Lemma 4.1.7).
5. Prove Lemma 4.1.8. (See the proof of Lemma 4.1.6.)
6. Define a representation $\rho_{\mathrm{C}<} : \Sigma^\omega \to \mathbb{R}$ by: $\rho_{\mathrm{C}<}(p) = x$, iff $\rho_\mathrm{C}(p) = x$ and $\overline{w}_0 < \overline{w}_1 < \ldots < x$ (w_i from Definition 4.1.5). Show $\rho_{\mathrm{C}<} \equiv \rho$.
7. Show that $\rho \equiv \rho_\mathrm{G}$ where

$$\rho_\mathrm{G}(p) = x :\Longleftrightarrow \begin{cases} \text{there are words } w_0, w_1 \ldots \in \mathrm{dom}(\nu_\mathbb{Z}) \\ \text{such that } p = \iota(w_0)\iota(w_1)\iota(w_2)\ldots \\ \text{and } \left| x - \frac{\nu_\mathbb{Z}(w_i)}{i+1} \right| < \frac{1}{i+1} \text{ for all } i\,. \end{cases}$$

8. Prove that the definitions of ρ, $\rho_<$ and $\rho_>$ are topologically and computationally robust (Theorem 4.1.11).

9◊ a) Show that multiplication by 3 is not continuous with respect to $\rho_{b,2}$.
◊ b) Show that neither addition nor multiplication are continuous with respect to $\rho_{b,n}$ for $n \geq 2$.

10. Prove Theorem 4.1.13.1 and 4.1.13.2.

♦11. The representations by infinite base-n fractions can be studied as members of a larger class of representations [Wei92a]:
Let $\nu : \mathbb{N} \to Q$ be a numbering of a dense subset $Q \subseteq \mathbb{R}$. Define a representation ϑ_ν of the real numbers by

$$\vartheta_\nu(p) = x :\iff (\forall i) \begin{cases} \nu(i) < x \implies p(i) = 0 \\ \nu(i) > x \implies p(i) = 1 \,. \end{cases}$$

Show:
a) If μ is a standard numbering of the finite base-n fractions, for example $\mu\langle i,j,k\rangle := (i-j)/n^k$, then $\rho_{b,n} \equiv \vartheta_\mu$.
b) For any two numberings $\mu : \mathbb{N} \to P$ and $\nu : \mathbb{N} \to Q$ of dense subsets of $\mathbb{R}$ we have:
- $\vartheta_\nu \leq_t \rho$, $\rho \not\leq_t \vartheta_\nu$,
- $\vartheta_\nu \leq \rho$, if ν and $\nu_{\mathbb{Q}}$ are r.e.-related,
- $Q \subseteq P \iff \vartheta_\mu \leq_t \vartheta_\nu$,
- $\nu \leq \mu \implies \vartheta_\mu \leq \vartheta_\nu$.

c) Define $\nu_0 : \mathbb{N} \to \mathbb{Q}$ by $\nu_0\langle i,j,k\rangle := (i-j)/(1+k)$. Then $\vartheta_{\nu_0} \equiv \rho_{b,2} \wedge \rho_{b,3} \wedge \ldots$, that is, ϑ_{ν_0} is the greatest lower bound of all $\rho_{b,n}$ (Definition 3.3.7).
d) Let ν and $\nu_{\mathbb{Q}}$ be r.e.-related. Then $f :\subseteq \mathbb{R} \to \mathbb{R}$ is (ϑ_ν, ρ)-computable, iff it is (ρ, ρ)-computable.
e) $f :\subseteq \mathbb{R} \to \mathbb{R}$ is $(\rho_{b,n}, \rho)$-computable, iff it is (ρ, ρ)-computable.

♦12. Prove Theorem 4.1.13.7 without using Exercise 4.1.11 [Her99b].

13. Let ρ_{Cn} be the naive Cauchy representation. Prove:
a) $\rho_{Cn} \not\leq_t \rho_<$,
b) ρ_{Cn} has the final topology $\{\emptyset, \mathbb{R}\}$,
c) there is a (ρ_{Cn}, ρ_{Cn})-computable function, which is not (ρ, ρ)-computable (hint: consider a constant function with value x_A (Example 1.3.2) where A is an r.e. non-recursive set),
♦ d) A real function is continuous, iff it is (ρ_{Cn}, ρ_{Cn})-continuous [BH00].

14. Prove the properties of the cut representations $\rho_\leq$ and $\rho_\geq$ stated in Example 4.1.14.2.

15. For an arbitrary notation $\mu :\subseteq \Sigma^* \to Q$ of a dense subset $Q \subseteq \mathbb{R}$ define the effective topological space $\mathbf{S}_\mu = (\mathbb{R}, \sigma_\mu, \nu_\mu)$ by $\nu_\mu(w) := [\mu(w); \infty)$ and $\delta_\mu := \delta_{\mathbf{S}_\mu}$ (cf. Example 4.1.14.2). For notations $\mu :\subseteq \Sigma^* \to Q$ and $\mu' :\subseteq \Sigma^* \to Q'$ show: $\delta_\mu \leq_t \delta_{\mu'}$, iff $Q' \subseteq Q$.

16. Prove the properties of the continued fraction representation stated in Example 4.1.14.3.

17. Show that there is neither a total nor an injective representation of the real numbers which is equivalent to $\rho_<$.

18. Complete the proof of Lemma 4.1.18.
◊19. Let δ and δ' be representations of an uncountable set M. Show that the set $\{(x,y) \mid x,y \in M, x = y\}$ is not (δ,δ')-open.
20. Show that the definition of ρ^n is computationally robust, that is, replacement of $\nu_{\mathbb{Q}}$ in Definition 4.1.17 by a notation ν_Q of a dense subset of $\mathbb{R}$ which is r.e.-related to it (Theorem 4.1.11) yields an equivalent representation.
◊21. Show that $\rho_<$ and $\rho_>$ are the restrictions of $\overline{\rho}_<$ and $\overline{\rho}_>$, respectively, to $\mathbb{R}$ and that ρ is equivalent to the restriction of $\overline{\rho}$ to $\mathbb{R}$. (See Definition 4.1.21.)
◊22. Show that the function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, $f(x) := 1/x^2$, is $(\overline{\rho}, \overline{\rho}_<)$-computable.

4.2 Computable Real Numbers

We will denote the set of computable (that is, ρ-computable) real numbers by $\mathbb{R}_c$. Informally, a real number x is computable, iff arbitrarily tight lower and arbitrarily tight upper rational bounds of x can be computed. This is formalized by each of the following characterizations.

Lemma 4.2.1. For any $x \in \mathbb{R}$ the following properties are equivalent:
1. x is ρ-computable.
2. [Tur36] x is $\rho_{b,n}$-computable, that is, the number x has a computable infinite base-n fraction ($n \in \mathbb{N}, n \geq 2$).
3. There is a computable function $g : \mathbb{N} \to \Sigma^*$ such that
$$|x - \nu_{\mathbb{Q}} g(n)| \leq 2^{-n} \quad \text{for all } n \in \mathbb{N} .$$
4. [Grz55] There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that
$$\left| \, |x| - \frac{f(n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for all } n \in \mathbb{N} .$$
5. [PER89] There are computable functions $s, a, b, e : \mathbb{N} \to \mathbb{N}$ with
$$\left| x - (-1)^{s(k)} \frac{a(k)}{b(k)} \right| \leq 2^{-N}, \text{ if } k \geq e(N), \quad \text{for all } k, N \in \mathbb{N} .$$

Proof: 1 $\iff$ 2: This follows from Theorem 4.1.13.

1 $\Longrightarrow$ 3: Let $\iota(u_0)\iota(u_1)\ldots$ be a computable ρ_C-name of x. Then $|x - \nu_{\mathbb{Q}}(u_n)| \leq 2^{-n}$ for all n. Define $g(n) := u_n$.

3 $\Longrightarrow$ 5: There are computable functions s, a, b with $\nu_{\mathbb{Q}} g(k) = (-1)^{s(k)} \frac{a(k)}{b(k)}$. Define $e(N) := N$.

5 $\Longrightarrow$ 4: From Property 5 we obtain $|\,|x| - a_N/b_N| \le 2^{-(N+2)} < 1/(2(N+1))$, where $a_N := a \circ e(N+2)$ and $b_N := b \circ e(N+2)$. There is a computable function $f : \mathbb{N} \to \mathbb{N}$ with $|a_N(N+1)/b_N - f(N)| \le 1/2$. We obtain

$$\left| |x| - \frac{f(N)}{N+1} \right| \le \left| |x| - \frac{a_N}{b_N} \right| + \left| \frac{a_N}{b_N} - \frac{f(N)}{N+1} \right| < 2\frac{1}{2(N+1)} \le \frac{1}{N+1} \; .$$

4 $\Longrightarrow$ 1: Assume $x \ge 0$. There is a computable function $g : \mathbb{N} \to \Sigma^*$ such that $\nu_{\mathbb{Q}} g(n) = f(2^{n+1})/(2^{n+1}+1)$. We obtain

$$|x - \nu_{\mathbb{Q}} g(n)| = \left| |x| - \frac{f(2^{n+1})}{2^{n+1}+1} \right| < \frac{1}{2^{n+1}+1} < 2^{-n-1} \; .$$

Define $p \in \Sigma^\omega$ by $p := \iota(g(0))\iota(g(1))\ldots$. Then p is computable, $|\nu_{\mathbb{Q}} g(n) - \nu_{\mathbb{Q}} g(m)| \le |\nu_{\mathbb{Q}} g(n) - x| + |x - \nu_{\mathbb{Q}} g(m)| \le 2^{-n}$ for $m > n$ and $\lim_{n\to\infty} \nu_{\mathbb{Q}} g(n) = x$, hence $\rho_{\mathrm{C}}(p) = x$. If $x < 0$, define g such that $\nu_{\mathbb{Q}} g(n) = -f(2^n)/(2^n+1)$. □

Every rational number a is computable (if $\nu_{\mathbb{Q}}(u) = a$, define $g(n) := u$ for all n in Lemma 4.2.1.3). In Example 1.3.1 we have shown that $\sqrt{2}$ and $\log_3 5$ are computable real numbers. Many other examples will follow from theorems below (Example 4.3.13.8). By Lemma 4.1.19, a vector $(x_1, \ldots, x_n)$ of real numbers is ρ^n-computable, iff all its components x_i are computable.

Definition 4.2.2 (modulus of convergence). *A function $e : \mathbb{N} \to \mathbb{N}$ is called a modulus of convergence of a sequence $(x_i)_{i\in\mathbb{N}}$, iff $|x_i - x_k| \le 2^{-n}$ for $i, k \ge e(n)$.*

If e is a modulus of convergence then e' with $e'(n) := \max_{k\le n} e(k)$ is a modulus of convergence, which is computable, if e is computable. Therefore we may assume in most cases that the modulus of convergence is non-decreasing. If e is a modulus of convergence then $|x - x_i| \le 2^{-n}$ for $i \ge e(n)$, where $x = \lim_{i\to\infty} x_i$. If e' is a function with $|x - x_i| \le 2^{-n}$ for $i \ge e'(n)$, then e with $e(n) := e'(n+1)$ is a modulus of convergence, which is computable, if e' is computable.

Therefore, it follows from Lemma 4.2.1.5 that the limit of any computable (more precisely $(\nu_{\mathbb{N}}, \nu_{\mathbb{Q}})$-computable) sequence of rational numbers with computable modulus of convergence is a computable real number. This observation can be generalized as follows:

Theorem 4.2.3. Let $(x_i)_{i\in\mathbb{N}}$ be a $(\nu_{\mathbb{N}}, \rho)$-computable sequence of real numbers with computable modulus of convergence $e : \mathbb{N} \to \mathbb{N}$. Then its limit $x = \lim_{i\to\infty} x_i$ is computable.

Proof: Since the sequence is $(\nu_{\mathbb{N}}, \rho_{\mathrm{C}})$-computable, for any $i, j \in \mathbb{N}$, a word u_{ij} can be computed such that $x_i = \rho_{\mathrm{C}}(\iota(u_{i0})\iota(u_{i1})\ldots)$. Define $v_i := u_{e(i+2),i+2}$ and $q := (\iota(v_0)\iota(v_1)\ldots)$. For all $k < m$ we obtain

$$\begin{aligned} |\overline{v}_k - x| &\le |\overline{u}_{e(k+2),k+2} - x_{e(k+2)}| + |x_{e(k+2)} - x| \\ &\le 2^{-k-2} + 2^{-k-2} \\ &\le 2^{-k-1} \end{aligned}$$

and $|\overline{v}_k - \overline{v}_m| \le |\overline{v}_k - x| + |\overline{v}_m - x| \le 2^{-k-1} + 2^{-m-1} \le 2^{-k}$. Therefore, $x = \rho_{\mathrm{C}}(q)$ and x is ρ_{C}-computable, since $q \in \Sigma^\omega$ is computable. □

Example 4.2.4.

1. Define $x_i := \sum_{k=0}^{i} 1/k!$. Then $\mathrm{e} = \lim_{i\to\infty} x_i$. Obviously the sequence $(x_i)_{i\in\mathbb{N}}$ is $(\nu_{\mathbb{N}}, \nu_{\mathbb{Q}})$-computable, hence $(\nu_{\mathbb{N}}, \rho)$-computable. Define $e(n) := n+1$. Then $|x_i - x_j| \le 2^{-n}$ for $i, j \ge e(n)$. By Theorem 4.2.3, the number e is computable.
2. A famous result by Leibnitz states $\pi/4 = 1 - 1/3 + 1/5 - 1/7 \ldots$. Define $x_i := \sum_{k=0}^{i} (-1)^k a_k$ where $a_k = 1/(2k+1)$. Then the sequence $(x_i)_{i\in\mathbb{N}}$ is computable. For $i \le j$ and even $j - i$,

   $$\begin{aligned} 0 &\le (a_i - a_{i+1}) + (a_{i+2} - a_{i+3}) + \ldots + (a_{j-2} - a_{j-1}) + a_j \\ &= (-1)^i \textstyle\sum_{k=i}^{j} (-1)^k a_k \\ &= a_i - (a_{i+1} - a_{i+2}) - \ldots - (a_{j-1} - a_j) \\ &\le a_i \end{aligned}$$

   and similarly for odd $j-i$, $0 \le (-1)^i \sum_{k=i}^{j} (-1)^k a_k \le a_i - a_j \le a_i$. In both cases, $|\sum_{k=i}^{j} (-1)^k a_k| \le a_i$. Define $e(n) := 2^n$. Then for $e(n) \le i < j$, $|x_j - x_i| = |\sum_{k=i+1}^{j} (-1)^k a_k| \le a_{i+1} = 1/(2i+3) < 1/i \le 2^{-n}$. By Theorem 4.2.3, the number $\pi/4$ is computable, hence π is computable.
3. By Example 1.3.2 for any set $A \subseteq \mathbb{N}$ of numbers, the sum

   $$x_A := \sum_{i\in A} 2^{-i}$$

   is a computable real number, iff A is a recursive set.
   Let A be recursively enumerable but not recursive. Then there is a computable injective function $f : \mathbb{N} \to \mathbb{N}$ with $A = \mathrm{range}(f)$. We obtain

   $$x_A := \sum_{j\in\mathbb{N}} 2^{-f(j)} = \lim_{n\to\infty} \sum_{j=0}^{n} 2^{-f(j)} .$$

   Therefore, $(a_n)_{n\in\mathbb{N}}$ with $a_n := \sum_{j=o}^{n} 2^{-f(j)}$ is a computable increasing sequence of rational numbers. Its limit x_A is $\rho_<$-computable. Since x_A is not computable, by Theorem 4.2.3 the sequence $(a_n)_{n\in\mathbb{N}}$ cannot have a computable modulus of convergence. Since the set A is not recursive, the enumerating function f cannot have a computable lower bound which is monotone and unbounded. Therefore, unforseeable values $f(n)$ are very small and terms $2^{-f(n)}$ are large. □

The $\rho_<$-computable numbers are also called *left-computable* or *left-r.e.* and the $\rho_>$-computable numbers are also called *right-computable* or *right-r.e.*. By Specker's example there is a left-computable real number which is not computable (Examples 1.3.2, 4.2.4.3).

Lemma 4.2.5. A real number x
1. is left-computable, iff $-x$ is right-computable,
2. is computable, iff it is left-computable and right-computable.

Proof: A direct proof is easy. Property 2 follows also from $\rho \equiv \rho_< \wedge \rho_>$ (Lemma 4.1.9). □

The computable real numbers are a countable set which, however, cannot be enumerated "effectively". We prove a "positive" version of this statement by diagonalization.

Theorem 4.2.6.
1. Let $(x_i)_{i\in\mathbb{N}}$ be a $(\nu_\mathbb{N}, \rho)$-computable sequence of real numbers. Then there is a computable real number x such that $x \neq x_i$ for all $i \in \mathbb{N}$.
2. There is no numbering or notation ν of the set $\mathbb{R}_c$ of the computable real numbers with r.e. domain such that $\nu \leq \rho$.

Proof: 1. We construct x by diagonalization. We may assume that the sequence is $(\nu_\mathbb{N}, \rho_C)$-computable. For any $i \in \mathbb{N}$ we can determine a sequence $q_i := \iota(u_{i0})\iota(u_{i1})\ldots$ with $x_i = \rho_C(q_i)$. Therefore, there is a computable function $g : \Sigma^* \to \Sigma^*$ with $g(0^i) = u_{i,2i+2}$. We obtain $|\nu_\mathbb{Q}\circ g(0^i) - x_i| \leq 2^{-2i-2}$. We compute a nested sequence $([\overline{u}_i; \overline{v}_i])_{i\in\mathbb{N}}$ of closed intervals with $\overline{v}_i - \overline{u}_i = 3^{-i}$ such that $x_i \notin [\overline{u}_i; \overline{v}_i]$ as follows:

$$(\overline{u}_0, \overline{v}_0) := (\nu_\mathbb{Q} \circ g(\lambda) + 1, \overline{u}_0 + 1)$$

$$(\overline{u}_{i+1}, \overline{v}_{i+1}) := \begin{cases} (\overline{u}_i, \overline{u}_{i+1} + \frac{1}{3} \cdot 3^{-i}) & \text{if } \nu_\mathbb{Q} \circ g(0^{i+1}) \geq \overline{u}_i + \frac{1}{2} \cdot 3^{-i} \\ (\overline{u}_i + \frac{2}{3} \cdot 3^{-i}, \overline{v}_i) & \text{otherwise.} \end{cases}$$

If $\nu_\mathbb{Q} \circ g(0^{i+1}) \geq \overline{u}_i + \frac{1}{2} \cdot 3^{-i}$, then $x_{i+1} \geq \overline{u}_i + \frac{1}{2} \cdot 3^{-i} - 2^{-2(i+1)-2} > \overline{u}_i + \frac{1}{3} \cdot 3^{-i}$, and if $\nu_\mathbb{Q} \circ g(0^{i+1}) \not\geq \overline{u}_i + \frac{1}{2} \cdot 3^{-i}$, then $x_{i+1} < \overline{u}_i + \frac{2}{3} \cdot 3^{-i}$. We obtain $x_{i+1} \notin [\overline{u}_{i+1}; \overline{v}_{i+1}]$. There is a computable sequence $i \mapsto w_i$ of words such that $\overline{\mathrm{I}^1}(w_i) = [u_i; v_i]$. Therefore, $q := \iota(w_0)\iota(w_1)\ldots$ is computable and $x := \rho^b(q)$ (ρ^b from Lemma 4.1.6) differs from all numbers x_i. Since $\rho^b \equiv \rho$, x is computable.

2. Assume that there is such a numbering $\nu :\subseteq \mathbb{N} \to \mathbb{R}_c$. There is a computable function $f : \mathbb{N} \to \mathbb{N}$ with $\mathrm{range}(f) = \mathrm{dom}(\nu)$. Then νf is a total numbering of $\mathbb{R}_c$ with $\nu f \leq \nu \leq \rho$. Therefore, $(\nu f(i))_{i\in\mathbb{N}}$ is a $(\nu_\mathbb{N}, \rho)$-computable sequence of all computable real numbers. This is impossible by Property 1.

If ν is a notation, there is a computable function $f : \mathbb{N} \to \Sigma^*$ with $\mathrm{range}(f) = \mathrm{dom}(\nu)$. Continue as above. □

We derive a notation of the computable real numbers canonically from the representation ρ using the standard notation $\xi^{*\omega}$ of the computable functions $f :\subseteq \Sigma^* \to \Sigma^\omega$ (Definition 2.3.4).

Definition 4.2.7. *Define the notation $\nu_\rho :\subseteq \Sigma^* \to \mathbb{R}_c$ of the computable real numbers by*

$$\nu_\rho(w) := \rho \circ \xi^{*\omega}_w(\lambda) \ .$$

For $w \in \mathrm{dom}(\nu_\rho)$, $\nu_\rho(w) = \rho(p)$ where $p \in \Sigma^\omega$ is the sequence computed by the Turing machine with code w on input λ. By the utm-theorem for $\xi^{*\omega}$, $\nu_\rho \leq \rho$. By Theorem 4.2.6 the domain $\mathrm{dom}(\nu_\rho) = \{w \in \Sigma^* \mid \xi^{*\omega}_w(\lambda) \in \mathrm{dom}(\rho)\}$ is not r.e.

Every countable subset $X \subseteq \mathbb{R}$ can be covered by arbitrarily small open sets. The (countable) set $\mathbb{R}_c$ of computable real numbers can be covered by arbitrarily small "r.e. open" sets [Spe59] (cf. Sect. 5.1).

Theorem 4.2.8. For each $N \in \mathbb{N}$ there is a computable sequence $i \mapsto w_i$ of words $w_i \in \mathrm{dom}(\mathrm{I}^1)$, such that

$$\mathbb{R}_c \subseteq U_N := \bigcup_{i\in\mathbb{N}} \mathrm{I}^1(w_i) \quad \text{and} \quad \sum_{i\in\mathbb{N}} \mathrm{length}(\mathrm{I}^1(w_i)) \leq 2^{-N} \ .$$

Proof: For each $i = \langle k, t\rangle$ define the word w_i as follows:
If the Turing machine with code $\nu_\Sigma(k)$ on input λ in t steps (but not in $(t-1)$ steps) writes an output $\iota(u_0)\iota(u_1)\ldots\iota(u_{k+N+4}) \in \Sigma^*$ with $|\overline{u}_j - \overline{u}_m| \leq 2^{-j}$ for $0 \leq j \leq m \leq k + N + 4$, then

$$\mathrm{I}^1(w_i) = \Big(\overline{u}_{k+N+4} - 2^{-k-N-3} \ ; \ \overline{u}_{k+N+4} + 2^{-k-N-3}\Big) \ ,$$

otherwise

$$\mathrm{I}^1(w_i) = \Big(0 \ ; \ 2^{-i-N-2}\Big) \ .$$

Suppose x is computable. Then $x = \rho_C(p)$ for some computable $p \in \Sigma^\omega$. There is some number k such that $\xi^{*\omega}_{\nu_\Sigma(k)}(\lambda) = p$. Then there is some t such that the machine with code $\nu_\Sigma(k)$ on input λ in t steps computes a prefix $\iota(u_0)\iota(u_1)\ldots\iota(u_{k+N+4}) \in \Sigma^*$ of p with $u_i \in \mathrm{dom}(\nu_\mathbb{Q})$. For $i = \langle k, t\rangle$ we

obtain $x \in \mathrm{I}^1(w_i)$. Therefore, $\mathbb{R}_c \subseteq \bigcup_{i\in\mathbb{N}} \mathrm{I}^1(w_i)$.
For each k there is at most one t such that the first condition holds, therefore,

$$\sum_{i\in\mathbb{N}} \mathrm{length}\big(\mathrm{I}^1(w_i)\big) \le \sum_{i\in\mathbb{N}} 2^{-i-N-2} + \sum_{k\in\mathbb{N}} 2^{-k-N-2} = 2^{-N} .$$

Since the function $(w,t) \mapsto v$, where v is the word which the machine with code w on input λ produces in t steps, is computable, the sequence $i \mapsto w_i$ is computable (cf. Lemma 2.1.5). □

Exercises 4.2.

◊ 1. Show that a real number x is computable, iff there are computable functions $f, g, h : \mathbb{N} \to \mathbb{N}$ with $|x - (f(n) - g(n))/(1 + h(n))| \le 2^{-n}$ for all $n \in \mathbb{N}$.
2. a) Define a $(\nu_{\mathbb{N}}, \rho_<)$-computable sequence $(x_i)_{i\in\mathbb{N}}$ which lists all $\rho_<$-computable real numbers.
   b) Construct by diagonalization a $\rho_>$-computable real number which is not $\rho_<$-computable.
3. Let $a : \mathbb{N} \to \mathbb{R}$ be a computable sequence of real numbers with infinite range. Then there is an *injective* computable sequence of real numbers $b : \mathbb{N} \to \mathbb{R}$ with $\mathrm{range}(a) = \mathrm{range}(b)$.
4. Let $(a_k)_{k\in\mathbb{N}}$ be a computable sequence of real numbers converging to 0.
   a) Show that the sequence $(a_k)_{k\in\mathbb{N}}$ has a computable modulus of convergence, if $a_k \le a_{k+1}$ for all k.
   b) Show that there is an increasing computable function $f : \mathbb{N} \to \mathbb{N}$ with $|a_{f(n)}| \le 2^{-n}$ for all n (hence $n \mapsto n$ is a modulus of convergence of the computable subsequence $n \mapsto a_{f(n)}$).
   c) Show that there is a computable sequence $(b_k)_{k\in\mathbb{N}}$ of real numbers converging to 0 which has no computable modulus of convergence. (Hint: define $b_k := 2^{-f(k)}$ where f is an injective enumeration of a non-recursive r.e. set.)
5. Show that the sequence $(a_n)_{n\in\mathbb{N}}$ in Example 4.2.4.3 is $(\nu_{\mathbb{N}}, \nu_{\mathbb{Q}})$-computable, $(\nu_{\mathbb{N}}, \rho)$-computable and $(\nu_{\mathbb{N}}, \rho_{\le})$-computable (Example 4.1.14.2).
6. Consider $n \in \mathbb{N}$, $n > 2$. Is $\sum_{i\in A} n^{-i}$ computable, if $A \subseteq \mathbb{N}$ is recursive (r.e.)?
7. Show that there is a sequence $(a_i)_{i\in\mathbb{N}}$ of rational numbers which is $(\nu_{\mathbb{N}}, \rho)$-computable but not $(\nu_{\mathbb{N}}, \nu_{\mathbb{Q}})$-computable. Hint: Let $a : \mathbb{N} \to \mathbb{N}$ be a computable injective function such that $\mathrm{range}(a)$ is not recursive. Define $x_n := 0$, if $n \notin \mathrm{range}(a)$, $x_n := 2^{-k}$, if $a(k) = n$.
8. Define a $(\nu_{\mathbb{Q}}, \rho)$-computable function $f : \mathbb{Q} \to \mathbb{R}$ such that $\mathrm{range}(f) \subseteq \mathbb{Q}$ and the restriction $f|^{\mathbb{Q}} : \mathbb{Q} \to \mathbb{Q}$ is not $(\nu_{\mathbb{Q}}, \nu_{\mathbb{Q}})$-computable. [Hau87]

9. Show that there is a computable sequence $y : \mathbb{N} \to \mathbb{R}$ of real numbers, such that the sequence $\mathrm{sgn} \circ y$ is not computable (where $\mathrm{sgn}(x) := 0,\ \text{if } x \le 0, 1$ otherwise).

♦10. By Taylor's theorem, $\sqrt{1+t} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \cdot t^k$ for all $t \in \mathbb{R}$ with $|t| < 1$. The series converges also for $|t| = 1$. For $t = -1$ we obtain

$$0 = 1 - \frac{1}{2} - \frac{1}{2 \cdot 4} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} - \dots .$$

Determine a modulus of convergence (find an expression in elementary functions and do not use summation $\sum$) for the sequence $i \mapsto x_i$ with $x_i := \sum_{k=0}^{i} \binom{\frac{1}{2}}{k} \cdot (-1)^k$. (Consult, for example, [BB85].)

◊11. Show that for every bounded $(\nu_{\mathbb{N}}, \rho_<)$-computable sequence $(x_i)_{i \in \mathbb{N}}$ of real numbers, $\sup_{i \in \mathbb{N}} x_i$ is $\rho_<$-computable.

12. There is a left-computable real number $x \in (0; 2)$ such that $x \neq \sum_{i \in A} 2^{-i}$ for every r.e. set $A \subseteq \mathbb{N}$. Hint: Let $B \subseteq \mathbb{N}$ be recursively enumerable but not recursive and define

$$y_B := \sum_{i \in B} 2^{-(2i+1)} + \sum_{i \notin B} 2^{-(2i+2)} .$$

13. Let $(x_i)_{i \in \mathbb{N}}$ be a computable sequence of real numbers such that $\sum_{i \in \mathbb{N}} |x_{i+1} - x_i|$ is finite. Show that x is the sum of a left-computable and a right-computable real number. There are real numbers of this type, which are neither left- nor right-computable. Left-computable and more general types of real numbers are investigated in [WZ98a].

14. Show that the multi-valued function $F :\subseteq \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{Q}$, defined by $\mathrm{R}_F := \{((x, y), a) \mid x < a < y\}$, is $(\rho_>, \rho_<, \nu_{\mathbb{Q}})$-computable.

15. The open set $U_N := \bigcup_{i \in \mathbb{N}} \mathrm{I}^1(w_i)$ from Theorem 4.2.8 containing all computable real numbers can be written as a disjoint union of open intervals. Let $K \subseteq U_N$ be the interval of this partition of U_N containing the (computable) number $0 \in \mathbb{R}$ and let $c := \sup(K)$ be its right-hand endpoint.
   a) Show c is not computable, that is, $c \notin \mathbb{R}_c$.
   b) Let c_n be the right-hand endpoint of the longest interval $J \subseteq \bigcup_{i=0}^{n} \mathrm{I}^1(w_i)$ with $0 \in J$. Show that $(c_n)_{n \in \mathbb{N}}$ is a non-decreasing computable sequence with $c = \sup_{n \in \mathbb{N}} c_n$. (Hence c is left-computable.)
   c) Show that $c_n \le a$ or $c_n \ge b$, if $n \ge i$ and $(a, b) := \mathrm{I}^1(w_i)$.
   d) Show that $f :\subseteq \mathbb{R} \rightrightarrows \mathbb{N}$, defined by

   $$\mathrm{R}_f := \{(x, n) \in \mathbb{R}_c \times \mathbb{N} \mid (\forall k > n) |c_k - x| \ge 2^{-n}\} ,$$

   is $(\rho, \nu_{\mathbb{N}})$-computable.

16. Effectivize Theorem 4.2.6.1. Define a multi-valued function

$$D : \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{R} \ \text{ by } \ \mathrm{R}_D := \{((x_0, x_1, \ldots,), x) \mid (\forall i)\ x \neq x_i\} .$$

   a) Show that D is $([\rho]^\omega, \rho)$-computable.
   b) Show that D has no $([\rho]^\omega, \rho)$-continuous choice function.

17. By Theorem 4.1.13.2, $x \in \mathbb{R}$ is $\rho_{\mathrm{b},2}$-computable, iff it is $\rho_{\mathrm{b},10}$-computable. By Theorem 4.1.13.4, $\rho_{\mathrm{b},2} \not\leq_{\mathrm{t}} \rho_{\mathrm{b},10}$. Show that there is a $[\rho_{\mathrm{b},2}]^\omega$-computable sequence $(x_0, x_1, \ldots)$ which is not $[\rho_{\mathrm{b},10}]^\omega$-computable [Mos57]. Hint: There are injective computable functions $a, b : \mathbb{N} \to \mathbb{N}$ such that $A := \mathrm{range}(a)$ and $B := \mathrm{range}(b)$ are recursively inseparable, that is, $A \cap B = \emptyset$ and for no recursive set C, $A \subseteq C$ and $B \subseteq \mathbb{N} \setminus C$ [Rog67, Odi89]. Let $\rho_{\mathrm{b},2}(\mathtt{0}\bullet d_1 d_2 \ldots) = 1/5$ and define

$$x_n := \begin{cases} \rho_{\mathrm{b},2}(\mathtt{0}\bullet d_1 d_2 \ldots) & \text{for } \ n \notin A \cup B \ , \\ \rho_{\mathrm{b},2}(\mathtt{0}\bullet d_1 d_2 \ldots d_k \mathtt{00} \ldots) & \text{for } \ a(k) = n \ , \\ \rho_{\mathrm{b},2}(\mathtt{0}\bullet d_1 d_2 \ldots d_k \mathtt{11} \ldots) & \text{for } \ b(k) = n \ . \end{cases}$$

4.3 Computable Real Functions

A real function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is computable (more precisely (ρ, ρ)-computable), iff some Type-2 machine transforms any ρ-name p of any $x \in \mathrm{dom}(f)$ to a ρ-name of $f(x)$, where a ρ-name of y is a list of all rational open intervals $J \in \mathrm{Cb}^{(1)}$ such that $y \in J$. Since equivalent representations induce the same computability, ρ can be replaced by any representation equivalent to it, for example by the Cauchy representation ρ_{C} (Lemma 4.1.6). Computable functions with n real arguments are realized accordingly by machines with n input tapes. Since the representation ρ is admissible with final topology $\tau_{\mathbb{R}}$ (Lemma 4.1.4), we obtain as a special case of our Main Theorem 3.2.11:

Theorem 4.3.1 (continuity). Every computable real function is continuous.

More precisely, every (ρ^n, ρ)-computable real function is continuous w.r.t. the real line topology $\tau_{\mathbb{R}}$. Many easily definable functions like the step function or the Gauß staircase (Fig. 1.3) are not (ρ, ρ)-computable, since they are not continuous. Some people reject TTE and similar approaches to computable analysis since they think that a reasonable computability theory for analysis should make such functions computable. This objection can be removed, since TTE admits various natural computable topological spaces which make the step function, the Gauß staircase and similar functions computable (for example the Gauß staircase is $(\rho, \rho_>)$-computable).

The following theorem lists some computable real functions.

Theorem 4.3.2 (some computable real functions). The following real functions are computable:

1. $(x_1, \ldots, x_n) \mapsto c$ (where $c \in \mathbb{R}$ is a computable constant),
2. $(x_1, \ldots, x_n) \mapsto x_i$ $(1 \leq i \leq n)$,
3. $x \mapsto -x$,
4. $(x, y) \mapsto x + y$,
5. $(x, y) \mapsto x \cdot y$,
6. $x \mapsto 1/x$,
7. $(x, y) \mapsto \min(x, y)$, $(x, y) \mapsto \max(x, y)$,
8. $x \mapsto |x|$,
9. every polynomial function in n variables with computable coefficients,
10. $(i, x) \mapsto x^i$ for $i \in \mathbb{N}$ and $x \in \mathbb{R}$ $(0^0 := 1)$.

Proof: We will use the representations ρ_{C} and ρ'''_{C} which are equivalent to ρ by Lemma 4.1.6.

1. Let M be a Type-2 machine with n input tapes which on every input computes some computable ρ-name of c. Then f_M realizes the constant function with value c.

2. Let M be a Type-2 machine with n input tapes which copies the i-th input tape to the output tape. Then f_M realizes the i-th projection.

3. There is a computable word function $f : \Sigma^* \to \Sigma^*$ such that $-\nu_{\mathbb{Q}}(w) = \nu_{\mathbb{Q}} f(w)$ for all $w \in \mathrm{dom}(\nu_{\mathbb{Q}})$. There is a Type-2 machine M, which transforms any input $p := \iota(u_0)\iota(u_1)\ldots \in \mathrm{dom}(\rho_{\mathrm{C}})$ (where $u_i \in \mathrm{dom}(\nu_{\mathbb{Q}})$) to the sequence $q := \iota(f(u_0))\iota(f(u_1))\ldots$. Obviously, $\rho_{\mathrm{C}}(q) = -x$, if $\rho_{\mathrm{C}}(p) = x$. Therefore, f_M realizes negation on $\mathbb{R}$.

4. Since addition on $\mathbb{Q}$ is $(\nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}})$-computable, there is a computable function $f :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^*$ such that $\overline{u} + \overline{v} = \nu_{\mathbb{Q}} f(u, v)$ for all $u, v \in \mathrm{dom}(\nu_{\mathbb{Q}})$. There is a Type-2 machine M, which transforms any input (p, q), $p := \iota(u_0)\iota(u_1)\ldots \in \mathrm{dom}(\rho'''_{\mathrm{C}})$ and $q := \iota(v_0)\iota(v_1)\ldots \in \mathrm{dom}(\rho'''_{\mathrm{C}})$ (where $u_i, v_i \in \mathrm{dom}(\nu_{\mathbb{Q}})$) to the sequence $r := \iota(y_0)\iota(y_1)\ldots$ with $y_i := f(u_{i+1}, v_{i+1})$. Since $\overline{y}_i = \overline{u}_{i+1} + \overline{v}_{i+1}$, we have

$$|\overline{y}_i - (\rho'''_{\mathrm{C}}(p) + \rho'''_{\mathrm{C}}(q))| \leq |\overline{u}_{i+1} - \rho'''_{\mathrm{C}}(p)| + |\overline{v}_{i+1} - \rho'''_{\mathrm{C}}(q)| \leq 2 \cdot 2^{-i-1} = 2^{-i} .$$

We obtain $r = f_M(p, q) \in \mathrm{dom}(\rho'''_{\mathrm{C}})$ and $\rho'''_{\mathrm{C}}(r) = x + y$. Therefore, f_M is a $(\rho'''_{\mathrm{C}}, \rho'''_{\mathrm{C}}, \rho'''_{\mathrm{C}})$-realization of addition.

5. Since multiplication on $\mathbb{Q}$ is $(\nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}})$-computable, there is a computable function $f :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^*$ such that $\overline{u} \cdot \overline{v} = \nu_{\mathbb{Q}} f(u, v)$ for all $u, v \in \mathrm{dom}(\nu_{\mathbb{Q}})$. There is a Type-2 machine M which transforms any input (p, q), $p := \iota(u_0)\iota(u_1)\ldots \in \mathrm{dom}(\rho'''_{\mathrm{C}})$ and $q := \iota(v_0)\iota(v_1)\ldots \in \mathrm{dom}(\rho'''_{\mathrm{C}})$, to the sequence $r := \iota(y_0)\iota(y_1)\ldots$ with $y_i := f(u_{m+i}, v_{m+i})$, where $m \in \mathbb{N}$ is the smallest number with $|\overline{u}_0| + 2 \leq 2^{m-1}$ and $|\overline{v}_0| + 2 \leq 2^{m-1}$. For all n we have

$$|\overline{u}_n| \leq |\overline{u}_n - \rho'''_{\mathrm{C}}(p)| + |\rho'''_{\mathrm{C}}(p) - \overline{u}_0| + |\overline{u}_0| \leq 2 + |\overline{u}_0| \leq 2^{m-1}$$

and accordingly $|\overline{v}_n| \leq 2^{m-1}$. With $x := \rho'''_C(p)$ and $y := \rho'''_C(q)$ we obtain

$$\begin{aligned}|\overline{y}_i - x \cdot y| &= |\overline{u}_{m+i} \cdot \overline{v}_{m+i} - x \cdot y| \\ &\leq |\overline{u}_{m+i} \cdot \overline{v}_{m+i} - \overline{u}_{m+i} \cdot y| + |\overline{u}_{m+i} \cdot y - x \cdot y| \\ &\leq |\overline{u}_{m+i} \cdot (\overline{v}_{m+i} - y)| + |(\overline{u}_{m+i} - x) \cdot y| \\ &\leq 2 \cdot 2^{m-1} \cdot 2^{-m-i} \\ &= 2^{-i} .\end{aligned}$$

We obtain $r = f_M(p, q) \in \mathrm{dom}(\rho'''_C)$ and $\rho'''_C(r) = x \cdot y$. Therefore, f_M is a $(\rho'''_C, \rho'''_C, \rho'''_C)$-realization of multiplication.

6. There is a Type-2 machine M which works as follows on input $p := \iota(u_0)\iota(u_1)\ldots \in \mathrm{dom}(\rho'''_C)$: First M searches for the smallest $N \in \mathbb{N}$ with $|\overline{u}_N| > 3 \cdot 2^{-N}$. As soon as such a number N has been found, M starts to write the sequence $r := \iota(y_0)\iota(y_1)\ldots$ with $\overline{y}_i = 1/\overline{u}_{2N+i}$.
Suppose that $x := \rho'''_C(p) \neq 0$. Then N exists and $|\overline{u}_k| > 2^{-N}$ for all $k \geq N$ and hence $|x| \geq 2^{-N}$. We obtain

$$\left|\overline{y}_i - \frac{1}{x}\right| = \left|\frac{1}{\overline{u}_{2N+i}} - \frac{1}{x}\right| = \frac{|x - \overline{u}_{2N+i}|}{|\overline{u}_{2N+i}| \cdot |x|} \leq 2^{-2N-i} \cdot 2^N \cdot 2^N = 2^{-i} .$$

Therefore, f_M is a (ρ'''_C, ρ'''_C)-realization of inversion. Notice that $f_M(p)$ does not exist, if $\rho'''_C(p) = 0$.

7. There is a Type-2 machine M which transforms any input (p, q), $p := \iota(u_0)\iota(u_1)\ldots \in \mathrm{dom}(\rho'''_C)$ and $q := \iota(v_0)\iota(v_1)\ldots \in \mathrm{dom}(\rho'''_C)$, to the sequence $r := \iota(y_0)\iota(y_1)\ldots$ with $\overline{y}_i = \min(\overline{u}_i, \overline{v}_i)$.
If $x = \min(x, y, \overline{u}_i, \overline{v}_i)$, then

$$|\overline{y}_i - \min(x, y)| = \min(\overline{u}_i, \overline{v}_i) - x \leq \overline{u}_i - x \leq 2^{-i} .$$

If $\overline{u}_i = \min(x, y, \overline{u}_i, \overline{v}_i)$, then

$$|\overline{y}_i - \min(x, y)| = \min(x, y) - \overline{u}_i \leq x - \overline{u}_i \leq 2^{-i} .$$

For the cases $y = \min(x, y, \overline{u}_i, \overline{v}_i)$ and $\overline{v}_i = \min(x, y, \overline{u}_i, \overline{v}_i)$, $|\overline{y}_i - \min(x, y)| \leq 2^{-i}$ can be concluded similarly. Therefore, f_M is a $(\rho'''_C, \rho'''_C, \rho''')$-realization of min. Since $\max(x, y) = (x + y) - \min(x, y)$, max is computable by Properties 3 and 4 and the composition theorem 3.1.6.

8. $|x| = \max(x, -x)$, apply Properties 3 and 7.

9. We apply the composition theorem 3.1.6. Every monomial f of degree 0 (that is, $f(x_1, \ldots, x_n) := c$) with computable constant c is computable by Property 1. Suppose that all monomials of degree k with computable coefficients are computable. If f_{k+1} is a monomial of degree $k+1$ with computable coefficient, then $f_{k+1}(x_1, \ldots, x_n) = f_k(x_1, \ldots, x_n) \cdot \mathrm{pr}_i(x_1, \ldots, x_n)$ for some monomial of degree k with computable coefficient and some i. By induction and Properties 2 and 5, f_{k+1} is computable.
Another easy induction shows that every polynomial function with computable coefficients is computable.

10. For $h(i,x) := x^i$ we have

$$h(0,x) = 1\ ,$$

$$h(n+1,x) = x \cdot h(n,x)\ .$$

If we define $f(x) := 1$ and $f'(n,y,x) := x \cdot y$, then f and f' are computable by Properties 1 and 5 above, and h is computable by Theorem 3.1.7.2 on primitive recursion. □

Example 4.3.3. The exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is computable. We use the estimation

$$\exp(x) = \sum_{i=0}^{N} \frac{x^i}{i!} + r_N(x)\ , \text{ where } \ r_N(x) \le 2 \cdot \frac{|x|^{N+1}}{(N+1)!}\ , \text{ if } \ |x| \le 1 + \frac{N}{2}\ .$$

Let M be a Type-2 machine which for any $p = \iota(u_0)\iota(u_1)\ldots \in \mathrm{dom}(\rho_{\mathrm{C}})$ computes a sequence $q = \iota(v_0)\iota(v_1)\ldots$, where for each n, v_n is determined as follows.

1. M determines the smallest $N_1 \in \mathbb{N}$ with $|\overline{u}_0| + 1 \le 1 + N_1/2$.
2. M determines the smallest $N \in \mathbb{N}$, $N \ge N_1$, with
$$2 \cdot \frac{|1 + N_1/2|^{N+1}}{(N+1)!} \le 2^{-n-2}\ .$$
3. M determines the smallest $m \in \mathbb{N}$ with
$$2^{-m} \cdot \sum_{i=1}^{N} \frac{i \cdot (1 + N_1/2)^{i-1}}{i!} \le 2^{-n-2}\ .$$
4. M determines $v_n \in \Sigma^*$ such that
$$\overline{v}_n = \sum_{i=0}^{N} \frac{\overline{u}_m^{\,i}}{i!}\ .$$

Assume $x = \rho_{\mathrm{C}}(p) = \rho_{\mathrm{C}}(\iota(u_0)\iota(u_1)\ldots)$. Then $|x| \le 1 + N_1/2$ and $|\overline{u}_m| \le 1 + N_1/2$. We obtain

$$\begin{aligned}
|\exp(x) - \overline{v}_n| &\le |\sum_{i=0}^{N} \frac{x^i}{i!} - \sum_{i=0}^{N} \frac{\overline{u}_m^{\,i}}{i!}| + |r_N(x)| \\
&\le |x - \overline{u}_m| \cdot \sum_{i=1}^{N} \frac{|x^{i-1} + x^{i-2}\overline{u}_m + \ldots + \overline{u}_m^{\,i-1}|}{i!} + 2^{-n-2} \\
&\le 2^{-m} \cdot \sum_{i=1}^{N} \frac{i \cdot (1 + N_1/2)^{i-1}}{i!} + 2^{-n-2} \\
&\le 2^{-n-2} + 2^{-n-2} \\
&\le 2^{-n-1}\ .
\end{aligned}$$

We obtain furthermore $\exp(x) = \rho_C(\iota(v_0)\iota(v_1)\ldots)$, since for $i < j$,

$$|\overline{v}_i - \overline{v}_j| \leq |\overline{v}_i - \exp(x)| + |\exp(x) - \overline{v}_j| \leq 2^{-i-1} + 2^{-j-1} \leq 2^{-i} .$$

Therfore, f_M realizes the exponential function. □

The computable real functions are closed under composition (Theorem 3.1.6) and under primitive recursion (Theorem 3.1.7). There are some other useful operations which map computable real functions to computable real functions.

Corollary 4.3.4. If $f, g :\subseteq \mathbb{R}^n \to \mathbb{R}$ are computable functions and $a \in \mathbb{R}$ is a computable number, then $x \mapsto a \cdot f(x)$, $x \mapsto f(x)+g(x)$, $x \mapsto f(x)\cdot g(x)$, $x \mapsto \max(f(x), g(x))$, $x \mapsto \min(f(x), g(x))$ and $x \mapsto 1/f(x)$ are computable functions.

Proof: By Theorem 3.1.6 the computable real functions are closed under composition. Apply Theorem 4.3.2. □

The join of two computable functions at a computable point is a computable function (Fig. 4.4).

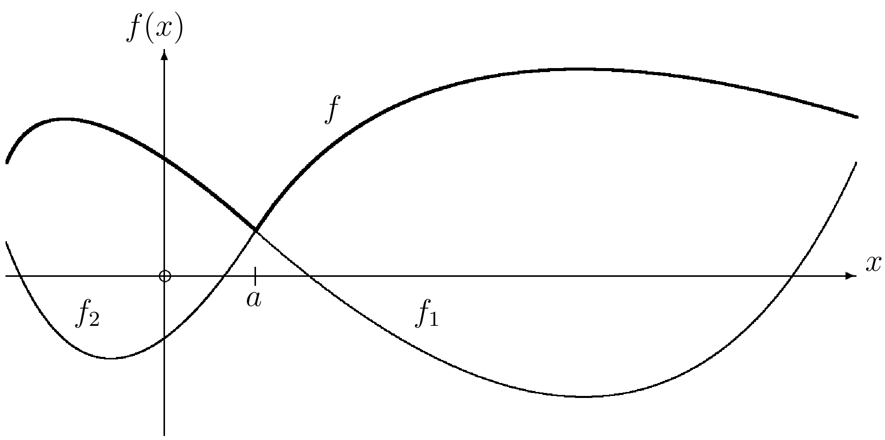

Fig. 4.4. The join f of f_1 and f_2 at a

Lemma 4.3.5 (join of functions). Let $f_1, f_2 :\subseteq \mathbb{R} \to \mathbb{R}$ be computable real functions and let $c \in \mathbb{R}$ be a computable real number. Then the function $f :\subseteq \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} f_1(x) & \text{if } x < c, \\ f_2(x) & \text{if } x > c, \\ f_1(a) & \text{if } x = c \text{ and } f_1(a) = f_2(a), \\ \text{div} & \text{otherwise,} \end{cases}$$

is computable.

Proof: First consider the case $c = 0$. We may assume that f_1 and f_2 are (ρ^a, ρ^a)-computable, ρ^a from Lemma 4.1.6. There are Type-2 machines M_1 and M_2 such that f_{M_1} and f_{M_2} realize f_1 and f_2, respectively. For $i = 1, 2$ let $M_i(p, k)$ be the output written by the machine M_i in k steps. For any $w \in \Sigma^*$ let $N(w) := (\lambda$, if w has no subword $\iota(w')$, its rightmost subword $\iota(w')$ otherwise). There is a Type-2 machine M which on input $p = \iota(u_0)\iota(u_1)\ldots \in \mathrm{dom}(\rho^a)$ operates in stages $k = 0, 1, 2\ldots$ as follows:
Stage k:

- If $0 \in \mathrm{I}^1(u_k)$ and if $N \circ M_1(p, k) = \lambda$ or $N \circ M_2(p, k) = \lambda$, then M goes to the next stage;
- If $0 \in \mathrm{I}^1(u_k)$ and if $N \circ M_1(p, k) = \iota(w_1)$ and $N \circ M_2(p, k) = \iota(w_2)$, then M writes some word $\iota(w)$, such that $\mathrm{I}^1(w)$ is the smallest interval with $\mathrm{I}^1(w_1) \cup \mathrm{I}^1(w_2) \subseteq \mathrm{I}^1(w)$;
- If $\mathrm{I}^1(u_k) < 0$, then M writes $N \circ M_1(p, k)$;
- if $\mathrm{I}^1(u_k) > 0$, then M writes $N \circ M_2(p, k)$.

Suppose that $x = \rho^a(p)$ and $f(x)$ exists. If $x < 0$, then M finally produces the output intervals of M_1 on input p. If $x > 0$, then M finally produces the output intervals of M_2 on input p. If $x = 0$, then M produces a combination of both outputs which converges to $f_1(0) = f_2(0)$. The result is always a ρ^b-name of $f(x)$, ρ^b from Lemma 4.1.6. Therefore, f is (ρ^a, ρ^b)-computable. If $c \neq 0$, apply the above join operation to the functions f'_1 and f'_2, $f'_1(x) := f_1(x + c)$ and $f'_2 := f_2(x + c)$, which are computable by Theorem 4.3.2, and shift the result f': $f(x) := f'(x - c)$. □

We turn now to functions on infinite sequences of real numbers. We use the representation $[\rho]^\omega$ (Definition 3.3.3) which is equivalent to $[\nu_{\mathbb{N}} \to \rho]_{\mathbb{N}}$ by Lemma 3.3.16. Projection and partial summation are computable:

Lemma 4.3.6 (sequences). For sequences $(x_0, x_1, \ldots)$ of real numbers

1. The projection $\mathrm{pr} : \big((x_0, x_1, \ldots), i\big) \mapsto x_i$ is $([\rho]^\omega, \nu_{\mathbb{N}}, \rho)$-computable;
2. The function $S_0 : \big((x_0, x_1, \ldots), i\big) \mapsto x_0 + x_1 + \ldots + x_i$ is $([\rho]^\omega, \nu_{\mathbb{N}}, \rho)$-computable;
3. the function $S : (x_0, x_1, \ldots) \mapsto (y_0, y_1, \ldots)$ where $y_i := x_0 + x_1 + \ldots + x_i$, is $([\rho]^\omega, [\rho]^\omega)$-computable.

Proof: 1. The function $(\langle p_0, p_1, \ldots\rangle, w) \mapsto p_{\nu_{\mathbb{N}}(w)}$ realizes the projection.

2. We apply Theorem 3.1.7 on primitive recursion. Define

$$h\big(0, (x_0, x_1, \ldots)\big) = x_0$$

$$h\big(n+1, (x_0, x_1, \ldots)\big) = h\big(n, (x_0, x_1, \ldots)\big) + x_{n+1} \ .$$

Since $f(x_0, x_1, \ldots) := x_0$ and $f'(n, y, (x_0, x_1, \ldots)) := y + x_{n+1}$ are computable by Property 1, h is computable by Theorem 3.1.7. Since $S_0\big((x_0, x_1, \ldots), i\big) = x_0 + x_1 + \ldots + x_i = h\big(i, (x_0, x_1, \ldots)\big)$, S_0 is $([\rho]^\omega, \nu_{\mathbb{N}}, \rho)$-computable.

3. By Property 2 and Theorem 3.3.15, S is $([\rho]^\omega, [\nu_\mathbb{N} \to \rho]_\mathbb{N})$-computable, and so $([\rho]^\omega, [\rho]^\omega)$-computable. □

The limits of converging sequences and series are computable. We add as a further variable a modulus $e : \mathbb{N} \to \mathbb{N}$ of convergence and use the representation $[\nu_\mathbb{N} \to \nu_\mathbb{N}]_\mathbb{N}$ (Definition 3.3.13).

Theorem 4.3.7 (limit of sequences and series of numbers). For sequences $(x_0, x_1, \ldots)$ of real numbers and modulus functions $e : \mathbb{N} \to \mathbb{N}$, the functions

$$L : \big((x_0, x_1, \ldots), e\big) \mapsto \lim_{i\to\infty} x_i \quad \text{and} \tag{4.1}$$

$$\mathrm{SL} : \big((x_0, x_1, \ldots), e\big) \mapsto \sum_{i\in\mathbb{N}} x_i \tag{4.2}$$

where $((x_0, x_1, \ldots), e) \in \mathrm{dom}(L)$, iff $(\forall j > i \geq e(n))|x_j - x_i| \leq 2^{-n}$, and $((x_0, x_1, \ldots), e) \in \mathrm{dom}(\mathrm{SL})$, iff $(\forall j \geq i \geq e(n))|x_i + \ldots + x_j| \leq 2^{-n}$, are $([\rho]^\omega, [\nu_\mathbb{N} \to \nu_\mathbb{N}]_\mathbb{N}, \rho)$-computable.

Proof: 4.1. We generalize the proof of Theorem 4.2.3. It suffices to show that the function is $([\rho_\mathrm{C}]^\omega, [\nu_\mathbb{N} \to \nu_\mathbb{N}]_\mathbb{N}, \rho_\mathrm{C})$-computable. There is a Type-2 machine M which on input $\big(\langle p_0, p_1, \ldots\rangle, q\big)$, $p_i = \iota(u_{i0})\iota(u_{i1})\ldots \in \mathrm{dom}(\rho_\mathrm{C})$, $e := [\nu_\mathbb{N} \to \nu_\mathbb{N}]_\mathbb{N}(q)$, writes the sequence $q := \iota(u_{e(2)2})\iota(u_{e(3)3})\iota(u_{e(4)4})\ldots$. Then f_M is a $([\rho_\mathrm{C}]^\omega, [\nu_\mathbb{N} \to \nu_\mathbb{N}]_\mathbb{N}, \rho_\mathrm{C})$-realization of L (cf. the proof of Theorem 4.2.3).

4.2. By Lemma 4.3.6, $S : (x_0, x_1, \ldots) \mapsto (y_0, y_1, \ldots)$, $y_i := \sum_{m\leq i} x_i$, is $([\rho]^\omega, [\rho]^\omega)$-computable. If for all $j \geq i \geq e(n)$, $|x_i + \ldots + x_j| \leq 2^{-n}$, then for all $j > i \geq e(n)$, $|y_j - y_i| = |x_{i+1} + \ldots + x_j| \leq 2^{-n}$. Therefore, $\mathrm{SL}\big((x_0, x_1, \ldots), e\big) = L\big(S(x_0, x_1, \ldots), e\big)$, L from 4.1 above, if $|x_i + \ldots + x_j| \leq 2^{-n}$ for all $j \geq i \geq e(n)$, and so SL is $([\rho]^\omega, [\nu_\mathbb{N} \to \nu_\mathbb{N}]_\mathbb{N}, \rho)$-computable. □

We apply Theorem 4.3.7 to show that the uniform limit of a fast converging computable sequence of real-valued functions is computable (Fig 4.5).

Theorem 4.3.8 (limit of sequences and series of functions). Let $\delta :\subseteq \Sigma^\omega \to M$ be a representation, let $X \subseteq M$. Let $(f_i)_{i\in\mathbb{N}}$ with $f_i :\subseteq M \to \mathbb{R}$ and $\mathrm{dom}(f_i) = X$ be a sequence of functions such that $(i, x) \to f_i(x)$ is $(\nu_\mathbb{N}, \delta, \rho)$-computable.

1. If there is a computable function $e : \mathbb{N} \to \mathbb{N}$ with $|f_j(x) - f_i(x)| \leq 2^{-n}$ for all $j > i \geq e(n)$ and $x \in X$, then the function $f :\subseteq M \to \mathbb{R}$, defined by $\mathrm{dom}(f) = X$ and $f(x) = \lim_{i\to\infty} f_i(x)$, is (δ, ρ)-computable.
2. If there is a computable function $e : \mathbb{N} \to \mathbb{N}$ with $|f_i(x) + \ldots + f_j(x)| \leq 2^{-n}$ for all $j \geq i \geq e(n)$ and $x \in X$, then the function $f :\subseteq M \to \mathbb{R}$, defined by $\mathrm{dom}(f) = X$ and $f(x) = \sum_{i\in\mathbb{N}} f_i(x)$, is (δ, ρ)-computable.

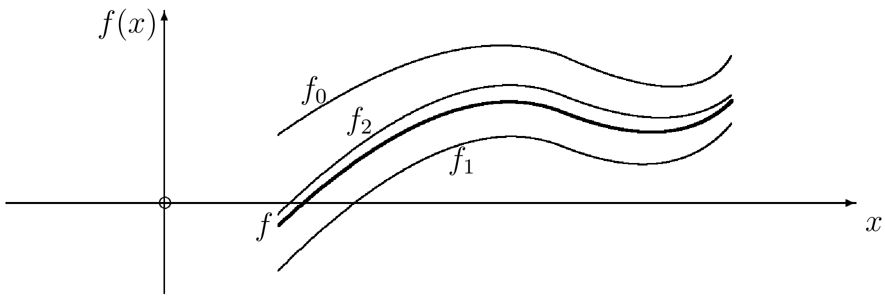

Fig. 4.5. Functions $f_0, f_1, f_2 \dots$ uniformly converging to f

Proof: 1. Since the function $(x, i) \mapsto f_i(x)$ is $(\delta, \nu_{\mathbb{N}}, \rho)$-computable, by Theorem 3.3.15.2 the function $F : x \mapsto (f_i(x))_{i\in\mathbb{N}}$ is $(\delta, [\nu_{\mathbb{N}} \to \rho]_{\mathbb{N}})$-computable and hence $(\delta, [\rho]^\omega)$-computable by Lemma 3.3.16. We obtain $f(x) = L(F(x), e)$ for all $x \in X$ where L is the limit operator from Theorem 4.3.7,4.1. The function f is (δ, ρ)-computable, since L is $([\rho]^\omega, [\nu_{\mathbb{N}} \to \nu_{\mathbb{N}}]_{\mathbb{N}}, \rho)$-computable and e is $[\nu_{\mathbb{N}} \to \nu_{\mathbb{N}}]_{\mathbb{N}}$-computable.

1. This follows correspondingly from Theorem 4.3.7.4.2. □

Lemma 4.3.6 and Theorems 4.3.7 and 4.3.8 can be generalized straightforwardly from $\mathbb{R}$ to $\mathbb{R}^n$.

Every complex number $z = x + iy \in \mathbb{C}$ has an absolute value $|z| = \sqrt{x^2 + y^2}$ and a norm $||z|| = \max(|x|, |y|)$ satisfying $||z|| \le |z| \le \sqrt{2} \cdot ||z||$. The set $\mathbb{C}$ of complex numbers can be identified with the set $\mathbb{R}^2$ of pairs of real numbers, $x + iy \leftrightarrow (x, y)$, with standard representation $[\rho]^2$. Then $\mathbb{C}^n$ is represented by $[[\rho]^2]^n \equiv [\rho]^{2n} \equiv \rho^{2n}$ (where we assume that the Cartesian product is associative, see Definitions 3.3.3, 4.1.17). We call a point $a \in \mathbb{C}$ computable, iff it is ρ^2-computable (iff it is (ρ, ρ)-computable), a function $f :\subseteq \mathbb{C}^n \to \mathbb{C}$ computable, iff it is (ρ^{2n}, ρ^2)-computable etc.. A complex-valued function is computable, iff its real part and its imaginary part are computable (Lemma 3.3.6).

Theorem 4.3.9 (computable complex functions). The complex functions $z \mapsto a$ (for computable $a \in \mathbb{C}$), $(z_1, z_2) \mapsto z_1 + z_2$, $(z_1, z_2) \mapsto z_1 \cdot z_2$, $z \mapsto 1/z$, $z \mapsto |z|$, $z \mapsto ||z||$, $z \mapsto \mathrm{Re}(z)$ and $z \mapsto \mathrm{Im}(z)$ are computable. Furthermore, every complex polynomial function with computable coefficients and the function $(j, z) \mapsto z^j$ are computable.

Proof: Consider the function $f : z \mapsto 1/z$. By Lemma 3.3.6 it suffices to show that the projections $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are (ρ, ρ, ρ)-computable. We have

$$f(x + iy) = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} \ ,$$

therefore, f is computable by Theorem 4.3.2. Computability of $|z|$ follows from computability of $x \mapsto \sqrt{x}$ for $x \in \mathbb{R}, x \geq 0$, which will be proved below (Example 4.3.13.6). The remaining proofs are left to the reader. □

Theorem 4.3.10 (sequences of complex numbers). Lemma 4.3.6 and Theorem 4.3.7 hold for sequences of complex numbers accordingly.

Every sequence $(a_j)_{j\in\mathbb{N}}$ of complex numbers defines a power series with coefficients $a_0, a_1, \ldots$ and a function $f :\subseteq \mathbb{C} \to \mathbb{C}$, defined by

$$f(z) := \sum_{j=0}^{\infty} a_j \cdot z^j \ .$$

The sum $f(z)$ of the series is defined for all z with $|z| < R$ and is not defined for all z with $|z| > R$ where $R := 1/\limsup_{j\to\infty} \sqrt[j]{|a_j|}$ is the *radius of convergence*. By Cauchy's estimate for every number $r < R$ there is a constant M such that

$$|a_j| \leq M \cdot r^{-j} \quad \text{for all} \quad j \in \mathbb{N} \ .$$

Neither the radius R of convergence nor a function h mapping each $r < R$ to an appropriate constant M for Cauchy's estimate can be computed from $(a_j)_{j\in\mathbb{N}}$ in general. For computing $f(z)$ from z and the sequence $(a_j)_{j\in\mathbb{N}}$ of coefficients further information about this sequence must be available. We will use a radius $r < R$ and a number M such that $|a_j| \leq M \cdot r^{-j}$ for all $j \in \mathbb{N}$.

We represent the set of sequences $j \mapsto a_j$ of complex numbers by $[\nu_\mathbb{N} \to \rho^2]_\mathbb{N}$ (Definition 3.3.13) or by $[\rho^2]^\omega$ (Definition 3.3.3) which are equivalent by Lemma 3.3.16.

Theorem 4.3.11 (power series). The function

$$P : \big((a_j)_{j\in\mathbb{N}}, r, M, z\big) \longmapsto \sum_{j=0}^{\infty} a_j \cdot z^j$$

defined for arguments with $|z| < r$ and $|a_j| \leq M \cdot r^{-j}$ for all j is

$$\big([\rho^2]^\omega, \nu_\mathbb{Q}, \nu_\mathbb{N}, \rho^2, \rho^2\big)\text{-computable.}$$

Proof: The multi-valued function $h :\subseteq \mathbb{Q} \times \mathbb{C} \rightrightarrows \mathbb{Q}$ with graph
$\mathrm{R}_h := \{(r, z, s) \mid |z| < s < r\}$ is $(\nu_\mathbb{Q}, [\rho]^2, \nu_\mathbb{Q})$-computable.
Now we show that the following variant Q of P which has a further input parameter s,

$$Q : \big((a_j)_{j\in\mathbb{N}}, r, s, M, z\big) \longmapsto \sum_{j=0}^{\infty} a_j \cdot z^j$$

defined for arguments with $|z| < s < r$ and $|a_j| \leq M \cdot r^{-j}$ for all j, is

$$\big([\rho^2]^\omega, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}, \nu_{\mathbb{N}}, \rho^2, \rho^2\big)\text{-computable.}$$

The function $(j, z) \mapsto z^j$ is $(\nu_{\mathbb{N}}, \rho^2, \rho^2)$-computable (Theorem 4.3.9). By the complex generalization of Lemma 4.3.6.1, the function $\big((a_j)_{j\in\mathbb{N}}, z, k\big) \mapsto a_k \cdot z^k$ is $([\rho^2]^\omega, \rho^2, \nu_{\mathbb{N}}, \rho^2)$-computable. Therefore, by Theorem 3.3.15.2 and Lemma 3.3.16,

$$G : \big((a_j)_{j\in\mathbb{N}}, z\big) \mapsto (a_j \cdot z^j)_{j\in\mathbb{N}} \ \text{ is } \ ([\rho^2]^\omega, \rho^2, [\rho^2]^\omega)\text{-computable.}$$

Next we determine a modulus of convergence. The function

$$H : (r, s, M, n) \mapsto \min\left\{m \in \mathbb{N} \mid M \cdot \left(\frac{s}{r}\right)^m \cdot \frac{r}{r-s} \leq 2^{-n}\right\}$$

$(r, s \in \mathbb{Q}, s < r, M, n \in \mathbb{N})$ is computable, and so by Theorem 3.3.15.2,

$$H' : (r, s, M) \mapsto e, \quad e(n) := H(r, s, M, n)\ ,$$

is $(\nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}, \nu_{\mathbb{N}}, [\nu_{\mathbb{N}} \to \nu_{\mathbb{N}}])$-computable. For $|z| \leq s < r$ and $k \geq j \geq e(n)$ we have

$$\begin{aligned} |a_j \cdot z^j + \ldots + a_k \cdot z^k| \leq \textstyle\sum_{k\geq j} |a_k| \cdot |z|^k &\leq \textstyle\sum_{k\geq j} M \cdot \left(\frac{s}{r}\right)^k \\ &= M \cdot \left(\frac{s}{r}\right)^j \cdot \frac{r}{r-s} \ \leq 2^{-n}\ . \end{aligned}$$

Let SL be the complex version of the summation operator from Theorem 4.3.7.4.2. Then

$$\sum_{i=0}^{\infty} a_i \cdot z^i = \mathrm{SL}\big(G((a_j)_{j\in\mathbb{N}}, z), H'(r, s, M)\big)\ ,$$

if $|z| \leq s < r$, and $|a_j| \leq M \cdot r^{-j}$ for all j. Therefore, the function Q is computable.
Combining machines for h and Q we can construct a machine computing P. □

For a computable sequence $(a_j)_{j\in\mathbb{N}}$ of complex numbers we obtain the following useful consequences:

Theorem 4.3.12. Let $(a_j)_{j\in\mathbb{N}}$ be a computable sequence of complex numbers and let $R := 1/\limsup_{j\to\infty} \sqrt[j]{|a_j|}$.

1. The function $f : z \mapsto \sum_{i=0}^{\infty} a_i \cdot z^i$ is computable on every closed ball $\{z \in \mathbb{C} \mid |z| \leq r\}$ with $r < R$.
2. Let $k \mapsto r_k$, and $k \mapsto M_k$ $(r_k \in \mathbb{Q}, M_k \in \mathbb{N})$ be computable sequences such that $|a_j| \leq M_k \cdot r_k^{-j}$ for all j, k.
   Then the function $f : z \mapsto \sum_{i=0}^{\infty} a_i \cdot z^i$ is computable on the open ball $\{z \in \mathbb{C} \mid |z| < \sup_{k\in\mathbb{N}} r_k\}$.

Proof: 1. There is some rational number r' such that $r < r' < R$. There is a Cauchy bound $M \in \mathbb{N}$ for r'. Then for all $|z| < r'$, $f(z) = P\big((a_j)_{j\in\mathbb{N}}, r', M, z\big)$, P from Theorem 4.3.11.

2. For input z first find some number k with $|z| < r_k$ and then compute $f(z) = P((a_i)_{i\in\mathbb{N}}, r_k, M_k, z)$, P from Theorem 4.3.11. □

The above theorems have many applications.

Example 4.3.13.

1. If $(a_j)_{j\in\mathbb{N}}$ is a computable sequence of complex numbers, $z_0 \in \mathbb{C}$ is computable and $r < R := 1/\limsup_{j\to\infty} \sqrt[j]{|a_j|}$, then $g :\subseteq \mathbb{C} \to \mathbb{C}$, defined by
$$g(z) := \sum_{j=0}^{\infty} a_j \cdot (z - z_0)^j \ ,$$
is computable on the disc $\{z \mid \ |z - z_0| \leq r\}$. For a proof consider the function f from Theorem 4.3.12. Then $g(z) = f(z - z_0)$, hence g is computable by Theorem 4.3.9.
2. The exponential function $\exp : \mathbb{C} \to \mathbb{C}$ can be defined by the power series
$$\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$
for all $z \in \mathbb{C}$. Since $\limsup_{j\to\infty} \sqrt[j]{1/j!} = 0$, the radius of convergence is $R = \infty$. By Theorem 4.3.12, for every $N \in \mathbb{N}$ the exponential function is computable on the disc $\{z \mid \ |z| \leq N\}$. This means, for every number N there is a machine M_N which computes exp on this disc.
There is also a single machine computing $\exp(z)$ for all $z \in \mathbb{C}$. To show this we apply Theorem 4.3.12.2. Consider $N \geq 1$. For $j \leq N$ we have
$$\frac{1}{j!} \leq 1 \leq N^{N-j} = N^N \cdot N^{-j} \ ,$$
and for $j > N$ we have
$$\frac{1}{j!} \leq \frac{1}{(N+1)\cdot(N+2)\cdot\ldots\cdot j} \leq \frac{1}{N^{j-N}} = N^N \cdot N^{-j} \ .$$
Define $r_k := k+1$ and $M_k := r_k^{r_k}$. By Theorem 4.3.12.2, the exponential function is computable on $\mathbb{C}$.
3. The trigonometric functions $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ are computable. For a proof use the identities $\sin(z) = (\exp(\mathrm{i}z) - \exp(-\mathrm{i}z))/(2\mathrm{i})$ and $\cos(z) = (\exp(\mathrm{i}z) + \exp(-\mathrm{i}z))/2$ and Example 2. In particular, the real trigonometric functions are computable.

4. For $|z| < 1$ we have

$$\log(1+z) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{z^j}{j} \; .$$

The radius of convergence is 1. The function $z \mapsto \log(1+z)$ is computable on the disc $\{z \mid \; |z| < 1\}$ (Exercise 4.3.12).

5. The real function $x \mapsto \log x$ is computable on the interval $(0;\infty)$: By Example 4.3.13.4 the number $\log 2 = \log(1+1/3) - \log(1-1/3)$ is computable. There is a machine M which on input x (more precisely, on input p with $\rho(p) = x > 0$) first determines some integer $d \in \mathbb{Z}$ with $1/2 < x \cdot 2^d < 3/2$ and then computes $-d \cdot \log 2 + \log(x \cdot 2^d)$. The function f_M realizes the function $x \mapsto \log x$. (The multi-valued function $g :\subseteq \mathbb{R} \rightrightarrows \mathbb{Z}$ with $\mathrm{R}_g = \{(x,d) \mid 1/2 < x \cdot 2^d < 3/2\}$ is $(\rho, \nu_{\mathbb{Z}})$-computable but has no $(\rho, \nu_{\mathbb{Z}})$-continuous choice function.)

6. The function $(x,y) \mapsto x^y$ for $x, y \in \mathbb{R}$ and $x > 0$ is computable: $x^y = \exp(y \cdot \log x)$. In particular, $x \mapsto \sqrt{x}$ is computable.

7. For real x with $|x| < 1$ we have

$$\arcsin x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \ldots \; .$$

The function arcsin is computable on $(-1;1)$ (Exercise 4.3.13).

8. Since computable functions map computable elements to computable ones, and since the rational numbers are computable, numbers like $\sqrt{2} = 2^{1/2}$, $\sqrt[m]{n}$ $(m, n > 1)$, $\mathrm{e} = \exp 1$, $\log 2$, $\log_a b = \log b / \log a$ $(a, b \in \mathbb{Q}$, $a, b > 0)$, $\pi = 6 \cdot \arcsin(1/2)$ and e^{π} are computable real numbers. □

Complex functions $f :\subseteq \mathbb{C} \to \mathbb{C}$ which can be defined by power series are called *analytic* [Ahl66]. We will discuss computability of analytic functions in Sect. 6.5.

Let $f : \mathbb{R} \to \mathbb{R}$ be a (total) computable real function and let $X \subseteq \mathbb{R}$ be a "very complicated" set. Then by definition the restriction $f\rfloor_X$ is also computable. But $f\rfloor_X$ has a computable extension with the simple domain $\mathbb{R}$. There is, however, a computable real function with very complicated domain, which has no computable extension.

Example 4.3.14 (a computable function with inherent G_δ-domain). Define the function $f :\subseteq \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} \sum\{2^{-i} \mid \mu(i) < x\} & \text{if } x \notin \mathbb{Q} \\ \text{div} & \text{otherwise,} \end{cases}$$

where $\mu\langle i,j,k\rangle := (i-j)/(1+k)$. Let M be a Type-2 machine which on input $p = \iota(u_0)\iota(u_1)\ldots \in \mathrm{dom}(\rho_{\mathrm{C}})$ operates in stages $i = 0, 1, 2, \ldots$ as follows. Let $\nu_{\mathbb{Q}}(v_{-1}) = 0$.

Stage i:
M searches for some m with $|\overline{u}_m - \mu(i)| > 2^{-m}$. If no such m exists, then the computation remains in Stage i forever. Otherwise, M prints $\iota(v_i)$ where $\overline{v}_i = \overline{v}_{i-1} + 2^{-i}$, if $\mu(i) < \overline{u}_m - 2^{-m}$, and $v_i = v_{i-1}$ otherwise.
Then, M produces an infinite output $q := \iota(v_0)\iota(v_1)\ldots \in \mathrm{dom}(\rho_C)$, iff $\rho_C(p) \notin \mathbb{Q}$, and in this case $\rho_C(q) = f \circ \rho_C(p)$. Therefore, f_M realizes the function f.
The function f has the domain $\mathrm{dom}(f) = \mathbb{R} \setminus \mathbb{Q}$ which is a G_δ-set, that is, it can be written as an intersection of a sequence of open sets: $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{i\in\mathbb{N}}(\mathbb{R} \setminus \{\mu(i)\})$. $\mathbb{R} \setminus \mathbb{Q}$ is even a "computable" G_δ-set (Exercise 4.3.17). Since for each rational number $a = \nu(k)$, $\lim_{x\to a+} f(x) - \lim_{x\to a-} f(x) = 2^{-k}$, the function f has no proper continuous extension and hence no proper computable extension. □

Every continuous partial real function has an extension with G_δ-domain [Kur66]. Every (ρ^n, ρ)-computable real function has a strongly (ρ^n, ρ)-computable extension. The domain of each strongly (ρ^n, ρ)-computable real function is a computable G_δ-set (Exercise 3.1.5 and Exercises 4.3.17 and 18).

In Chap. 9 we will discuss several other definitions for computable real functions and compare them with the definitions given here. As a special case of Corollary 3.2.13, every computable or continuous function from $\mathbb{R}^n$ to a discrete space is constant.

Lemma 4.3.15. Let μ be a notation of a set M, $M \neq \emptyset$. If a function $f : \mathbb{R}^n \to M$ is (ρ^n, μ)-continuous, then f is a constant function.

Proof: The final topology $\tau_{\mathbb{R}^n}$ of the admissible representation ρ^n of $\mathbb{R}^n$ is connected. Apply Corollary 3.2.13. □

Corollary 4.3.16.
1. Every $(\rho^n, \nu_\mathbb{N})$-continuous or -computable function $f : \mathbb{R}^n \to \mathbb{N}$ is constant.
2. Every $(\rho^n, \nu_\mathbb{Q})$-continuous or -computable function $f : \mathbb{R}^n \to \mathbb{Q}$ is constant.
3. Every $(\rho^n, \delta_\mathbb{B})$-continuous or -computable function $f : \mathbb{R}^n \to \mathbb{B}$ is constant ($\delta_\mathbb{B}$ from Definition3.1.2).

Proof: The first two statements are immediate. Consider $i \in \mathbb{N}$. The function $H_i : h \mapsto h(i)$ is $(\delta_\mathbb{B}, \nu_\mathbb{N})$-computable, hence $H_i \circ f$ is $(\rho^n, \nu_\mathbb{N})$-continuous, and so constant. Therefore, f is constant. □

Exercises 4.3.

◊ 1. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a computable function and let $c \in \mathbb{R}$ be a computable constant. Define $g(x) := f(x, c)$ for all x. Show that g is computable.

◊ 2. The Gauß staircase is $(\rho, \rho_>)$-computable. Its restriction to $\mathbb{R} \setminus \mathbb{Z}$ is (ρ, ρ)-computable.

◊ 3. Use Exercise 3.1.4 to show that $\emptyset$ and $\mathbb{R}$ are the only ρ-decidable subsets or $\mathbb{R}$.

◊ 4. Show that in Theorem 4.3.2 Property 3 follows from Properties 1,2 and 5.

5. Show that the real functions $x \mapsto |x|$, $x \mapsto ||x||$, $(x, y) \mapsto d(x, y)$ and $(x, y) \mapsto d^{\mathrm{e}}(x, y)$ for $x, y \in \mathbb{R}^n$ are computable.

6. Let $a_0, b_0, \ldots, a_n, b_n \in \mathbb{Q}$ with $a_0 < \ldots < a_n$. Show that the *rational polygon* $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} b_0 & \text{if } x < a_0 \\ b_i + (x - a_i)(b_{i+1} - b_i)/(a_{i+1} - a_i) & \text{if } a_i \leq x < a_{i+1} \\ b_n & \text{if } a_n < x\ , \end{cases}$$

is computable.

7. Show that the function

$$((x_i)_{i\in\mathbb{N}}, n) \mapsto \prod_{i=0}^{n} x_i$$

is $([\rho]^\omega, \nu_{\mathbb{N}}, \rho)$-computable.

8. Show that Lemma 4.3.6, Theorem 4.3.7 and Theorem 4.3.8 hold for $\mathbb{R}^n$ replacing $\mathbb{R}$ (and, in particular, for complex numbers).

9. Let $(f_i)_{i\in\mathbb{N}}$ with $f_i : \mathbb{R} \to \mathbb{R}$ be a sequence of real functions with
   a) $(i, x) \mapsto f_i(x)$ is $(\nu_{\mathbb{N}}, \rho, \rho)$-computable,
   b) there is a computable function $e : \mathbb{N}^2 \to \mathbb{N}$ with $|f_i(x) - f_j(x)| \leq 2^{-n}$ for all $i, j \geq e(n, k)$ and $|x| < k$.

   Show that the sequence converges to a computable function $f : \mathbb{R} \to \mathbb{R}$.

10. Complete the proof of Theorem 4.3.9.

11. Let $a, b \in \mathbb{R}$, $a < b$, a right-computable and b left-computable, a and b not computable. Show that there is a real function $f :\subseteq \mathbb{R} \to \mathbb{R}$ such that f is strictly increasing, $\mathrm{dom}(f) = (0; 1)$, $\mathrm{range}(f) = (a; b)$, f and f^{-1} are computable. The computable function f has a (unique) continuous extension to $[0; 1]$ which is not computable. (Hint: consider Example 4.2.4.3; let f be an infinite polygon.)

12. Prove that $z \mapsto \log(1 + z)$ is computable on the open disc $\{z \mid |z| < 1\}$.

13. Prove that $x \mapsto \arcsin x$ is computable on the open interval $\{x \mid |x| < 1\}$.

14. Show that the real exponential function $x \mapsto \mathrm{e}^x$ is $(\rho_<, \rho_<)$-computable.

15. (Sorting real numbers)
    a) Show that the function $f : \mathbb{R}^n \to \mathbb{R}^n$, defined by $f(x_1, \ldots, x_n) := (y_1, \ldots, y_n)$ such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$ and $y_1 \leq y_2 \leq \ldots \leq y_n$, is computable.
    b) Show that the multi-valued function $f : \mathbb{R}^2 \rightrightarrows \mathbb{N}$, defined by $\mathrm{R}_f := \{((x_1, x_2), i) \mid x_i = \min(x_1, x_2)\}$, is not $(\rho^2, \nu_{\mathbb{N}})$-continuous.

c) Show that the function

$$(x_1, \ldots, x_n) \mapsto \pi, \quad x_i \in \mathbb{R},\ x_i \neq x_j \text{ for } \ 1 \leq i, j \leq n\ ,$$

where π is the permutation of $\{1, \ldots, n\}$ such that $x_{\pi(1)} < \ldots < x_{\pi(n)}$, is (ρ^n, ν)-computable (where ν is a canonical notation of the permutations of $\{1, \ldots, n\}$).

16. Let $0 < a < b$ be left-computable real numbers. Show that there is a $(\nu_{\mathbb{N}}, \nu_{\mathbb{Q}})$-computable sequence of rational numbers a_i with $b - a = \lim_{i\to\infty} a_i$.
17. Show that there is some r.e. set $A \subseteq \Sigma^* \times \Sigma^*$ with $\mathbb{R} \setminus \mathbb{Q} = \bigcap_u \bigcup_v \{\mathrm{I}^1_v \mid (u, v) \in A\}$ (that is, $\mathbb{R} \setminus \mathbb{Q}$ is a "computable" G_δ-set).

♦18. [Wei93] Call $X \subseteq \mathbb{R}$ a computable G_δ-set, iff $X = \bigcap_u \bigcup_v \{\mathrm{I}^1_v \mid (u, v) \in A\}$ for some r.e. set $A \subseteq \Sigma^* \times \Sigma^*$.
   a) Show that every computable real function $f :\subseteq \mathbb{R} \to \mathbb{R}$ has a strongly (ρ, ρ)-computable extension (Exercise 3.1.5).
   b) Show that $\mathrm{dom}(f)$ is a computable G_δ-set for every strongly (ρ, ρ)-computable real function $f : \mathbb{R} \to \mathbb{R}$.
   c) Show that for every computable G_δ-set X there is a strongly (ρ, ρ)-computable real function $f : \mathbb{R} \to \mathbb{R}$ with $X = \mathrm{dom}(f)$.
   d) Let $X \subseteq \mathbb{N}$ be a Π_2-subset, that is, $X = \{x \in \mathbb{N} \mid (\forall i)(\exists k)(x, i, k) \in B\}$ for some decidable set $B \subseteq \mathbb{N}^3$. Show that X is a computable G_δ-subset of $\mathbb{R}$. (If X is r.e. , then X, $\mathbb{N} \setminus X$ and $\mathbb{R} \setminus X$ are computable G_δ-subsets of $\mathbb{R}$.)

♦19. For a function $f : \mathbb{R}^m \to \mathbb{R}^n$, the following properties are equivalent:
   a) f is (ρ^m, ρ^n)-computable,
   b) the set $\{(u, v) \mid f[\overline{\mathrm{I}}^m(u)] \subseteq \mathrm{I}^n(v)\}$ is r.e. ,
   c) $f^{-1}[\mathrm{I}^n(v)] = \bigcup \{\mathrm{I}^m(u) \mid (u, v) \in B\}$ for some r.e. set $B \subseteq \Sigma^* \times \Sigma^*$

   (Theorem 3.2.14).

20. Show that every continuous, $(\rho^n, \mathrm{id}_{\Sigma^\omega})$-continuous or $(\rho^n, \mathrm{id}_{\Sigma^\omega})$-computable function $f : \mathbb{R}^n \to \Sigma^\omega$ is constant.

◊21. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be computable. Show that the function $(n, x) \mapsto f^n(x)$ is $(\nu_{\mathbb{N}}, \rho^n, \rho^n)$-computable. Hint: Apply Theorem 3.1.7.

22. For $c \in \mathbb{C}$ define $f_c : \mathbb{C} \to \mathbb{C}$ by $f_c(z) := z^2 + c$. Show that the function $(n, c) \mapsto f^n_c(0)$ is $(\nu_{\mathbb{N}}, \rho^2, \rho^2)$-computable. Hint: Apply Theorem 3.1.7.
23. Show that every continuous (and hence (ρ^n, ρ)-continuous) function $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{range}(f) \subseteq \mathbb{Q}$ is constant.