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The Mountain Pass Theorem and Applications

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Introduction

The *Mountain Pass Theorem (MPT)* by Ambrosetti and Rabinowitz (1973) is a celebrated result in nonlinear analysis with many applications in particular to partial differential equations. In the first section we develop some concepts of the calculus of variations and prove the MPT via the *Deformation Lemma*. In the second section we are concerned with a first application of the MPT to prove the existence of a solution of the semilinear problem

$$\begin{cases} -\Delta u(x) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } u|_{\partial\Omega}, \end{cases}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$, with Theorem 2 being our main result. In the following section we investigate nonlinear elliptic PDEs involving the p-Laplacian $\Delta_p u := \nabla\left(|\nabla u|^{p-2}\nabla u\right)$, where $|\nabla u|^{p-2} = \left\{\left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2\right\}^{\frac{p-2}{p}}$. The PDEs are of the form

$$-div(a(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = f(x,u),\ x \in \Omega,$$

and Ω an unbounded domain. In the beginning of section three we assume

$$f(x,u) = g(x)u^{\alpha},\ p-1 < \alpha < p^* - 1 \text{ and } 1 < p < n,$$

and prove existence using the MPT and $C^{1,\delta}$ regularity of the solutions (Theorem 3). We continue by showing that $\alpha = p^* - 1$ is a "critical exponent" by giving a proof of nonexistence if $\alpha = p^* - 1$. In the last part of section three we study the limit case $n = p$ (spatial dimension equals the exponent of the Sobolev space $W^{1,p}(\Omega)$). In this case the "critical exponent" is determined by the Trudinger inequality (and its generalizations). We prove $C^{1,\delta}$- regularity for the subcritical case (Theorem 3.5), i.e. for

$$\lim_{u\to\infty} \frac{f(x,u)}{e^{|u|^{\mu}}} = 0 \text{ for some } 0 < \mu < \frac{n}{n-1}, \quad \text{uniformly on } \mathbb{R}^n$$

and show that a weak solution exists if $f(x,u)$ is allowed to grow as $e^{\frac{n}{n-1}|u|}$ (the critical case) (Theorem 3.9). In the Appendix we give a short account of important properties of Sobolev spaces, especially the Sobolev embedding Theorem and the Rellich-Kondrachov Theorem.

1 Deformation Lemma and Mountain Pass Theorem

In the original proof of the Mountain Pass theorem, Ambrosetti and Rabinowitz used the *Deformation Lemma*. We choose a similar method, but instead of proving the Deformation Lemma directly we will give a quantitative version from which it follows. First we need some definitions. Throughout the next two sections we will take X to be a Banach space with norm $\|\cdot\|$. Unless stated otherwise we will mean convergence in the norm of X when taking about convergence of functionals on X.

Definition 1.1 (Frechét differentiability).
*Let Y be a Banach space and $u \in U \subset X$ with U open. $I : X \to Y$ is said to be **Frechét differentiable** at x, if a linear operator $A \in L(X,Y)$ exists such that*

$$I[u+\phi] - I[u] - Au = o(\|\phi\|).$$

*A is called **Frechét derivative** of I at u and is denoted by $I'[u]$. Suppose I is Frechét differentiable in U, then we call $I' : U \to L(X,Y)$ the Frechét derivative of I in U and say that I is a C^1-functional iff*

I' is continuous.
In the rest of the text we will mean Frechét differentiability when we talk about the differentiability of a functional.

Definition 1.2 (Palais-Smale condition).
*Let $I : X \to \mathbb{R}$ be a C^1- functional. We say that I satisfies the **Palais-Smale condition (PS)**, if any sequence $\{u_n\}$ in X such that $\{I[u_n]\}$ is bounded and $\{I'[u_n]\} \to 0$, has a convergent subsequence. A sequence with $\{I[u_n]\}$ bounded and $\{I'[u_n]\} \to 0$ is called **Palais-Smale sequence** .*

We also make use of the following weaker form introduced in [9]:
Definition 1.3 (local Palais-Smale condition).
*A C^1- functional $I : X \to \mathbb{R}$ satisfies the **local Palais-Smale condition** $(PS)_c$ at the point $c \in \mathbb{R}$ if any sequence $\{u_n\}$ in X with $\{I[u_n]\} \to c$ and $\{I'[u_n]\} \to 0$, has a convergent subsequence.*
This means that the set of critical points of I at level c is compact. If I satisfies (PS) then it does satisfy $(PS)_c$ for all $c \in \mathbb{R}$ (which lie in the closure of the image of I), but the converse does not hold.

Some examples:

- The function $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^3$ satisfies the (PS) condition. In fact if X is a finite-dimensional Banach space $I \in C^1(X, \mathbb{R})$ and $|I| : X \to \mathbb{R}$ is coercive (i.e. tends to $+\infty$ if $\|x\|_X$ tends to $+\infty$, then I satisfies (PS) (see Prop. 2.1 in [16]).

- For $X = \mathbb{R}$, $I[u] = \sin u$, satisfies $(PS)_c$ for all c in $\mathbb{R}\backslash\{-1, 1\}$. At 1 (resp. -1) the $(PS)_c$ condition fails, since one can construct a sequence $\{x_i\}$ of certain increasing odd integer multiplies of $\frac{\pi}{2}$ such that $I[x_i] = 1$ (resp. -1) $\forall i$, then $I'[x_i] = \cos x_i = 0$ for all i, contradicting $(PS)_c$ since x_i does not have a convergent subsequence.

- The functional
$$I[u] = \int_\Omega \left[|\nabla u(x)|^2 - \frac{|u^+(x)|^p}{p} - \frac{\mu |u^+(x)|^q}{q} \right],$$
for the problem
$$-\Delta u = u^{p-1} + \mu u^{q-1} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$
$\Omega \subset \mathbb{R}^n$ a smooth domain, $u \in W_0^{1,2}(\Omega)$ and $n \geq 4$, $p = \frac{2n}{n-2}$, $0 < q < p$, $\mu > 0$, satisfies $(PS)_c$ for all $c < \frac{\Gamma^{n/2}}{n}$, where Γ is the Sobolev embedding constant of the injection of $W_0^{1,2}(\Omega)$ into $L^p(\Omega)$ (see [8]).

We will need the following concept introduced by Palais ([30]). In the following K will denote the set of critical points of a functional I, i.e. $K := \{t \in X \mid I'[t] = 0\}$.
Definition 1.4 (pseudo-gradient vector field).
*Let $I \in C^1(X, \mathbb{R})$. We call $v \in X$ a **pseudo-gradient vector** of I at $u \in X\backslash K$ if v satisfies:*

$$(i) \qquad \|v\| \leq 2\|I'[u]\|,$$

$$(ii) \qquad \|I'[u]\|^2 \leq \langle I'[u], v\rangle,$$

*where $\langle\cdot,\cdot\rangle$ denotes the duality pairing. If $F : X\backslash K \to X$ is locally Lipschitz continuous and $F(x)$ is a pseudo-gradient vector for all $x \in X\backslash K$, then F is called a **pseudo-gradient vector field** of I.*
Note that any convex combination of pseudo-gradient vectors (resp. fields) is again a pseudo-gradient vector (field). We follow Rabinowitz [31], Lemma A.2 and show that there is always such a $v \in X$ as long as I is C^1.

Definition 1.5 (paracompactness).
*A topological space is called **paracompact** if every open cover has a locally finite refinement.*
By the *Theorem of Stone* (see [32] for a proof) metric spaces are paracompact.

Lemma 1.6. *If I is a C^1- functional, there exists a pseudo-gradient vector field on the set of regular points (i.e. $u \in X$ with $I'[u] \neq 0$) of I.*

Proof. Let us denote the set of regular values of I in X by $\tilde{X}$, i.e. $\tilde{X} = X \backslash K$. Then for every $u \in \tilde{X}$ there is a $y \in X$ with $\|y\| = 1$ and

$$\langle I'[u], y \rangle > \frac{2}{3}\|I'[u]\|.$$

Thus $v := \frac{3}{2}\|I'[u]\|y$ is a pseudo-gradient vector of I at u, since

$$\begin{aligned} (i) \qquad & \frac{3}{2}\|I'[u]\|\|y\| < 2\|I'[u]\|, \text{ and} \\ (ii) \qquad & \|I'[u]\|^2 \leq \left\langle I'[u], \frac{3}{2}\|I'[u]\|y \right\rangle. \end{aligned}$$

Since I' is continuous, there is an open neighbourhood V_u of u, such that v is a pseudo-gradient vector for every $w \in V_u$. Clearly $\{V_u \mid u \in \tilde{X}\}$ is an open cover of $\tilde{X}$, and since $\tilde{X}$ as a metric space is paracompact, we can take a locally finite refinement of this cover, $\{Q_k\}$ say. We set $\rho_k(u) := dist(u, \tilde{X}\backslash Q_k)$, then ρ_k is Lipschitz continuous and $\rho_k(u) = 0$ if $u \notin Q_k$. Define

$$\beta_k(u) := \frac{\rho_k(u)}{\sum_i \rho_i(u)}.$$

Every Q_k lies in some V_u, which we denote by V_{u_k} for all k, then $v_k = \frac{3}{2}\|I'[u_k]\|y_k$ is a pseudo-gradient vector of I in Q_k. Set

$$B(u) := \sum_k v_k \beta_k(u),$$

therefore, since $0 \leq \beta_k(u) \leq 1$ and $\sum_k \beta_k(u) = 1$ for all $u \in \tilde{X}$, $B(u)$ is a convex combination of pseudo-gradient vectors. Since B is also locally Lipschitz continuous it is a pseudo-gradient vector field of I.

We give the following variant of the Deformation Lemma as stated by Willem [37]. Set $I_a := \{u \in X \mid I[u] \leq a\}$.

Theorem 1.7 (Quantitative version of the Deformation Lemma). *Let $U \subset X, \delta > 0$ and $U_\delta := \{u \in X \mid dist(u, U) \leq \delta\}$. Suppose $I : X \to \mathbb{R}$ is a C^1-functional and there exists $c \in \mathbb{R}$, such that for some $\varepsilon > 0$ and $u \in I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap U_{2\delta}$*

$$\frac{4\varepsilon}{\delta} \leq \|I'[u]\|. \tag{1}$$

Then there is a continuos deformation $\eta \in C([0, 1] \times X, X)$ with

$$\begin{aligned} (i) \qquad & \eta(0, u) = u \qquad \forall u \in X, \\ (ii) \qquad & \eta(t, u) = u \qquad \forall u \notin I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap U_{2\delta}, \ \forall t \in [0, 1], \\ (iii) \qquad & \eta(1, I_{c+\varepsilon} \cap U) \subset I_{c-\varepsilon} \cap U_\delta, \\ (iv) \qquad & \eta(t, \cdot) \quad \text{is a homeomorphism} \quad \forall t \in [0, 1]. \end{aligned}$$

Proof. The proof follows [16]. Since I is a C^1-functional there is a pseudo-gradient vector field on $\tilde{X}$ and by assumption (1) $I^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap U_{2\delta} \subset \tilde{X}$. Define the locally Lipschitz continuous funciton $V : X \to \mathbb{R}$,

$$V(u) = \begin{cases} 1 & \text{on } I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap U_\delta, \\ 0 & \text{on } \overline{X \backslash I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap U_{2\delta}}. \end{cases}$$

Then $W : X \to X$

$$W(u) = \begin{cases} -\frac{V(u)v(u)}{\|v(u)\|} & \text{on } I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap U_{2\delta}, \\ 0 & \text{on } \overline{X \backslash I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap U_{2\delta}}, \end{cases}$$

too is locally Lipschitz and bounded, thus the boundary value problem

$$\begin{cases} \frac{df}{dt} = W(f), \\ f(0) = u \end{cases}$$

has a unique solution $f(\cdot,u)$ for every $u \in X$ defined on the maximal interval $(t^-(u), t^+(u))$. We show that $t^{\pm}(u) = \pm\infty$. Suppose, by contradiction, $t^+(u) < +\infty$. Let t_n be a sequence with $t_n \to t^+(u)$ and $t_n < t^+(u)$. Integrating the ODE leads to

$$\|f(t_{n+1}, u) - f(t_n, u)\| \le |t_{n+1} - t_n|,$$

since $\|W(u)\| \le 1$ on X. Thus $f(t_n, u)$ is a Cauchy sequence and hence converges to some $\tilde{u} \in X$. Taking $\tilde{u}$ as new initial data in the ODE, the solution is a continuation of $f(t, u)$ to values $t > t^+(u)$ contradicting the maximality. $t^-(u) = -\infty$ is proved in the same way. Let $f(\cdot, u)$ be a solution on $[0, \infty[$, and define $\eta : [0,1] \times X \to X$

$$\eta(t, u) = f(\delta t, u).$$

From the definition of W it is clear that (i) and (ii) hold true for η. We show that (iii) is satisfied: For $t > 0$

$$\|f(t,u) - u\| = \| \int_0^t W(f(s,u))\, ds\| \le \int_0^t \|W(f(s,u))\|\, ds \le t,$$

and therefore $f(t, U) \subset U_\delta$ for all $t \in [0, \delta]$.

$$\frac{d}{dt} I[f(t,u)] = \langle I'[f(t,u)], f'(t,u)\rangle = \langle I'[f(t,u)], W(f(t),u)\rangle \le 0,$$

from the definition of f and (1). Thus for $u \in I_{c+\varepsilon} \cap U$, and if $I[f(t,u)] < c - \varepsilon$ for some $t \in [0, \delta[$, it follows that $f(\delta, u) \subset I_{c-\varepsilon} \cap U_\delta$. If such a "t" does not exist we have

$$c - \varepsilon \le I[f(t,u)] \le I[f(0,u)] = I[u] \le c + \varepsilon,$$

and so $f(t,u) \in I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap U_\delta \quad \forall t \in [0, \delta[$. Using (1) we get

$$\begin{aligned} I[f(\delta,u)] = I[u] + \int_0^\delta \frac{d}{dt} I[f(s,u)]\, ds = I[u] + \int_0^\delta \langle I'[f(s,u)], W(f(s,u))\rangle\, ds \\ = I[u] + \int_0^\delta \left\langle I'[f(s,u)], \frac{v(f(s,u))}{\|v(f(s,u))\|} \right\rangle ds \le c + \varepsilon - \int_0^\delta \frac{\|I'[f(s,u)]\|^2}{\|v(w(s,u))\|}\, ds, \\ \le c + \varepsilon - \frac{1}{2}\int_0^\delta \|I'[f(s,u)]\|\, ds \le c + \varepsilon - \frac{4\varepsilon}{2} = c - \varepsilon, \end{aligned}$$

and thus $f(\delta, u) \subset I_{c-\varepsilon} \cap U_\delta$.

We obtain the "standard" Deformation Lemma as a corollary:

Corollary 1.8 (Deformation Lemma). *Let $c \in \mathbb{R}$ and $I : X \to \mathbb{R}$ be a C^1-functional satisfying $(PS)_c$. If c is a regular value of I then, for a given $\varepsilon' > 0$ and some $\varepsilon \in (0, \varepsilon')$, there is $\eta \in C([0,1] \times X, X)$ such that:*

$$\begin{array}{lll} (i) & \eta(0,u) = u & \forall u \in X, \\ (ii) & \eta(t,u) = u & \forall u \notin I^{-1}([c-\varepsilon', c+\varepsilon']),\ \forall t \in [0,1], \\ (iii) & \eta(1, I_{c+\varepsilon} \cap U) \subset I_{c-\varepsilon}, & \\ (iv) & \eta(t,\cdot) \quad \textit{is a homeomorphism} & \forall t \in [0,1]. \end{array}$$

Proof. There are $\tilde{\varepsilon}, \tilde{\delta} > 0$ such that we have $\|I[u]\| \ge \tilde{\delta}$ for $u \in I^{-1}([c - \tilde{\varepsilon}, c + \tilde{\varepsilon}])$, because otherwise one would have a sequence $\{u_n\}$ with $c - \frac{1}{n} \le I[u_n] \le c + \frac{1}{n}$ and $\|I'[u_n]\| \le \frac{1}{n}$, thus c would be a critical value of I. So by Theorem 1.7 with $U = X$, ε' small enough and, $\tilde{\delta} = \frac{4\varepsilon}{\delta}$ we get the result.

The great strength of the quantitative version of the Deformation Lemma, lies in the fact that we do not assume the "a priori" compactness of the $(PS)_c$ condition. This allows us to apply the MPT to functionals which do not satisfy the (PS) or $(PS)_c$ condition. We thus get a sequence $\{u_n\}$ with $\{I[u_n]\} \to c$ and $\{I'[u_n]\} \to 0$ of which we prove afterwards that it converges to a nontrivial solution. Similarly Ekeland's Variational Principle can be used instead of the Deformation Lemma to obtain this kind of machinery.

We are now able to prove the Mountain Pass theorem.

Theorem 1.9 (The Mountain Pass Theorem). *Let X be a Banach space, $I : X \to \mathbb{R}$ a C^1-functional satisfying the Palais-Smale condition and $I[0] = 0$.* Suppose

$$\begin{aligned} (A) \quad & \exists\, \rho, \alpha > 0 \text{ such that } I|_{\partial B_\rho} \geq \alpha, \\ (B) \quad & \exists\, e \in X \backslash B_\rho \text{ with } I[e] \leq 0, \end{aligned}$$

where B_ρ denotes the ball of radius ρ around 0.
Then I has a critical value $c \geq \alpha$ and c is characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I[u],$$

where

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

Proof. $c < \infty$, and for $\gamma \in \Gamma$ $\gamma([0,1]) \cap \partial B_\rho \neq \emptyset$. Therefore

$$\max_{u \in \gamma([0,1])} I[u] \geq \inf_{v \in \partial B_\rho} I[v] \geq \alpha,$$

and thus $c \geq \alpha$. Suppose that c is a regular value of I. Then, by Corollary 1.8, for $\varepsilon' = \alpha$ there is $\varepsilon \in (0, \varepsilon')$ and a deformation η with $\eta(1, I_{c+\varepsilon}) \subset I_{c-\varepsilon}$. By the definition of c, we can choose a $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma([0,1])} I[u] \leq c + \varepsilon.$$

$h(t) = \eta(1, \gamma(t))$ is in $C([0,1], X)$ and $h(0) = 0$ since $\gamma(0) = 0$, $I[0] = 0 \leq \alpha \leq c - \varepsilon'$ and (i) of Corollary 1.8, $h(1) = e$ follows similarly. So $h \in \Gamma$ and thus

$$\max_{h \in \gamma([0,1])} I[u] \geq c,$$

which contradicts $h([0,1]) \subset I_{c-\varepsilon}$.

Note that we could have used the weaker $(PS)_c$ condition instead of (PS), with c being the c in the proof.

Remark. The name of the theorem comes from the intuitive interpretation in two dimensions. Imagine a valley around $I[0] = 0$ surrounded by a mountain range. v with $I[v] \leq 0$ is another valley outside the mountain range. To travel from 0 to v we must pass over the mountains and the theorem gives us the mountain pass with the smallest elevation.

2 A Semilinear Problem

We will now apply the Mountain Pass theorem to

$$\begin{cases} -\Delta u(x) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } u|_{\partial\Omega}, \end{cases} \tag{2}$$

Equations like this occur frequently in physics. For example standing waves of the nonlinear Schrödinger equation are described in such a way. These are solutions $\Phi(x,t) = e^{i\alpha t} u(x)$ of

$$i \frac{\partial \Phi}{\partial t} = \nabla \Phi + g(|\Phi|)\Phi,$$

and therefore u satisfies

$$\Delta u + g(|u|)u + \alpha u = 0.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. We will assume the following properties to hold:

(i) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and for all x there are constants $c_1, c_2 \geq 0$, such that for all t

$$|f(x,t)| \leq c_1|t|^p + c_2, \quad 1 \leq p < \frac{2n}{n-2} - 1 \text{ if } n \geq 3, \text{ and } 1 \leq p < \infty \text{ if } n = 2,$$

(ii) For $F(x,u) := \int_0^u f(x,t)\,dt$ there is a $\beta > p$ with

$$\beta F(x,t) \leq t f(x,t)$$

for all $(x,t) \in \Omega \times \mathbb{R}$, and

(iii) For all $x \in \Omega$

$$f(x,t) = o(|t|) \ as\ t \to 0.$$

Dirichlet's principle tells us that a weak solution $u \in W_0^{1,2}(\Omega)$ can be obtained as the minimizer of the functional

$$I[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx =: I_1[u] - I_2[u].$$

We are going to use the norm $\|u\| := \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ on $W_0^{1,2}(\Omega)$, which by the Poincaré inequality is equivalent to the usual norm. We prove:

Theorem 2. *If (i),(ii),(iii) hold, then (2) has a nontrivial weak solution.*

The following technical lemma, depicts a basic property of the more general concept of Nemytskii (superposition) operators (see [11]).

Lemma 2.1 $\Omega \subset \mathbb{R}^n$ *a bounded domain, suppose the following properties hold for the function g:*

(i')

$$g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}),$$

(ii') For some constants $c_1', c_2' \geq 0,\ r, s \geq 1$

$$|g(x,t)| \leq c_1' + c_2'|t|^{\frac{r}{s}},$$

then $u(x) \mapsto g(x, u(x))$ *is in* $C(L^r(\Omega), L^s(\Omega))$.

Proof. For $u \in L^r(\Omega)$

$$\int_\Omega |g(x,u(x))|^s\,dx \leq \int_\Omega (c_1' + c_2'|u(x)|^{\frac{r}{s}})^s\,dx \leq c_3' \int_\Omega (1 + |u(x)|^r)dx,$$

and so $g : L^r(\Omega) \to L^s(\Omega)$. To prove the continuity we can assume $u = 0$ and $g(x,0) = 0$ since g is continuous at u if and only if $f(x,\phi(x)) = g(x,\phi(x) + u(x)) - g(x,u(x))$ is continuous at $\phi = 0$. From (i'); there is a δ' for any given ε' such that $|t| \leq \delta'$ implies $|g(x,t)| \leq \varepsilon'$. Suppose again $u \in L^r(\Omega)$ and $\|u\|_{L^r(\Omega)} \leq \delta$, with δ free for now, define $\Omega_1 := \{x \in \overline{\Omega} \mid |u(x)| \leq \gamma\}$ then

$$\int_{\Omega_1} |g(x,u(x))|^s\,dx \leq {\varepsilon'}^s \mu(\Omega),$$

where $\mu(\Omega)$ denotes the (Lebesgue) measure of Ω. Choose ε' in such a way that ${\varepsilon'}^s \mu(\Omega) \leq \left(\frac{\varepsilon}{2}\right)^s$. The same argument as above for $\overline{\Omega} \backslash \Omega_1$ gives us

$$\int_{\overline{\Omega} \backslash \Omega_1} |g(x,u(x))|^s\,dx \leq c_3'(\mu(\overline{\Omega} \backslash \Omega_1) + \delta^r).$$

Since

$$\delta^r \geq \int_{\overline{\Omega} \backslash \Omega_1} |u(x)|^r \geq \gamma^r \mu(\overline{\Omega} \backslash \Omega_1),$$

we have $\mu(\overline{\Omega}\backslash\Omega_1) \leq \left(\frac{\delta}{\gamma}\right)^r$, and therefore

$$\int_{\overline{\Omega}\backslash\Omega_1} |g(x,u(x))|^s \, dx \leq c_3'\delta^r \left(1+\frac{1}{\gamma^r}\right).$$

Choose δ small enough to get $c_3'\delta^r\left(1+\frac{1}{\gamma^r}\right) \leq \left(\frac{\varepsilon}{2}\right)^s$ and the proof is complete.

Lemma 2.2 $I[\cdot]$ *is weakly continuous.*

Proof. Let $(u_i) \rightharpoonup u$ in $W_0^{1,2}(\Omega)$, then (u_i) is bounded in $W^{1,2}(\Omega)$ and since $p+1 < \frac{2n}{n-2}$, by the Rellich-Kondrachov theorem, $(u_i) \to u$ in $L^{p+1}(\Omega)$.

$$|I[u_i]-I[u]| \leq \int_\Omega |F(x,u_i)-F(x,u)| \, dx,$$

with the Mean Value theorem and assumption (i) we get

$$|I[u_i]-I[u]| \leq \int_\Omega (c_1|u_i|^p + c_2)(u_i-u)dx \leq \left(\int_\Omega (u_i-u)^{p+1}dx\right)^{\frac{1}{p+1}} \left(\int_\Omega (c_1|u_i|^p+c_2)^{\frac{p+1}{p}} dx\right)^{\frac{p}{p+1}},$$

and thus $I[u_i] \to I[u]$.

Remark. In the proof of the lemma we used the inequality $p+1 < \frac{2n}{n-2}$. This does only make sense for $n > 2$, but since the compact embedding $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ holds true for $n = 1,2$ and all $q \in [p,\infty)$ the argument extends to this cases. We make it a convention to ignore the cases $n \leq 2$ when stating an inequality like the above.

Lemma 2.3 $I[\cdot]$ *is in* $C^1(\Omega\times\mathbb{R},\mathbb{R})$.

Proof. The Proof follows Rabinowitz [31], Proposition B.10. Clearly $I_1[u] = \frac{1}{2}\|u\|^2$ is continuously (Frechét) differentiable with $I_1'[u](\phi) = \int_\Omega \nabla u\cdot\nabla\phi dx$. We have to show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left|\int_\Omega F(x,u+\phi) - \int_\Omega F(x,u) - \int_\Omega f(x,u)\phi\right| < \varepsilon\|\phi\|$$

for all $\phi \in W_0^{1,2}(\Omega)$ with $\|\phi\| \leq \delta$. Define

$$\Omega_1 := \{x\in\Omega \mid |u(x)| \geq \gamma\}, \quad \Omega_2 := \{x\in\Omega \mid |\phi(x)| \geq \kappa\}, \quad \Omega_3 := \{x\in\Omega \mid |u(x)|\leq\gamma,\ |\phi(x)|\leq\kappa\}.$$

Use the Mean Value theorem and assumption (i) to get

$$\int_{\Omega_1} |F(x,u+\phi)-F(x,u)| \, dx \leq \int_{\Omega_1} (c_1 + c_2(|u(x)|+|\phi(x)|)^p)\,|\phi(x)| \, dx$$

$$\leq c_1\mu(\Omega_1)^{\frac{n+2}{2n}}\|\phi\|_{L^{\frac{2n}{n-2}}(\Omega_1)} + c_3\mu(\Omega_1)^{\frac{1}{s}}\left(\|u\|^p_{L^{p+1}(\Omega_1)} + \|\phi\|^p_{L^{p+1}(\Omega_1)}\right)\|\phi\|_{L^{\frac{2n}{n-2}}(\Omega_1)},$$

where $\frac{n-2}{2n}+\frac{p}{p-1}+\frac{1}{s} = 1$ and we used the Hölder inequality. Then, via the Sobolev inequality, we have

$$\int_{\Omega_1} |F(x,u+\phi)-F(x,u)| \, dx \leq c_4\|\phi\|\left(\mu(\Omega_1)^{\frac{n+2}{2n}} + \mu(\Omega_1)^{\frac{1}{s}}(\|u\|^p+\|\phi\|^p)\right).$$

Using the same argument as above:

$$\int_{\Omega_1} |f(x,u)\phi| \, dx \leq c_5\|\phi\|\left(\mu(\Omega_1)^{\frac{n+2}{2n}} + \mu(\Omega_1)^{\frac{1}{s}}\|u\|^p\right).$$

Since $\mu(\Omega_1)^{\frac{n+2}{2n}}, \mu(\Omega_1)^{\frac{1}{s}} \to 0$ as $\gamma\to\infty$ we can choose γ sufficiently large, such that for $\|\phi\|\leq 1$,

$$c_6\left(\mu(\Omega_1)^{\frac{n+2}{2n}} + \mu(\Omega_1)^{\frac{1}{s}}(\|u\|^p+1)\right) \leq \frac{\varepsilon}{3}.$$

In a very similar way:

$$\int_{\Omega_2} |F(x,u+\phi) - F(x,u) - f(x,u)\phi|\, dx \le \left(\int_{\Omega_2} (c_1 + c_2(|u(x)| + |\phi(x)|)^p)^{\frac{p+1}{p}}\ dx \right)^{\frac{p}{p+1}} \|\phi\|_{L^{p+1}(\Omega_2)}$$

$$\le c_3 \left(1 + \|u\|^p + \|\phi\|^p\right) \left(\int_{\Omega_2} |\phi(x)|^{p+1} \left(\frac{|\phi(x)|}{\kappa} \right)^{\frac{2n}{n-2}-p-1} dx \right)^{\frac{1}{p+1}}$$

$$\le c_4 \left(1 + \|u\|^p + \|\phi\|^p\right) \kappa^{\frac{p+1-\frac{2n}{n-2}}{p+1}} \|\phi\|^{\frac{\frac{2n}{n-2}}{p+1}}.$$

Thus, by choosing δ small enough

$$c_4 \left(2 + \|u\|^p\right) \kappa^{\frac{p+1-\frac{2n}{n-2}}{p+1}} \|\phi\|^{\frac{\frac{2n}{n-2}}{p+1}-1} \le \frac{\varepsilon}{3}.$$

Finally note that, since $F \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ given any ε_1, γ_1 there is a $\kappa_1 = \kappa_1(\varepsilon_1, \gamma_1)$ such that

$$|F(x,u+\phi) - F(x,u) - f(x,u)\phi| \le \varepsilon_1 |\phi|,$$

with $x \in \overline{\Omega},\ |u| \le \gamma_1,\ |\phi| \le \kappa_1$. By setting $\gamma_1 = \gamma$ and $\kappa_1 \le \kappa$ we get

$$\int_{\Omega_3} |F(x,u+\phi) - F(x,u) - f(x,u)\phi|\, dx \le \varepsilon_1 \|\phi\|_{L^p(\Omega_3)} \|\chi_{(\Omega_3)}\|_{L^{p'}(\Omega)},$$

where, as usual, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\chi_{(\Omega_3)}$ the characteristic function of Ω_3. Now choose ε_1 such that $\varepsilon_1 \|\chi_{(\Omega_3)}\|_{L^{p'}(\Omega)} \le \frac{\varepsilon}{3}$, and by putting together the estimates over $\Omega_1, \Omega_2, \Omega_3$ the differentiability follows.

Let $(u_i) \to u$ in $W_0^{1,2}(\Omega)$, then

$$\|I'[u_i] - I'[u]\|_{W^{-1,2'}(\Omega)} = \sup_{\|\phi\| \le 1} \left| \int_\Omega f(x,u_i(x)) - f(x,u(x))\phi(x)\, dx \right|$$

$$\le C \|f(x,u_i(x)) - f(x,u(x))\|_{L^{\frac{p+1}{p}}(\Omega)},$$

From assumption (i) we know that

$$|f(x,u(x))| \le c_1 |u(x)|^{\frac{sp}{s}} + c_2,$$

and by taking $s = \frac{p+1}{p}$ the result follows from Lemma 2.1.

We use the following lemma in the verification of the Palais-Smale condition.
Lemma 2.4 *$I_2' : W_0^{1,2}(\Omega) \to W^{-1,2'}(\Omega)$ is compact.*

Proof. Let $\{u_i\}$ be a bounded sequence in $W_0^{1,2}(\Omega)$. We get the compact embedding $W_0^{1,2}(\Omega) \to L^q(\Omega)$ for all $q \in [1, \frac{2n}{n-2}]$, from the Rellich-Kondrachov theorem (see Appendix). Thus $\{u_i\}$ has a Cauchy subsequence, $\{u_{ij}\}$ say, in $L^q(\Omega)$. Note that

$$|I'[u_{ij}]\phi - I'[u_{il}]\phi| = \left| \int_\Omega \left(f(x,u_{ij}) - f(x,u_{il}) \right) \phi\, dx \right|,$$

therefore it follows from the assumptions on f that $I'[u_{ij}]$ is Cauchy in $W^{-1,2'}(\Omega)$.

Proof (of Theorem 2). We prove first that the Palais-Smale condition holds for $I[\cdot]$. Suppose $\{u_i\} \subset W_0^{1,2}(\Omega)$ with $I'[u_i] \to 0$ as $i \to \infty$ and $|I[u_i]| \le C$. Then

$$C \ge I[u_i] \ge \frac{1}{2}\|u_i\|^2_{W_0^{1,2}(\Omega)} - \frac{1}{\beta} \int_\Omega f(x,u_i)u_i dx \ge \left(\frac{1}{2} - \frac{1}{\beta} \right) \|u_i\|^2_{W_0^{1,2}(\Omega)} + \frac{1}{\beta} I'[u_i]u_i,$$

and thus $\{u_i\}$ is bounded. Since $W_0^{1,2}(\Omega)$ is reflexive there exists a weakly convergent subsequence (see for example [39]). $I'[u] = I_1'[u] + I_2'[u]$ with I_1' linear, invertible and I_2' compact, it follows $\{(u_i) + {I_1'}^{-1} I_2'[u_i]\} \to 0$ as $i \to \infty$. $\{u_i\}$ is bounded and since $I_2'[\cdot]$ is compact (Lemma 2.4), $\{{I_1'}^{-1} I_2'[u_i]\}$ is relatively compact, i.e. has a convergent subsequence, then $\{u_i\}$ too has a convergent subsequence.

Now to prove the remaining conditions of the MPT: (A) from (i) we know that for every $\varepsilon > 0$ there is a $\delta > 0$ with

$$|F(x,t)| \leq \frac{\varepsilon |t|^2}{2}$$

for a.e. $x \in \Omega$ and all $t \in B_\delta(0)$. On the other hand (i) and (ii) imply

$$|F(x,t)| \leq C|t|^p$$

for all $t \in \mathbb{R}^n \backslash B_\delta(0)$. Thus

$$|F(x,t)| \leq \frac{\varepsilon |t|^2}{2} + C|t|^p$$

for every $t \in \mathbb{R}^n$ and a.e. $x \in \Omega$.

$$\left| \int_\Omega F(x,u)dx \right| \leq \frac{\varepsilon}{2} \int_\Omega |u|^2 dx + C \int_\Omega |u|^p dx.$$

Now use the Sobolev inequality to get

$$\left| \int_\Omega F(x,u)dx \right| \leq \frac{\varepsilon}{2} \|u\|^2_{W_0^{1,2}(\Omega)} + C\|u\|^p_{W_0^{1,2}(\Omega)}.$$

Then, for $\|u\|_{W_0^{1,2}(\Omega)}$ small enough,

$$I[u] \geq \frac{1}{2}\|u\|_{W_0^{1,2}(\Omega)} - C\varepsilon \|u\|_{W_0^{1,2}(\Omega)},$$

and therefore $I[u] \geq c > 0$ for some c. To prove that (B) holds, note that we have $\frac{\beta}{t} \leq \frac{f(x,t)}{F(x,t)}$ by condition (ii), integrating from s_0 to s, with $s_0 \leq s$, gives

$$F(x,s) \geq F(x,s_0) s^\beta {s_0}^\beta.$$

Thus

$$I[ut] \leq \frac{t^2}{2} \int_\Omega |\nabla u|^2 dx - Ct^\beta \int_\Omega u^\beta dx,$$

therefore $I[tu] \to -\infty$ as $t \to \infty$. The Mountain Pass theorem now tells us that (1) has a nontrivial weak solution.

Some remarks on the case $n = 2$ and critical growth:
For $n = 2$ we assumed that the nonlinearity has at most arbitrary polynomial growth. We can weaken this assumption. In the case $n = p$ the Rellich-Kondrachov theorem gives us, for $\Omega \subset \mathbb{R}^n$, the embedding

$$W^{1,p}(\Omega) \subset L^q(\Omega) \quad \forall q \in [p, +\infty).$$

This fails for $q = +\infty$. To see this one has only to consider the following example given by Bernhard Ruf, $u(x) = \log(1 - \log(|x|))$ in $\Omega = B_1(0)$. u is in $W^{1,2}(\Omega)$ since

$$\int_\Omega |\nabla u|^2 dx = 2\pi \int_0^1 r \left| \frac{d}{dr} \log(1 - \log r) \right|^2 dr = 2\pi \int_0^1 r \left| \frac{1}{1 - \log r} \frac{-1}{r} \right|^2 dr$$

$$= 2\pi \int_0^1 \frac{1}{(1 - \log r)^2} \frac{1}{r} dr < \infty,$$

but clearly $\log(1 - \log(|x|)) \notin L^\infty(\Omega)$.
An equivalent way of stating the Sobolev embedding is

$$\sup_{u \in W^{1,2}(\Omega),\ \|u\|_{W^{1,2}(\Omega)} \leq 1} \int_\Omega |u|^p dx < +\infty \quad \text{for } 1 \leq p \leq 2^*,$$

and the supremum is infinite for $p > 2^*$.
We state our question using this new notion: what is the maximal growth of $f(u) : \mathbb{R} \to \mathbb{R}_+$ such that

$$\sup_{u \in W_0^{1,2}(\Omega),\ \|u\|_{W_0^{1,2}(\Omega)} \leq 1} \int_\Omega f(u)\, dx < +\infty\ ?$$

The answer is given by the Trudinger-Moser inequality ([26], [36]): For a bounded domain $\Omega \subset \mathbb{R}^n$

$$\sup_{u \in W_0^{1,n}(\Omega),\ \|u\|_{W_0^{1,2}(\Omega)} \leq 1} \int_\Omega \left(e^{\alpha |u|^{\frac{n}{n-1}}} - 1 \right) dx \leq +\infty,$$

for all $\alpha \leq \alpha_n = n\,(\omega_{n-1})^{\frac{1}{n-1}}$, where ω_{n-1} denotes the surface area of the n-sphere and the supremum can be attained. For $\alpha > \alpha_n$ the supremum is infinite.
Thus $f(u)$ can actually behave like $e^{4\pi |u|^2}$, when $|u| \to \infty$ (see also [25]).

3 A nonlinear p-Laplacian problem

Partial Differential Equations involving the p-Laplacian operator are of importance in fluid-mechanics especially non-Newtonian fluid flows ([3]) and fluid flows through porous media ([2]. The operator appears also in equations describing singular solutions to the Einstein-Yang-Mills equations ([5]). We will consider the elliptic problem

$$\begin{cases} -div(a(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = g(x)u^\alpha, \\ x \in \Omega \subset \mathbb{R}^n,\ u|_{\partial\Omega} = 0,\ \lim_{|x|\to\infty} u = 0,\ p-1 < \alpha < p^* - 1 \end{cases} \tag{3}$$

with $1 < p < n$ and Ω an exterior domain (i.e. the interior of the complement of a bounded domain with $C^{1,\delta}$ boundary. $p^* = \frac{np}{n-p}$ denotes the Sobolev critical exponent.

Various studies about problems similar to this one have appeared. We mention [12],[40],[18],[20] and [6]. All of this works focus on unbounded domains. This fact poses a problem mostly because we lose the compact Sobolev embedding.

For the coefficients a and b we will assume $0 < a_0 \leq a(x) \in L^\infty \cap C(\overline{\Omega})$ and $0 < b_0 \leq b(x) \in L^\infty \cap C(\overline{\Omega})$. We take E to be the completion of C_0^∞ under the norm $\|u\| := (\int_\Omega a(x)|\nabla u|^p + b(x)|u|^p dx)^{\frac{1}{p}}$. Set $f(x,u) := g(x)u^\alpha$, we then assume:

(i) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$,

(ii) $0 \leq g(x)$ (not identically equal to zero), $g(x) \in L^\infty \cap L^{p_0}(\Omega)$ with $p_0 := \frac{np}{np-(\alpha+1)(n-p)}$,

(iii) there is a $\beta > p$ such that for $F(x,u) := \int_0^u f(x,t)dx$, $\quad \beta F(x,u) \leq uf(x,u)$ for $(x,u) \in \Omega \times \mathbb{R}^+$.

Our main Result will be:
Theorem 3. *Under the assumptions (i), (ii), (iii) (3) has a positive decaying solution $u \in C^{1,\delta}(\overline{\Omega} \cap B_r(0))$ for every $r > 0$, and $\delta \in (0,1)$.*

The energy functional of (3) is

$$I[u] = \frac{1}{p}\|u\|^p - \int_\Omega F(x,u)dx =: I_1[u] - I_2[u].$$

. We want to apply the MPT to obtain a weak solution as critical point of $I[u]$, i.e. as $u \in E$ with

$$I'[u](\phi) = \int_\Omega (a(x)|\nabla u|^{p-2}\nabla u \nabla\phi) + b(x)|u|^{p-2}u\phi - f(x,u)\phi dx = 0$$

for all $\phi \in E$. Thus we start by proving the assumptions of the MPT. The biggest difficulty of handling the unbounded domain, is the loss of compact Sobolev embedding. In the next proofs we will often treat the cases $\Omega_k = \{x \in \Omega \mid |x| \le k \in \mathbb{N}\}$ and $\Omega \backslash \Omega_k$ separately. The work over Ω_k will be similar to that already done in section 2. In the proof of the following lemma we make use of [31], Prop. B.10; [40], Lemma 1; and [38], Thm. 2.10.

Lemma 3.1 $I_2[\cdot]$ *is weakly continuous and continuously differentiable on E with* $I_2'[u](\phi) = \int_\Omega f(x,u)\phi dx$ *for* $\forall \phi \in E$.

Proof. Assume $u_i \to u$ weakly in E. Note that

$$|I_2[u_i] - I_2[u]| \le \int_{\Omega_k} |F(x,u_i) - F(x,u)| + \int_{\Omega\backslash\Omega_k} g(|u_i|^{\alpha+1} + |u|^{\alpha+1}).$$

Since $\frac{1}{p_0} + \frac{\alpha+1}{p^*} = 1$, we can use the Hölder inequality to yield

$$|I_2[u_i] - I_2[u]| \le \int_{\Omega_k} |F(x,u_i) - F(x,u)| + \left(\int_{\Omega\backslash\Omega_k} g^{p_0}\right)^{\frac{1}{p_0}} \left\{ \left(\int_{\Omega\backslash\Omega_k} |u_i|^{p^*}\right)^{\frac{\alpha+1}{p^*}} + \left(\int_{\Omega\backslash\Omega_k} |u|^{p^*}\right)^{\frac{\alpha+1}{p^*}} \right\}.$$

By applying the Sobolev inequality we have

$$|I_2[u_i] - I_2[u]| \le \int_{\Omega_k} |F(x,u_i) - F(x,u)| + C\|g\|_{L^{p_0}(\Omega\backslash\Omega_k)} \left\{ \|u_i\|^{\alpha+1} + \|u\|^{\alpha+1} \right\}.$$

Because $\{u_i\}$ is bounded in E, $\{u_i \mid_{\Omega_k}\}$ is bounded in $W^{1,p}(\Omega_k)$ for all k. The Rellich-Kondrachov theorem gives us the compact embedding $W^{1,p}(\Omega_k) \hookrightarrow L^q(\Omega_k)$ for $1 \le q < p^*$ and thus there is a converging subsequence $\{u_{i_j}\} \to u$ in $L^q(\Omega_k)$. Because of $F(x,t) \le \frac{1}{\alpha+1} g(x)|t|^{\alpha+1}$ from the Vitali convergence theorem follows $\int_{\Omega_k} F(x,u_{i_j}) \to \int_{\Omega_k} F(x,u)$ for all k. Since $\int_\Omega (\chi_{\Omega\backslash\Omega_k} g)^{p_0} \to 0$ as $k \to \infty$ we have $I_2[u_i] \to I_2[u]$ for sufficiently large k.
To prove the differentiability we have to show: for any fixed $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \int_\Omega F(x,u+\phi) - \int_\Omega F(x,u) - \int_\Omega f(x,u)\phi \right| < \varepsilon \|\phi\|$$

for all $\phi \in E$ with $\|\phi\| \le \delta$. Using the Mean-Value-Theorem we get

$$\left| \int_{\Omega\backslash\Omega_k} F(x,u+\phi) - \int_{\Omega\backslash\Omega_k} F(x,u) - \int_{\Omega\backslash\Omega_k} f(x,u)\phi \right| \le \int_{\Omega\backslash\Omega_k} g \left\{ (|u| + |\phi|)^\alpha |\phi| + |u|^\alpha |\phi| \right\}$$

$$\le C \int_{\Omega\backslash\Omega_k} g\{|u|^\alpha |\phi| + |\phi|^{\alpha+1}\},$$

since $(|u| + |\phi|)^\alpha \le 2^\alpha (|u|^\alpha + |\phi|^\alpha)$ (from $(|x| + |y|)^\alpha \le |x|^\alpha + |y|^\alpha$ for $0 < \alpha \le 1$ and the convexity of $|\cdot|^\alpha$ for $1 \le \alpha$). We use the Hölder inequality

$$\le C \left(\int_{\Omega\backslash\Omega_k} g^{p_0}\right)^{\frac{1}{p_0}} \left\{ \left(\int_{\Omega\backslash\Omega_k} |u|^{p^*}\right)^{\frac{\alpha}{p^*}} \left(\int_{\Omega\backslash\Omega_k} |\phi|^{p^*}\right)^{\frac{1}{p^*}} + \left(\int_{\Omega\backslash\Omega_k} |\phi|^{p^*}\right)^{\frac{\alpha+1}{p^*}} \right\}$$

and then (by the Sobolev inequality)

$$\le C\|g\|_{L^{p_0}(\Omega\backslash\Omega_k)} \left\{ \|u\|^\alpha + \|\phi\|^\alpha \right\} \|\phi\|.$$

Since $\int_\Omega (\chi_{\Omega\backslash\Omega_k} g)^{p_0} \to 0$ as $k \to \infty$ we have

$$C\|g\|_{L^{p_0}(\Omega\backslash\Omega_k)} \left\{ \|u\|^\alpha + \|\phi\|^\alpha \right\} \|\phi\| \le \frac{\varepsilon}{2} \|\phi\|.$$

for sufficiently large k.
To prove the differentiability over Ω_k, we define the following three subsets of Ω_k:

$$\Omega_{k1} := \{x \in \Omega_k \mid |u(x)| \ge \gamma\}, \quad \Omega_{k2} := \{x \in \Omega_k \mid |\phi(x)| \ge \kappa\}, \quad \Omega_{k3} := \{x \in \Omega_k \mid |u(x)| \le \gamma,\ |\phi(x)| \le \kappa\}.$$

Then
$$\left| \int_{\Omega_{k1}} F(x, u+\phi) - \int_{\Omega_{k1}} F(x,u) - \int_{\Omega_{k1}} f(x,u)\phi \right| \le C \int_{\Omega_{k1}} g \left\{ |u|^\alpha |\phi| + |\phi|^{\alpha+1} \right\}$$
$$\le C_1 \int_{\Omega_{k1}} \left\{ |u|^\alpha |\phi| + |\phi|^{\alpha+1} \right\},$$
similar as above. We get
$$C_1 \int_{\Omega_{k1}} \left\{ |u|^\alpha |\phi| + |\phi|^{\alpha+1} \right\} \le C_2 \{ \|u\|^\alpha_{L^{\alpha(p^*)'}(\Omega_{k1})} \|\phi\| + \|\phi\|^{\alpha+1} \},$$
by applying the Hölder and Sobolev inequality (note that $\alpha(p^*)' = \frac{\alpha p^*}{p^*-1} < p^*$) .
$$\int_{\Omega_{k1}} |u|^{\alpha(p^*)'} \le \|u\|^{\alpha(p^*)'}_{L^{\alpha+1}(\Omega_{k1})} \|\chi_{\Omega_{k1}}\|_{L^{\left(\frac{\alpha+1}{\alpha(p^*)'}\right)'}(\Omega_{k1})},$$
again by the Hölder inequality. Since
$$\infty > \int_{\Omega_{k1}} |u|^p \ge \int_{\Omega_{k1}} \gamma^p \ge \gamma^p \mu(\Omega_{k1})$$
, we have $\mu(\Omega_{k1}) \to 0$ as $\gamma \to \infty$, which implies
$$\|u\|^\alpha_{L^{\alpha(p^*)'}(\Omega_{k1})} \to 0$$
as $\gamma \to \infty$. One can use the same argument for $|\phi|^{\alpha+1}$ instead of $|u|^\alpha|\phi|$ and then choose γ sufficiently large, such that
$$\left| \int_{\Omega_{k1}} F(x, u+\phi) - \int_{\Omega_{k1}} F(x,u) - \int_{\Omega_{k1}} f(x,u)\phi \right| \le \frac{\varepsilon}{6}\|\phi\|.$$
For Ω_{k2}:
$$\left| \int_{\Omega_{k2}} F(x, u+\phi) - F(x,u) - f(x,u)\phi \; dx \right| \le C \int_{\Omega_{k2}} \left\{ |u|^\alpha |\phi| + |\phi|^{\alpha+1} \right\} \; dx$$
$$\le C \{ \|u\|^\alpha_{L^{\alpha+1}(\Omega_{k2})} + \|\phi\|^\alpha_{L^{\alpha+1}(\Omega_{k2})} \} \|\phi\|_{L^{\alpha+1}(\Omega_{k2})}.$$
Note that
$$\lim_{\|\phi\|_{W_0^{1,p}(\Omega_{k2})} \to 0} \frac{\|\phi\|_{L^{\alpha+1}(\Omega_{k2})}}{\|\phi\|_{L^{p^*}(\Omega_{k2})}} = 0,$$
since, for every $\theta,\ 0 < \theta < 1$,
$$\int_{\Omega_{k2}} \left(\frac{|\phi|}{\theta \|\phi\|_{L^{p^*}(\Omega_{k2})}} \right)^{\alpha+1} dx \le \int_{\Omega_{k2}} \left(\frac{|\phi|}{\|\phi\|_{L^{p^*}(\Omega_{k2})}} \right)^{p^*} \left(\frac{\|\phi\|_{L^{p^*}(\Omega_{k2})}}{\kappa} \right)^{p^*-\alpha-1} \left(\frac{1}{\theta} \right)^{\alpha+1} dx,$$
and thus, by choosing $\|\phi\|_{L^{p^*}(\Omega_{k2})}$ small enough,
$$\int_{\Omega_{k2}} \left(\frac{|\phi|}{\theta \|\phi\|_{L^{p^*}(\Omega_{k2})}} \right)^{\alpha+1} dx \le \int_{\Omega_{k2}} \left(\frac{|\phi|}{\|\phi\|_{L^{p^*}(\Omega_{k2})}} \right)^{p^*} dx \le 1.$$
Therefore we have
$$\|\phi\|_{L^{\alpha+1}(\Omega_{k2})} \le C_1 \theta_1 \|\phi\|_{W_0^{1,p}(\Omega_{k2})},$$
and can choose θ_1 small enough to yield
$$\left| \int_{\Omega_{k2}} F(x, u+\phi) - \int_{\Omega_{k2}} F(x,u) - \int_{\Omega_{k2}} f(x,u)\phi \right| \le \frac{\varepsilon}{6}\|\phi\|.$$
Since $F(x,u) \in C^1(\overline{\Omega} \times \mathbb{R})$ there is a κ_1 for any given $\gamma_1, \varepsilon_1 > 0$, such that
$$|F(x, u+\phi) - F(x,u) - f(x,u)\phi| \le \varepsilon_1 |\phi|,$$

for all $x \in \overline{\Omega}$ if $|u| \leq \gamma_1$ and $|\phi| \leq \kappa_1$. Taking $\gamma_1 = \gamma$ and $\kappa_1 \leq \kappa$ and integrating gives:

$$\left| \int_{\Omega_{k3}} F(x, u+\phi) - F(x,u) - f(x,u)\phi \, dx \right| \leq \varepsilon_1 \int_{\Omega_{k3}} |\phi| \, dx \leq \varepsilon_1 C \|\phi\|,$$

choosing ε_1 such that $\varepsilon_1 C = \frac{\varepsilon}{6}$, finally gives us our desired result

$$\left| \int_\Omega F(x, u+\phi) - \int_\Omega F(x,u) - \int_\Omega f(x,u)\phi \right| < \varepsilon \|\phi\|$$

for all $\phi \in E$ with $\|\phi\| \leq \delta$.

To see that $I_2'[u] : E \to E^*$ is continuous apply the same procedure as above to

$$|I_2'[u_i]\phi - I_2'[u]\phi| \leq \int_{\Omega_k} |f(x,u_i) - f(x,u)||\phi| + \int_{\Omega \setminus \Omega_k} g \left\{ |u_j|^\alpha + |u|^\alpha \right\} |\phi| \tag{4}$$

for all $\phi \in E$.

Lemma 3.2 *$I_2'[\cdot]$ is a compact map.*

Proof. Let $\{u_i\}$ be a bounded sequence in E. The compact embedding $W^{1,p}(\Omega_k) \hookrightarrow L^q(\Omega_k)$ for $1 \leq q < p^*$ implies that there is a Cauchy subsequence $\{u_{ij}\}$ in $L^q(\Omega_k)$. Plugging u_{ii}, u_{ij} into (4) it follows, in a similar way as above, that $I_2'[u_{ii}]$ is Cauchy.

In the following Lemma we will use a variant of *Moser's iteration technique* to prove the boundedness of the weak solutions. The idea behind this is to obtain an inequality like

$$\|u\|_{L^{\delta\beta}(\Omega)} \leq C \|u\|_{L^\beta(\Omega)},$$

for all $\beta \in [1, \infty]$, $u \in L^p(\Omega)$, $\delta \in (1, \infty)$, and some constant $C > 0$; which we then can iterate: Setting $\beta = p$ we obtain

$$\|u\|_{L^{\delta p}(\Omega)} \leq \|u\|_{L^p(\Omega)},$$

for some $\delta p > p$. In the second step we take $\beta = \delta p$, etc.. This process yields

$$\|u\|_{L^{p\delta^k}(\Omega)} \leq C' \|u\|_{L^p(\Omega)},$$

for some new constant C', and letting $k \to \infty$ we showed the boundedness of u

$$\|u\|_{L^\infty(\Omega)} \leq C'' \|u\|_{L^p(\Omega)}.$$

Lemma 3.3 *If $u \in E$ is a critical point of I, then $u \in L^q(\Omega)$ for all $p \leq q \leq \infty$, and $\lim_{|x| \to \infty} u = 0$.*

Proof. We adapt [18], Thm. 1.2 and [40], Lemma 2. To prove the L^∞- boundedness of u we choose a Moser-iterative approach. We can assume $u \geq 0$ since the same argument works for $u^+ = \max\{u(x), 0\}$ and $u^- = \min\{-u(x), 0\}$. Set $u_L(x) := \min\{u(x), L\}$ for $L \in \mathbb{R}, L \geq 0$, then $(u_L)^i \in E$ for all $i \in \mathbb{R}, i \geq 1$.

$$I'[u_L]((u_L)^i) = \int_\Omega (a(x)|\nabla u_L|^{p-2} \nabla u_L \nabla (u_L)^i) + b(x)|u_L|^{p-2} u_L (u_L)^i dx = \int_\Omega f(x, u_L)(u_L)^i dx = 0$$

Since $g(x)u^\alpha \leq \|g\|_{L^\infty(\Omega)} |u|^\alpha$ we obtain

$$i \int_\Omega (u_L)^{i-1} |\nabla u_L|^p \, dx \leq \frac{\|g\|_{L^\infty(\Omega)}}{\|a\|_{L^\infty(\Omega)}} \int_\Omega (u)^{\alpha+i} \, dx.$$

Now use the identity $(u_L)^{i-1}|\nabla u_L|^p = (\frac{p}{i+p-1})^p |\nabla (u_L)^{\frac{i+p-1}{p}}|^p$ followed by the Sobolev inequality to give

$$\left(\int_\Omega (u_L)^{\frac{n(i+p-1)}{n-p}} \right)^{\frac{n-p}{n}} \leq C \int_\Omega (u)^{\alpha+i}.$$

Set $i = i_0 = 1+\mu$, $\quad \mu = p^* - \alpha - 1$, $\quad q_0 = \frac{n(i_0+p-1)}{n-p}$ and by letting $L \to \infty$ we get $u \in L^{q_0}(\Omega)$. Consider i_0, q_0 as our starting point of an iteration procedure. For the second step set $i_1 = 1 + \mu + \frac{n}{n-2}\mu$ and $q_1 = \frac{n(i_1+p-1)}{n-p}$ and thus $u \in L^{q_1}(\Omega)$. Continuing this process for $i_k = 1 + \mu + \frac{n}{n-2}\mu + ... + (\frac{n}{n-2})^j \mu$, $q_j = \frac{n(i_j+p-1)}{n-p}$ this gives us $u \in L^q(\Omega)$ for $\frac{np}{n-p} \le q < \infty$ from which $u \in L^q(\Omega)$ $p \le q < \infty$ follows by the interpolation inequality . To get the L^∞ estimate of u, set $i = kp+1$ and as above

$$\frac{kp+1}{(k+1)^p} \int_\Omega |\nabla (u_L)^{(k+1)}|^p dx \le \int_\Omega g(x)(u)^{\alpha+kp+1} dx.$$

By applying the Sobolev inequality we have

$$\frac{kp+1}{C^p(k+1)^p} \left(\int_\Omega (u_L)^{(k+1)p^*} dx \right)^{\frac{p}{p^*}} \le \int_\Omega g(x)(u)^{\alpha+kp+1} dx.$$

Now fix a $q > p^*$ and set

$$t = \frac{p^* q p}{\alpha(q-p^*) + q + (p-1)p^*}.$$

Then $\frac{t}{p} > 1$ and $t < p^*$. In the integral write $u^{\alpha+kp+1}$ as $u^{\alpha+1-p+(k+1)p}$ and use the Hölder inequality, since

$$\frac{1}{p_0} + \frac{\alpha+1-p}{q} + \frac{p}{t} = 1.$$

We get

$$\int g(x)u^{\alpha+kp+1}dx = \int g(x)u^{\alpha+1-p}u^{(k+1)p}dx \le \left(\int (g(x))^{p_0} dx \right)^{\frac{1}{p_0}} \left(\int u^q dx \right)^{\frac{\alpha+1-p}{q}} \left(\int u^{(k+1)t} dx \right)^{\frac{p}{t}}.$$

As we have already shown $u \in L^q(\Omega)$ for our fixed q, therefore it exists a constant C_1 independent of L and k such that

$$\left(\int_\Omega (u_L)^{(k+1)p^*} dx \right)^{\frac{p}{p^*}} \le C_1 \frac{(k+1)^p}{kp+1} \left(\int u^{(k+1)t} dx \right)^{\frac{p}{t}}.$$

This is equivalent to

$$\|u_L\|_{L^{(k+1)p^*}(\Omega)} \le C_1{}^{\frac{1}{p(k+1)}} \frac{(k+1)^{\frac{1}{k+1}}}{(kp+1)^{\frac{1}{p(k+1)}}} \|u\|_{L^{(k+1)t}(\Omega)}.$$

Choose $k = k_1 = \frac{p^*}{t} - 1$ to get

$$\|u_L\|_{L^{(k_1+1)p^*}(\Omega)} \le C_1{}^{\frac{1}{p(k_1+1)}} \frac{(k_1+1)^{\frac{1}{k_1+1}}}{(k_1p+1)^{\frac{1}{p(k_1+1)}}} \|u\|_{L^{p^*}(\Omega)}.$$

Use Fatou's Lemma and the fact that $\lim_{L\to\infty} u_L(x) = u(x)$ to yield

$$\|u\|_{L^{(k_1+1)p^*}(\Omega)} \le C_1{}^{\frac{1}{p(k_1+1)}} \frac{(k_1+1)^{\frac{1}{k_1+1}}}{(k_1p+1)^{\frac{1}{p(k_1+1)}}} \|u\|_{L^{p^*}(\Omega)}.$$

Now choose $k_n = (\frac{p^*}{t})^n - 1$ for our iteration process. Then

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \le C_1{}^{\frac{1}{p(k_n+1)}} \frac{(k_n+1)^{\frac{1}{k_n+1}}}{(k_np+1)^{\frac{1}{p(k_n+1)}}} \|u\|_{L^{(k_{n-1}p^*)}(\Omega)}$$

for all $n \in \mathbb{N}$. It follows

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \le C_1{}^{\frac{1}{p}\sum_{i=1}^n \frac{1}{k_i+1}} \prod_{i=1}^n \left\{ \frac{(k_i+1)^{\frac{1}{k_i+1}}}{(k_ip+1)^{\frac{1}{p(k_i+1)}}} \right\} \|u\|_{L^{p^*}(\Omega)}.$$

Since

$$\left\{ \frac{(k+1)}{(kp+1)^{\frac{1}{p}}} \right\}^{\frac{1}{\sqrt{k+1}}} > 1,$$

for all $k > 0$, and

$$\lim_{k\to\infty} \left\{ \frac{(k+1)}{(kp+1)^{\frac{1}{p}}} \right\}^{\frac{1}{\sqrt{k+1}}} = 1,$$

there is a constant $C_2 > 0$ independent of $n \in \mathbb{N}$ with

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \le C_1^{\frac{1}{p}\sum_{i=1}^n \frac{1}{k_i+1}} C_2^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} \|u\|_{L^{p^*}(\Omega)}.$$

Note that $\frac{1}{k_i+1} = \left(\frac{t}{p^*}\right)^i$, $\frac{1}{\sqrt{k_i+1}} = \left(\sqrt{\frac{t}{p^*}}\right)^i$ and $\frac{t}{p^*} < \sqrt{\frac{t}{p^*}} < 1$, therefore

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \le C_3\|u\|_{L^{p^*}(\Omega)}$$

for all $n \in \mathbb{N}$ and some constant $C_3 > 0$ independent of n. Letting n tend to infinity we get $u \in L^\infty(\Omega)$. Thus we have proved the first part of the theorem. The second part follows by Theorem 1 of Serrin's [33], which gives us

$$\|u\|_{L^\infty(B_1(x))} \le C_4\|u\|_{L^{p^*}(B_2(x))}$$

for $B_2(x) \subseteq \Omega$, with C_4 independent of x and $C_4 = C_4(n,p)$ from which implies the decay of u.

Proof (of Theorem 3). First we will show that the assumptions for the Mountain Pass theorem are met. For (A):

$$I[u] \ge \frac{1}{p}\int_\Omega a_0|\nabla u|^p + b_0|u|^p - \frac{1}{\alpha+1}\int_\Omega g|u|^{\alpha+1} \ge \frac{1}{p}\int_\Omega a_0|\nabla u|^p + b_0|u|^p - C\|u\|^{\alpha+1}$$
$$= \frac{1}{p}\|u\|^p - C\|u\|^{\alpha+1}$$

since $g(x) \in L^{p_0}(\Omega)$. Therefore, and since $\alpha+1 > p$, for $\|u\|$ small enough, $\|u\| = \rho$ say, there is a $c > 0$ with $I[u] \ge c$ for all $\|u\| = \rho$.

Now to show (B):

$$I[t^{\frac{1}{p}}u] \le \frac{t}{p}\int_\Omega a_0|\nabla u|^p + b_0|u|^p\,dx - t^{\frac{\alpha+1}{p}}C\|u\|^{\alpha+1},$$

thus $I[t^{\frac{1}{p}}u] \to -\infty$ as $t \to +\infty$.

Finally for the Palais-Smale condition: Suppose $\{u_i\} \subset E$ with $\{I[u_i]\} \le C$ and $\{I'[u_i]\} \to 0$ in E^*. Then

$$I[u_i] \ge \frac{1}{p}\int_\Omega a(x)|\nabla u_i|^p + b(x)|u_i|^p dx - \frac{1}{\beta}\int_\Omega f(x,u_i)u_i dx$$
$$= \left(\frac{1}{p} - \frac{1}{\beta}\right)\int_\Omega a(x)|\nabla u_i|^p + b(x)|u_i|^p dx + \frac{1}{\beta}\left(\int_\Omega a(x)|\nabla u_i|^p + b(x)|u_i|^p dx - \int_\Omega f(x,u_i)u_i dx\right)$$
$$= \left(\frac{1}{p} - \frac{1}{\beta}\right)\|u_i\|^p + \frac{1}{\beta}I'[u_i](u_i),$$

thus $\{u_i\}$ is bounded. Since I_2' is compact, there is a subsequence of $\{u_i\}$, $\{u_{i_j}\}$ say, such that $\{I_2'[u_{i_j}]\}$ is a Cauchy sequence in E^*. To prove that $\{u_{i_j}\}$ is Cauchy in E we will need the following inequality, which can be found in [19] or [23] .

$$|x-y|^p \le \begin{cases} (|x|^{p-2}x - |y|^{p-2}y)\cdot(x-y) & \text{if } p \ge 2, \\ \left(|x|^{p-2}x - |y|^{p-2}y\cdot(x-y)\right)^{\frac{p}{2}}\left(|x|^p + |y|^p\right)^{\frac{2-p}{p}} & \text{if } 1 < p < 2 \end{cases}$$

for $x, y \in \mathbb{R}^n$.

$$\int_\Omega a(x)\left(|\nabla u_{i_j}|^{p-2}\nabla u_{i_j} - |\nabla u_{i_l}|^{p-2}\nabla u_{i_l}\right)\cdot(\nabla u_{i_j} - \nabla u_{i_l}) + b(x)\left(|u_{i_j}|^{p-2}u_{i_j} - |u_{i_l}|^{p-2}u_{i_l}\right)\cdot(u_{i_j} - u_{i_l})dx$$
$$\le |I'[u_{i_j}](u_{i_j} - u_{i_l})| + |I'[u_{i_l}](u_{i_l} - u_{i_j})| + \left|\int_\Omega (f(x,u_{i_j}) - f(x,u_{i_l}))(u_{i_j} - u_{i_l})dx\right|$$

$$\leq C\left(\|I'[u_{ij}]\|_{E^*} + \|I'[u_{il}]\|_{E^*} + \|I_2'[u_{ij}] - I_2'[u_{il}]\|_{E^*}\right)$$

with $C = C(n,p)$. Thus $\{u_{ij}\}$ is Cauchy in E , the Palais-Smale condition holds. The conditions for the Mountain Pass theorem are met, therefore we get the existence of a nontrivial weak solution. From Lemma 3.3, we know u decays, and setting $\phi = u^-$ in $I'[u](\phi)$ one sees $u \geq 0$ in Ω. Using the strong or weak Harnack inequality from [35], it follows that u is positive. Since the structural assumptions for the regularity results [22], (or [34]) hold, we get $u \in C^{1,\delta}(\overline{\Omega} \cap B_R(0))$, $\forall R > 0$.

On the importance of the critical exponent: In what follows we briefly discuss the necessity of the critical exponent condition, i.e. $p - 1 < \alpha < p^* - 1$.
Assuming $\alpha < p-1$ instead we can establish nonnegativity, regularity and decay of the, at this point hypothetical, solution in the same way as above. To prove existence we use a local minimization argument (note that since $I[\cdot]$ is weakly continuous the minimum is attained): Since

$$I[u] \geq \frac{1}{p}\|u\|^p - C\|u\|^{\alpha+1},$$

I is bounded below and thus has a critical point u with $I[u] = \inf\{I[v] \mid v \in E\}$. It remains to show that u is nontrivial, but since

$$I[t\phi] = \frac{t^p}{p}\|\phi\|^p - \frac{t^{\alpha+1}}{\alpha+1}\int_\Omega g(x)|\phi|^{\alpha+1}dx < 0,$$

for $t > 0$ small and some $\phi \in C_0^\infty$, this is clear.
We show that the restriction $\alpha < p^* - 1$ can not simply be removed. The proof of Theorem 3 works in the same way when we consider the problem over $\mathbb{R}^n$ instead of Ω, the only change being the regularity result we use ($u \in C^{1,\delta}(B_r(0))$, $\forall r > 0$) which can be found in [34]. Consider the special case

$$-div(|\nabla u|^{p-2}\nabla u) + b_0|u|^{p-2}u = |u|^{\alpha-2}u,$$

with $\alpha = p^*$. This equation has no solution. We use the following Pohozaev type identity (see [41]):

$$\int_{\mathbb{R}^n}|\nabla u|^p dx = \int_{\mathbb{R}^n}|u|^{p^*}dx - \frac{b_0 p^*}{p}\int_{\mathbb{R}^n}|u|^p dx.$$

If u is a solution then by definition

$$\int_{\mathbb{R}^n}|\nabla u|^p dx = \int_{\mathbb{R}^n}|u|^{p^*}dx - b_0\int_{\mathbb{R}^n}|u|^p dx.$$

Putting this two identities together one sees that u must be the trivial solution. If however a subcritical perturbation of $|u|^{p^*}u$, i.e. $f(x,u)$ continuous with $\lim_{u\to\infty}\frac{f(x,u)}{|u|^{p^*}u} = 0$, is added to the right side of aboves equation a nontrivial solution exists ([42]).

The limit case $n = p$: In analogy to the semilinear problem above the Trudinger inequality becomes of crucial importance when $n = p$. Consider the problem

$$\begin{cases} -div(|\nabla u|^{n-2}\nabla u) + b(x)|u|^{n-2}u = f(x,u),\ x \in \mathbb{R}^n, \\ \lim_{|x|\to\infty} u = 0. \end{cases} \tag{5}$$

Let the space E be the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm $\|u\|^n = \int_{\mathbb{R}^n}|\nabla u|^n + b(x)|u|^n dx$. We define the functional $I : E \to \mathbb{R}$

$$I[u] = \frac{1}{n}\|u\| - \int_{\mathbb{R}^n} F(x,u)dx.$$

The fact that $I[\cdot]$ is well defined (i.e. that $F(x,u) \in L^1(\mathbb{R}^n)$) will become true when we impose our assumptions on the problem. We say $f(x,u)$ has subcritical growth if

$$\lim_{u\to\infty}\frac{f(x,u)}{e^{|u|^\mu}} = 0 \text{ for some } 0 < \mu < \frac{n}{n-1}, \quad \text{uniformly on } \mathbb{R}^n.$$

If this condition is satisfied proving the existence of a weak solution is not that different than for $p < n$ (see for example [15]). We give a regularity result for the subcritical case and the existence of weak solutions for the critical case. For both proofs the following lemma will be the main ingredient:

Lemma 3.4. *If $n \geq 2$, $0 < \gamma \leq \frac{n}{n-1}$, $\alpha > 0$ (if $\gamma = \frac{n}{n-1}$, also assume $\alpha < n(\omega_{n-1})^{\frac{1}{n-1}}$), then*

$$\int_{\mathbb{R}^n} \exp(\alpha|u|^\gamma) - \Phi_{n-2}(\alpha, u)\, dx < \infty,$$

for all $u \in W^{1,n}(\mathbb{R}^n)$,

$$\Phi_{n-2}(\alpha, u) := \sum_{k=0}^{n-2} \frac{\alpha^k}{k!}|u|^{\gamma k}.$$

If $\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq K$, $\|u\|_{L^p(\mathbb{R}^n)} \leq M$, with $K, M < \infty$, then there is a constant $C(K, M, n, \alpha, \gamma)$ depending on K, M, n, α, γ alone, such that

$$\int_{\mathbb{R}^n} \exp(\alpha|u|^\gamma) - \Phi_{n-2}(\alpha, u)\, dx \leq C(K, M, n, \alpha, \gamma).$$

The Lemma is motivated by generalizations of the Trudinger inequality ([21]), and similar results can be found in [29],[13],[14]. The proof uses the important technique of Schwarz symmetrization, which was already used by Trudinger in the proof of the orginal inequality.

Proof. We first state some properties of the Schwarz symmetrization. For nonnegative $u \in L^n(\mathbb{R}^n)$ there is a unique radial function $u^* \in L^n(\mathbb{R}^n)$,

$$\lambda(\{x \mid u^*(x) \geq c\}) = \lambda(\{x \mid u(x) \geq c\}),$$

for all $c > 0$, u^* decreasing in $|x|$, and $\exists R_c$ such that $\{x \mid u^*(x) \geq c\} = B_{R_c}(0)$. If $u \in W_0^{1,n}(\mathbb{R}^n)$ then $u^* \in W_0^{1,n}(\mathbb{R}^n)$ and also $\|\nabla u^*\|_{L^n(\mathbb{R}^n)} \leq \|\nabla u\|_{L^n(\mathbb{R}^n)}$. For every $f : [0, \infty) \to [0, \infty)$, with $f(0) = 0$ we have

$$\int_{\mathbb{R}^n} f(u^*(x))\, dx = \int_{\mathbb{R}^n} f(u(x))\, dx.$$

We can concentrate on the critical case, i.e. $\gamma = \frac{n}{n-1}$ and $\alpha < n(\omega_{n-1})^{\frac{1}{n-1}}$ and further assume $u \geq 0$. From what is stated above one sees that

$$\int_{\mathbb{R}^n} \exp\left(\alpha|u|^{\frac{n}{n-1}}\right) - \Phi_{n-2}(\alpha, u)\, dx = \int_{\mathbb{R}^n} \exp\left(\alpha|u^*|^{\frac{n}{n-1}}\right) - \Phi_{n-2}(\alpha, u^*)\, dx$$

$$\leq \int_{|x|<r} \exp\left(\alpha|u^*|^{\frac{n}{n-1}}\right)\, dx + \int_{|x|\geq r} \exp\left(\alpha|u^*|^{\frac{n}{n-1}}\right) - \Phi_{n-2}(\alpha, u^*)\, dx.$$

Like in [29] we use the following inequalities for $a, b \geq 0$

$$(a+b)^{\frac{n}{n-1}} \leq a^{\frac{n}{n-1}} + b^{\frac{n}{n-1}} + C\{n\}\, a^{\frac{1}{n-1}} b,$$

some positive constant C depending on n, and

$$a^q b^{q'} \leq \varepsilon a + \varepsilon b^{-\frac{q}{q'}},$$

for $\frac{1}{q'} + \frac{1}{q} = 1$ and all $\varepsilon > 0$. By using this and the fact that $u^* - u^*(rx_0) \in W^{1,n}(B_r(0))$, where x_0 is a fixed vector of unit length, we obtain

$$|u^*|^{\frac{n}{n-1}} \leq (1+\varepsilon)(u^*(x) - u^*(rx_0))^{\frac{n}{n-1}} + \left(C\{n\}^{\frac{n}{n-1}}\varepsilon^{\frac{1}{1-n}} + 1\right) u^*(rx_0)^{\frac{n}{n-1}},$$

and thus

$$\int_{|x|<r} \exp\left(\alpha|u^*|^{\frac{n}{n-1}}\right)\, dx$$
$$\leq \exp\left(\left(C\{n\}^{\frac{n}{n-1}}\varepsilon^{\frac{1}{1-n}} + 1\right) u^*(rx_0)^{\frac{n}{n-1}}\right) \int_{|x|<r} \exp\left(\alpha(1+\varepsilon)|u^*(x) - u^*(rx_0)|^{\frac{n}{n-1}}\right)\, dx.$$

Letting $(1+\varepsilon)\alpha < n(\omega_{n-1})^{\frac{1}{n-1}}$, then by the Trudinger inequality

$$\int_{|x|<r} \exp\left(\alpha|u^*|^{\frac{n}{n-1}}\right)\, dx \leq C'\{n\}\frac{\omega_{n-1}r^n}{n} \exp\left(\left(C\{n\}^{\frac{n}{n-1}}\varepsilon^{\frac{1}{1-n}} + 1\right) u^*(rx_0)^{\frac{n}{n-1}}\right).$$

We use a generalization to Sobolev spaces of Strauss' *Radial Lemma*, due to Lions ([24]):

$$|u^*(x)| \le |x|^{-\frac{n-1}{n}} C''\{n\} \|u^*\|_{L^n(\mathbb{R}^n)}^{\frac{n}{n-1}} \|\nabla u^*\|_{L^n(\mathbb{R}^n)}^{\frac{1}{n}} \quad \text{a.e..}$$

Since $\|\nabla u^*\|_{L^n(\mathbb{R}^n)} \le \|\nabla u\|_{L^n(\mathbb{R}^n)}$ and $\|u^*\|_{L^n(\mathbb{R}^n)} = \|u\|_{L^n(\mathbb{R}^n)}$ (see e.g. [17], § 6.3) we get

$$\begin{aligned}&\int_{|x|<r} \exp\left(\alpha |u^*|^{\frac{n}{n-1}}\right)\, dx \\ &\quad \le C'\{n\} \frac{\omega_{n-1} r^n}{n} \exp\left(\left(\left(C\{n\}^{\frac{n}{n-1}} \varepsilon^{\frac{1}{1-n}} + 1\right) |x_0 r|^{-1} C'''\{n\} \|u(rx_0)\|_{L^n(\mathbb{R}^n)}^{\left(\frac{n}{n-1}\right)^2} \|\nabla u(rx_0)\|_{L^n(\mathbb{R}^n)}^{\frac{1}{n-1}}\right)\right).\end{aligned}$$

Finally, using the assumptions on $\|\nabla u\|_{L^n(\mathbb{R}^n)}$ and $\|u\|_{L^n(\mathbb{R}^n)}$, this yields

$$\int_{|x|<r} \exp\left(\alpha |u^*|^{\frac{n}{n-1}}\right)\, dx \le A\{n,\varepsilon\} \frac{\omega_{n-1} r^n}{n} \exp\left(\frac{M^{\left(\frac{n}{n-1}\right)^2} K^{\frac{1}{n-1}}}{r}\right),$$

for $A\{n,\varepsilon\}$ some constant.
To handle the integral over $|x| \ge r$ note that

$$\begin{aligned}&\int_{|x|\ge r} \exp\left(\alpha |u^*|^{\frac{n}{n-1}}\right) - \Phi_{n-2}(\alpha, u^*)\, dx \\ &\qquad = \frac{\alpha^{n-1}}{(n-1)!} \int_{|x|\ge r} |u^*|^n\, dx + \frac{\alpha^n}{n!} \int_{|x|\ge r} |u^*|^{\frac{n^2}{n-1}}\, dx + \sum_{k=n+1}^{\infty} \frac{\alpha^k}{k!} \int_{|x|\ge r} |u^*|^{k\frac{n}{n-1}}\, dx,\end{aligned} \tag{6}$$

and since for all $k \ge n+1$

$$\int_{|x|\ge r} \frac{1}{|x|^k}\, dx = \omega_{n-1} \int_r^\infty \frac{1}{t^{k-n-1}}\, dx = \frac{\omega_{n-1}}{r^{k-n}},$$

we can use the Radial Lemma again, to yield

$$\sum_{k=n+1}^{\infty} \frac{\alpha^k}{k!} \int_{|x|\ge r} |u^*|^{k\frac{n}{n-1}}\, dx \le \omega_{n-1} \sum_{k=n+1}^{\infty} \frac{\alpha^k}{k!} \left(\frac{M^{k\left(\frac{n}{n-1}\right)^2} K^{k\frac{1}{n-1}}}{r^{k-n}}\right).$$

From $\frac{n^2}{n-1} > n$ and the embedding $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^s(\mathbb{R}^n)$ for all $s \in [n, +\infty)$, the result follows.

Theorem 3.5 ([13]). *If $u \in W^{1,n}(\mathbb{R}^n)$ is a weak solution to (5), and $f(x,u) \in C(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, $b(x) \in C(\mathbb{R}^n, \mathbb{R})$ with $b(x) \ge b > 0$ and*

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{n-1}} = 0 \quad \textit{uniformly in } \mathbb{R}^n,$$

then $u \in L^\infty(\mathbb{R}^n)$ and there is $q > n$ and $R_q > 0$, $C > 0$ such that for all $R \ge R_q$

$$\|u\|_{L^\infty(|x|\ge R)} \le C \|u\|_{L^q(|x|\ge R/2)} < \infty.$$

Note that from Theorem 3.5 follows, by Tolksdorf's regularity result mentioned above that $\lim_{|x|\to\infty} u = 0$ and $u \in C^{1,\beta}_{loc}(\mathbb{R}^n)$ for some $0 < \beta < 1$.
Proof. By the assumptions on $f(x,u)$ we have

$$|f(x,u)| \le \delta |u|^{n-1} + C_\delta \sum_{l=l_0}^{\infty} \frac{|u|^{\mu l + n - 1}}{l!},$$

for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}$. Let u be a weak solution, then set once more $u^+ = \max\{u(x), 0\}$ and $u_M(x) = \min\{u(x), M\}$ for $M \in \mathbb{R}, M > 0$. Choose $\eta \in C^\infty(\mathbb{R}^n)$, such that for R,r with $0 < r \le \frac{R}{2}$,

$$\eta = \begin{cases} 1 & \text{if } |x| \ge R, \\ 0 & \text{if } |x| \ge R - r, \end{cases}$$

and also $|\nabla\eta| \le \frac{2}{r}$, $0 \le \eta \le 1$. Use $u^+u_M^{+n(i-1)} \in W^{1,n}(\mathbb{R}^n)$ for $i \ge 1$ as a test function and then since u is a weak solution

$$\int_{\mathbb{R}^n} |\nabla u^+|^n u_M^{+n(i-1)} dx + \ n(i-1)\int_{\mathbb{R}^n} |\nabla u_M^+|^n u_M^{+n(i-1)} dx + \ b\int_{\mathbb{R}^n} u^{+n} u_M^{+n(i-1)} dx$$
$$\le C\sum_{l=l_0}^{\infty} \frac{1}{l!}\int_{\mathbb{R}^n} u^{+\mu l+n-1} u_M^{+n(i-1)} u^+ dx.$$

Setting $W = u^+u_M^{+n(i-1)}$ this implies

$$\int_{\mathbb{R}^n} |\nabla W|^n \ dx \le Ci^n \sum_{l=l_0}^{\infty} \int_{\mathbb{R}^n} u^{+\mu l}|W|^n \ dx,$$

and by the Hölder inequality with $\frac{\mu}{\mu+\varepsilon} + \frac{\varepsilon}{\mu+\varepsilon} = 1$ for an ε such that $\mu + \varepsilon < \frac{n}{n-1}$ we get

$$\int_{\mathbb{R}^n} |\nabla W|^n \ dx \le Ci^n \sum_{l=l_0}^{\infty} \left(\int_{\mathbb{R}^n} u^{+(\mu+\varepsilon)l} \ dx\right)^{\frac{\mu}{\mu+\varepsilon}} \|W^n\|_{L^{\frac{\mu+\varepsilon}{\varepsilon}}(\mathbb{R}^n)}.$$

Therefore Lemma 3.4 yields

$$\|\nabla W\|_{L^n(\mathbb{R}^n)} \le Ci\|W\|_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)}.$$

Now we use Nirenberg's inequality (see appendix)

$$\|\nabla^j u\|_{L^q(\mathbb{R}^n)} \le C\|\nabla^m u\|^{\alpha}_{L^r(\mathbb{R}^n)}\|u\|^{1-\alpha}_{L^p(\mathbb{R}^n)},$$

with $j = 0,\ m = 1,\ p = \frac{n(\mu+\varepsilon}{\varepsilon},\ r = n$ and some $0 < \alpha < 1$ to get the iteration inequality

$$\|W\|_{L^q(\mathbb{R}^n)} \le \|W\|_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)},$$

$q > \frac{n(\mu+\varepsilon)}{\varepsilon}$ and the L^∞ bound $\|u^+\|_{L^\infty} < \infty$, respectively $\|u^-\|_{L^\infty} < \infty$, follows by Moser iteration. For all $\delta > 0$ we have, by plugging in $\eta^n u^+ u_M^{+n(i-1)}$ as a test function,

$$\int_{\mathbb{R}^n} |\nabla u^+|^n \eta^n u_M^{+n(i-1)} \ dx + n(i-1)\int_{\mathbb{R}^n} |\nabla u_M^+|^n u_M^{+n(i-1)}\eta^n dx + \ b\int_{\mathbb{R}^n} u^{+n}u_M^{+n(i-1)}\eta^n dx$$
$$+ \ n\int_{\mathbb{R}^n} |\nabla u^+|^{n-1}|\nabla\eta|\eta^{n-1}u^+u_M^{+n(i-1)} dx \le \int_{\mathbb{R}^n} f(x,u^+)u^+u_M^{+n(i-1)}\eta^n dx.$$

Using Lemma 3.4 again gives

$$\int_{\mathbb{R}^n} f(x,u^+)u^+u_M^{+n(i-1)}\eta^n \ dx \le \delta\int_{\mathbb{R}^n} u^{+n}u_M^{+n(i-1)}\eta^n dx + C_\delta\|\eta u^+ u_M^{+i-1}\|^n_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)}.$$

Choosing δ small enough we get

$$\int_{\mathbb{R}^n} |\nabla u^+|^n\eta^n u_M^{+n(i-1)} \ dx + n(i-1)\int_{\mathbb{R}^n} |\nabla u_M^+|^n u_M^{+n(i-1)} dx + \ C\int_{\mathbb{R}^n} u^{+n}u_M^{+n(i-1)} dx$$
$$\le \ n\int_{\mathbb{R}^n} |\nabla u^+|^{n-1}|\nabla\eta|\eta^{n-1}u^+u_M^{+n(i-1)} dx + b\|\eta u^+u_M^{+i-1}\|^n_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)},$$

$C > 0$ a constant, and using Young's inequality on the first term of the right hand side yields

$$n\int_{\mathbb{R}^n} |\nabla u^+|^{n-1}|\nabla\eta|\eta^{n-1}u^+u_M^{+n(i-1)} dx + b\|\eta u^+u_M^{+i-1}\|^n_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)}$$
$$\le \kappa\int_{\mathbb{R}^n} |\nabla u^+|^n\eta^n u_M^{+n(i-1)} \ dx + C'\int_{\mathbb{R}^n} |\nabla\eta|^n u^{+n}u_M^{+n(i-1)} \ dx + C''\|\eta u^+u_M^{+i-1}\|^n_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)},$$

where C', $C'' > 0$ are constants which may depend on $\kappa > 0$ and κ is arbitrary. From the last two inequalities one gets

$$\int_{\mathbb{R}^n} |\nabla u^+|^n\eta^n u_M^{+n(i-1)} \ dx + n(i-1)\int_{\mathbb{R}^n} |\nabla u_M^+|^n u_M^{+n(i-1)}\eta^n dx + \ C\int_{\mathbb{R}^n} u^{+n}u_M^{+n(i-1)}\eta^n dx$$
$$\le C'''\left\{\int_{\mathbb{R}^n} |\nabla\eta|^n u^{+n}u_M^{+n(i-1)} \ dx + \|\eta u^+u_M^{+i-1}\|^n_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)}\right\}, \qquad (7)$$

by taking κ to be sufficiently small. Writing out $\nabla\left(\eta u^+ u_M^{+i-1}\right)$ explicitly and using (7) we have

$$\int_{\mathbb{R}^n} |\nabla\left(\eta u^+ u_M^{+i-1}\right)|^n \, dx + \|\eta u^+ u_M^{+i-1}\|^n_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)} \leq C i^n \left\{ \int_{\mathbb{R}^n} |\nabla \eta|^n u^{+n} u_M^{+n(i-1)} \, dx + \|\eta u^+ u_M^{+i-1}\|^n_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)} \right\}. \tag{8}$$

Finally we use the properties of the bump function to give

$$(8) \leq C i^n \left(\frac{2^n\left(R^n - (R-r)^n\right)}{r^n}\right) \|u^+ u_M^{+(i-1)}\|^n_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(|x|\geq R-r)}.$$

Now one can use Nirenberg's inequality again and get, for $\frac{n(\mu+\varepsilon)}{\varepsilon} < s$

$$\|\eta u^+ u_M^{+(i-1)}\|_{L^s(\mathbb{R}^n)} \leq C\left(\|\nabla\left(\eta u^+ u_M^{+(i-1)}\right)\|_{L^n(\mathbb{R}^n)} + \|\eta u^+ u_M^{+(i-1)}\|_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(\mathbb{R}^n)}\right) \leq C' i \left(\frac{2^n\left(R^n-(R-r)^n\right)}{r^n}\right)^{1/n} \|u^+ u_M^{+(i-1)}\|_{L^{\frac{n(\mu+\varepsilon)}{\varepsilon}}(|x|\geq R-r)}.$$

Taking the limit $M \to \infty$

$$\|u^+\|_{L^{si}(\mathbb{R}^n)} \leq C'^{1/i} i^{1/i} \left(\frac{2^n R^n}{r^n}\right)^{1/ni} \|u^+\|_{L^{\frac{ni(\mu+\varepsilon)}{\varepsilon}}(|x|\geq R-r)},$$

and setting $\alpha := \frac{s\varepsilon}{n(\mu+\varepsilon)}$, $i = \alpha^m$ and $r_m = \frac{R}{2^{m+1}}$ yields

$$\|u^+\|_{L^{\alpha^m s}(|x|\geq R-r_{m+1})} \leq C'^{1/\alpha^m} i^{1/\alpha^m} 2^{\frac{n(m+2)}{\alpha^m n}} \|u^+\|_{L^{\alpha^{m-1}s}(|x|\geq R)},$$

from which the Theorem follows.

We now come to the critical case. Because of the lack of compactness we can not show the (PS)-condition for $I[\cdot]$. To compensate for this, we instead prove that a sequence $u_i \in W^{1,n}(\mathbb{R}^n)$ with $I[u_i] \to c$ and $I'[u_i] \to 0$ as $i \to \infty$, obtained from the MPT, does converge to a nontrivial solution. This approach, introduced by Willem ([37]), can be considered "a posteriori" compared to the "a priori" compactness condition employed before, and is of great importance in modern applications of the MPT. We demonstrate this technique following [29].

We impose the following assumptions on problem (5):

(a) $f \in C(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ and for $c_1, c_2, \alpha > 0$ constants

$$|f(x,u)| \leq c_1 |u|^{n-1} + c_2 \sum_{i=n-1}^{\infty} \frac{\alpha^i}{i!} |u|^{\frac{ni}{n-1}}$$

(b) For some $\beta > n$ and $\forall u > 0$,

$$\beta F(x,u) \leq u f(x,u),$$

(c) For constants C_1, $C_2 > 0$, $\forall u \geq C_1$,

$$0 < F(x,u) \leq C_2 f(x,u),$$

(d)

$$\lim_{u\to+\infty} u f(x,u) \exp\left(-\alpha |u|^{\frac{n}{n-1}}\right) \geq C_3 > 0,$$

uniformly on all compact sets of $\mathbb{R}^n$,

(e) $b(x) \in C(\mathbb{R}^n, \mathbb{R})$, $b(x) \geq b > 0$, and b is *coercive*, i.e. $b(x) \to \infty$ for $|x| \to \infty$,

(f)

$$\limsup_{u\to 0^+} \frac{nF(x,u)}{|u|^n} < \lambda_1(n) = \inf_{E\ni u\neq 0} \frac{\|u\|^n}{\|u\|_{L^n(\mathbb{R}^n)}}, \quad \text{uniformly in } \mathbb{R}^n.$$

We actually do not have to assume that $b(x) \in L^\infty$, if we instead let E be the subspace of $W^{1,n}$ defined by

$$E = \left\{ u \in W^{1,n}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} b(x)|u|^n \, dx < \infty \right\}.$$

To make up for the lack of compactness we prove the following Lemma, which gives us an upper bound on the *Mountain Pass level* c:

Lemma 3.6. *If assumptions (a) - (e) are satisfied there is a $r > 0$ such that*

$$\max\{I[t\overline{\mathfrak{M}}_k(x,r)] \mid t \geq 0\} < \frac{n^{n-2}\omega_{n-1}}{\alpha^{n-1}},$$

where $\mathfrak{M}_k(x,r)$ denotes the ***Moser sequence*** *introduced in [26]*

$$\mathfrak{M}_k(x,r) = \frac{1}{\omega_{n-1}^{1/n}} \begin{cases} (\log k)^{\frac{n-1}{n}} & \text{if} \quad |x| \leq \frac{r}{k}, \\ \frac{\log\left(\frac{r}{|x|}\right)}{(\log k)^{\frac{1}{n}}} & \text{if } \frac{r}{k} \leq |x| \leq r, \\ 0 & \text{if } |x| \geq r, \end{cases}$$

and $\overline{\mathfrak{M}}_k(x,r) = \frac{\mathfrak{M}_k(x,r)}{\|\mathfrak{M}_k(x,r)\|}$.

Proof. Note that $\mathfrak{M}_k(x,r) \in W^{1,n}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |\nabla \mathfrak{M}_k(x,r)|^n \, dx = 1.$$

Also

$$\overline{\mathfrak{M}}_k(x,r) = \frac{\log n}{\omega_{n-1}^{1/(n-1)}} + a_k, \quad \forall |x| \leq \frac{r}{k},$$

where $a_k \geq 0$ is a bounded sequence. Suppose that

$$\max\{I[t\overline{\mathfrak{M}}_k(x,r)] \mid t \geq 0\} \geq \frac{n^{n-2}\omega_{n-1}}{\alpha^{n-1}}, \quad \forall k$$

where we fixed the r such that $\frac{n^{n-1}}{\alpha^{n-1} r^n} < C_3$. We prove that this leads to a contradiction. The fact that

$$I[tu] \to -\infty \text{ as } t \to \infty \text{ , } \forall u \geq 0 \text{ with compact support and } u \not\equiv 0,$$

is shown exactly as we did above. Then, for given k, we can take $t_k > 0$ so that

$$I[t_k\overline{\mathfrak{M}}_k(x,r)] = \max\{I[t\overline{\mathfrak{M}}_k(x,r)] \mid t \geq 0\}, \tag{9}$$

$$I[t_k\overline{\mathfrak{M}}_k(x,r)] = \frac{t_k^n}{n} - \int_{\mathbb{R}^n} F\left(x, t_k\overline{\mathfrak{M}}_k(x,r)\right) \, dx \geq \frac{n^{n-2}\omega_{n-1}}{\alpha^{n-1}},$$

by our assumption and thus $t_k^n \geq \frac{n^{n-1}\omega_{n-1}}{\alpha^{n-1}}$. From (9) we get

$$t_k^n = \int_{\mathbb{R}^n} f\left(x, t_k\overline{\mathfrak{M}}_k(x,r)\right) t_k\overline{\mathfrak{M}}_k(x,r) \, dx,$$

by taking the derivative $\frac{d}{dt_k}$. (d) gives

$$uf(x,u) \geq (C_3 - \delta) \exp\left(\alpha |u|^{\frac{n}{n-1}}\right),$$

for fixed $\delta > 0$, $\exists C_\delta > 0$, $\forall |u| \geq C_\delta$, and $\forall |x| \leq r$. Integrating this inequality and using the "cut-off" property of the Moser sequence yields

$$t_k^n \geq (C_3 - \delta) \int_{|x| \leq \frac{r}{k}} \exp\left(\alpha |t_k\overline{\mathfrak{M}}_k(x,r)|^{\frac{n}{n-1}}\right) \, dx = \frac{(C_3-\delta)\omega_{n-1} r^n}{nk^n} \exp\left(\frac{\alpha t_k^{n/(n-1)} \log k}{\omega_{n-1}^{1/(n-1)}} + \alpha t_k^{n/(n-1)} a_k\right). \tag{10}$$

Dividing both sides by t_k^n and taking the limit $k \to \infty$ one sees that the sequence t_k must be bounded. We rewrite (10) as

$$t_k^n \geq \frac{(C_3 - \delta)\omega_{n-1} r^n}{n} \exp\left(\left(\frac{\alpha t_k^{n/(n-1)}}{\omega_{n-1}^{1/(n-1)} n} - 1 \right) n \log k + \alpha t_k^{n/(n-1)} a_k \right),$$

and since $t_k^n \geq \frac{n^{n-1}\omega_{n-1}}{\alpha^{n-1}}$

$$t_k^n \to \frac{n^{n-1}\omega_{n-1}}{\alpha^{n-1}}, \quad \text{as } k \to \infty,$$

follows.

Define the sets

$$\Omega_k := \left\{ x \in B_r(0) \mid C_\delta \leq t_k \overline{\mathfrak{M}}_k(x,r) \right\}, \quad \Gamma_k := B_r(0) \backslash \Omega_k.$$

By repeating the steps leading up to (10) and adding integrals over the above sets, we get the improved lower bound

$$\begin{aligned} t_k^n &\geq (C_3 - \delta) \int_{\Omega_k} \exp\left(\alpha |t_k \overline{\mathfrak{M}}_k(x,r)|^{\frac{n}{n-1}} \right) dx + \int_{\Gamma_k} t_k \overline{\mathfrak{M}}_k(x,r) f\left(x, t_k \overline{\mathfrak{M}}_k(x,r)\right) dx \\ &= (C_3 - \delta) \int_{|x| \leq r} \exp\left(\alpha |t_k \overline{\mathfrak{M}}_k(x,r)|^{\frac{n}{n-1}} \right) dx - (C_3 - \delta) \int_{\Gamma_k} \exp\left(\alpha |t_k \overline{\mathfrak{M}}_k(x,r)|^{\frac{n}{n-1}} \right) dx \\ &\qquad + \int_{\Gamma_k} t_k \overline{\mathfrak{M}}_k(x,r) f\left(x, t_k \overline{\mathfrak{M}}_k(x,r)\right) dx. \end{aligned}$$

Because of $\overline{\mathfrak{M}}_k(x,r) \to 0$, as $k \to \infty$ on $B_r(0)$, the set of points $x \in B_r(0)$ with $t_k \overline{\mathfrak{M}}_k(x,r) \geq C_\delta$ gets arbitrarily "thin". Therefore the last two terms converge to $(C_3 - \delta)\frac{\omega_{n-1} r^n}{n}$ and 0 respectively, and using $t_k^n \geq \frac{\omega_{n-1} n^{n-1}}{\alpha}$ again as well as the properties of $\overline{\mathfrak{M}}_k(x,r)$ we get

$$\int_{|x| \leq \frac{r}{k}} \exp\left(\alpha |t_k \overline{\mathfrak{M}}_k(x,r)|^{\frac{n}{n-1}} \right) dx \geq \int_{|x| \leq \frac{r}{k}} \exp\left(n \omega_{n-1}^{\frac{1}{n-1}} |\overline{\mathfrak{M}}_k(x,r)|^{\frac{n}{n-1}} \right) dx = \frac{\omega_{n-1} r^n \exp(n \omega_{n-1}^{\frac{1}{n-1}} a_k)}{n}.$$

Finally we have

$$\int_{\frac{r}{n} \leq |x| \leq r} \exp\left(n \omega_{n-1}^{\frac{1}{n-1}} |\overline{\mathfrak{M}}_k(x,r)|^{\frac{n}{n-1}} \right) dx \to r^n \omega_{n-1},$$

as $k \to \infty$ by computing the integral using a change of variable (see [29] for details). Putting this together in the limit yields

$$\lim_{k \to \infty} t_k^n = \frac{n^{n-1}\omega_{n-1}}{\alpha^{n-1}} \geq (C_\delta - \delta) \frac{\omega_{n-1} r^n}{n} \lim_{k \to \infty} \exp\left(a_k n \omega_{n-1}^{1/(n-1)} \right) - (C_\delta - \delta) \frac{\omega_{n-1} r^n}{n} + (C_\delta - \delta)\, \omega_{n-1} r^n.$$

Thus

$$\frac{n^{n-1}\omega_{n-1}}{\alpha^{n-1}} \geq (C_\delta - \delta)\, \omega_{n-1} r^n,$$

which in turn implies

$$\frac{n^{n-1}}{\alpha^{n-1} r^n} \geq C_\delta,$$

a contradiction.

Lemma 3.7 (Compact Embedding). *If $b(x)$ satisfies (e), then $E \hookrightarrow L^q(\mathbb{R}^n)$, $\forall q \in [n, \infty)$ is compact.*

Proof. The proof is a generalization of Costa [10]. Assume $u_k \rightharpoonup 0$ in E, we show $u_k \to 0$ in $L^q(\mathbb{R}^n)$, for $\forall q \in [n, \infty)$. Let $\|u\| \leq C$, for a positive constant C. Since b is coercive we can choose, for a given $\varepsilon > 0$, a $R > 0$, such that $b(x) \geq \frac{2C^q}{\varepsilon}$ for all $|x| \geq R$. Restricting u_k to $B_R(0)$, we have $u_k \rightharpoonup 0$ in $W^{1,n}(B_R(0))$. Then the compact embedding $W^{1,n}(B_R(0)) \hookrightarrow L^q(B_R(0))$ for all $q \in [n, \infty)$ implies

$$\int_{B_R(0)} |u_m|^q dx \leq \frac{\varepsilon}{2}, \quad \text{for some } m_0 \in \mathbb{N},\ \forall m \geq m_0.$$

$$\frac{2}{\varepsilon}\int_{\mathbb{R}^n\setminus B_R(0)}|u_m|^q dx \le \frac{1}{C^q}\int_{\mathbb{R}^n\setminus B_R(0)} b(x)|u_m|^q dx \le \frac{1}{C^q}\|u_m\|^q \le 1,$$

and therefore $\|u_m\|^q_{L^q(\mathbb{R}^n)} \le \varepsilon$.

Remark. *This is the only time we will use that b is coercive. Assumption (e) can be exchanged for various other ones, for example, a(x) postive, continuous and $(a(x))^{-1} \in L^{\frac{1}{n-1}}(\mathbb{R}^n)$, see [4], or a(x) radially symmetric ([24], Thm. II.1.) .*

We show that the remaining assumption (A) of the MPT holds true:
Lemma 3.8. *If assumptions (a) -(f) hold, then $I[\cdot]$ satisfies, for some constants ζ, $\rho > 0$,*

$$I|_{\partial B_\rho} \ge \zeta > 0.$$

Proof. We can choose δ, ε in such a way that (f) implies

$$|F(x,u)| \le \frac{(\lambda_1(n)-\varepsilon)\,|u|^n}{n},$$

for all $|u| \le \delta$. By (a) we also have

$$|F(x,u)| \le C|u|^q \sum_{i=n-1}^{\infty} \frac{\gamma^i}{i!}|u|^{\frac{ni}{n-1}},$$

for $|u| \ge \delta$, and $q > n$ and some constants $C = C(\delta, q), \gamma > 0$. Putting this estimates together yields

$$|F(x,u)| \le \frac{(\lambda_1(n)-\varepsilon)\,|u|^n}{n} + C|u|^q \sum_{i=n-1}^{\infty} \frac{\gamma^i}{i!}|u|^{\frac{ni}{n-1}}.$$

We use

$$\int_{\mathbb{R}^n} |u|^q \sum_{i=n-1}^{\infty} \frac{\gamma^i}{i!}|u|^{\frac{ni}{n-1}}\, dx \le C(n,\gamma)\|u\|^q,$$

for $\|u\| \le K$. This can be shown in a similar way as Lemma 3.4 using symmetrization . Thus

$$I[u] \ge \frac{1}{n}\|u\|^n - \frac{(\lambda_1(n)-\varepsilon)\,||u||^n_{L^n(\mathbb{R}^n)}}{n} - C(n,\gamma)\|u\|^q,$$

for $\|u\| \le K$. We see, using the continuous embedding $E \hookrightarrow L^n$, and $\lambda_1(n) = \inf_{E\ni u\ne 0} \frac{\|u\|^n}{\|u\|_{L^n(\mathbb{R}^n)}}$ $\ge b$, that we can choose ρ, $\|u\| = \rho$, small enough, such that $I[u] \ge \zeta > 0$, for some ζ.
Now that we have verified the Mountain Pass geometry we can prove:

Theorem 3.9. *If the conditions (a) - (f) are satisfied, problem (5) has a weak nontrivial solution.*

Proof. The MPT ensures the existence of a sequence $\{u_k\} \subset E$ with $I'[u_k] \to 0$ and $I[u_k] \to c$ for $k \to \infty$, $c > 0$. We prove that this sequence converges to a weak nontrivial solution. By construction

$$\left|\int_{\mathbb{R}^n} |\nabla u_k|^{n-2}\nabla u_k \nabla\phi + b(x)|u_k|^{n-2}u_k\phi\, dx - \int_{\mathbb{R}^n} f(x,u_k)\phi\, dx\right| \le \varepsilon\|\phi\|, \tag{11}$$

for $\forall\phi \in E$ and $\varepsilon > 0$. By (b) we get

$$\left(\frac{\beta}{n}-1\right)\|u_k\| - \int_{\mathbb{R}^n} \beta F(x,u_k) - f(x,u_k)u_k\, dx \le C + \varepsilon\|u_k\|,$$

with C a positive constant. Thus $\{u_k\}$ is bounded in E, and also

$$\int_{\mathbb{R}^n} f(x,u_k)u_k\, dx \le C, \qquad \int_{\mathbb{R}^n} F(x,u_k)\, dx \le C.$$

From the compact embedding, given by Lemma 3.7 we know that

$$u_k \rightharpoonup u \text{ in } E; \quad u_k \to u \text{ in } L^q(\mathbb{R}^n),\ \forall q \in [n,\infty); \quad u_k(x) \to u(x) \text{ a.e. in } \mathbb{R}^n;$$

by passing down to a subsequence (since E is reflexive). Using this, one can show that

$$f(x,u_k) \to f(x,u) \text{ in } L^1(B_R(0)); \text{ and } |\nabla u_k|^{n-2}\nabla u_k \rightharpoonup |\nabla u|^{n-2}\nabla u \text{ in } \left(L^{n/(n-1)}(B_R(0))\right)^n;$$

for all $R > 0$. The proof of the latter is quite lengthy, involving the theory of distributions . We omit this step and instead refer to [28] or [14]. Taking the limit in (11), we see that u_k converges to a weak solution u. It is left to show, u is nontrivial. Assume $u \equiv 0$, then $F(x,u_k) \to 0$ in $L^1(B_R(0))$ for all $R > 0$, by the generalized Lebesgue Dominated Convergence Theorem (see for example Coroll. 4.14. in [17]). Since

$$\int_{\mathbb{R}^n} |F(x,u_k)|\, dx \le C \int_{\mathbb{R}^n} \sum_{i=n-1}^{\infty} \frac{\gamma^i}{i!}|u|^{\frac{ni}{n-1}}\, dx,$$

for some positive constants C, γ, by (a), plugging in the symmetrization of u as in the proof of Lemma 3.4 and using the Radial Lemma once more, one sees that $F(x,u_k) \to 0$ in $L^1(\mathbb{R}^n)$.

$$\frac{1}{n}\int_{\mathbb{R}^n} |\nabla u_k|^n + b(x)|u_k|^n\, dx - \int_{\mathbb{R}^n} F(x,u_k)\, dx \to c \text{ as } n \to \infty,$$

and the two last terms converge to zero in the limit, thus

$$\frac{1}{n}\int_{\mathbb{R}^n} |\nabla u_k|^n\, dx \to c.$$

From the upper bound on the Mountain Pass Level given by Lemma 3.6, we get

$$\|\nabla u_k\|^n_{L^n(\mathbb{R}^n)} \le nc + \delta < \frac{n^{n-1}\omega_{n-1}}{\alpha^{n-1}} + \delta,$$

for some $\delta > 0$. By repeating the argument of Lemma 3.4, we get

$$\int_{\mathbb{R}^n} \sum_{i=n-1}^{\infty} \frac{\alpha^i}{i!}|u_k|^{\frac{ni}{n-1}}\, dx \le C'$$

C' a positive constant, $\forall k$. Bounding

$$\int_{\mathbb{R}^n} f(x,u_k)u_k\, dx,$$

from above by (a), and using Hölder's inequality, we see that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} |f(x,u_k)|^q\, dx = 0,$$

for $q > 1$ close enough to 1. Then, since

$$\int_{\mathbb{R}^n} |\nabla u_k|^n + b(x)|u_k|^n\, dx - \int_{\mathbb{R}^n} f(x,u_k)u_k\, dx \to 0, \text{ as } k \to \infty,$$

$$\int_{\mathbb{R}^n} |\nabla u_k|^n\, dx \to 0,$$

a contradiction. u is nontrivial.

Remark on the necesssity of the assumptions (a) - (f). It is curious that one can show the same result assuming only (a) and $b(x) \in C(\mathbb{R}^n,\mathbb{R})$, $b_1 \le b(x) \le b_2$, by using *Ekeland's variational principle* instead of the MPT, see [14].

Open problem. In contrast to the analog semilinear case, to this point there seems to be no work on nonexistence of higher regularity solutions or regularity results for problem (5), when $n \le p < \infty$ and the growth of the nonlinearity is critical or supercritical.

4 Appendix: Sobolev Spaces

We state some basic definitions and theorems from the theory of Sobolev spaces. We will not prove any of the following results, but instead refer to specific literature.

Definition A.1 (weak derivative).
$\Omega \subset \mathbb{R}^n$ an open set. For $u \in L^1_{loc}(\mathbb{R}^n)$ and α a multi-index,

$$\int_\Omega \nabla^\alpha \varphi(x) u(x)\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x) u(x)\, dx \quad \forall \varphi \in C_c^\infty,$$

*if $v_\alpha \in L^1_{loc}(\mathbb{R}^n)$ exists, it is the **weak derivative** $\nabla^\alpha u$ of u and it is unique.*

Definition A.2 (Sobolev space).
The Sobolev spaces $W^{m,p}(\Omega)$ are defined as

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) \mid \nabla^\alpha u \in L^p(\Omega),\ \forall 0 \le |\alpha| \le m\}.$$

The spaces $W^{m,p}(\Omega)$ are equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} := \left(\sum_{0 \le |\alpha| \le m} \|u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

- By $W_0^{m,p}(\Omega)$ we denote the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$. (Also, $C_c^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$ for $1 \le p < \infty$ if Ω is bounded with Lipschitz boundary);
- $W^{1,p}(\Omega)$ is a Banach space for $1 \le p \le \infty$, reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$;
- $W^{m,2}(\Omega) = H^m(\Omega)$ are Hilbert spaces, with $\langle u, v\rangle_{H^m(\Omega)} := \sum_{0 \le |\alpha| \le m} \langle \nabla^\alpha u, \nabla^\alpha v\rangle_{L^2(\Omega)}$ the scalar product.

Theorem A.2 (weak product rule).
Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $1 \le p < \infty$, then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\nabla(uv) = v\,(\nabla u) + u\,(\nabla v).$$

Theorem A.3 (Sobolev's embedding theorem).
For $1 \le p < \infty$, then

$$\begin{aligned}
&W^{1,p}(\mathbb{R}^n) \subset L^{p*}(\mathbb{R}^n), && \frac{1}{p*} = \frac{1}{p} - \frac{1}{n}, && \text{if } p < n;\\
&W^{1,p}(\mathbb{R}^n) \subset L^{q}(\mathbb{R}^n), && \forall q \in [p, \infty), && \text{if } p = n;\\
&W^{1,p}(\mathbb{R}^n) \subset L^{\infty}(\mathbb{R}^n), && && \text{if } p > n;
\end{aligned}$$

and all these injections are continuous. The statement is still true, if we exchange $\mathbb{R}^n$ with a bounded, open set $\Omega \subset \mathbb{R}^n$, having a Lipschitz boundary.
Actually this theorem is not only due to Sobolev, but rather a collection of results by Sobolev, Nirenberg, Gagliardo and Morrey.

Theorem A.4 (Rellich-Kondrachov). If $\Omega \subset \mathbb{R}^n$ is bounded, with C^1-boundary, then the following injections are compact

$$\begin{aligned}
&W^{1,p}(\Omega) \subset L^{q}(\Omega), && \forall q \in [1, p*), && \frac{1}{p*} = \frac{1}{p} - \frac{1}{n}, && \text{if } p < n;\\
&W^{1,p}(\Omega) \subset L^{q}(\Omega), && \forall q \in [p, \infty), && && \text{if } p = n;\\
&W^{1,p}(\Omega) \subset C(\overline{\Omega}), && && && \text{if } p > n.
\end{aligned}$$

Theorem A.5 (Nirenberg's inequality).
Let $u \in L^p(\mathbb{R}^n)$, and $\nabla^m u \in L^r(\mathbb{R}^n)$, then for C a constant depending only on n, α, p, r, m, j

$$\|\nabla^j u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla^m u\|^{\alpha}_{L^r(\mathbb{R}^n)} \|u\|^{1-\alpha}_{L^p(\mathbb{R}^n)},$$

for

$$\frac{1}{q} = \frac{j}{n} + \alpha \left(\frac{1}{r} - \frac{m}{n} \right) + (1-\alpha) \frac{1}{p},$$

and all α with $\frac{j}{m} \leq \alpha \leq 1$, and the following exceptional cases:

- *If $j = 0$, $rm < n$, $p = \infty$, then we have to assume additionally that u tends to zero at infinity or that $u \in L^{\tilde{p}}$, for some finite $\tilde{p} > 0$.*
- *If $1 < r < \infty$, and $m - j - \frac{n}{r}$ a nonnegative integer, than the theorem holds only for $\alpha < 1$.*

Remark. *Note that Nirenberg's inequality implies the Sobolev embedding: Set $\alpha = 1$, then Theorem A.5 states as*

$$\|u\|_{L^{p*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)},$$

which is frequently referred to as ***Gagliardo-Nirenberg-Sobolev inequality****.*

This definitions and properties can be found in many textbooks, see for example [1] or [7].

References

[1] R. A. Adams and John J. F. Fournier, Sobolev Spaces, *Academic Press*, 2003.

[2] N. Ahmed, D.K. Sunada, Nonlinear Flow in Porous Media, J. Hydraulics Division Proc. Am. Soc. Civil Eng. **95** (1969), 1847-1857.

[3] G. Astrita, G. Marrucci, Principles of Non-Newtonian Fluid Mechanics, *McGraw-Hill, New York*(1974).

[4] S. Aouaoui, Three nontrivial Solutions for some Elliptic Equation involving the N-Laplacian, *Electronic Journal of Qualitative Theory of Differential Equations* **2** (2015), 1-12.

[5] R. Bartnik , J. McKinnon, Particle-like Solutions of the Einstein-Yang-Mills Equations, *Phys. Rev. Lett.* **61** (1988), 141- 144.

[6] M.-F. Bidaut-Veron, Local and Global Behaviour of Solutions of Quasilinear Equations of Emden-Fowler Type, *Arch. Rational Mech. Anal.* **107** (1989), 293-324.

[7] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, *Springer-Verlag*, 2011.

[8] H. Brezis and L. Nirenberg, Positive solutions of Nonlinear Elliptic Equations involving Critical Sobolev Exponents, *Comm. Pure Appl. Math.* **36** (1983), 437–477.

[9] H. Brezis, J.M. Coron, and L. Nirenberg, Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz, *Commun. Pure Appl. Math.* **33** (1980), 667–684.

[10] D. G. Costa, On a Nonlinear Elliptic Problem in $\mathbb{R}^n$, *Center for Mathematical Sciences, Univ. Winconsin, CMS-Tech.* Summary Rep. 93-13, 1993.

[11] D. G. De Figueiredo, Lectures on the Ekeland Variational Principle with Applications and Detours, *Tata Institute of Fundamental Research* 1989.

[12] P. Drabek, Nonlinear Eigenvalues for p-Laplacian in $\mathbb{R}^n$, *Math. Nachr.* **173** (1995), 132-139.

[13] Li Gongbao, Some Properties of Weak Solutions of Nonlinear Scalar Field Equations, *Annales Academia Scientiarum Fennice* **14** (1989), 27-36.

[14] Li Gongbao and Huan-Song Zhou, The Existence of a Weak Solution of Inhomogeneous Quasilinear Elliptic Equation with Critical Growth Conditions, *Acta Mathematica Sinica, New Series* **11** (1995), 146-155.

[15] Li Gongbao and Zhu Xiping, Nontrivial Solutions of Nonlinear Scalar Field Equations with Strong Nonlinearity, *Acta Math. Sci.* **8** (1988), 431-448.

[16] Y. Jabri, The Mountain Pass Theorem - Variants, Generalizations and some Applications, *Cambridge University Press* 2003.

[17] O. Kavian, Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques, *Springer-Verlag, Paris* 1993.

[18] A. El Khalil, S. El Manouni and M. Quanan, On Some Nonlinear Elliptic Problems for p-Laplacian in $\mathbb{R}^n$, *Nonlinear differ. equ. appl.* **15** (2008), 295-307.

[19] S. Kichenassamy and L. Veron, Singular Solutions of the p-Laplace Equation, *Math. Ann.* **275** (1985), 599-615.

[20] G. Li, S. Yan, Eigenvalue Problems for Quasilinear Elliptic Equations on $\mathbb{R}^n$, *Comm. Partial Differential Equations* **14** (1989),1291-1314.

[21] Yuxiang Li and B. Ruf, A sharp Trudinger-Moser Type Inequality for Unbounded Domains in $\mathbb{R}^n$, *Indiana Univ. Math. J.* **57** (2008), 451-480.

[22] G. M. Lieberman, Boundary Regularity For Solutions Of Degenerate Elliptic Equations, *Nonlinear Analysis. Theory, Methods and Applications,* **12** (1988), 1203-1219.

[23] P. Lindqvist, On the Equation $div(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, *Amer. Math. Soc.* **109** (1990), 157-164.

[24] P-L. Lions, Symétrie et Compacité dans les Espaces de Sobolev, *Journal of Functional Analysis* **49**(1982), 315-334.

[25] R. Magnanini, S. Sakaguchi and A. Alvino, Geometric Properties for Parabolic and Elliptic PDE's, *Springer INdAm Series 2*, Springer, Milan, 2013.

[26] J. Moser, A Sharp Form of an Inequality by N. Trudinger, *Indiana Univ. Math.* **20** (1971), 1077–1092.

[27] L. Nirenberg, On Elliptic Partial Differential Equations, *Annali della Scuola Normale Superiore di Pisa* **13** (1959), 115-162.

[28] Joao Marcos B. Do Ó, Semilinear Dirichlet Problems For the N-Laplacian in $\mathbb{R}^n$ with Nonlinearities in the Critical Growth Range, *Differential and Integral Equations* **9** (1996) ,967-979.

[29] Joao Marcos B. Do Ó, N-Laplacian Equations in $\mathbb{R}^n$ with Critical Growth, *Abstr. Appl. Anal.* **2** (1997), 301–315.

[30] R. S. Palais, Critical Point Theory and the Minimax Principle, *Nonlinear Funct. Anal. App.* **15** (1970), 185-212.

[31] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, *American Mathematical Society* 1984.

[32] M. E. Rudin, A New Proof that Metric Spaces are Paracompact, *Amer. Math. Society* **20** (1969), 603.

[33] J. Serrin, Local Behavior of Solutions of Quasi-Linear Equations, *Acta Math.* **111** (1964), 247-302.

[34] P. Tolksdorf, Regularity for More General Class of Quasilinear Equations, *J. Diff. Equations* **51** (1984), 126-150.

[35] N. S. Trudinger, On Harnack Type Inequalities as their Applications to Quasilinear Elliptic Equations, *Comm. Pure App. Math.* **20** (1967), 721-747.

[36] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, *J. Math. Mech.* **17** (1967), 473–483.

[37] M. Willem, Lectures on Critical Point Theory, *Trabalho de mathematica* **199** (1983).

[38] F. Yongqiang, Existence of Solutions for p(x)-Laplacian problem on an unbounded domain, *Topological Methods in Nonlinear Analysis* **30** (2007),235-249.

[39] K. Yosida, Functional Analysis and its Application, Springer-Verlag, 1965.

[40] Lao Sen Yu, Nonlinear p-Laplacian Problems on unbounded Domains, *American Mathematical Society* **115** (1992), 1037-1045.

[41] Huan-Song Zhou, Solutions for a Quasilinear Elliptic Equation with Critical Sobolev Exponent and Perturbation on $\mathbb{R}^n$, *Differential and Integral Equations* **13** (2000), 595-612.

[42] X. P. Zhu and J. F. Yang, The Quasilinear Elliptic Equation on Unbounded Domain Involving Critical Sobolev Exponent, *J. Partial Diff. Equa.* **2** (1989), 65–75.