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Chaotic solution C_0-semigroups of partial differential equations

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Abstract

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Ich erkläre, dass ich die Bachelorarbeit selbstständig und ohne unzulässige Inanspruchnahme Dritter verfasst habe. Ich habe dabei nur die angegebenen Quellen und Hilfsmittel verwendet und die aus diesen wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht. Die Versicherung selbstständiger Arbeit gilt auch für enthaltene Zeichnungen, Skizzen oder graphische Darstellungen. Die Bachelorarbeit wurde bisher in gleicher oder ähnlicher Form weder derselben noch einer anderen Prüfungsbehörde vorgelegt und auch nicht veröffentlicht. Mit der Abgabe der elektronischen Fassung der endgültigen Version der Bachelorarbeit nehme ich zur Kenntnis, dass diese mit Hilfe eines Plagiatserkennungsdienstes auf enthaltene Plagiate geprüft werden kann und ausschließlich für Prüfungszwecke gespeichert wird.

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Chapter 1

Introduction

Strongly continuous (operator) semigroups have been an object of mathematical research for several decades now and their application to differential equations have led to new insights about the existence and structure of solutions especially of partial differential equations. The approach used is to rewrite a problem of a given partial differential equation as an abstract Cauchy problem. The strategy which is then followed is to show that a given differential operator is the generator of a strongly continuous semigroup. Without knowing the exact semigroup that is generated, which is possible only in some cases, one is able to derive statements based on the properties of the generator of the semigroup.

An interesting question with regard to solutions of partial differential equation is if the solution is stable as $t \to \infty$. This can be translated to the question of stability of the generated semigroup. The heat equation in one dimension

$$v_t(x,t) = d v_{xx}(x,t), \qquad d > 0$$

on an interval (0,1) with Dirichlet boundary conditions is an example of an equation with a stable solution. The generated semigroup converges uniformly to 0. In contrast, the equation

$$\begin{aligned} u_t(x,t) &= a u_{xx}(x,t) + b u_x(x,t) + c u(x,t) \\ u(0,t) &= 0 \text{ for } t \geqslant 0 \\ u(x,0) &= f(x) \text{ for } x \geqslant 0, f \in X \end{aligned}$$

is (Devaney- or topologically) chaotic for suitable a, b, and c. We will later on define exactly what we understand by a chaotic semigroup.
In this thesis we prove that two modified versions of the heat semigroup on an interval are indeed stable and that the second equation above is chaotic. To show this we will need a number of preliminaries. In chapter 2 we will give an introduction to semigroup theory. Two highlights of this chapter will be the well-known Hille-Yosida theorem and the Lumer-Phillips theorem, both stating conditions under which a given operator will generate a strongly continuous semigroup.

In chapter 3 we will introduce criteria for stability, hypercyclicity and chaoticity of semigroups. This chapter will close with the spectral conditions for chaotic semigroups, derived by Desch, Schappacher and Webb.

In the fourth and final chapter we will see the application of semigroup to partial differential equations, especially looking at stability or chaoticity of its solutions. We will close this chapter with an outlook on how to approach a coupled system of stable and chaotic solutions.

We assume that the reader has knowledge about elementary functional analysis and is familiar with elementary facts about partial differential equations. In the annex we have summarized needed definitions and theorems from functional analysis, complex analysis and operator theory, with references to proofs in the literature.

Note: This file is the corrected version of the original theses. Minor typos have been eliminated and mistakes have been corrected. Corrected parts are in red.

Chapter 2

Introduction to C_0-semigroups

In this chapter we will introduce strongly continuous semigroups of operators and some of their properties. We will look at operators which generate such semigroups and answer the question under which circumstances will an operator generate a semigroup. In the second half of this chapter, we will introduce some special type of semigroups and look at what happens if generators of semigroups are perturbed by another operator.

2.1 Elementary properties of C_0-semigroups

Definition 2.1. *We call a family* $(T(t))_{t\geqslant 0}$ *of bounded linear operators on a Banach space* X *a strongly continuous (or* C_0*-) semigroup if it satisfies the following semigroup properties:*

$$(SP) \qquad \begin{cases} T(t+s) = T(t)T(s) & \textit{for all } t, s \geqslant 0 \\ T(0) = I \end{cases}$$

Furthermore, we require that the orbit maps $\xi_x : t \mapsto \xi_x(t) := T(t)x$ *are continuous from* $\mathbb{R}_+$ *to* X *for all* $x \in X$.

If the semigroup properties also hold for $t \in \mathbb{R}$ then $(T(t))_{t\in\mathbb{R}}$ is called a strongly continuous group on X. In the following we will assume that $T(t)$ is a semigroup, unless otherwise stated, omit the index $t \geqslant 0$, and use the short notation $T(t)$.

The following proposition establishes equivalences to the strong continuity property of semigroups following [EN00, p. 38]:

Proposition 2.2. *The following assertions are equivalent for a semigroup* $T(t)$ *on a Banach space* X.

(a) $T(t)$ *is strongly continuous.*

(b) $\lim_{t\downarrow 0} T(t)x = x$ *for all* $x \in X$.

(c) There exist $\delta > 0, M \geqslant 1$*, and a dense subset* $D \in X$ *such that the following properties hold*

(i) $\|T(t)\| \leqslant M$ *for all* $t \in [0, \delta]$,

(ii) $\lim_{t\downarrow 0} T(t)x = x$ *for all* $x \in D$.

Proof: The implication (a) $\Rightarrow$ (c.ii) follows immediately from the semigroup properties. Next, we prove (a) $\Rightarrow$ (c.i) by contradiction. Assume there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$

which converges to zero such that $\|T(\delta_n)\| \to \infty$ as $n \to \infty$. By the uniform boundedness principle (see annex A3) there also exists an $x \in X$ such that $(\|T(\delta_n)x\|)_{n\in\mathbb{N}}$ is unbounded which is impossible since the semigroup is continuous at t=0.
To show that (c) $\Rightarrow$ (b), we define $K := \{t_n : n \in \mathbb{N}\} \cup \{0\}$ for an arbitrary sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ converging to $t = 0$. Then $K \in [0,\infty)$ is sequentially compact and hence compact. With (c) we see that $T(\cdot)_{|K}$ is bounded and that $T(\cdot)_{|K}x$ is continuous for all $x \in D$. These conditions allow us to apply annex 5(b) and we obtain

$$\lim_{n\to\infty} T(t_n)x = x$$

for all $x \in X$.
Finally, we show that (b) $\Rightarrow$ (a). Let $t_0 > 0$ and $x \in X$. By the semigroup properties and properties of the operator norm we see

$$\lim_{h\downarrow 0} \|T(t_0+h)x - T(t_0)x\| \leqslant \|T(t_0)\| \cdot \lim_{h\downarrow 0} \|T(h)x - x\| = 0,$$

proving right continuity. Now let $h < 0$. The estimate

$$\|T(t_0+h)x - T(t_0)x\| \leqslant \|T(t_0+h)\| \cdot \|x - T(-h)x\|$$

implies left continuity as long as $\|T(t)\|$ is uniformly bounded for $t \in [0, t_0]$. This holds for some small interval $[0,\delta]$ by the uniform boundedness principle and also on each compact interval thanks to the semigroup properties. □
Strongly continuous semigroups are exponentially bounded on compact interval as the following proposition shows [EN00, p. 39]:

Proposition 2.3. *For every strongly continuous semigroup* $T(t)$ *there exist constants* $w \in \mathbb{R}$ *and* $M \geqslant 1$ *such that*

$$\|T(t)\| \leqslant Me^{wt}$$

for all $t \geqslant 0$

Proof: We write $t = s + n$ for $n \in \mathbb{N}$ and $0 \leqslant s < 1$. We choose an $M \geqslant 1$ such that $\|T(s)\| \leqslant M$. Then we can estimate

$$\begin{aligned}\|T(t)\| &\leqslant \|T(s)\| \cdot \|T(1)\|^n \leqslant M^{n+1} \\ &= Me^{n \log M} \\ &\leqslant Me^{wt}\end{aligned}$$

with $w := \log M$ and for all $t \geqslant 0$. □

For a strongly continuous semigroup $T(t)$ we call

$$\omega_0 := \omega_0(T(t)) = \inf\{w \in \mathbb{R} : \exists M_w \geqslant 1 \text{ s.t. } \|T(t)\| \leqslant M_w e^{wt} \text{ for all } t \geqslant 0\}$$

its *growth bound*. A semigroup is called *bounded* if $w = 0$ and $M \geqslant 1$, and *contractive (or a contraction semigroup)* if $w = 0$ and $M = 1$ is possible.

Before concluding this section, we will introduce the useful construction of rescaling a semigroup, that will allow us to e.g. lower the growth bound of a semigroup to zero (by setting $\mu = -\omega_0$ and $\alpha = 1$ in the definition below) [EN00, p. 43].

Definition 2.4. *Let* $T(t)$ *be a* C_0*-semigroup on a Banach space X. For any* $\mu \in \mathbb{C}$ *and* $\alpha > 0$*, we can define a rescaled semigroup* $S(t)$ *by*

$$S(t) := e^{\mu t} T(\alpha t)$$

for all $t \geqslant 0$.

2.2 Generators of C_0-semigroups and their resolvents

2.2.1 Generators of semigroups

Definition 2.5. *Let* $T(t)$ *be a* (C_0)*-semigroup on* X*. The (infinitesimal) generator* A *of* T *is defined by*

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \frac{d}{dt} T(t)x|_{t=0}$$

The domain $D(A)$ *of* A *is the set of all* $x \in X$ *for which the limit exists. Note that* $D(A)$ *is a linear subspace of* X*.*

The following lemma, based on [EN00, p. 50], will prove some useful properties of generators of semigroups. We will make frequent use of these properties throughout the text.

Lemma 2.6. *A generator* $(A, D(A))$ *of a* (C_0)*-semigroup* $(T(t))_{t \geqslant 0}$ *has the following properties*

(1) $A : D(A) \subseteq X \to X$ *is a linear operator.*

(2) If $x \in D(A)$ *then* $T(t)x \in D(A)$ *and*

$$\frac{d}{dt} T(t)x = T(t)Ax = AT(t)x \text{ for all } t \geqslant 0$$

(3) For every $t \geqslant 0$ *one has*

$$\begin{aligned} T(t)x - x &= A \int_0^t T(s)x \, ds \qquad \text{if } x \in X \\ &= \int_0^t T(s)Ax \, ds \qquad \text{if } x \in D(A) \end{aligned}$$

Proof: Statement (1) is proved by the linearity of the limit. Let $x, y \in X, \lambda \in \mathbb{C}$ then

$$A(\lambda(x+y)) = \lim_{t \to 0} \frac{\lambda T(t)(x+y) - \lambda(x+y)}{t} = \lambda \lim_{t \to 0} \frac{T(t)x - x}{t} + \lambda \lim_{t \to 0} \frac{T(t)y - y}{t} = \lambda Ax + \lambda Ay.$$

For assertion (2) let $x \in D(A)$. We know that $\frac{1}{h}(T(t+h)x - T(t)x)$ converges to $T(t)Ax$ as $h \downarrow 0$. Hence

$$\lim_{h \downarrow 0} \frac{1}{h}(T(h)T(t)x - T(t)x)$$

exists and therefore also $T(t)x \in D(A)$. We conclude $AT(t)x = T(t)Ax$. Finally, to prove assertion (3) let $x \in X$ and $t \geqslant 0$. We obtain

$$\begin{aligned}\frac{1}{h}\left(T(h)\int_0^t T(s)x\,ds - \int_0^t T(s)x\,ds\right) &= \frac{1}{h}\int_0^t T(s+h)x\,ds - \frac{1}{h}\int_0^t T(s)x\,ds\\ &= \frac{1}{h}\int_h^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^t T(s)x\,ds\\ &= \frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^h T(s)x\,ds\end{aligned}$$

which converges to $T(t)x - x$ as $h \downarrow 0$. This proves the first part of the statement. If $x \in D(A)$ then the functions $s \mapsto T(s)\frac{T(h)x-x}{h}$ converge uniformly on $[0,t]$ to $T(s)Ax$. Hence we obtain

$$\lim_{h\downarrow 0}\frac{1}{h}(T(h)-I)\int_0^t T(s)x\,ds = \lim_{h\downarrow 0}\int_0^t T(s)\frac{1}{h}(T(h)-I)x\,ds = \int_0^t T(s)Ax\,ds.$$

This proves the second part of assertion (3). □

Based on the lemma above, one can prove the following theorem about the properties of generators [EN00, p. 51].

Theorem 2.7. *The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Proof: We first show that A is closed. We consider a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ for which $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$. By the lemma 2.6, we know that $T(t)x_n - x_n = \int_0^t T(s)Ax_n\,ds$ for $t > 0$. The uniform convergence of $T(\cdot)Ax_n$ on $[0,t]$ for $n \to \infty$ implies that $T(t)x - x = \int_0^t T(s)y\,ds$. If we multiply both sides by $1/t$ and take the limit $t \downarrow 0$ we see that $x \in D(A)$ and $Ax = y$. Hence, A is closed. The previous lemma implies that elements of $1/t\int_0^t T(s)x\,ds$ belong to $D(A)$. The strong continuity of $T(t)$ implies $\lim_{t\downarrow 0} 1/t\int_0^t T(s)x\,ds = x$ for every $x \in X$. Therefore $D(A)$ is dense in X.
For the uniqueness, we assume that there is a second strongly continuous semigroup $S(t)$ with the same generator $(A, D(A))$. We define the function $u : [0,t] \to X$ by $u(s) := T(t-s)S(s)x$. This function is differentiable and we can obtain its derivative by the product rule:

$$\begin{aligned}\frac{d}{ds}u(s) &= \left(\frac{d}{ds}T(t-s)\right)S(s)x + T(t-s)\frac{d}{ds}(S(s)x)\\ &= -AT(t-s)S(s)x + T(t-s)AS(s)x = 0.\end{aligned}$$

□

We will briefly return to rescaled semigroups and a practical lemma which will need again later.

Lemma 2.8. *A rescaled semigroup* $S(t) := e^{\mu t}T(\alpha t)$ *with* $\mu \in \mathbb{C}$ *and* $\alpha > 0$ *has the generator* $B = \alpha A + \mu I$ *with* $D(B) = D(A)$.

Proof: We derive the rescaled semigroup:

$$\frac{d}{dt}S(t)x = \frac{d}{dt}e^{\mu t}T(\alpha t)x = \mu e^{\mu t}T(\alpha t)x + \alpha e^{\mu t}AT(\alpha t)x$$

With $t \to 0$ we obtain $Bx = (\mu I + \alpha A)x$. We see that $x \in D(B)$ if and only of $x \in D(A)$. $\square$

Lemma 2.9. *Let* $(A, D(A))$ *be the generator of a* C_0*-semigroup T(t). Then the following identities hold [EN00, p. 55]:*

$$\begin{aligned} e^{-\lambda t}T(t)x - x &= (A-\lambda)\int_0^t e^{-\lambda s}T(s)x\,ds \qquad \textit{if } x \in X, \\ &= \int_0^t e^{-\lambda s}T(s)(A-\lambda)x\,ds \qquad \textit{if } x \in D(A). \end{aligned}$$

Proof: If $\lambda = 0$ then the identity follows immediately from 2.6(3). Now let $\lambda \neq 0$. We rescale the semigroup and define $\tilde{T}(t) = e^{-\lambda t}T(t)$. With lemma 2.8, the semigroup $\tilde{T}$ has the generator $B = A - \lambda$. Applying 2.6(3) will now deliver the desired identity. $\square$

2.2.2 Resolvents of generators

Generators of semigroups are closely linked to their resolvents. We recap that a resolvent is defined as

$$R(\lambda, A) := (\lambda - A)^{-1}.$$

Further we define the resolvent set

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A : D(A) \to X \text{ is bijective}\}.$$

We denote the complement $\sigma(A) := \mathbb{C}\backslash\rho(A)$ as the spectrum of A.The following theorem provides some properties of resolvents [EN00, p. 55].

Theorem 2.10. *Let* $T(t)$ *be a strongly continuous semigroup on a Banach space X. Let the constants* $w \in \mathbb{R}, M \geqslant 1$ *be such that* $\|T(t)\| \leqslant Me^{wt}$ *for all* $t \geqslant 0$*. For the generator* $(A, D(A))$ *of* $T(t)$ *the following properties hold.*

(1) If $\lambda \in \mathbb{C}$ *such that* $R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)x\,ds$ *exists for all* $x \in X$ *then* $\lambda \in \rho(A)$ *and* $R(\lambda, A) = R(\lambda)$.

(2) If Re $\lambda > w$ *then* $\lambda \in \rho(A)$*, and the resolvent* $R(\lambda, A) = R(\lambda)$ *as in (1).*

(3) For all Re $\lambda > w$ *we have* $\|R(\lambda, A)\| \leqslant M/(\text{Re}\,\lambda - w)$.

Note: The following formula for $R(\lambda, A)$ is called the integral representation of the resolvent:

$$R(\lambda, A)x = \lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\,ds \text{ for all } x \in X$$

Proof: (1) By rescaling the semigroup we may assume that $\lambda = 0$. For any $x \in X$ and $h > 0$ we have

$$\begin{aligned}\frac{T(h)-I}{h}R(0)x &= \frac{T(h)-I}{h}\int_0^\infty T(s)x\,ds\\ &= \frac{1}{h}\int_0^\infty T(s+h)x\,ds - \frac{1}{h}\int_0^\infty T(s)x\,ds\\ &= \frac{1}{h}\int_h^\infty T(s)x\,ds - \frac{1}{h}\int_0^\infty T(s)x\,ds\\ &= -\frac{1}{h}\int_0^h T(s)x\,ds\end{aligned}$$

Letting $h \downarrow 0$ we receive $AxR(0) = -x$, or $AR(0) = -I$ and $\operatorname{ran}R(0) \subseteq D(A)$. At the same time we have for $x \in D(A)$.

$$\lim_{t\to\infty}\int_0^t T(s)x\,ds = R(0)x$$

and

$$\lim_{t\to\infty} A\int_0^t T(s)x\,ds = \lim_{t\to\infty}\int_0^t T(s)Ax\,ds = R(0)Ax$$

Since A is a closed operator we know that $R(0)Ax = AR(0)x$ and therefore $R(0) = (-A)^{-1}$. Hence, λ is part of the resolvent set of A.
(2) and (3) follow from the following norm estimate:

$$\left\|\int_0^t e^{\lambda s}T(s)ds\right\| \leqslant M\int_0^t e^{(w-\operatorname{Re}\lambda)s}ds.$$

If $\operatorname{Re}\lambda > w$ then the exponent on the right-hand side is negative and the term on the right converges to $M/(\operatorname{Re}\lambda - w)$ as $t \to \infty$. □

Corollary 2.11. *The spectrum $\sigma(A)$ of the generator of a strongly continuous semigroup is located in some left half plane.*

Proof: From 2.10 (2) immediately follows that $\lambda \leqslant w$ for all $\lambda \in \sigma(A)$. □

Before closing this section, we will introduce the series expansion of the resolvent which we will need later [EN00, p. 240].

Proposition 2.12. *For a closed operator $A : D(A) \subset X \to X$ and for $\mu \in \rho(A)$ we have*

$$R(\lambda, A) = \sum_{n=0}^{\infty}(\mu-\lambda)^n R(\mu, A)^{n+1}$$

for all $\lambda \in \mathbb{C}$ with $|\mu - \lambda| < 1/\|R(\mu, A)\|$.

Proof: For a $\lambda \in \mathbb{C}$ we can write

$$\lambda - A = \mu - A + \lambda - \mu = [I - (\mu-\lambda)R(\mu, A)](\mu - A).$$

The term $[I - (\mu - \lambda)R(\mu, A)]$ is invertible whenever $(\mu - \lambda) < \|R(\mu, A)\|^{-1}$. In that case, the operator is bijective. We can then obtain the resolvent as

$$R(\lambda, A) = R(\mu, A)[I - (\mu-\lambda)R(\mu, A)]^{-1} = \sum_{n=0}^{\infty}(\mu-\lambda)^n R(\mu, A)^{n+1}.$$

□

2.3 Hille-Yosida generation theorem

In the 1940s, Einar Hille and Kōsaku Yosida worked in parallel on the question which linear operators can generate a C_0-semigroup. Both came up with an answer independently from each other of what is today know as the Hille-Yosida theorem.
The idea of proving that an operator will generate a semigroup is a very powerful one. Even without knowing the explicit semigroup that the operator generates, one can make use of special properties of the semigroup and its generating operator. We will see an application of this later on in the text in the context of partial differential equations. Before stating the theorem, we will introduce Yosida approximants which will be needed in the proof.

Definition 2.13. *The operators*

$$A_n := nAR(n, A) = n^2R(n, A) - nI$$

are called the Yosida approximants. The equation holds by the definition of the resolvent.

For the Hille-Yosida theorem we will also need a small technical lemma, which we will not prove here, see [EN00, p. 73] for details.

Lemma 2.14. *Let* $(A, D(A))$ *be a closed and densely defined operator. We assume that* $w \in \mathbb{R}$ *and* $M > 0$ *such that* $[w, \infty) \subset \rho(A)$ *and* $\|\lambda R(\lambda, A)\| \leqslant M$ *for all* $\lambda \geqslant w$. *Then following statements hold:*

(1) $\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x$ *for all* $x \in X$.

(2) $\lim_{\lambda\to\infty} \lambda AR(\lambda, A)x = \lim_{\lambda\to\infty} \lambda R(\lambda, A)Ax = Ax$ *for all* $x \in D(A)$.

Now we have all the definitions and tools to prove the actual Hille-Yosida generation theorem. Theorem and proof follow [EN00, p. 73].

Theorem 2.15 (Hille-Yosida)**.** *For a linear operator* $(A, D(A))$ *on a Banach space* X*, the following properties are all equivalent.*

(1) $(A, D(A))$ *generates a strongly continuous contraction semigroup.*

(2) $(A, D(A))$ *is closed, densely defined, and for every* $\lambda > 0$ *one has* $\lambda \in \rho(A)$ *and* $\|\lambda R(\lambda, A)\| \leqslant 1$.

(3) $(A, D(A))$ *is closed, densely defined, and for every* $\lambda \in \mathbb{C}$ *with* $Re\lambda > 0$ *one has* $\lambda \in \rho(A)$ *and* $\|R(\lambda, A)\| \leqslant 1/Re\lambda$.

Proof: (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) : By 2.7 we know that the generator of a strongly continuous semigroup is closed and densely defined. By theorem 2.10 (3) with $M = 1$ (contraction semigroup!) and $w = 0$ we know that $\lambda \in \rho(A)$ and furthermore

$$\|R(\lambda, A)\| \leqslant \frac{1}{Re\lambda}$$

which proofs (3). For (2) we multiply with λ which does not change the direction of the inequality.

(2) $\Rightarrow$ (1) We begin by considering the continuous semigroups

$$T_n(t) := e^{tA_n}, \qquad t \geqslant 0$$

Since the A_n converge pointwise on $D(A)$ by lemma 2.14 it is reasonable to assume that the limit of these semigroups with $n \to \infty$ exists for each $x \in X$ and that the limit is a strongly continuous semigroup on X, too, with the generator $(A, D(A))$. We will prove these assumptions one by one.
Each $T_n(t)$ is a contraction semigroup since

$$\|T_n(t)\| \leqslant e^{-nt}e^{\|n^2R(n,A)\|t} \leqslant e^{-nt}e^{nt} = 1$$

The second inequality holds due to the condition in (2). Pointwise convergence for a densely defined operator implies uniform convergence (see annex A4). We know that the generator of a semigroup is densely defined by theorem 2.7 hence it is sufficient to prove convergence of $T_n(t)$ on $D(A)$.
Using the fundamental theorem of calculus applied to the functions

$$s \mapsto T_m(t-s)T_n(s)x, \qquad s \in [0,t], x \in D(A), m, n \in \mathbb{N}$$

we find that

$$\begin{aligned} T_n(t)x - T_m(t)x &= \int_0^t \frac{d}{ds}(T_m(t-s)T_n(s)x)ds \\ &= \int_0^t T_m(t-s)T_n(s)(A_n x - A_m x)ds \end{aligned}$$

Due to the contractivity we see that

$$\|T_n(t)x - T_m(t)x\| \leqslant t\|A_n x - A_m x\|.$$

By Lemma 2.14 (2) the sequence $(A_n)x$ is a Cauchy sequence for each $x \in D(A)$. Therefore, $T_n(t)x$ converges also uniformly on finite intervals $[0, t_0]$.
The limit $T(t)x := \lim_{n\to\infty} T_n(t)x$ exists for all $x \in X$, it satisfies the semigroup property and consists of contractions. For each $x \in D(A)$ the orbit map $\xi : t \mapsto T(t)x, 0 \leqslant t \leqslant t_0$ is continuous which is sufficient by 2.2 so that $T(t)$ is strongly continuous. It remains to show that this semigroup has the generator $(A, D(A)$.
We denote $(B, D(B))$ the generator of $T(t)$ and fix an $x \in D(A)$. On each compact interval $[0, t_0]$ the functions

$$\xi_n : t \mapsto T_n(t)x$$

converges uniformly to $\xi(\cdot)$ as shown above while its derivatives

$$\dot{\xi}_n : t \mapsto T_n(t)A_n x$$

converges uniformly to

$$\eta : t \mapsto T(t)Ax.$$

Therefore, ξ is differentiable with $Ax = \eta(0) = \dot{\xi}(0) = Bx$ for $x \in D(A)$, which implies that $D(A) \subset D(B)$.
We now select a $\lambda > 0$ with $\lambda \in \rho(A)$ by assumption. Then $\lambda I - A$ is a bijection from $D(A)$ onto X by definition of the resolvent set. Similarly, we have established above that B is the generator of a contraction semigroup and thus by 2.10, $\lambda \in \rho(B)$. Therefore $\lambda I - B$ is also a bijection from $D(B)$ onto X. Since $\lambda I - A$ and $\lambda I - B$ coincide on $D(A)$ we conclude that $D(B) = D(A)$ and hence $B = A$.
(3) $\Rightarrow$ (1) The proof is identical to (2) $\Rightarrow$ (1) except that for the last step we select a

$\tilde{\lambda} = \text{Re}\,\lambda > 0$ with $\lambda \in \mathbb{C}$. Hence we have bijections $\tilde{\lambda}I - A$ from $D(A)$ onto X and $\tilde{\lambda}I - B$ from $D(B)$ onto X so that again $B = A$. □

The Hille-Yosida theorem describes the prerequisites of operators to generate a contraction semigroup. By making use of the fact that we can rescale semigroups, we can apply the Hille-Yosida theorem to obtain prerequisites to obtain quasi-contractive semigroups, i.e. $\|T(t)\| \leqslant e^{wt}$ for some $w \in \mathbb{R}$ and $t \geqslant 0$.

Let now $T(t)$ be a quasi-contractive semigroup. We can rescale this semigroup to obtain a contraction semigroup $S(t) := e^{-wt}T(t)$ for $t \geqslant 0$. The generator of the rescaled semigroup is $B = A - w$. We can now formulate the generation theorem for quasi-contractive semigroups:

Corollary 2.16. *Let $w \in \mathbb{R}$. For a linear operator $(A, D(A))$ on a Banach space X, the following properties are all equivalent.*

(1) $(A, D(A))$ generates a strongly continuous quasi-contractive semigroup, i.e $\|T(t)\| \leqslant e^{wt}$ for $t \geqslant 0$.

(2) $(A, D(A))$ is closed, densely defined, and for every $\lambda > w$ one has $\lambda \in \rho(A)$ and $\|(\lambda - w)R(\lambda, A)\| \leqslant 1$.

(3) $(A, D(A))$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with Re $\lambda > w$ one has $\lambda \in \rho(A)$ and $\|R(\lambda, A)\| \leqslant 1/\text{Re}\,\lambda - w$.

The Hille-Yosida theorem has later on been generalised by Feller, Miyadera and Phillips in 1952, eliminating the requirement of having a generator that generates a contraction semigroup. This comes at the cost of working with growth estimates of the n-th powers of the resolvents which can be hard to check. Since we will usually be able to reduce the problem to the case of contraction semigroups, we will state the generalisation below, but we will not prove it. For a detailed proof see [EN00, p. 77].

Theorem 2.17 (Generation theorem by Feller, Miyadera, Phillips)**.** *For a linear operator $(A, D(A))$ on a Banach space X and the constants $w \in \mathbb{R}$, $M \geqslant 1$ the following properties are all equivalent.*

(1) $(A, D(A))$ generates a strongly continuous semigroup satisfying $\|T(t)\| \leqslant Me^{wt}$ for $t \geqslant 0$

(2) $(A, D(A))$ is closed, densely defined, and for every $\lambda > w$ one has $\lambda \in \rho(A)$ and $\|(\lambda - w)R(\lambda, A)^n\| \leqslant M$ for all $n \in \mathbb{N}$.

(3) $(A, D(A))$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with Re $\lambda > w$ one has $\lambda \in \rho(A)$ and $\|R(\lambda, A)^n\| \leqslant M/(\text{Re}\lambda - w)^n$.

2.4 Dissipative operators and contraction C_0-semigroups

Definition 2.18. *A linear operator $(A, D(A))$ on a Banach space X is called dissipative if*

$$\|(\lambda I - A)x\| \geqslant \lambda\|x\|$$

for all $\lambda > 0$ and $x \in D(A)$.

Definition 2.19. *A dissipative operator A for which $R(I - A) = X$ holds is called m-dissipative. [Paz83]*

Remark: Alternatively we can define an operator as m-dissipative if $R(\lambda I - A) = X$ holds for all $\lambda > 0$. This can be seen as follows: If A is a dissipative operator, then so is μA for all $\mu > 0$. This allows multiplying the left side of the equation in the definition above with an $\lambda > 0$.
We will now introduce properties of dissipative operators ([EN00, p. 98])

Proposition 2.20. *Let* $(A, D(A))$ *be a dissipative operator. Then the following properties hold:*

(1) $\lambda - A$ *is injective for all* $\lambda > 0$ *and*

$$\|(\lambda - A)^{-1}z\| \leqslant \frac{1}{\lambda}\|z\|$$

for all z in $ran(\lambda - A) := (\lambda - A)D(A)$.

(2) $\lambda - A$ *is surjective for some* $\lambda > 0$ *if and only if it is surjective for each* $\lambda > 0$. *In that case, one has* $(0, \infty) \subset \rho(A)$.

(3) A is closed if and only if $ran(\lambda - A)$ *is closed for some (hence all)* $\lambda > 0$.

(4) If ran(A) $\subseteq \overline{D(A)}$, *then* A *is closable. The closure* $\overline{A}$ *is dissipative as well and* $ran(\lambda - \overline{A}) = \overline{ran(\lambda - A)}$

Proof: (1) follows from the definition 2.18.

For (2) we will assume that $(\lambda_0 - A)$ is surjective for an $\lambda_0 > 0$. By (1) we see that $\lambda_0 \in \rho(A)$ and $\|R(\lambda_0, A)\| \leqslant 1/\lambda_0$. The conditions of the series expansion of the resolvent (see 2.12) are fulfilled for the interval $(0, 2\lambda_0)$, hence the series expansion yields that $(0, 2\lambda_0) \subset \rho(A)$. The dissipativity of A yields that $\|R(\lambda, A)\| \leqslant 1/\lambda$ for $0 < \lambda < 2\lambda_0$. By the same argument, we see that $\lambda - A$ is surjective for all $\lambda > 0$, hence $(0, \infty) \subset \rho(A)$.

For (3), we see that A is closed if and only if $\lambda - A$ is closed for some and thus all $\lambda > 0$. This is equivalent to $(\lambda - A)^{-1} : \text{ran}(\lambda - A) \to D(A)$ being closed. By (1) we know that $(\lambda - A)^{-1}$ is bounded and by the closed graph theorem (see annex A6) we know that it is closed if and only if its domain, i.e. $\text{ran}(\lambda - A)$, is closed.

(4) An operator is closable if and only if for each sequence $(x_n)_{n \in \mathbb{N}} \subset D(A)$ with $x_n \to 0$ we have $y = 0$ for $Ax_n \to y$. The definition of the dissipative operator (2.18) implies

$$\|\lambda(\lambda - A)x_n + (\lambda - A)w\| \geqslant \lambda\|\lambda x_n + w\|$$

if $w \in D(A)$ and for all $\lambda > 0$. Letting $n \to \infty$ and dividing both sides by λ we obtain

$$\| - y + w - 1/\lambda Aw\| \geqslant \|w\|$$

Letting $\lambda \to \infty$ the inequality yields

$$\| - y + w\| \geqslant \|w\|.$$

We can now chose a $w \in D(A)$ arbitrarily close to $y \in \overline{\text{ran}(A)}$ so that $0 \geqslant \|y\|$ and hence $y = 0$.
It remains to show that also $\overline{A}$ is dissipative. For a closed operator there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset D(A)$ with $x_n \to x \in D(\overline{A})$ and $Ax_n \to \overline{A}x$ when $n \to \infty$. The dissipativity of A and continuity of the norm imply that $\|(\lambda - \overline{A})x\| \geqslant \lambda\|x\|$ for all $\lambda > 0$. Thus, $\overline{A}$ is dissipative. Finally, $\text{ran}(\lambda - A)$ is dense in $\text{ran}(\lambda - \overline{A})$ and thus by (3) we know that also $\text{ran}(\lambda - \overline{A})$ is closed in X which proves (4). □

We will now characterize dissipative operators via duality sets which will facilitate the work in L^p-spaces:

Definition 2.21 (Duality set). *Let X be a Banach space and X′ its dual space. For every* $x \in X$ *we call*

$$\mathcal{J}(x) := \left\{x' \in X' : \langle x, x'\rangle = \|x\|^2 = \|x'\|^2\right\}$$

its duality set [EN00, p. 87].

Duality sets allow us to characterize a dissipative operator in an another way.

Proposition 2.22. *A linear operator* A *is dissipative if for each* $x \in D(A)$ *there exists an* $x' \in \mathcal{J}(x)$ *such that* $\langle Ax, x'\rangle \leqslant 0$ *or* $Re\langle Ax, x'\rangle \leqslant 0$ *if the underlying space is complex.*

Proof: See [EN00, p. 88].
Remark: If X is a Hilbert space, then we find by the Riesz-Fréchet representation theorem (see A8) that $\mathcal{J} = \{x\}$ for each $x \in H$. Proof see [Hun+13, p. 33]. Using this fact, we find that the condition in proposition 2.22 simplifies to $\mathrm{Re}\langle Ax, x\rangle \leqslant 0$.

Example 2.23. *Let* $X = L^2(\mathbb{R})$ *and the differential operator* A *defined as* $Au = u'$ *with* $D(A) = W^{1,2}(\mathbb{R})$. *We obtain by integration by parts*

$$\langle Au, u\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} u'u dx = \frac{u^2}{2}$$

Unless we introduce additional boundary conditions, the operator A *is generally not dissipative. If we e.g. restrict the space to* $X = L^2([0,1],\mathbb{R})$ *and* $D(A) = W^{1,2}([0,1],\mathbb{R})$ *and set* $u(1) = 0$ *then* A *is dissipative. The same holds for the half-line* $X = L^2([0,\infty),\mathbb{R})$ *and* $D(A) = W^{1,2}([0,\infty),\mathbb{R})$ *if* $u(0) = 0$ *and* $u(t) \to 0$ *for* $t \to \infty$*:*

$$\langle Au, u\rangle_{L^2([0,\infty))} = \int_0^\infty u'u\, dx = \Big(\lim_{t\to\infty} u(t) - u(0)\Big) - \int_0^\infty u'u\, dx$$

which simplifies to

$$2\langle Au, u\rangle_{L^2([0,\infty))} = \lim_{t\to\infty} u(t) = 0$$

as $u \in L^2$.

We close this section with the important Lumer-Phillips theorem that shows dissipative operators can generate contraction semigroups ([EN00, p. 99]):

Theorem 2.24 (Lumer-Phillips). *For a densely defined, dissipative operator* $(A, D(A))$ *on a Banach space X the following statements are equivalent.*

(1) The closure $\overline{A}$ *of* A *generates a contraction semigroup.*

(2) ran$(\lambda I - A)$ *is dense in* X *for some (hence all)* $\lambda > 0$.

Proof: (1) ⇒ (2) From the Hille-Yosida generation theorem (2.15) we can conclude that $\mathrm{ran}(\lambda - \overline{A}) = X$ for all $\lambda > 0$. By proposition 2.20 we have $\mathrm{ran}(\lambda - \overline{A}) = \overline{\mathrm{ran}(\lambda - A)}$ and thus obtain (2).

(2) ⇒ (1) Since $\mathrm{ran}(\lambda I - A)$ is dense in X, we can conclude that $(\lambda - \overline{A})$ is surjective. By proposition 2.20 we know that $(0,\infty) \subset \rho(A)$. Dissipativity of A implies the estimate $\|R(\lambda, \overline{A})\| \leqslant 1/\lambda$ for all $\lambda > 0$ which is one of the requirements of the Hille-Yosida generation theorem (2.15). □

2.5 Analytic C_0-semigroups

In this section we will introduce analytic semigroups. This class of semigroups will be helpful to make statements about the behaviour of semigroups e.g. when it comes to perturbations of generators. As we will need only a few facts about analytic semigroups later on, this introduction is cursory without proofs. We refer to [EN00] chapter II.4 - from where definitions are taken - for proofs and more details. We will begin our overview with sectorial operators which, as we will see, are closely linked to the notion of analytic semigroups.

Definition 2.25 (Sectorial operator). *A closed linear operator $(A, D(A))$ with dense domain $D(A)$ on a Banach space X is called sectorial (of angle δ) if there exists $0 < \delta \leqslant \pi/2$ such that the sector*

$$\Sigma_{\pi/2+\delta} := \left\{\lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + \delta\right\} \setminus \{0\}$$

is contained in the resolvent set $\rho(A)$, and if for each $\varepsilon \in (0, \delta)$ there exists $M_\varepsilon \geqslant 1$ such that

$$\|R(\lambda, A)\| \leqslant \frac{M_\varepsilon}{|\lambda|} \text{ for all } 0 \neq \lambda \in \overline{\Sigma}_{\pi/2+\delta-\varepsilon}.$$

We will see that sectorial operators can be generators of analytic semigroups. What we understand exactly by an analytic semigroup will be defined next.

Definition 2.26 (Analytic semigroup). *Let $(A, D(A))$ be a sectorial operator of angle δ. A family of bounded, linear operators $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ is called an analytic semigroup (of angle δ) if*

(1) $T(0) = I$ *and* $T(z_1 + z_2) = T(z_1)T(z_2)$ *for all* $z_1, z_2 \in \Sigma_\delta$.

(2) The map $z \mapsto T(z)$ *is analytic in* Σ_δ.

(3) $\lim_{\Sigma_{\delta'}\ni z\to 0} T(z)x = x$ *for all* $x \in X$ *and* $0 < \delta' < \delta$.

If in addition

(4) $\|T(z)\|$ *is bounded in* $\Sigma_{\delta'}$ *for all* $0 < \delta' < \delta$, *then* $T(z)$ *is called a bounded analytic semigroup.*

Proposition 2.27 (Representation of analytic semigroups). *Let $(A, D(A))$ be a sectorial operator of angle δ and define $T(0) := I$ and operators $T(z)$ for $z \in \Sigma_\delta$ by*

$$T(z) := \frac{1}{2\pi i}\int_\gamma e^{\mu z} R(\mu, A)\, d\mu$$

where γ is a piecewise smooth curve in $\Sigma_{\pi/2+\delta}$ going from $\infty e^{-i(\pi/2+\delta')}$ to $\infty e^{i(\pi/2+\delta')}$ for some $\delta' \in (|\arg z|, \delta)$. Then the family of maps $(T(z))$ is an analytic semigroup and the operator $(A, D(A))$ is its generator.

Proof. See [EN00, p. 97 ff] □

Similar to the Hille-Yosida theorem, we can characterize those generators that generate analytic semigroups by the following theorem.

Theorem 2.28. *For an operator $(A, D(A))$ on a Banach space X, the following statements are equivalent.*

(1) A *generates a bounded analytic semigroup* $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ *on* X.

(2) There exists $\theta \in (0, \pi/2)$ *such that the operators* $e^{\pm i\theta}A$ *generate bounded strongly continuous semigroups on* X.

(3) A *generates a bounded strongly continuous semigroup* $T(t))_{t\geqslant 0}$ *on* X *such that ran*$(T(t)) \subset D(A)$ *for all* $t > 0$, *and* $M := \sup_{t>0} \|tAT(t)\| < \infty$.

(4) A *generates a bounded strongly continuous semigroup* $T(t))_{t\geqslant 0}$ *on* X *and there exists a constant* $C > 0$ *such that* $\|R(r + is, A)\| \leqslant C/|s|$ *for all* $r > 0$ *and* $0 \neq s \in \mathbb{R}$.

(5) A *is sectorial.*

Proof. See [EN00, p. 101] □

Corollary 2.29. *If* A *is a normal operator on a Hilbert space* H *satisfying*

$$\sigma(A) \subseteq \{z \in \mathbb{C} : \arg(-z) < \delta\}$$

for $\delta \in [0, \pi/2)$, *then* A *generates a bounded analytic semigroup. [EN00, p. 105]*

A normal operator is an operator for which $AA^* = A^*A$ holds. A self-adjoint operator is obviously normal.

2.6 Perturbation of semigroups

In the previous sections we have - at least partially - answered the question under which conditions operators generate a semigroup. In a next step, it seems only naturally to ask what happens if we perturb a semigroup-generating operator $A : D(A) \subset X \to X$ by another operator $B : D(B) \subset X \to X$. In other words, does the sum of $A + B$ again generate a strongly continuous semigroup? We will see that the answer is yes, under certain conditions.
We start our discussion with bounded operators and the bounded perturbation theorem ([EN00, p. 158]):

Theorem 2.30 (Bounded Perturbation Theorem)**.** *Let* $(A, D(A))$ *be the generator of a strongly continuous semigroup* $T(t)$ *on a Banach space* X *satisfying* $\|T(t)\| \leqslant Me^{wt}$ *for all* $t \geqslant 0$, $w \in \mathbb{R}$ *and* $M \geqslant 1$. *Provided that* $B \in \mathcal{L}(X)$, *the sum* $C := A + B$ *with* $D(C) := D(A)$ *generates a strongly continuous semigroup.*

Proof. We first assume that $w = 0$ and $M = 1$, so that $T(t)$ becomes a contraction semigroup. The spectrum of the generator of a contraction semigroup is located in the left half plane with $\sigma(A) \leqslant 0$ (see e.g. corollary IV.2.4 [EN00, p. 252]). Then $\lambda \in \rho(A)$ for all $\lambda > 0$. We can decompose $\lambda - C$ as

$$\lambda - C = \lambda - A - B = (I - BR(\lambda, A))(\lambda - A)$$

We can conclude from the bijectivity of $\lambda - A$ that also $\lambda - C$ is bijective and hence $\lambda \in \rho(C)$ if and only if $I - BR(\lambda, A)$ is invertible in $\mathcal{L}(X)$. In that case we obtain

$$R(\lambda - C) = R(\lambda - A)[I - BR(\lambda - A)]^{-1}$$

Next we choose $\operatorname{Re}\lambda > \|B\|$. By the Hille-Yosida generation theorem (2.15(3)) we have

$$\|BR(\lambda, A)\| \leqslant \|B\| \, \|R(\lambda, A)\| \leqslant \frac{\|B\|}{\operatorname{Re}\lambda} < 1$$

and conclude that $\lambda \in \rho(C)$. By using the Neumann series we obtain

$$R(\lambda, C) = R(\lambda - A) \sum_{n=0}^{\infty} (BR(\lambda, A))^n.$$

We estimate, again by using the Hille-Yosida theorem for the resolvent of A

$$\|R(\lambda, C)\| \leqslant \frac{1}{\operatorname{Re}\lambda} \cdot \frac{1}{1 - \frac{\|B\|}{\operatorname{Re}\lambda}} = \frac{1}{\operatorname{Re}\lambda - \|B\|}$$

for all $\operatorname{Re}\lambda > \|B\|$. By corollary 2.16, C generates a quasi-contractive strongly continuous semigroup $S(t)$.
We return to the general case of $w \in \mathbb{R}$ and $M \geqslant 1$. By rescaling, we can assume that $w = 0$. We introduce a new norm

$$|||x||| := \sup_{t \geqslant 0} \|T(t)x\|$$

on X. This new norm satisfies

$$\|x\| \leqslant |||x||| \leqslant M\|x\|$$

and turns $T(t)$ into a contraction semigroup. We find

$$|||Bx||| \leqslant M\|B\| \cdot \|x\| \leqslant M\|B\| \cdot |||x|||$$

for all $x \in X$. The sum $C = A + B$ generates thus a strongly continuous semigroup. □

In the unbounded case, the situation is more complicated. Results can be found if the unperturbed operator A is the generator of a analytic semigroup or if A is dissipative. In both cases we need additional requirements for the perturbing semigroup B - it must be A-bounded. The definition follows [EN00, p. 169].

Definition 2.31. *Let* X *be a Banach space and* $A : D(A) \subset X \to X$ *a linear operator on* X. *An operator* $B : D(B) \subset X \to X$ *is called* A*-bounded if* $D(A) \subseteq D(B)$ *and if there exist constants* $a, b \in \mathbb{R}_+$ *such that*

$$\|Bx\| \leqslant a\|Ax\| + b\|x\| \tag{2.1}$$

for all $x \in D(A)$. *The* A*-bound of* B *is defined as*

$$a_0 := \inf\{a \geqslant 0 : \exists b \in \mathbb{R}_+ \textit{ s.t. } 2.1 \textit{ holds}\} \tag{2.2}$$

The following two theorems give us particularly nice results for the perturbation of operators if the perturbed operator A generates either an analytic semigroup or a contraction semigroup.

Theorem 2.32. *Let the operator* $(A, D(A))$ *generate an analytic semigroup* $(T(z))_{z \in \Sigma_\delta \cup \{0\}}$ *on a Banach space* X. *Then we can find a constant* $\alpha > 0$ *such that* $(A + B, D(A))$ *generates an analytic semigroup for every* A*-bounded operator* B *with the* A*-bounded* $a_0 < \alpha$.

Proof. See [EN00, p. 176] □

Theorem 2.33. *Let the operator* $(A, D(A))$ *generate an contractive semigroup on a Banach space* X*. If the operator* $(B, D(B))$ *is dissipative and* A*-bounded with* $a_0 < 1$ *then* $(A + B, D(A))$ *generates a contraction semigroup.*

Proof. See [EN00, p. 173] □

Chapter 3

Stability of C_0-semigroups

In this chapter we will introduce the notions of stability, hypercyclicity and chaoticity of semigroups. We will finally derive spectral conditions for a chaotic behaviour of semigroups and apply these insights in the following chapter when we will look at concrete applications to partial differential equations.

3.1 Stability

There are several notions of stability of a C_0-semigroup. We will limit this section to a cursory overview the most relevant definitions and theorems for our purposes, based on [EN00, p. 296].

Definition 3.1. *A strongly continuous semigroup $T(t)$ is called*

(1) uniformly exponentially stable if there exists an $\varepsilon > 0$ such that

$$\lim_{t\to\infty} e^{\varepsilon t}\|T(t)\| = 0, \tag{3.1}$$

(2) uniformly stable if

$$\lim_{t\to\infty} \|T(t)\| = 0, \tag{3.2}$$

(3) strongly stable if

$$\lim_{t\to\infty} \|T(t)x\| = 0 \quad \textit{for all } x \in X. \tag{3.3}$$

The question which conditions a generator of a semigroup must fulfil to be a stable has been a question of relatively recent research. If the underlying space of the generator is a Hilbert space, we are able to characterize the stability of a semigroup more easily than if we operate on e.g. Banach space, which might even be non-reflexive which would require to look at the dual of generators (theorem by Arendt-Batty-Lyubich-Vũ). However, we will limit ourselves to Hilbert spaces in this section.

Theorem 3.2. *Let $(A, D(A))$ be a self-adjoint operator on a Hilbert space H such that $\langle Ax, x\rangle \leqslant 0$ for all $x \in D(A)$. Then the following assertions are equivalent:*

(1) The semigroup generated by A is strongly stable.

(2) 0 is not an eigenvalue of A.

Proof. See [EN00, p. 324]. □

Theorem 3.3. *Let* A *generate a strongly continuous semigroup* $T(t)$ *on a Hilbert space* H. *Then* $T(t)$ *is uniformly exponentially stable if and only if there exists a constant* $M > 0$ *such that*

$$\|R(\lambda, A)\| < M \quad \text{for all } \lambda \text{ with } \mathrm{Re}\,\lambda > 0.$$

Proof. See [Eis10, p. 97] □

3.2 Hypercyclic and chaotic semigroups

We start our considerations about hypercyclic and chaotic semigroups with some elementary definitions as they can be found e.g. in [GP11] or [DSW97]. In this section, X will denote a Banach space, $T(t)$ a strongly continuous semigroup on X, and A its infinitesimal generator.

Definition 3.4. *A periodic point is an* $x \in X$ *for which a* $t > 0$ *exists such that* $T(t)x = x$. *We denote the set of all periodic points by* X_p.

Definition 3.5. *A semigroup* $T(t) : X \to X$ *is called topologically transitive if for any pair* U, V *of non-empty open subsets of* X *there exists some* $t \geqslant 0$ *such that* $T(t)(U) \cap V \neq \emptyset$.

Definition 3.6. *A semigroup* $T(t)$ *is called hypercyclic if there exists* $x \in X$ *such that the orbit* $T(t)x$ *is dense in* X. *We call* x *a hypercyclic vector. If additionally* X_p *is dense in* X *then we call the semigroup chaotic.*

Furthermore we will introduce two notations which we will use later on [DSW97]:

X_∞ will denote the set of all $x \in X$ such that for each $\varepsilon > 0$ there exist some $w \in X$ and some $t > 0$ with $\|w\| < \varepsilon$ and $\|T(t)w - x\| < \varepsilon$.

X_0 will denote the set of all $x \in X$ such that $\lim_{t\to\infty} T(t)x = 0$.

As there are different notions of chaoticity, we would like to point out that this thesis refers to the chaoticity as defined first by Devaney [Dev03, p. 50]. Devaney-chaos (also referred to as topological chaos) requires that the semigroup $T(t)$ is topologically transitive. In definition 3.6 we have only required that $T(t)$ is hypercyclic. The Birkhoff transitivity theorem for semigroups shows that $T(t)$ is hypercyclic if and only if it is transitive and hence we may use the notion of Devaney-chaos. Before proving this theorem, we will need one auxiliary theorem.

Theorem 3.7 (Conejero-Müller-Peris)**.** *If* $T(t)$ *is a hypercyclic semigroup on a Banach space* X *for an* $x \in X$, *then* $T(t_0)$ *is a hypercyclic semigroup for every* $t_0 > 0$.

Proof: See [CMP07].

Theorem 3.8 (Birkhoff transitivity theorem for semigroups, [DSW97])**.** *Let* $T(t)$ *be a strongly continuous semigroup on a separable Banach space* X. *Then the following assertions are equivalent:*

(1) $T(t)$ *is hypercyclic.*

(2) For all $y, z \in X$ *and all* $\varepsilon > 0$ *there exist some* $v \in X$ *and some* $t > 0$ *such that* $\|y - v\| < \varepsilon$ *and* $\|z - T(t)v\| < \varepsilon$.

(3) *For all* $\varepsilon > 0$ *there exists a dense subset* $D \subset X$ *such that for all* $z \in D$ *there exists a dense subset* $D' \subset X$ *such that for all* $y \in D'$ *there exist* $v \in X$ *and* $t > 0$ *such that* $\|y - v\| < \varepsilon$ *and* $\|z - T(t)v\| < \varepsilon$.

Proof: (1) $\Rightarrow$ (2) Let the orbit of $T(t)x$ is dense in X. By 3.7 we see that for each $s > 0$, $T(t)x, t > s$ is also hypercyclic and thus dense. Take an $y, z \in X$, there exists some $s > 0$ such that $\|y - T(s)x\| < \varepsilon$ and some $u > s$ such that $\|z - T(u)x\| < \varepsilon$. We now put $v = T(s)x$ and $u = s + t$. By the semigroup properties $T(u)x = T(s + t)x = T(t)T(s)x = T(t)v$ and hence (2) is obtained.

(2) $\Rightarrow$ (1): Let $\{z_1, z_2, z_3, ...\}$ be a dense sequence in X. We will now construct the sequences $\{y_1, y_2, y_3, ...\} \subset X$ and $\{t_1, t_2, t_3, ...\} \subset [0, \infty)$ inductively: We put $y_1 = z_1$ and t_1. Now we find y_n and t_n for $n > 1$ such that

$$\begin{aligned} \|y_n - y_{n-1}\| &\leqslant \frac{2^{-n}}{\sup\{\|T(t_j)\| \mid j < n\}}, \\ \|z_n - T(t_n)y_n\| &\leqslant 2^{-n}. \end{aligned}$$

The first inequality implies that $\|y_n - y_{n-1}\| \leqslant 2^{-n}$ which in turn means that the sequence y_n has a limit x. We find that

$$\begin{aligned} \|z_n - T(t_n)x\| &\leqslant \|z_n - T(t_n)y_n\| + \|T(t_n)\| \, \|y_n - x\| \\ &\leqslant \|z_n - T(t_n)y_n\| + \sum_{i=n+1}^{\infty} \|T(t_n)\| \, \|y_i - y_{i-1}\| \qquad (*) \\ &\leqslant 2^{-n} + \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n} + 2^{-n} = 2^{-n+1} \qquad (**) \end{aligned}$$

For $(*)$ we use the inequality $\|y_n - x\| \leqslant \sum_{i=n+1}^{\infty} \|y_i - y_{i-1}\|$ and for $(**)$ we employ the geometric series:

$$\sum_{i=n+1}^{\infty} 2^{-i} = \sum_{i=0}^{\infty} 2^{-i} - \sum_{i=0}^{n} 2^{-i} = 2 - \sum_{i=0}^{n} 2^{-i} = 1 - \sum_{i=1}^{n} 2^{-i} = 2^{-n}$$

We can now find an arbitrarily large n such that for $z \in X$ and an $\varepsilon > 0$ so that we have $\|z_n - z\| < \varepsilon/2$. If we also choose n large enough so that $2^{-n+1} < \varepsilon/2$ we find

$$\|T(t_n)x - z\| \leqslant \|z - z_n\| + \|z_n - T(t_n)x\| < \varepsilon.$$

This shows that $T(t)x$ is dense and by definition $T(t)$ is hypercyclic.

(2) $\Rightarrow$ (3) We put $D = D' = X$.

(3) $\Rightarrow$ (2) Since D and D' are dense subsets of X we can find elements close enough so that (2) still holds. To show this we keep $\varepsilon > 0$ and $z \in X$. We can find now a $\tilde{z} \in D$ such that $\|z - \tilde{z}\| < \varepsilon/2$ and respectively a $\tilde{y} \in D'$ with $\|y - \tilde{y}\| < \varepsilon/2$. We now choose t and v as in (3). Furthermore, instead of ε we employ $\varepsilon/2$ and obtain with the help of the triangle inequality

$$\begin{aligned} \|T(t)v - z\| &\leqslant \|T(t)v - \tilde{z}\| + \|\tilde{z} - z\| < \varepsilon, \\ \|v - y\| &\leqslant \|v - \tilde{y}\| + \|\tilde{y} - y\| < \varepsilon. \end{aligned}$$

□

We will close this section with a theorem that allows us to find sufficient conditions for the hypercyclicity of a semigroup.

Theorem 3.9 ([DSW97]). *Let $T(t)$ be a strongly continuous semigroup on a separable Banach space X. If both X_∞ and X_0 are dense subsets, then $T(t)$ is hypercyclic.*

Proof. We apply 3.8(iii) with $D = X_\infty$ and $D' = X_0$. Let $z \in X_0$ and $y \in X_\infty$. Due to the density of X_∞ there is a $\|w\| < \varepsilon$ such that for each $\varepsilon > 0$ and arbitrarily large $t > 0$ we have

$$\|T(t)w - z\| < \frac{\varepsilon}{2}.$$

For t large enough we have $\|T(t)y\| < \varepsilon/2$ since $y \in X_\infty$. Setting $v = y + w$ we find that

$$\begin{aligned}\|z - T(t)v\| \leqslant \|z - T(t)w\| + \|T(t)y\| &< \varepsilon/2 \\ \|y - v\| = \|w\| &< \varepsilon.\end{aligned}$$

□

Before we move on, we would briefly like to explain why hypercyclic and chaotic behaviour of operators cannot occur in a finite dimensional setting. Every finite dimensional Banach space is isomorphic to some $\mathbb{C}^n$. We know that every operator T on $\mathbb{C}^n$ must have one or more eigenvalues λ_j. The adjoint operator T^* must consequently also have one or more eigenvalues $\overline{\lambda_j}$. By the following lemma from [BM09, p. 11], we see that no operator on $\mathbb{C}^n$ can be hypercyclic (and thus also not chaotic).

Lemma 3.10. *Let T be a hypercyclic operator. Then its adjoint T^* has no eigenvalues.*

Proof. Let T be hypercyclic for $x \in X$. We assume by contradiction that its adjoint operator T^* has the eigenvalue λ thus $T^*x^* = \lambda x^*$ for some eigenvalue $x^* \in X^*, x^* \neq 0$. For any $n \geqslant 0$ we have

$$\langle T^n x, x^* \rangle = \langle x, (T^*)^n x^* \rangle = \lambda^n \langle x, x^* \rangle.$$

The hypercyclicity of T for x implies that the left-hand side is dense in $\mathbb{K}$. However, the set $\{\lambda^n \langle x, x^* \rangle, n \in \mathbb{N}\}$ is not dense in $\mathbb{K}$, hence no $x \in X$ can be a hypercyclic vector.

□

3.3 Spectral conditions for chaotic semigroups

In 1997, Desch, Schappacher and Webb published an article in which they laid down spectral conditions of the generators of chaotic semigroups ([DSW97]). These conditions help to establish the chaoticity of semigroups as we will see in chapter 4. For our purposes, it suffices to state the main theorem of their paper. The proof will follow the article by Desch, Schappacher and Webb and add a few details left out in their paper.

Theorem 3.11 (Desch, Schappacher, Webb). *Let X be a separable Banach space and let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $T(t)$ on X. Let U be an open subset of the point spectrum of A which intersects the imaginary axis. For each eigenvalue $\lambda \in U$ let x_λ be a non-zero eigenvector, i.e. $Ax_\lambda = \lambda x_\lambda$. We define for each functional $\phi \in X'$ a function $F_\phi : U \to \mathbb{C}$ by $F_\phi(\lambda) = \langle \phi, x_\lambda \rangle$. If for each $\phi \in X'$ the function F_ϕ is analytic and if F_ϕ does not vanish identically on U unless $\phi = 0$, then $T(t)$ is chaotic.*

Proof: We will define a set $Y_V = \operatorname{span}\{x_\lambda \,|\, \lambda \in V\}$, where $V \subset U$ is an arbitrary subset admitting a cluster point in U. The idea of the proof is to show first that Y_V is dense in X. In a next step we will show that Y_V is included in X_∞ and X_0, hence these two must be dense in X and by 3.9 we know that $T(t)$ must be hypercyclic. Last, we will show that also X_p is dense and thus $T(t)$ is chaotic.

We will prove that Y_V is dense in X by contradiction. We assume that there exist a $\phi \neq 0$ and $\tilde{F}_\phi = \langle \phi, x \rangle = 0$ for all $x \in Y_V$. Since Y_V is a linear subspace, there exists an $\phi \in X'$ by the Hahn-Banach theorem (see A7) with $\tilde{F}_\phi = F_\phi|_{Y'_V}$. By assumption, V has a cluster point in U and F_ϕ is analytic, which implies by the identity theorem for analytic functions (see A9) that $F_\phi \equiv 0$ on U. This contradicts the assumption.

Before showing that the inclusions $Y_V \subset X_0$, $Y_V \subset X_\infty$ and $Y_V \subset X_p$ hold, we will make one observation. We multiply both sides of the identities 2.9 with $e^{\lambda t}$ and obtain

$$\begin{aligned} T(t)x - e^{\lambda t}x &= (A-\lambda)\int_0^t e^{\lambda(t-s)}T(s)x\, ds \qquad &&\text{if } x \in X, \\ &= \int_0^t e^{\lambda(t-s)}T(s)(A-\lambda)x\, ds \qquad &&\text{if } x \in D(A). \end{aligned}$$

If λ is an eigenvalue and $x \in D(A)$ it follows that $(A-\lambda)x = 0$ and hence $T(t)x = e^{\lambda t}x$ and $T(t)e^{-\lambda}x = x$.

We will now show that X_0 is dense. We choose V_1 to be a subset of $\{\lambda \in U \,|\, \mathrm{Re}(\lambda) < 0\}$ admitting a cluster point in U and $Y_{V_1} = \mathrm{span}\{x_\lambda \,|\, \lambda \in V_1\}$. For any $y_\lambda \in Y_{V_1}$ with a fixed λ we find

$$\lim_{t\to\infty} T(t)y_\lambda = \lim_{t\to\infty} e^{\lambda t}y_\lambda = 0$$

since $\mathrm{Re}(\lambda) < 0$. We conclude that $Y_{V_1} \subset X_0$. Since Y_{V_1} is dense in X, also X_0 must be dense in X.

Next, we will show that X_∞ is dense. We define the subset V_2 of $\{\lambda \in U \,|\, \mathrm{Re}(\lambda) > 0\}$, again admitting a cluster point in U, and $a x_\lambda \in Y_{V_2} = \mathrm{span}\{x_\lambda \,|\, \lambda \in V_2\}$. We find that

$$\sum_{j=1}^{n} a_j x_{\lambda_j} = T(t)\sum_{j=1}^{n}\left(a_j e^{-\lambda_j t}x_{\lambda_j}\right).$$

We define $w = \sum_{j=1}^{n}\left(a_j e^{-\lambda_j t}x_{\lambda_j}\right)$. Since $\mathrm{Re}(\lambda) > 0$ the individual terms converge to zero when $t \to \infty$. Hence, for $t > 0$ large enough we find any $\varepsilon > 0$ such that $\|w\| < \varepsilon$. Furthermore, we see that $T(t)w = \sum_{j=1}^{n}\left(a_j x_{\lambda_j}\right)$ and therefore $\|T(t)w - \sum_{j=1}^{n}\left(a_j x_{\lambda_j}\right)\| < \varepsilon$. This shows that $Y_{V_2} \subset X_\infty$ and by the same argumentation as above X_∞ is dense in X.

Finally, we prove that X_p is dense. Let V_3 be the subset of $\{\lambda \in U \,|\, \mathrm{Re}(\lambda) = 0\}$ where each λ has a rational imaginary part and such that the sequence $\{\lambda_j\}$ has a cluster point. We can now write each λ_j in the form $i\frac{p_j}{q}$ with $p_j \in \mathbb{Z}$ and $q \in \mathbb{N}\setminus\{0\}$. Let now $a x_\lambda \in Y_{V_3} = \mathrm{span}\{x_\lambda \,|\, \lambda \in V_3\}$. For all $t = 2\pi q > 0$ we have

$$T(t)\sum_{j=1}^{n} a_j x_{\lambda_j} = \sum_{j=1}^{n} e^{\lambda t}a_j x_{\lambda_j} = \sum_{j=1}^{n} e^{\frac{itp_j}{q}} a_j x_{\lambda_j} = \sum_{j=1}^{n} a_j x_{\lambda_j}$$

Hence $Y_{V_3} \subset X_p$ and thus X_p is dense in X. □

The article by Desch, Schappacher, Webb and their criteria for chaoticity has proven to be helpful when verifying if semigroups are indeed chaotic or not. The criteria in 3.11 were later on generalised by Banasiak and Moszyński who showed that if we remove the non-degeneracy conditions, the semigroup $T(t)$ is still chaotic on a smaller, still infinite-dimensional subspace of X, which is $T(t)$-invariant [BM05]. By $T(t)$-invariance we mean that $T(t)\tilde{X} \subseteq \tilde{X}$ for all $t \geqslant 0$, $\tilde{X}$ being a closed subspace of X.

Chapter 4

Application of semigroup theory to differential equations

4.1 The abstract Cauchy problem

Definition 4.1. *The following problem*

$$(\mathrm{ACP}) \qquad \begin{cases} \dfrac{d}{dt}u(t) = Au(t) & \textit{for } t \geqslant 0 \\ u(0) = x \end{cases}$$

is called the abstract Cauchy problem.

If we consider $(A, D(A))$ to be the generator of a C_0-semigroup $T(t)$ on X, then

$$u(t, x) = T(t)x$$

is the unique solution of the ACP as long as $x \in D(A)$. This is easily seen by the properties of the generator of a semigroup. We remember that $\frac{d}{dt}T(t)u(t) = T(t)Au(t) = AT(t)u(t)$. If we allow any $x \in X$, the unique solution of the corresponding integral equation is

$$u(t) = A\int_0^t u(s)ds + x, \ t \geqslant 0.$$

Deriving $u(t)$ with respect to t and setting $t = 0$ respectively gives the ACP in the form above and shows that the integral equation holds. This solution is called a classical solution if $x \in D(A)$ and a mild solution if $x \in X$. We call $T(t)$ the solution semigroup of the ACP.
We call the abstract Cauchy problem (ACP) well posed if the domain $D(A)$ is dense and for each $x \in D(A)$ there exists a unique classical solution that depends continuously on the initial value x. The following theorem establishes that an operator is indeed a generator of a C_0-semigroup if the ACP is well posed [BKR17, p. 124].

Theorem 4.2. *A closed linear operator A on a Banach space X is the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ if and only if the abstract Cauchy problem (ACP) is well posed.*

Proof. Let A be a closed linear operator and the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$. By theorem 2.7 $D(A)$ is dense and A determines the semigroup uniquely, hence the ACP is well-posed.

For the converse we assume the ACP to be well-posed. Then the ACP has a unique classical solution
$$u(t,x) = T(t)x$$
with $x \in D(A)$ and $t \geqslant 0$. All $T(t)$ are bounded operators on X and linear - due to the linearity of A and the uniqueness of u. We see that the semigroup property holds as follows:
$$\begin{aligned} T(t+s)x &= u(t+s,x) \\ &= u(t,u(s,x)) \\ &= T(t)T(s)x \end{aligned}$$
for every $x \in D(A)$ and $t,s \geqslant 0$. Furthermore, we see that $T(0)x = u(0,x) = x$.

We observe that $\|T(t)\|$ is uniformly bounded on every compact interval $[0,\tau]$. To see why that is, we assume that there exists a sequence $(t_k) \subset [0,\tau]$ with $\|T(t_k)\| \to \infty$ as $k \to \infty$. We could choose an $(x_k) \in D(A)$ converging to 0 such that
$$\|u(t_k,x_k)\| = \|T(t_k)x_k\| \geqslant 1$$
which contradicts the assumptions on u.

The mapping $t \to T(t)x$ is continuous for every x in the dense set $D(A)$ and proposition 2.2(c) implies that $T(t)$ is a strongly continuous semigroup on X. Let B be the generator of $T(t)$. Since A and B are both closed, have dense domains and coincide on the $T(t)$ invariant set $D(A)$ they are equal. □

4.2 An example of a stable solution: the heat equation

The simplified, n-dimensional heat equation is the partial differential equation
$$u(x,t)_t = \Delta u(x,t) \text{ on } \mathbb{R}^n \times (0,\infty),$$
where Δ is the Laplace operator with the domain $D(\Delta) = W^{2,2}(\mathbb{R}^n)$.
It is a well known result that the closure of Laplace operator generates a strongly continuous semigroup
$$T(t)f(s) := (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{-|s-r|^2}{4t}} f(r)dr$$
on the domain $X = L^p(\mathbb{R}^n), 1 \leqslant p < \infty$ for $t > 0, s \in \mathbb{R}^n$ and $f \in X$. Proof see [EN00, p. 69].This semigroup is often referred to a heat semigroup or Gaussian semigroup.
If we impose boundary conditions on the domain of the generator, the semigroup can often not be stated explicitly anymore. We have to conclude properties of the generated semigroup from the properties of the generator. We will look at two concrete examples on one-dimensional intervals on which we impose Dirichlet boundary conditions: Let $X = L^2([0,1],\mathbb{C})$ and $d > 0$:
$$\begin{aligned} v_t(x,t) &= dv_{xx}(x,t) && \text{on } [0,1]\times(0,\infty) \\ v(0,t) = v(1,t) &= 0 && \text{for } t \geqslant 0 \\ v(x,0) &= f(x) && \text{for } x \in [0,1] \text{ with } f \in X. \end{aligned} \tag{4.1}$$

In addition, we will consider a section problem: Let $X = L^2([0,1],\mathbb{C})$, $d > 0$ and $e < 0$:

$$\begin{aligned} w_t(x,t) &= dw_{xx}(x,t) + ew(x,t) && \text{on } [0,1]\times(0,\infty) \\ w(0,t) = w(1,t) &= 0 && \text{for } t \geqslant 0 \\ w(x,0) &= f(x) && \text{for } x \in [0,1] \text{ with } f \in X. \end{aligned} \tag{4.2}$$

We can rewrite the problems 4.1 and 4.2 with operators as follows:

$$\begin{aligned} Af &= f'' && D(A) = \{f \in W^{2,2}([0,1],\mathbb{C}) | f(0) = f(1) = 0\} \\ Bf &= f && D(B) = \{f \in L^2([0,1],\mathbb{C}) | f(0) = f(1) = 0\} \\ C_1 &= dA \\ C_2 &= dA + eB \end{aligned}$$

We will show that the closure of A generates a contraction semigroup by the Lumer-Phillips theorem (2.24). Hence, we need to show that A is dissipative and densely defined and that $\operatorname{ran}(\lambda - A)$ is dense in X. The dissipativity of A can be shown via the duality set (2.22). By integration of parts we obtain

$$\langle Af, f\rangle_{L^2([0,1])} = \int_0^1 \Delta f\, f\, dx = -\int_0^1 (\nabla f)^2\, dx \leqslant 0. \tag{4.3}$$

$(A, D(A))$ is densely defined in X, due to the fact that $D(A) = W^{2,2}([0,1]) \subset W^{0,2}[0,1] = L^2([0,1])$. What remains to be proved is that $\operatorname{ran}(\lambda - A)$ is dense in X for any $\lambda > 0$. We will make use of the fact that the image of a dense subset remains dense under a surjective continuous function. We will use a similar approach as Engel and Nagel ([EN00, p. 94]. By using λ^2 we ensure that it is positive as required by the Lumer-Phillips theorem. We see that $\lambda^2 - A$ is surjective if for every $g \in W^{2,2}$ there exists a function $f \in W^{2,2}$ with $f(0) = 0$ and $f(1) = 0$ such that $\lambda^2 f - f'' = g$. Such a function can be found. Hence $\operatorname{ran}(\lambda^2 - A)$ is dense and therefore $(\overline{A}, D(\overline{A}))$ generates a contraction semigroup. The above argumentation also holds for the generator dA as long as $d > 0$ so that the closure of C_1 is the generator of a contraction semigroup, too.

We will now turn to the operator C_2. By using perturbation theorem 2.32 we will show that C_2 generates an analytic semigroup if dA generates an analytic semigroup and if B is an A-bounded operator with the the A-bound $a_0 < \alpha$. We will start by proving that dA generates an analytic semigroup.

We recall that an operator A^* is called an adjoint operator if $\langle A^* f, g\rangle = \langle f, Ag\rangle$. The operator A is called self-adjoint if $Af = A^* f$ and $D(A) = D(A^*)$. We verify that dA with $d \in \mathbb{R}$, $d > 0$ is indeed self-adjoint

$$\langle f, dAg\rangle_{L^2([0,1])} = d\int_0^1 \bar{f}\, g''\, dx = d\left(\bar{f}\, g'\Big|_0^1 + \int_0^1 \overline{f'}\, g'\, dx\right) = d\int_0^1 \overline{f'}\, g'\, dx$$

$$\langle dAf, g\rangle_{L^2([0,1])} = d\int_0^1 \overline{f''}\, g\, dx = d\left(\overline{f'}\, g\Big|_0^1 + \int_0^1 \overline{f'}\, g'\, dx\right) = d\int_0^1 \overline{f'}\, g'\, dx$$

with the domain $D(A) = \{f \in W^{2,2}([0,1])\}$. Self-adjoint operators are normal and with corollary 2.29 we see that dA generates a bounded analytic semigroup (Remember that A generates is a contraction semigroup, which implies that $\sigma(A) \leqslant 0$).

In a next step we verify that eB is dA-bounded, i.e. that $\|Bf\| \leqslant a\|Af\| + b\|f\|$ holds. With $eBf = ef$ we find

$$\|ef\| \leqslant a\|dAf\| + b\|f\|.$$

Let $b = |e|$, then the inequality holds for any $a \geqslant 0$, hence the A-bound of B is 0. Furthermore, $W^{2,2}[0,1] \subset L^2[0,1]$, also $D(dA) \subseteq D(eB)$. By theorem 2.32 $(C_2, D(C_1))$ generates an analytic semigroup.

We will now turn to the stability of the semigroups generated by the generators C_1 and C_2. It was shown that C_1 is both a self-adjoint and dissipative operator on a Hilbert space. By 3.2, C_1 generates a strongly stable semigroup. Using the same theorem, we can show that also C_2 generates a strongly stable semigroup. For this we will quickly verify that C_2 is indeed also self-adjoint and fulfils the condition $\langle C_2 f, f\rangle_{L^2[0,1]} \leqslant 0$. We prove dissipativity first:

$$\langle C_2 f, f\rangle_{L^2([0,1])} = \int_0^1 d\Delta f\, f + ef^2\, dx = -d\int_0^1 (\nabla f)^2\, dx + e\int_0^1 f^2 dx \leqslant 0. \tag{4.4}$$

We know that the right hand side is smaller than zero from equation 4.3 (remember that $d > 0$) and the fact that $e < 0$, so that also the second term must be negative. We verify self-adjointness:

$$\begin{aligned}
\langle f, dAg + eBg\rangle_{L^2([0,1])} &= \int_0^1 \bar{f}\,(dg'' + eg)\, dx \\
&= d\int_0^1 \bar{f}g''\, dx + e\int_0^1 \bar{f}g\, dx \\
&= d\bar{f}\,g'\Big|_0^1 + d\int_0^1 \overline{f'}\,g'\, dx + e\int_0^1 \bar{f}g\, dx \\
&= d\int_0^1 \overline{f'}\,g'\, dx + e\int_0^1 \bar{f}g\, dx
\end{aligned}$$

$$\begin{aligned}
\langle dAf + eBf, g\rangle_{L^2([0,1])} &= \int_0^1 (d\overline{f''} + e\bar{f})g\, dx \\
&= d\int_0^1 \overline{f''}\,g\, dx + e\int_0^1 \bar{f}g\, dx \\
&= d\,\overline{f'}\,g\Big|_0^1 + d\int_0^1 \overline{f'}\,g'\, dx + e\int_0^1 \bar{f}g\, dx \\
&= d\int_0^1 \overline{f'}\,g'\, dx + e\int_0^1 \bar{f}g\, dx.
\end{aligned}$$

Furthermore, $D(C_2) = D(C_2^*)$ holds. Hence, also C_2 generates a strongly stable semigroup.

4.3 An example of a chaotic semigroup

The following example is taken from the paper of Desch, Schappacher and Webb which shows a chaotic solution semigroup of a partial differential equation [DSW97, p. 807]. We

will prove that the generated semigroup is indeed chaotic by verifying the conditions of theorem 3.11.
Let $X = L^2([0,\infty),\mathbb{C})$. We consider the following partial differential equation:

$$\begin{aligned} u_t(x,t) &= au_{xx}(x,t) + bu_x(x,t) + cu(x,t), \\ u(0,t) &= 0 && \text{for } t \geqslant 0 \\ u(x,0) &= f(x) && \text{for } x \geqslant 0 \text{ with } f \in X \end{aligned} \tag{4.5}$$

Let $a, b, c > 0$ and $c < b^2/(2a) < 1$. We claim that the solution semigroup to this problem is chaotic.
Proof: We re-write the partial differential equations in terms of the following operators:

$$\begin{aligned} A_1 f &= f'' & D(A_1) &= \{f \in W^{2,2}([0,\infty),\mathbb{C}) | f(0) = 0\} \\ B_1 f &= f' & D(B_1) &= \{f \in W^{1,2}([0,\infty),\mathbb{C})\} \\ A &= aA_1 + bB_1 + cI \end{aligned}$$

The defined operator A_1 is the generator of a contraction semigroup and so is aA_1 for any $a > 0$. The operator B_1 is a dissipative operator by example 2.23 and taking into account the remark in 2.19 so is bB_1. In order to apply theorem 2.33 we need to verify if bB_1 is (aA_1)-bounded with $a_0 < 1$, i.e.

$$\|b_1 Bf\| \leqslant \alpha \|aA_1 f\| + \beta \|f\|.$$

In example III.2.2 Engel and Nagel (see [EN00, p. 169] for the full proof) show that an operator $\tilde{B} = \frac{d}{dx}$ with $D(\tilde{B}) := W^{1,p}(\Omega)$ is $\tilde{A}$-bounded with $\tilde{A} = \frac{d^2}{dx^2}$ on $D(A) = W^{2,p}(\Omega)$ where $\Omega \subseteq \mathbb{R}$. The $\tilde{A}$-bound $a_0 = 0$. They estimate:

$$\|\tilde{B}f\|_p \leqslant \frac{9}{\varepsilon}\|f\|_p + \varepsilon\|\tilde{A}f\|_p.$$

Let $\tilde{A} = aA_1$ and $\tilde{B} = bB_1$ with $a, b > 0$ and $b < \infty$ we find

$$\|B_1 f\|_p \leqslant \frac{9}{b\varepsilon}\|f\|_p + \frac{a\varepsilon}{b}\|A_1 f\|_p.$$

Choosing $\varepsilon > 0$ small enough, one sees that the A_1-bound is still zero. Furthermore, $D(A_1) \subset D(B_1)$, since $W^{m,p}(\Omega) \subset W^{k,p}(\Omega)$ for all $k \leqslant m$. By theorem 2.33 $aA_1 + bB_1$ generates a contraction semigroup. Since clearly $cI \in \mathcal{L}$, the bounded perturbation theorem (2.30) shows that A is the generator of a strongly continuous semigroup.

In a next step we will verify the criteria of 3.11 to show that the semigroup is chaotic. For this, we define the slotted disk

$$U = \left\{\lambda \in \mathbb{C} \,\middle|\, \left|\lambda - \left(c - \frac{b^2}{4a}\right)\right| \leqslant \frac{b^2}{4a}, \mathrm{Im}(\lambda) \neq 0 \text{ if } \mathrm{Re} \leqslant c - \frac{b^2}{4a}\right\}$$

As $c < b^2/(2a) < 1$ and $a, b, c > 0$ we see that U intersects the imaginary axis. We look at the eigenvalue problem $Af = \lambda f$ or

$$af_\lambda'' + bf_\lambda' + cf_\lambda = \lambda f_\lambda \quad \text{and} \quad f_\lambda(0) = 0$$

A solution for this problem is

$$f_\lambda(x) = e^{-(b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right).$$

The boundary condition is quickly verified to hold. We can verify the solution by deriving f_λ and multiplying with a, b and c respectively:

$$\begin{aligned}
cf_\lambda(x) =& ce^{-(b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right)\\
bf'_\lambda(x) =& -\frac{b^2}{2a}e^{-(b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right) + be^{-(b/2a)x}\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \cos\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right)\\
af''_\lambda(x) =& \frac{b^2}{4a}e^{-(b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right) - \frac{b}{2}e^{-(b/2a)x}\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \cos\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right)\\
&- \frac{b}{2}e^{-(b/2a)x}\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \cos\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right)\\
&- ae^{-(b/2a)x}\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right).
\end{aligned}$$

It is quickly verified that the cosine terms cancel out when inserting the derivatives into the differential equation. We will look at the remaining terms and see that:

$$\left(c - \frac{b^2}{2a} + \frac{b^2}{4a} - a\left(\frac{c-\lambda}{a} - \frac{b^2}{4a^2}\right)\right)\left(e^{-(b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right)\right) = \lambda f_\lambda.$$

To ensure that f_λ are indeed eigenvectors of A with eigenvalues λ we must show that $f_\lambda \in X$ and that $f''_\lambda \in X$ so that $f_\lambda \in D(A)$. We estimate f_λ by using the well-known fact that the sine function can be estimated from above by the exponential function, hence

$$\begin{aligned}
|f_\lambda(x)| &\leqslant e^{-(b/2a)x} \exp\left[\left|\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right| x\right]\\
&\leqslant \exp\left[\frac{x}{\sqrt{a}}\left[-\sqrt{\frac{b^2}{4a}} + \sqrt{\left|\lambda - \left(c - \frac{b^2}{4a}\right)\right|}\right]\right] \xrightarrow[x\to\infty]{} 0.
\end{aligned}$$

As the last term in the exponent is smaller than zero, also $|f_\lambda(x)| \to 0$ as $x \to +\infty$, hence $f_\lambda \in L^2([0,\infty),\mathbb{C}) = X$. Similarly it can be verified that $f''_\lambda \in X$:

$$\begin{aligned}
f''_\lambda(x) =& \frac{b^2}{4a^2}e^{-(b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right)\\
&- \frac{b}{a}e^{-(b/2a)x}\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \cos\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right)\\
&- e^{-(b/2a)x}\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right).
\end{aligned}$$

The first and third term of the right hand side go to zero as $x \to \infty$ by a similar argument as above which leaves us with the middle term. We use the fact that $\cos(\cdot) < 1$ and obtain:

$$\begin{aligned}
|f''_\lambda(x)| &\leqslant \left[\frac{1}{\sqrt{a}}\left[-\sqrt{\frac{b^2}{a}}+\sqrt{\left|\lambda-\left(c-\frac{b^2}{4a}\right)\right|}\right]\right] e^{-(b/2a)x} \\
&\leqslant \left[\frac{1}{\sqrt{a}}\left[-\sqrt{\frac{b^2}{a}}+\sqrt{\frac{b^2}{4a}}\right]\right] e^{-(b/2a)x} \\
&= -\frac{b}{2a}e^{-(b/2a)x} \to 0
\end{aligned}$$

as $x \to +\infty$.

Choose an $\phi \in X' = X$ and $\lambda \in U$, we define $F_\phi(\lambda)$ as required by theorem 3.11:

$$F_\phi(\lambda) = \langle \phi, F_\lambda \rangle = \int \phi(x) e^{-(b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a}-\frac{b^2}{4a^2}}\right) dx. \tag{4.6}$$

We rename the first part of the integrant in 4.6

$$\psi(x) = \begin{cases} e^{-(b/2a)x}\phi(x) & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases}$$

which yields

$$F_\phi(\lambda) = \int \psi(x) \sin\left(x\sqrt{\frac{c-\lambda}{a}-\frac{b^2}{4a^2}}\right) dx.$$

Using the identity:

$$\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$$

we obtain

$$F_\phi(\lambda) = \frac{1}{2i}\left[\int \psi(x) e^{ix\sqrt{\frac{c-\lambda}{a}-\frac{b^2}{4a^2}}}\, dx - \int \psi(x) e^{-ix\sqrt{\frac{c-\lambda}{a}-\frac{b^2}{4a^2}}}\, dx\right]. \tag{4.7}$$

This is equivalent to writing F_ϕ in terms of the Fourier transform $\tilde{\psi}$ of ψ:

$$F_\phi(\lambda) = \frac{1}{2i}\left[\tilde{\psi}\left(-\sqrt{\frac{c-\lambda}{a}-\frac{b^2}{4a^2}}\right) - \tilde{\psi}\left(\sqrt{\frac{c-\lambda}{a}-\frac{b^2}{4a^2}}\right)\right]. \tag{4.8}$$

The integrals in 4.7 converge absolutely and the square root is analytic for $\lambda \in U$. Hence, $F_\phi(\lambda)$ depends analytically on $\lambda \in U$. Assume that F_ϕ vanishes on the whole set U then by analycity we can see from 4.8 that

$$\tilde{\psi}(\mu) = \tilde{\psi}(-\mu) \quad \text{for all } \mu \in \mathbb{R}.$$

Consequently, $\tilde{\psi}$ is an even function which implies that also ψ is even, since the Fourier transform is even if and only if the function is even (see e.g. [Vre11, p. 167]). As $\psi = 0$ on the negative half-line, ψ must vanish everywhere, implying that $\phi = 0$.

This satisfies the criteria by Desch, Schappacher and Webb (theorem 3.11) and infers that the semigroup generated by A is indeed chaotic.

4.4 Outlook

We have shown in the previous section that equation 4.5 has a chaotic solution semigroup. Furthermore, we know that the equations 4.1 and 4.2 have a stable solution semigroup. It would be interesting to consider the case of coupling these equations on their boundary on the y-axis and mirroring equation 4.2 on the y-axis (the case 4.1 is included by allowing $e = 0$). This would lead to the following problem:

$$\begin{aligned}
w_t(x,t) &= dw_{xx}(x,t) + ew(x,t) && \text{on } [-1,0] \times [0,\infty) \\
u_t(x,t) &= au_{xx}(x,t) + bu_x(x,t) + cu(x,t) && \text{on } [0,\infty) \times [0,\infty) \\
w(-1,t) &= 0 \\
w(0,t) &= u(0,t) \\
w_x(0,t) &= u_x(0,t)
\end{aligned}$$

Similarly to our examples, we rewrite this problem in terms of operators:

$$\begin{aligned}
A_1 f &= f'' & D(A_1) &= \{f \in W^{2,2}([0,\infty),\mathbb{C})\} \\
A_2 g &= g'' & D(A_2) &= \{g \in W^{2,2}([-1,0],\mathbb{C}) \mid g(-1) = 0\} \\
B_1 f &= f' & D(B_1) &= \{f \in W^{1,2}([0,\infty),\mathbb{C})\}
\end{aligned}$$

This would lead to the coupled operator $A(f,g) = (aA_1 + bB_1 + cI,\, dA_2 + eI)$ which domain would be $D(A) = \{W^{2,2}([0,\infty),\mathbb{C}) \oplus W^{2,2}([-1,0],\mathbb{C}) \mid g(-1) = 0; f(0) = g(0); f'(0) = g'(0)\}$. Similar as in the last section, one would be interested in the point spectrum of the operator A which would lead to the eigenvalue problem $Af_\lambda = \lambda f_\lambda$, respectively $Ag_\lambda = \lambda g_\lambda$ keeping the boundary conditions as defined above:

$$dg_\lambda'' + eg_\lambda = \lambda g_\lambda \qquad x \in [-1,0] \tag{4.9}$$

$$af_\lambda'' + bf_\lambda' + cf_\lambda = \lambda f_\lambda \qquad x \in [0,\infty). \tag{4.10}$$

Solving this system of equations would lead to an equation for f with parameters a, b, c, d, e, λ. The solution of these differential equation can be found in annex 2. Similarly as in chapter 4.3 one would need to verify that indeed $f'' \in L^2$. We assume that this would lead to conditions for the parameters under which $f'' \in L^2$ holds. In that case, one could prove that the coupled operator will indeed generate a chaotic semigroup.

Annex 1: Review functional analysis, complex analysis and operator theory

In this chapter, we will gather some definitions and results from functional analysis and operator theory which will be used in the main part of this thesis.

A 1 (Elementary definitions). *Let X, Y be Banach spaces.*

- *A compact operator T maps any bounded subset of a X to a relatively compact set in Y.*
- *A densely defined linear operator T is a linear operator that is defined on a dense linear subspace $D(T)$ of X and takes values in Y: $D(T) \subseteq X \to Y$.*

A 2. *Let T be a bounded linear operator on a Banach space X and let X' be its dual space. A family of operators $(T_i)_{i \in I} \subset \mathcal{L}(X)$ converges to $T \in \mathcal{L}(X)$ if and only if:*

- $\|T(t) - T\| \to 0$ *(uniform operator topology)*
- $\|T(t)x - Tx\| \to 0 \ \forall x \in X$ *(strong operator topology, SOT)*
- $|\langle T(t)x - Tx, x' \rangle| \to 0 \ \forall x \in X, x' \in X'$ *(weak operator topology, WOT)*

A 3 (Uniform boundedness principle). *Let K be a subset of $\mathcal{L}(X)$. Then the following properties are equivalent*

(1) K *is bounded for the SOT, i.e.* $\sup \|T(x)\| < \infty$ *for all* $x \in X$ *and* $T \in K$

(2) K *is uniformly bounded, i.e.* $\sup \|T\| < \infty$ *for all* $T \in K$

For a proof, see e.g [Wer18, p. 157].

A 4. *Let X, Y be a Banach space, $T \in \mathcal{L}(X, Y)$ and $(T_k) \subset \mathcal{L}(X, Y)$ be sequences bounded in norm. Then the following statements are equivalent [BKR17, p. 333].*

(1) $T_k x \to Tx$ *for all* $x \in X$ *(Convergence in SOT).*

(2) There is a dense subspace $D \subset X$ so that for all $x \in D$ we have pointwise convergence $T_k x \to Tx$ in X.

(3) For every compact set $K \subset X$ and all $x \in K$ we have uniform convergence $T_k x \to Tx$ in X.

A 5 (Compact sets in Banach spaces). *Let X be a Banach space and F be a function from a compact set $K \subset \mathbb{R}$ into $\mathcal{L}(X)$. Then the following assertions are equivalent.*

(1) F *is continuous for the SOT, i.e.* $K \ni t \mapsto F(t)x \in X$ *are continuous for every* $x \in X$.

(2) F *is uniformly bounded on* K *and the maps* $K \ni t \mapsto F(t)x \in X$ *are continuous for all* x *in some dense* $D \subset X$.

(3) F *is continuous for the topology of uniform convergence on compact subsets of* X*, i.e.*

$$K \times C \ni (t, x) \mapsto F(t)x \in X$$

is uniformly continuous for every compact set C *in* X*.*

For a proof see [EN00, p. 37].

A 6 (Closed graph theorem)**.** *Let* X *and* Y *be Banach spaces. If* $T \in \mathcal{L}(X, Y)$ *and closed then* T *is continuous.*

For a proof, see e.g [Wer18, p. 174].

A 7 (Hahn Banach)**.** *Let* X *be a normed vector space and* U *a linear subspace of* X*. Then for each linear functional* $u : U \to \mathbb{K}$ *there exists a continuous linear functional* $x : X \to \mathbb{K}$ *such that* $x\big|_U = u$ *and* $\|x\| = \|u\|$*.*

See e.g. [Wer18, p. 109].

A 8 (Riesz-Fréchet representation theorem)**.** *Let* H *be a Hilbert space and let* H' *denote its dual space. For each continuous functional* $\phi \in H'$ *exists exactly one* $y \in H$ *such that* $\phi(x) = \langle x, y \rangle$ *for all* $x \in H$*. Furthermore,* $\|y\|_H = \|\phi\|_{H'}$*.*

See e.g. [Wer18, p. 246].

A 9 (Identity theorem for analytic functions)**.** *Let* $f : D \to \mathbb{C}$ *be an analytic function on a region* D*. Suppose that* $f(z_n) = 0$ *for a sequence* $\{z_n\} \in D$ *where* $z_n \to z \in D$*, i.e. it has a cluster point. Then* f *is identically zero in* D*.*

For a proof see e.g. [Sim15, p. 54].

Annex 2: Solution to the eigenvalue problem in chapter 4.4

The solution for equation of 4.9 is:

$$g_\lambda(x) = C_1 e^{x\sqrt{\frac{\lambda-e}{d}}} + C_2 e^{-x\sqrt{\frac{\lambda-e}{d}}}$$

if $\lambda - e \geqslant 0$. The boundary condition $g(-1) = 0$ yields

$$C_1 = -C_2 e^{\left(2\sqrt{\frac{\lambda-e}{d}}\right)},$$

hence

$$g_\lambda(x) = C_2\left(-e^{\left(2\sqrt{\frac{\lambda-e}{d}}\right)} e^{\left(x\sqrt{\frac{\lambda-e}{d}}\right)} + e^{\left(-x\sqrt{\frac{\lambda-e}{d}}\right)}\right), \text{ and}$$

$$g'_\lambda(x) = C_2\left(\sqrt{\frac{\lambda-e}{d}}\right)\left(-e^{\left(2\sqrt{\frac{\lambda-e}{d}}\right)} e^{\left(x\sqrt{\frac{\lambda-e}{d}}\right)} - e^{\left(-x\sqrt{\frac{\lambda-e}{d}}\right)}\right)$$

We note that $g(0) = C_2\left(-e^{\left(2\sqrt{\frac{\lambda-e}{d}}\right)} + 1\right)$ and $g'(0) = C_2\left(\sqrt{\frac{\lambda-e}{d}}\right)\left(-e^{\left(2\sqrt{\frac{\lambda-e}{d}}\right)} - 1\right)$. Assuming $\lambda - e < 0$ we obtain

$$\hat{g}_\lambda(x) = C_1 \cos\left(x\sqrt{\frac{e-\lambda}{d}}\right) + C_2 \sin\left(x\sqrt{\frac{e-\lambda}{d}}\right).$$

In that case, the boundary condition $g(-1) = 0$ yields

$$C_1 = -C_2 \tan\left(-\sqrt{\frac{e-\lambda}{d}}\right)$$

which leads to

$$\hat{g}_\lambda(x) = C_2\left[-\tan\left(-\sqrt{\frac{e-\lambda}{d}}\right)\cos\left(x\sqrt{\frac{e-\lambda}{d}}\right) + \sin\left(x\sqrt{\frac{e-\lambda}{d}}\right)\right], \text{ and}$$

$$\hat{g}'_\lambda(x) = C_2\left(\sqrt{\frac{e-\lambda}{d}}\right)\left[\tan\left(-\sqrt{\frac{e-\lambda}{d}}\right)\sin\left(x\sqrt{\frac{e-\lambda}{d}}\right) + \cos\left(x\sqrt{\frac{e-\lambda}{d}}\right)\right].$$

We note that $\hat{g}_\lambda(0) = -C_2 \tan\left(-\sqrt{\frac{e-\lambda}{d}}\right)$ and $\hat{g}'_\lambda(0) = C_2\left(\sqrt{\frac{e-\lambda}{d}}\right)$.

Next, we look at equation 4.10. The general solution is

$$f_\lambda(x) = C_3 e^{\left(-b/2a+\sqrt{\frac{b^2}{4a^2}-\frac{c-\lambda}{a}}\right)x} + C_4 e^{\left(-b/2a-\sqrt{\frac{b^2}{4a^2}-\frac{c-\lambda}{a}}\right)x}$$

if $\left(\frac{b^2}{4a^2} - \frac{c-\lambda}{a}\right) \geqslant 0$. The first derivative is:

$$\begin{aligned} f'_\lambda(x) =& C_3 \left(-b/2a + \sqrt{\frac{b^2}{4a^2} - \frac{c-\lambda}{a}}\right) e^{\left(-b/2a+\sqrt{\frac{b^2}{4a^2}-\frac{c-\lambda}{a}}\right)x} \\ &+ C_4 \left(-b/2a - \sqrt{\frac{b^2}{4a^2} - \frac{c-\lambda}{a}}\right) e^{\left(-b/2a-\sqrt{\frac{b^2}{4a^2}-\frac{c-\lambda}{a}}\right)x}. \end{aligned}$$

If $x = 0$ then $f_\lambda(0) = C_3 + C_4$ and $f'_\lambda(0) = C_3 \left(-b/2a + \sqrt{\frac{b^2}{4a^2} - \frac{c-\lambda}{a}}\right) + C_4 \left(-b/2a - \sqrt{\frac{b^2}{4a^2} - \frac{c-\lambda}{a}}\right)$. Assuming $\left(\frac{b^2}{4a^2} - \frac{c-\lambda}{a}\right) < 0$, the general solution of 4.10 becomes

$$\hat{f}_\lambda(x) = C_3 e^{(-b/2a)x} \cos\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right) + C_4 e^{(-b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right).$$

The first derivative is

$$\begin{aligned} \hat{f}'_\lambda =& - C_3 \frac{b}{2a} e^{(-b/2a)x} \cos\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right) - C_3 \left(\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right) e^{(-b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right) \\ & - C_4 \frac{b}{2a} e^{(-b/2a)x} \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right) + C_4 \left(\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right) e^{(-b/2a)x} \cos\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right). \end{aligned}$$

We find that $\hat{f}_\lambda(0) = C_3$ and $\hat{f}'_\lambda(0) = -C_3 \frac{b}{2a} + C_4 \left(\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right)$.

List of symbols

$\|\cdot\|$	norm
$D(A)$	domain of A
Im	imaginary part
$\mathcal{L}(X)$	set of bounded linear operators on X
$L^p(\Omega)$	Lebesgue space over Ω
$R(\lambda, A)$	resolvent, i.e. $(\lambda - A)^{-1}$
$\mathrm{ran}(A)$	range of A
Re	real part
$\rho(A)$	resolvent set of A
$\sigma(A)$	spectrum of A
$\sigma_p(A)$	point spectrum of A
T^*	adjoint operator of T
$W^{k,p}(\Omega)$	Sobolev space, i.e. subset of functions $f \in L^p(\Omega)$ such that f and its mixed partial weak derivatives up to the order k are in L^p
X'	dual space of X
X_p	set of periodic points in X

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