Li-Yau inequalities for general non-local diffusion equations via reduction to the heat kernel

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joint work with Frederic Weber

The classical Li-Yau inequality

Suppose $u : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ solves $\partial_t u - \Delta u = 0$. Then

$$-\Delta(\log u) \leq \frac{d}{2t} \quad \text{in } (0,\infty) \times \mathbb{R}^d. \tag{1}$$

Since $\partial_t(\log u) - \Delta(\log u) = |\nabla(\log u)|^2$, this is equivalent to

$$\partial_t(\log u) \geq |\nabla(\log u)|^2 - \frac{d}{2t} \quad \text{in } (0,\infty) \times \mathbb{R}^d. \tag{2}$$

- This extends to complete d-dimensional Riemannian manifolds M with $Ric(M) \geq 0$ (Li, Yau, Acta Math. 1986).

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- This extends to complete d-dimensional Riemannian manifolds M with $Ric(M)\ge 0$ (Li, Yau, Acta Math. 1986).
- (2) is sharp, one has equality for $u(t,x)=(4\pi t)^{-d/2}\exp\left(\frac{-|x|^2}{4t}\right)$.
- Integration of (2) over a path connecting (t_1,x_1) and (t_2,x_2) with $0<t_1<t_2$ gives the sharp Harnack estimate

$$u(t_1,x_1)\le u(t_2,x_2)\Big(\frac{t_2}{t_1}\Big)^{d/2}\exp\Big(\frac{|x_1-x_2|^2}{4(t_2-t_1)}\Big).$$

How can we prove Li-Yau? Basic idea: $v := \log u$ solves

$$\partial_t v - \Delta v = |\nabla v|^2, \tag{3}$$

by the chain rule $\Delta H(u) = H'(u)\Delta u + H''(u)|\nabla u|^2$ with $H = \log$.
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So we need $-\Delta v \leq \frac{d}{2t}$. Apply Δ to (3) and use Bochner's identity.

$$\begin{aligned}\partial_t \Delta v - \Delta(\Delta v) &= \Delta\big(|\nabla v|^2\big) \\ &= 2\nabla v \cdot \nabla \Delta v + 2|\nabla^2 v|^2_{HS} \quad \Big(+ 2Ric(\nabla v, \nabla v)\Big).\end{aligned}$$

Now, $|\nabla^2 v|^2_{HS} \geq \frac{1}{d}(\Delta v)^2$ (a CD-inequality), and thus

$$\partial_t \Delta v - \Delta(\Delta v) \geq 2\nabla v \cdot \nabla \Delta v + \frac{2}{d}(\Delta v)^2$$

$\omega(t) := -\frac{d}{2t}$ solves $\partial_t \omega = \frac{2}{d}\omega^2$. Comparison arg. $\hookrightarrow \Delta v \geq \omega$

The fractional heat equation

Let $\beta \in (0, 2)$. We now consider

$$\partial_t u + (-\Delta)^{\frac{\beta}{2}} u = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^d. \tag{4}$$

The fractional Laplacian can be defined by

$$-(-\Delta)^{\frac{\beta}{2}} u(x) = c_{\beta,d} \ \mathrm{p.v.} \int_{\mathbb{R}^d} \frac{u(y) - u(x)}{|x - y|^{d+\beta}} \, dy.$$

Question: Does a Li-Yau inequality hold for (4)? If yes, how does the bound depend on time?
Addressed in the survey by Garofalo, *Fractional thoughts...* (2019), as a key open problem.

Difficulties:

- The chain rule fails for the fractional Laplacian.

- The fractional Laplacian has infinite dimension, i.e. the Bakry-Émery curvature condition $CD(0, n)$ is violated for any $n \in (0, \infty)$, see Spener, Weber, Z., Comm. PDE (2020). Note that Δ in $\mathbb{R}^d$ satisfies $CD(0, d)$.

Reduction to the heat kernel for the heat equation

Lemma 1: Let $Pf(x) = \int_{\mathbb{R}^d} H(x,y)f(y)\,dy$, for sufficiently regular, positive functions H and f. Then

$$\int_{\mathbb{R}^d} \left|\nabla_x \log H(x,y)\right|^2 H(x,y)f(y)\,dy \geq \left|\nabla \log Pf(x)\right|^2 Pf(x). \quad (5)$$

Proof: By Hölder's inequality we have

$$\begin{aligned}(\partial_{x_i} Pf(x))^2 &= \Big(\int_{\mathbb{R}^d} \partial_{x_i} H(x,y)f(y)\,dy\Big)^2 \\ &\leq \int_{\mathbb{R}^d} \frac{(\partial_{x_i} H(x,y))^2}{H(x,y)} f(y)\,dy \int_{\mathbb{R}^d} H(x,y)f(y)\,dy,\end{aligned}$$

which directly leads to (5) by summing up and employing the chain rule for the gradient ($\nabla(\log g) = \frac{\nabla g}{g}$).

For the heat kernel $H(t,x,y) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}}$ we have

$$-\Delta_x\big(\log H(t,x,y)\big) = \frac{d}{2t} =: \varphi(t). \tag{6}$$

For any positive solution u of the heat equation,

$$\partial_t u - u\Delta(\log u) = u|\nabla(\log u)|^2.$$

In particular,

$$\partial_t H(t,x,y) + H(t,x,y)\varphi(t) = H(t,x,y)|\nabla_x(\log H(t,x,y))|^2$$

Consider a positive solution $u(t,x) = \int_{\mathbb{R}^d} H(t,x,y)u_0(y)\,dy$ of the heat equation (Widder's type theorem!). Then

$$\begin{aligned}\partial_t u(t,x) + \varphi(t)u(t,x) &= \int_{\mathbb{R}^d} \big(\partial_t H(t,x,y) + \varphi(t)H(t,x,y)\big)\,u_0(y)\,dy \\ &= \int_{\mathbb{R}^d} \big(|\nabla_x(\log H(t,x,y))|^2 H(t,x,y)\big)\,u_0(y)\,dy \\ &\geq |\nabla(\log u(t,x))|^2 u(t,x) \qquad \text{(by Lemma 1)} \\ &= \partial_t u(t,x) - u(t,x)\Delta(\log u(t,x)).\end{aligned}$$

Hence

$$-\Delta(\log u(t,x)) \leq \varphi(t) = \frac{d}{2t}.$$

Note that we only need $-\Delta_x\big(\log H(t,x,y)\big) \leq \varphi(t)$.

Conclusion: Li-Yau for heat kernel implies Li-Yau for pos. solutions.

Reduction to the heat kernel for the fractional heat equation

Question: Is there a similar argument for the fractional heat equation (FHE)?

Let u be a positive solution of the FHE. Then

$$\partial_t(\log u) + (-\Delta)^{\frac{\beta}{2}}(\log u) = \Psi_\Upsilon(\log u),$$

where

$$\Psi_\Upsilon(v)(x) = c_{\beta,d} \int_{\mathbb{R}^d} \frac{\Upsilon\big(v(y) - v(x)\big)}{|x-y|^{d+\beta}}\, dy$$

with $\Upsilon(z) = e^z - 1 - z$ (see Dier, Kassmann, Z., Ann. Sc. Norm. Super. Pisa (2021)). Equivalently,

$$\partial_t u + u(-\Delta)^{\frac{\beta}{2}}(\log u) = u\Psi_\Upsilon(\log u).$$

Non-local log-transform: follows from a simple rule for differences

$$\log u(y) - \log u(x) = \frac{u(y)-u(x)}{u(x)} - \Big(\frac{u(y)}{u(x)} - 1 - \big[\log u(y) - \log u(x)\big]\Big)$$
$$= \frac{u(y)-u(x)}{u(x)} - \Upsilon\big(\log u(y) - \log u(x)\big),$$

with $\Upsilon(z) = e^z - 1 - z$. Thus, with $L := -(-\Delta)^{\frac{\beta}{2}}$,

$$L(\log u)(x) = \frac{Lu(x)}{u(x)} - \underbrace{c_{\beta,d} \int_{\mathbb{R}^d} \frac{\Upsilon\big(\log u(y) - \log u(x)\big)}{|x-y|^{d+\beta}}\, dy}_{\Psi_\Upsilon(\log u)(x)}$$

Non-local analogue of

$$\Delta(\log u)(x) = \frac{\Delta u(x)}{u(x)} - \big|\nabla(\log u(x))\big|^2$$

Recall the key inequality from the local case

$$\int_{\mathbb{R}^d} \left|\nabla_x \log H(x,y)\right|^2 H(x,y) f(y)\, dy \geq \left|\nabla \log Pf(x)\right|^2 Pf(x),$$

where $Pf(x) = \int_{\mathbb{R}^d} H(x,y) f(y)\, dy$ and H and f are sufficiently regular, positive functions.

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Lemma 2: (Weber, Z. (2021)) Let P, H, f be as before. Then

$$\int_{\mathbb{R}^d} \Psi_\Upsilon(\log H(\cdot,y))(x) H(x,y) f(y)\, dy \geq \Psi_\Upsilon(\log Pf)(x) Pf(x). \tag{7}$$

Proof: Use the convexity of $r \mapsto \Upsilon(\log r) = r - \log r - 1$.

Positive (strong) solutions u of the FHE can be expressed as

$$u(t,x) = \int_{\mathbb{R}^d} G^{(\beta)}(t, x-y) u_0(y)\, dy, \tag{8}$$

where $G^{(\beta)}$ is the fund. sol. of the FHE, see Barrios, Peral, Soria, Valdinoci, ARMA (2014).

Set $H(t,x,y) = G^{(\beta)}(t, x-y)$. Using Lemma 2 we can argue as before to see the implication

$$(-\Delta)^{\frac{\beta}{2}}(\log G^{(\beta)})(t,x) \leq \varphi(t) \Rightarrow (-\Delta)^{\frac{\beta}{2}}(\log u)(t,x) \leq \varphi(t).$$

Question: For which function φ do we have

$$(-\Delta)^{\frac{\beta}{2}}(\log G^{(\beta)})(t,x) \leq \varphi(t) \quad ?$$

Lemma 3: For all $\beta \in (0,2)$, $t>0$, and $x \in \mathbb{R}^d$, we have

$$(-\Delta)^{\frac{\beta}{2}}(\log G^{(\beta)})(t,x) \leq \frac{C_{LY}(\beta,d)}{t}, \tag{9}$$

where the finite constant $C_{LY}(\beta,d)>0$ is given by

$$C_{LY}(\beta,d) = \frac{c_{\beta,d}}{2} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\log\left(\frac{\Phi_\beta(y)^2}{\Phi_\beta(y+\sigma)\Phi_\beta(y-\sigma)}\right)}{|\sigma|^{d+\beta}} \mathrm{d}\sigma, \tag{10}$$

with $\Phi_\beta(y) = G^{(\beta)}(1,y)$, $y \in \mathbb{R}^d$.

Note that $G^{(\beta)}(t,x) = t^{-\frac{d}{\beta}}\Phi_\beta\left(xt^{-\frac{1}{\beta}}\right)$. $C_{LY}(1,d) = \frac{d(d+1)}{2B\left(\frac{d+1}{2},\frac{1}{2}\right)}$.
$C_{LY}(\beta,d)$ is the smallest constant among all $C>0$ satisfying

$$(-\Delta)^{\frac{\beta}{2}}(\log G^{(\beta)})(t,x) \leq \frac{C}{t}, \quad t>0,\ x\in\mathbb{R}^d.$$

Proof:

$$(-\Delta)^{\frac{\beta}{2}}(\log G^{(\beta)})(t,x)$$
$$= \frac{c_{\beta,d}}{2}\int_{\mathbb{R}^d} \frac{2\log G^{(\beta)}(t,x) - \log G^{(\beta)}(t,x+h) - \log G^{(\beta)}(t,x-h)}{|h|^{d+\beta}} dh$$
$$= \frac{c_{\beta,d}}{2}\int_{\mathbb{R}^d} \frac{\log\left(\frac{G^{(\beta)}(t,x)^2}{G^{(\beta)}(t,x+h)\,G^{(\beta)}(t,x-h)}\right)}{|h|^{d+\beta}} dh.$$

Using $G^{(\beta)}(t,x) = t^{-\frac{d}{\beta}}\Phi_\beta\big(xt^{-\frac{1}{\beta}}\big)$ and setting $y = xt^{-\frac{1}{\beta}}$, we get that

$$\int_{\mathbb{R}^d} \frac{\log\left(\frac{G^{(\beta)}(t,x)^2}{G^{(\beta)}(t,x+h)G^{(\beta)}(t,x-h)}\right)}{|h|^{d+\beta}} dh = \frac{1}{t}\int_{\mathbb{R}^d} \frac{\log\left(\frac{\Phi_\beta(y)^2}{\Phi_\beta(y+\sigma)\Phi_\beta(y-\sigma)}\right)}{|\sigma|^{d+\beta}} d\sigma,$$

where we have substituted $\sigma = ht^{-\frac{1}{\beta}}$.

Next, we write

$$J(y) := \int_{\mathbb{R}^d} \frac{\log\left(\frac{\Phi_\beta(y)^2}{\Phi_\beta(y+\sigma)\Phi_\beta(y-\sigma)}\right)}{|\sigma|^{d+\beta}} d\sigma$$
$$= \int_{\mathbb{R}^d \setminus B_1(0)} \dots \, d\sigma + \int_{B_1(0)} \dots \, d\sigma =: J_1(y) + J_2(y).$$

To see boundedness of $J_1(y)$ we use

$$G^{(\beta)}(t,x) \asymp \frac{t}{\left(t^{\frac{2}{\beta}} + |x|^2\right)^{\frac{d+\beta}{2}}}, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d,$$

for $J_2(y)$ we also use $\log r \le r - 1$, apply Taylor and employ the bounds

$$|\nabla \Phi_\beta(x)| \lesssim \frac{1}{|x|^{d+\beta+1}}, \quad \|\nabla^2 \Phi_\beta(x)\| \lesssim \frac{1}{|x|^{d+\beta+2}}, \quad |x| \ge 1.$$

Theorem 4: (Weber, Z. (2021))
Let $\beta \in (0,2)$ and $u : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ be a strong solution of the FHE

$$\partial_t u + (-\Delta)^{\frac{\beta}{2}} u = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^d.$$

Then the Li-Yau type inequality

$$\big(-\Delta\big)^{\frac{\beta}{2}} (\log u)(t,x) \leq \frac{C_{LY}(\beta,d)}{t} \tag{11}$$

holds for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$.

Reduction principle for more general non-local operators

Setting: (M, d) metric space, $\mathcal{B}(M)$ the Borel σ-algebra on M. We consider

$$Lf(x) = \int_{M\setminus\{x\}} \big(f(y) - f(x)\big) k(x, dy), \tag{12}$$

where the kernel is s.t. $k(x, \cdot)$ defines a σ-finite measure on $\mathcal{B}(M \setminus \{x\})$ for any $x \in M$ and $f : M \to \mathbb{R}$ is s.t. the integral exists. We also include the singular case (as with the fractional Laplacian), replace then $\int_{M\setminus\{x\}}$ by $\lim\limits_{\varepsilon\to 0^+} \int_{M\setminus B_\varepsilon(x)}$.

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Important example: L is generator of a continuous-time Markov chain on a discrete (countable) state space M with natural graph structure; M is the set of vertices.

$$Lf(x) = \sum_{y\in M} k(x, y)\big(f(y) - f(x)\big).$$

Setting $\Psi_\Upsilon(f)(x) = \int_{M\setminus\{x\}} \Upsilon\big(f(y) - f(x)\big) k(x, dy)$, we find again that for (suitable) positive functions f

$$L(\log f) = \frac{Lf}{f} - \Psi_\Upsilon(\log f).$$

The key lemma, Lemma 2, extends to the more general case.

Lemma 5: (Weber, Z. (2021))
Let $Pf(x) = \int_M H(x, y) f(y)\, d\nu(y)$, where $\nu : \mathcal{B}(M) \to [0, \infty]$ is a σ-finite measure, and f and H are positive functions satisfying appropriate measurability and integrability conditions. Then we have for any $x \in M$ that

$$\int_M \Psi_\Upsilon(\log H(\cdot, y))(x) H(x, y) f(y) d\nu(y) \geq \Psi_\Upsilon(\log Pf)(x) Pf(x).$$

Theorem 6: (Weber, Z. (2021))
Let $u : [0, \infty) \times M \to (0, \infty)$ be a solution of $\partial_t u - Lu = 0$ of the form (p and u_0 are positive)

$$u(t,x) = \int_M p(t,x,y) u_0(y) \, d\mu(y).$$

Assume certain technical assumptions and the estimate

$$-L\big(\log p(t,\cdot,y)\big)(x) \le \varphi(t,x), \tag{13}$$

for all $(t,x) \in (0,\infty) \times M$ and μ−a.e. $y \in M$, where $\varphi : (0,\infty) \times M \to \mathbb{R}$. Then

$$-L(\log u(t,\cdot))(x) \le \varphi(t,x) \tag{14}$$

holds true for all $(t,x) \in (0,\infty) \times M$.

A discrete example

Consider the (unweighted) complete graph K_n. Here $M = \{1, 2, \ldots, n\}$ and $Lf(x) = \sum_{y \in M} k(x,y)\big(f(y) - f(x)\big)$, where $k(x,y) = 1$ for all $x, y \in M$, $x \neq y$. The heat kernel reads

$$p(t,x,y) = \begin{cases} \frac{1-e^{-nt}}{n} & : x \neq y, \\ \frac{1+(n-1)e^{-nt}}{n} & : x = y. \end{cases}$$

$$-L\big(\log p(t,\cdot,y)\big)(x) = \begin{cases} (n-1)\log\left(\frac{1+(n-1)e^{-nt}}{1-e^{-nt}}\right) & : x = y \\ \log\left(\frac{1-e^{-nt}}{1+(n-1)e^{-nt}}\right) & : x \neq y. \end{cases}$$

Thus $-L\big(\log p(t,\cdot,y)\big)(x)$ is maximal if $x = y$ for any fixed $t > 0$. We obtain the sharp Li-Yau inequality

$$-L\big(\log u(t,\cdot)\big)(x) \leq (n-1)\log\left(\frac{1+(n-1)e^{-nt}}{1-e^{-nt}}\right).$$

Harnack inequalities

Question: Do the non-local Li-Yau inequalities lead to Harnack inequalities?

Note that the Li-Yau inequality $-L(\log u(t,\cdot))(x) \leq \varphi(t,x)$ is equivalent to the differential Harnack inequality

$$\partial_t \log u(t,x) \geq \Psi_\Upsilon(\log u)(t,x) - \varphi(t,x), \quad t > 0,\, x \in M. \quad (15)$$

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Discrete case (graphs): yes, see Dier, Kassmann, Z., Ann. Sc. Norm. Super. Pisa (2021), where (15) is used with $\varphi(t)$.

See also Bauer, Horn, Lin, Lippner, Mangoubi, Yau, J. Differential Geom. (2015) and Münch, J. Math. Pures Appl. (2018)

The Harnack inequality for the fractional heat equation

Turns out to be much more involved! We have to replace integration along continuous paths with appropriate jumps. By means of the Li-Yau inequality

$$(-\Delta)^{\frac{\beta}{2}}(\log u)(t,x) \leq \frac{C_{LY}(\beta,d)}{t}, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d$$

one can show

Theorem 7: (Weber, Z. (2021))
Let $\beta \in (0,2)$ and $u : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ be a strong solution to the FHE. Then there exists a constant $C = C(\beta,d) > 0$ s.t. for all $0 < t_1 < t_2 < \infty$ and $x_1, x_2 \in \mathbb{R}^d$ there holds

$$u(t_1,x_1) \leq u(t_2,x_2)\Big(\frac{t_2}{t_1}\Big)^{C_{LY}} \exp\left(C\left[1 + \frac{|x_1-x_2|^{\beta+d}}{(t_2-t_1)^{1+\frac{d}{\beta}}}\right]\right). \quad (16)$$

Known results on the parabolic Harnack inequality for the space fractional heat equation:

• Bass, Levin, Trans. Amer. Math. Soc. (2002); Chen, Kumagai, Stochastic Process. Appl. (2003): local solutions, probabilistic methods

• Chang-Lara, D'avila, J. Differential Equations (2016): local solutions in a rough setting, purely analytic proof

• Bonforte, Sire, Vázquez, Nonlinear Anal. (2017); Dier, Kemppainen, Siljander, Z., Math. Z. (2020): global solutions, estimates based on fundamental solution, no time gap is required between t_1 and t_2

Proof: For any $s \in [t_1, t_2]$ we have

$$\log \frac{u(t_1, x_1)}{u(t_2, x_2)} = -\int_{t_1}^{s} \partial_t \log u(t, x_1) dt + \log \frac{u(s, x_1)}{u(s, x_2)} - \int_{s}^{t_2} \partial_t \log u(t, x_2) dt.$$

Combining this with the differential Harnack inequality

$$\partial_t \log u(t, x) \geq \Psi_\Upsilon(\log u)(t, x) - \frac{C_{LY}(\beta, d)}{t}, \quad t > 0,\ x \in \mathbb{R}^d,$$

gives

$$\log \frac{u(t_1, x_1)}{u(t_2, x_2)} \leq \int_{t_1}^{t_2} \frac{C_{LY}(\beta, d)}{t} \mathrm{d}t + \log \frac{u(s, x_1)}{u(s, x_2)} - \int_{t_1}^{s} \Psi_\Upsilon(\log u)(t, x_1) dt - \int_{s}^{t_2} \Psi_\Upsilon(\log u)(t, x_2) dt,\ s \in [t_1, t_2]. \tag{17}$$

Setting $v = \log u$ and

$$f(t) := v(t,x_1) - v(t,x_2) - \int_{t_1}^{t} \Psi_\Upsilon(v)(t,x_1)dt - \int_{t}^{t_2} \Psi_\Upsilon(v)(t,x_2)dt,$$

for $t \in [t_1, t_2]$, the idea is to choose s s.t. $f(s) = \min_{t\in[t_1,t_2]} f(t)$. We select a suitable (positive) weight function $\eta(t)$ and estimate

$$\begin{aligned}\min_{t\in[t_1,t_2]} f(t) &\le \frac{1}{\int_{t_1}^{t_2}\eta(t)dt}\int_{t_1}^{t_2}\eta(t)f(t)dt\\
&= \frac{1}{\int_{t_1}^{t_2}\eta(t)dt}\Big[\int_{t_1}^{t_2}\Big(\eta(t)A_1(t) - \Psi_\Upsilon(v)(t,x_1)\int_t^{t_2}\eta(\tau)d\tau\Big)dt\\
&\qquad + \int_{t_1}^{t_2}\Big(\eta(t)A_2(t) - \Psi_\Upsilon(v)(t,x_2)\int_{t_1}^{t}\eta(\tau)d\tau\Big)dt\Big],\end{aligned}$$

$$A_1(t) = \frac{1}{|Q_t|}\int_{Q_t}\big(v(t,x_1) - v(t,y)\big)dy,\ A_2(t) = \frac{1}{|Q_t|}\int_{Q_t}\big(v(t,y) - v(t,x_2)\big)dy.$$

Here, assuming that $|x_1 - x_2| \leq 1$, we take $Q_t = B_{r(t)}(x_1)$ with radius

$$r(t) = \Big(\frac{\omega_d c_{\beta,d}}{1+\alpha}(t_2 - t)\Big)^{\frac{1}{\beta}},$$

with fixed parameter $\alpha > \frac{1}{2}\max\{0, \frac{d}{\beta} - 1\}$. Setting $t_* = \frac{t_1+t_2}{2}$, we choose the weight function

$$\eta(t) = \begin{cases} (t-t_1)^\alpha & , t \in [t_1, t_*) \\ (t_2 - t)^\alpha & , t \in [t_*, t_2]. \end{cases}$$

Recall that $\Upsilon(z) = e^z - 1 - z$ and

$$\Psi_\Upsilon(v)(t,x) = c_{\beta,d} \int_{\mathbb{R}^d} \frac{\Upsilon\big(v(t,y) - v(t,x)\big)}{|x-y|^{d+\beta}}\, dy.$$

Use $z \leq \Upsilon(-z) + 1$, $z \in \mathbb{R}$, for the A_1-term, and $\Upsilon(z) \geq \frac{1}{2}z^2$, $z \geq 0$, for the A_2-term. General case: scaling argument.

Reference

Weber, Z.: Li-Yau inequalities for general non-local diffusion equations via reduction to the heat kernel. Math. Ann. (2022), https://doi.org/10.1007/s00208-021-02350-z

THANK YOU FOR YOUR ATTENTION!

The Γ-calculus of Bakry and Émery (1985)

Let $\mathcal{L}$ be a Markov generator. For $u, v \in D(\mathcal{L})$ suff. smooth,

$$\Gamma(u,v) = \frac{1}{2}\big(\mathcal{L}(uv) - u\mathcal{L}v - v\mathcal{L}u\big), \quad \text{(carré du champ operator)}$$

$$\Gamma_2(u,v) = \frac{1}{2}\big(\mathcal{L}\Gamma(u,v) - \Gamma(u,\mathcal{L}v) - \Gamma(v,\mathcal{L}u)\big). \quad \text{(Gamma deux)}$$

Set $\Gamma(v) = \Gamma(v,v)$ and $\Gamma_2(v) = \Gamma_2(v,v)$.

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$$\Gamma(u,v) = \frac{1}{2}\big(\mathcal{L}(uv) - u\mathcal{L}v - v\mathcal{L}u\big), \quad \text{(carré du champ operator)}$$

$$\Gamma_2(u,v) = \frac{1}{2}\big(\mathcal{L}\Gamma(u,v) - \Gamma(u,\mathcal{L}v) - \Gamma(v,\mathcal{L}u)\big). \quad \text{(Gamma deux)}$$

Set $\Gamma(v) = \Gamma(v,v)$ and $\Gamma_2(v) = \Gamma_2(v,v)$. $\mathcal{L}$ is said to satisfy the curvature-dimension condition $CD(\rho, d)$, for $\rho \in \mathbb{R}$ and $d \in (0,\infty]$, if for every function v from a sufficiently rich subspace of $D(\mathcal{L})$,

$$\Gamma_2(v) \geq \frac{1}{d}(\mathcal{L}v)^2 + \rho\,\Gamma(v), \quad \mu - \text{a.e.}, \tag{18}$$

where μ is a fixed invariant and reversible measure for $e^{\mathcal{L}t}$.
See Bakry, Gentil, Ledoux: Analysis and Geometry of Markov diffusion operators, Springer 2014.