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Dynamical systems, Invariance Conditions and Applications

Master Thesis

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Abstract

The concept of an invariant set is of great importance in the theory of dynamical systems. An invariant set D can be described as follows: if a solution enters, or starts in D, then it stays in D. In this thesis, the main focus lies on dynamical systems generated by a system of non-linear differential equations. It is of great interest to identify and construct such invariant sets, in order to decipher the structure of a solution, especially when a closed-form is not available. This thesis is divided into four parts. In the first part (Chapter 2), important notions and theorems of the qualitative theory of differential equations are being treated, and the definition of a dynamical system is being introduced. It turns out that proving the global existence of solutions is a recurrent obstacle. In the second part (Chapter 3), the concept of invariance is formally defined, and criteria to identify invariant sets are presented and also applied to specific examples. More complicated sets are treated at the end of this Chapter, showing how complicated the structures of invariant sets can be. In the third part (Chapter 4), stability concepts are being presented. The linearization of a system around an equilibrium is being studied. Moreover, the concept of a Lyapunov function is introduced. Finally, in the fourth part (Chapter 5), methods to construct invariant sets are studied. While the concept of Lyapunov functions can be directly applied to construct invariant sets, the relationship between stability and invariance is rarely treated in the literature. More research and analysis are needed in this area.

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$\mathbb{N}$	natural numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$B_r(x_0)$	open ball with radius r and centre $x_0 \in \mathbb{R}^n$ with respect to a norm.
$\overline{B}_r(x_0)$	closed ball with radius r and centre $x_0 \in \mathbb{R}^n$ with respect to a norm.
$dist(x, D)$	distance from a point $x \in \mathbb{R}^n$ to a set $D \subset \mathbb{R}^n$with $dist(x, D) = \inf\limits_{d \in D} \|x - d\|$
$intD$	interior of a set $D \subset \mathbb{R}^n$
∂D	boundary of a set $D \subset \mathbb{R}^n$
$\overline{D}$	closure of a set $D \subset \mathbb{R}^n$
$(\cdot\|\cdot)$	scalar product on $\mathbb{R}^n$
$\|\cdot\|$	absolute value of a real number as well as a non-specified norm defined on $\mathbb{R}^n$
$C^k(I; \mathbb{R}^n)$	space of k-times continuously differentiable functions $u : I \to \mathbb{R}^n$, $I \subset \mathbb{R}$, $k \in \mathbb{N}_0 \cup \{\infty\}$
P	projection in $\mathbb{R}^n$
Δ_p	discrete p-Laplacian Δ_p: $f \mapsto \mathcal{I}(\left\|\mathcal{I}^T f\right\|^{p-2} \mathcal{I}^T f)$
$\widetilde{\Delta_p}$	continuous p-Laplacian $\widetilde{\Delta_p}$: $f \mapsto \nabla(\|\nabla f\|^{p-2} \nabla f)$
Φ	fundamental matrix of a linear system of ODEs
$Re\lambda_i$	real part of a complex number λ_i
$\preceq$	a symmetric negative semi-definite matrix Q is denoted by $Q \preceq 0$
$\succ$	a symmetric positive definite matrix Q is denoted by $Q \succ 0$
$\mathcal{G}$	graph $\mathcal{G} = (V, E)$ with set of vertices V, and set of edges E
$\mathcal{I}$	incidence matrix of a graph $\mathcal{G}$
trA	trace of a matrix A
$\det A$	determinant of a matrix A

1 Introduction

The principal goal in dynamical systems theory (sometimes known as non-linear dynamics or chaos theory), is to understand the behaviour of states in a system, namely the population of a country, the density of a compound in a chemical solution, or the position of a particle in a physical system. Historically, the modern theory of dynamical systems derives from the foundations laid by H.J. Poincaré (1854- 1912) on the qualitative analysis of non-linear differential equations. His work includes the study of periodic motions and definitions of stability (see Section 4.1.1). The formal study of dynamical systems involves studying mathematical models derived from various fields, such as physics, chemistry, biology or economics. For example, one can formulate problems from classical mechanics as a dynamical system, where the position and velocity of a particle are the state variables. In this case, one can write the dynamical system down as a system of differential equations, which will be the sole focus in this thesis. Thus, only dynamical systems derived from (non-linear) systems of ordinary differential equations (ODEs) are being considered. The essential definitions and theorems treating existence, uniqueness and positivity of solutions for a system of ODEs of the form (1) are addressed in Section 2.2.

Inherent in the concept of dynamical systems is the primary focus on the "qualitative theory", due to the fact, that exact solutions for complex systems are often too challenging to solve or even fully interpret. One takes the following approach instead: the system of differential equations which model some natural or technical phenomenon, may still deliver pieces of information without the demand of an exact solution. Dynamical systems theory does not propose specific models of the reality; it is broadly speaking, a set of methods to decipher systems of ODEs possessing the form

$$\dot{x}_i(t) = f_i(t, x_1, x_2, \dots, x_n) \tag{1}$$

with $t \in \mathbb{R}, x \in \mathbb{R}^n$, where $x_1, x_2, \dots, x_n$ are called the state variables. The main purpose is naturally the solving of (1), to obtain the so-called orbits (or solution curves of (1))

$$x(t) = (x_1(t), x_2(t), \dots, x_n(t)) \quad t \in [0, \infty).$$

In the world of dynamical systems, the methods accentuate the considerations of all orbits, especially the description of qualitative properties mentioned above: namely the existence of solutions, rather than derivation of explicit closed-form expressions or in the field of numerical mathematics, approximations.

Roughly speaking, a set $D \subset \mathbb{R}^n$ is invariant with respect to a system (1), if for every orbit $x(t)$ and $t_0 \in \mathbb{R}$,

$$x(t_0) \in D \Rightarrow x(t) \in D \text{ for all } t \geq t_0.$$

Now, only a few non-linear ODEs can be solved analytically. However, one can deduce a great deal from the study of the linearization of systems around an **equilibrium** (or other **invariant sets**), considered in Section 4.1. For instance, the linearization of the system about an equilibrium $\tilde{x} = (\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t))$, where $f_i(\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)) = 0$ for $i = 1, \ldots, n$ leads to the linear system $\dot{y} = Df(\tilde{x})y$, with $Df(\tilde{x})$ being the $n \times n$ Jacobian matrix evaluated at the equilibrium point. One can say, that linear theory is important in the study of non-linear dynamical systems since the stability of solutions of non-linear systems can often be derived from the stability of their linearization. In summary, the goal of the qualitative theory is to understand the behaviour of the solutions through geometrical and topological lenses (further details on the development and history of the concept of dynamical systems over the years can be found in [20, p 115-138] for example).

The concept of invariance is formally introduced on general sets $D \in \mathbb{R}^n$ in Chapter 3 and considered on convex sets in Section 3.2. A central aim in this chapter is to find criteria proving the invariance of a giving set $D \subset \mathbb{R}^n$. Further classes of more complicated and complex invariant sets are being considered in Section 3.3. In any non-linear dynamical system, the "skeletal structure" of the global dynamical behaviour is built on the invariant sets of the system, such as:

1. Equilibria or fixed points;
2. Periodic orbits;
3. The connecting orbits or invariant sets amongst them.

Therefore, one central enquiry in dynamical systems theory is to get a grasp of the **existence** and **structure** of **invariant sets**. One can even show the presence of invariant sets with complicated structures in systems with chaotic behaviour, briefly treated in Chapter 3.3.3. The stability properties, and especially the concept of the Lyapunov stability, and how they relate to the idea of invariance, are being treated in Chapter 4. Finally, some methods and tools for the construction of invariant sets are being presented in Chapter 5.

2 Differential Equations and Dynamical Systems

2.1 Notations

In this thesis, the standard terminology known from topology and mathematical analysis shall be used. As usual, $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ represent the natural, real and complex numbers, respectively. Moreover, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
The open and closed balls with radius r and centre $x_0 \in \mathbb{R}^n$ are denoted by $B_r(x_0)$ and $\overline{B}_r(x_0)$. The distance from a point $x \in \mathbb{R}^n$ to a set $D \subset \mathbb{R}^n$ is denoted by $dist(x, D) := \inf\limits_{d \in D} |x - d|$. Furthermore, $intD$, ∂D and $\overline{D}$ shall be the interior, the boundary and the closure of a set $D \in \mathbb{R}^n$, respectively.
The scalar product in $\mathbb{R}^n$ is denoted by $(\cdot|\cdot)$; the absolute value of a real number as well as a non-specified norm defined on $\mathbb{R}^n$ shall be designated with $|\cdot|$.
Let the interval $I \subset \mathbb{R}, k \in \mathbb{N}_0 \cup \{\infty\}$, then $C^k(I; \mathbb{R}^n)$ designates the space of the k-times continuously differentiable functions $u: I \to \mathbb{R}^n$. For $k = 0$, one sets $C(I; \mathbb{R}^n) := C^0(I; \mathbb{R}^n)$.
A projection P in $\mathbb{R}^n$ satisfies $P^2 = P$ and induces with $x = Px + (x - Px)$ a decomposition of $\mathbb{R}^n$ in a direct sum. Finally, the dot denotes the derivative with respect to the time t.

2.2 Differential Equations

Let $G \subset \mathbb{R}^{n+1}$ be an open set with $(t_0, x_0) \in G$, $f: G \to \mathbb{R}^n$ continuous and locally Lipschitz in x. The following initial value problem (IVP) for a system of first-order ODEs

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{2}$$

is being considered in this section, if not specified otherwise.

If f does not explicitly depend on t, i.e. $f: \Omega_0 \to \mathbb{R}^n$ with $\Omega_0 \subset \mathbb{R}^n$, then one has a system of autonomous ordinary differential equations (ODEs) given by

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0. \end{cases} \tag{3}$$

2.2.1 Existence and Uniqueness of Solutions

Before diving into the concept of dynamical systems, this subsection is dedicated to fundamental theorems and lemmata from the qualitative theory of differential equations.

Astonishingly, the existence of a local solution requires only the continuity of $f(t,x)$ in (2), as treated in the following theorem.

Theorem 2.1 (Peano's existence theorem, 1886)**.** Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be an open set, $f\colon \Omega \to \mathbb{R}^n$ continuous and $(t_0, x_0) \in \Omega$ an initial value. Then, the IVP defined in (2) possesses a local solution $\phi\colon [t_0 - \alpha, t_0 + \beta] \to \mathbb{R}^n$, with suitable $\mathbb{R} \ni \alpha, \beta > 0$.

Proof. The proof can be found in [62, Satz 2.3]. □

Peano's result ensures a **local solution** for the IVP (2) if $f(t,x)$ is at least continuous in a neighbourhood of (t_0, x_0). One speaks of **local existence**, which is unfortunately not suited for the remainder of this thesis.

Example 2.2. [25, Section 4.2] Consider the ODE $\dot{x} = 2\sqrt{|x|}$. One can see that $x(t) = 0$ is a solution on $(-\infty, \infty)$, but also $x(t) = x^2$ on $[0, \infty)$ and $x(t) = -x^2$ on $(-\infty, 0]$. But $x \mapsto x^2$ is not a maximal solution on $\mathbb{R}^+$, because it can be extended to a solution on $\mathbb{R}$ through

$$x \mapsto \begin{cases} x^2, & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

or also

$$x \mapsto \begin{cases} -(x-a)^2 & \text{if } x \leq a, \\ 0 & \text{if } a \leq x \leq b, \\ (x-b)^2 & \text{if } b \leq x. \end{cases}$$

This entails, that there is an endless number of solutions for $\dot{x} = 2\sqrt{|x|}$ with initial value $x(t_0) = x_0$.

The previous example shows that the plain existence of a solution for (2) is not sufficient, hence one also needs **uniqueness**. To this end, the notion of **Lipschitz continuity** (see, e.g. [4, Definition 1.17]) shall be reminded:

Definition 2.3. *Let $G \subset \mathbb{R}^{n+1}$. A continuous function $f\colon G \to \mathbb{R}^n$ is called **locally Lipschitz** with respect to x, if for every open set $K \subset G$ and $(t_0, x_0) \in K$ there is a closed ball $\bar{B}_r(x_0)$, an $\alpha > 0$ with $[t_0 - \alpha, t_0 + \alpha] \times \bar{B}_r(x_0) \subset K$ and a Lipschitz constant $L(t_0, x_0) > 0$ fulfilling*

$$|f(t,x) - f(t,\tilde{x})| \leq L(t_0, x_0)\, |x - \tilde{x}|\,, \text{ with } |t - t_0| \leq \alpha, \quad x, \tilde{x} \in \bar{B}_r(x_0).$$

Remark 1. In the case, that the Lipschitz constant does not depend on $(t_0, x_0) \in K$, f is called **globally Lipschitz**.

With the notion of a locally Lipschitz function, it is now possible to formulate an essential result on the existence and **uniqueness** of solutions of (2).

Theorem 2.4 (Picard-Lindelöf theorem)**.** Let the function $f\colon G \to \mathbb{R}^n$ with $f \in C(G;\mathbb{R}^n)$ be locally Lipschitz in x on an open set $G \subset \mathbb{R} \times \mathbb{R}^n$. Then for every $(t_0, x_0) \in G$, there exists a unique solution of the IVP (2) in some open interval containing t_0.

Proof. The proof can be found in [4, Section 1.2.3]. □

At first glance, the Lipschitz property seems a little bit too demanding, but this apprehension is unfounded, as can be seen in the following proposition.

Proposition 2.5. *Let $G \subset \mathbb{R} \times \mathbb{R}^n$ and $f\colon G \to \mathbb{R}^n$ be of the class $C^1(G;\mathbb{R}^n)$ (i.e. continuously differentiable), then f is locally Lipschitz in x.*

Proof. The proof can be found in [4, Proposition 1.19]. □

Theorem 2.4 establishes that Lipschitz continuity of f guarantees the unique existence of a solution of (2) on an interval containing t_0. The following questions now come to mind:

1. Is it possible to extend the solution to a larger interval?
2. If so, is there a "greatest" interval that supports the solution?

However, before discussing this topic any further, it shall be noted that one can put solutions together. For instance, let $x_1(t)$ be the solution on the interval $[t_0, t_1]$ and $x_2(t)$ the solution on $[t_1, t_2]$, satisfying $x_1(t_1) = x_2(t_1)$. Now, one has

$$\dot{x}_1(t_1) = f(t_1, x_1(t_1)) \underset{x_1(t_1)=x_2(t_1)}{=} f(t_1, x_2(t_1)) = \dot{x}_2(t_1)$$

and therefore the joint function

$$x(t) = \begin{cases} x_1(t), & \text{for } t \in [t_0, t_1] \\ x_2(t), & \text{for } t \in [t_1, t_2] \end{cases} \tag{4}$$

is continuously differentiable on $[t_0, t_2]$.

This property and the uniqueness of the solutions of (2) lead to the following (see [48, Def. 2.3.1]) notion.

Definition 2.6. *Let $t_\pm(t_0, x_0) \in \mathbb{R}$ be defined by*

$$\begin{aligned} t_+ &= t_+(t_0, x_0) = \sup\{t_1 \geq t_0 | \ \textit{a solution } x_1 \textit{ exists for } (2) \textit{ on } [t_0, t_1]\}, \\ t_- &= t_-(t_0, x_0) = \sup\{t_2 \leq t_0 | \ \textit{a solution } x_2 \textit{ exists for } (2) \textit{ on } [t_2, t_0]\}. \end{aligned}$$

The intervals $[t_0, t_+), (t_-, t_0]$ or (t_-, t_+) are called ***maximal interval of existence*** *to* *the right, to the left, or per se, respectively. The* ***maximal solution*** *of* (2) *is defined by* $x(t) = x_1(t)$ *for all* $t \in [t_0, t_1]$ *and by* $x(t) = x_2(t)$ *on* $[t_2, t_0]$*, respectively. Hence, one has* $x(t) \in C^1((t_-, t_+); \mathbb{R}^n)$.

Under the assumptions of Theorem 2.4, the maximal interval of a solution $x\colon I \to \mathbb{R}^n$ of (2) is the largest open interval (t_-, t_+) where the existence of a solution coincides with $x \in I$. But it should be noted, that for an interval J with $I \subset J$, it is relatively open in J, i.e. $I = O \cap J$ for some open set $O \subset \mathbb{R}$ (for a proof, see [35, Proposition 4.4]). One can readily suspect, that the maximal interval of a solution will unfortunately not always be $\mathbb{R}$, as can be clearly (cf. [4, Example 1.45]) seen in the following example.

Example 2.7. Let $f(t, x) := x^2$ and $x(t_0) = x_0$ in Equation (2). One can readily see, that the solutions are (for $c \in \mathbb{R}$)

- $x(t) = 0 \quad$ for $x_0 = 0$ with the maximal interval $\mathbb{R}$,
- $x(t) = \dfrac{1}{t_0 - t + \frac{1}{x_0}}$ with the maximal intervals $\begin{cases} (-\infty, t_0 + \frac{1}{x_0}), & \text{for } x_0 > 0 \\ (t_0 + \frac{1}{x_0}, \infty), & \text{for } x_0 < 0. \end{cases}$

The possible situations, which can arise for a maximal interval of existence, are summarized in the following theorem.

Theorem 2.8 (Extension Theorem)**.** A solution $x(t) \in C^1([t_0, t_1); \mathbb{R}^n), t_1 > t_0$ for (2) can be continued into the future until the boundary of the phase space Ω, if and only if there exists a continuation $\tilde{x} \in C^1([t_0, t_+); \mathbb{R}^n)$ of $x(t)$ with $t_1 \leq t_+ \leq \infty$ such that $\tilde{x}$ also satisfies (2). One can characterize the point t_+ on the right-hand side according to the three following cases:

1. $t_+ = \infty$: $x(t)$ is therefore a global solution to the right (see Example 2.7, case $x_0 < 0$).
2. $t_+ < \infty$ and $\lim\limits_{t \to t_+} |\tilde{x}(t)| = \infty$. This is the case of a so-called "blow up", as can be seen in Example 2.7, case $x_0 > 0$.
3. $t_+ < \infty$ and $\lim\limits_{t \to t_+} dist((t, \tilde{x}(t)), \partial\Omega) = 0$, i.e. the solution "collapses" in finite time at the boundary of Ω.

The three situations are analogous in the case of t_-.

Proof. The proof and further details can be found in [15, Definition 2.3, 48, Satz 2.3.2]. □

Now, Picard-Lindelöf's Theorem 2.4 secures a unique **local** solution for (2) and Theorem 2.8 describes its maximal interval of existence. Sometimes useful to obtain a global existence of a solution for (2) is the following result.

Proposition 2.9. *Let $G = J \times \mathbb{R}^n, J \subset \mathbb{R}$ open, $(t_0, x_0) \in G$ and $f: G \to \mathbb{R}^n$ continuous, locally Lipschitz in x. Further, let f be linearly bounded, i.e.*

$$|f(t,x)| \leq r(t)\,|x| + s(t) \tag{5}$$

for all $t \in J, x \in \mathbb{R}^n$ and continuous functions $r, s: J \to \mathbb{R}_+$. The solution of (2) then exists globally.

Proof. For the proof, see [37, Satz 2.11]. □

Example 2.10. One considers the IVP

$$\dot{x}(t) = \sqrt{|t|}\,|\cos(t)| + g(t) \text{ with } x(0) = x_0, \tag{6}$$

g being a continuous function $g: J \to \mathbb{R}_+$, $J \subset \mathbb{R}$. For instance, one can take $g(t) = t^4 + t^2 + e^t$. Let $f(t,x) := \sqrt{|t|}\,|\cos(t)| + g(t)$. One can clearly see, that f is continuous, and that

$$|f(t,x) - f(t,\tilde{x})| \leq \sqrt{|t|}\,||\cos(x)| - |\cos(\tilde{x})|| \leq \sqrt{|\tau|}\,|x - \tilde{x}|$$

holds for $t \leq \tau$. It follows with Theorem 2.4 and Definition 2.6 that there exists a unique maximal solution for (6). Now, if one takes $r(t) = 0$ and $s(t) = \sqrt{|t|} + g(t)$,

$$f(t,x) \leq r(t)\,|x| + s(t)$$

holds. It follows with Proposition 2.9, that the solution of (6) exists globally.

Models originating in the life sciences, for example from Volterra-Lotka and Kermack-McKendrick, usually do not satisfy the conditions of Proposition 2.9. Therefore, one needs a criterion for proving global existence to the right for (2).

Proposition 2.11. *Let $G = J \times \mathbb{R}^n, J \subset \mathbb{R}$ open, $(t_0, x_0) \in G$ and $f: G \to \mathbb{R}^n$ continuous, locally Lipschitz in x. A constant $\omega \geq 0$ shall exist satisfying*

$$(f(t,x)|x) \leq \omega\,|x|_2^2 \tag{7}$$

for all $(t,x) \in G$. Then, all the solutions of (2) exist globally to the right.

Proof. The proof can be found in [48, Korollar 2.5.3]. □

An interesting application of Proposition 2.11 can be found in an article from Mugnolo [44], concerning the discrete p-Laplace operator.

Example 2.12. (The discrete p-Laplacian) An essential notion in the graph theory is the signed incidence matrix, defined as follows (see, e.g. [26, Definition 1.1.17]):

Definition 2.13. *(Signed incidence matrix) Let $\mathcal{G}$ be a finite oriented graph, m and n the number of edges and vertices, respectively. The signed incidence matrix is an $n \times m$ matrix $\mathcal{I}$, such that*

$$\mathcal{I}_{ij} = \begin{cases} 0, & \text{if vertex } v_i \text{ and edge } e_j \text{ are not incident,} \\ 1, & \text{if the vertex } v_i \text{ is the head of edge } e_j, \\ -1, & \text{if the vertex } v_i \text{ is the tail of edge } e_j. \end{cases}$$

Let $\mathcal{D}$ be the diagonal degree matrix, and $\mathcal{A}$ be the adjacency matrix of a finite undirected graph. The discrete Laplacian is a notable notion in the world of the graph theory and was introduced by Kirchhoff as

$$\Delta := D - A.$$

He then discovered that for any orientation of the graph, its related signed incidence matrix $\mathcal{I}$ satisfies

$$\Delta = \mathcal{I}\mathcal{I}^T.$$

Let $\mathcal{G}$ be a finite (oriented) graph, $\mathcal{I}$ his associated signed incidence matrix, $2 \leq p < \infty$ and as usual f a real function of several variables. Now, the considerations above motivate the definition (see [44, Equation 1.4]) of the discrete p-Laplacian

$$\Delta_p \colon f \mapsto \mathcal{I}(\left|\mathcal{I}^T f\right|^{p-2} \mathcal{I}^T f), \tag{8}$$

in clear analogy with the p-Laplacian in the continuum (see, e.g. [2, Section 12.5])

$$\widetilde{\Delta_p} \colon f \mapsto \nabla(|\nabla f|^{p-2} \nabla f).$$

The discrete p-Laplacian is an essential concept in graph theory and can be viewed as the discrete version of the p-Laplace operator. One can define the system of ODEs

$$\frac{dx}{dt} = -\Delta_p x(t), \tag{9}$$

and set $f(t, x(t)) := -\Delta_p x(t)$. Now, if $G = J \times \mathbb{R}^n, J \subset \mathbb{R}$ open, $(t_0, x_0) \in G$, then $f \colon G \to \mathbb{R}^n$ is obviously continuous and locally Lipschitz in x. One has

$$(f(t,x)|x) = -\left(\mathcal{I}(\left|\mathcal{I}^T x\right|^{p-2} \mathcal{I}^T x)|x\right)$$

$$
\begin{aligned}
&= -\left(\left|\mathcal{I}^T x\right|^{p-2} \mathcal{I}\mathcal{I}^T x | x\right) \\
&= -\left(\left|\mathcal{I}^T x\right|^{p-2} \mathcal{I}^T x | \mathcal{I}^T x\right) \\
&= -\left|\mathcal{I}^T x\right|^{p-2} \left(\mathcal{I}^T x | \mathcal{I}^T x\right) \\
&= -\left|\mathcal{I}^T x\right|^{p-2} \left|\mathcal{I}^T x\right|^{2} \\
&= -\left|\mathcal{I}^T x\right|^{p} \\
&\leq 0.
\end{aligned}
$$

Now, setting $w = 0$, one recognizes that the prerequisites of Proposition 2.11 are satisfied. Therefore, the solutions of (9) exist globally to the right.

It has to be said, that neither of the criterions 2.9 and 2.11 has proven itself in practice, and should therefore not be further pursued (see [53, page 119] for more details). An advantageous method will be treated later on in Section 4.2.

2.2.2 Linear Equations with Constant Coefficients

This Section, based on [48, Section 3.3.2, 13, Section 2.1.4, 58, Chapter 6] serves as a brief reminder of the theory of solutions of linear equations (with constant coefficients). One considers the IVP defined through

$$\dot{x} = Ax, \quad x(t_0) = x_0 \tag{10}$$

with the constant matrix $A \in \mathbb{R}^{n\times n}$ and $x(t) \in \mathbb{R}^n$. Rather than computing explicitly the solutions of the linear system (10) directly, it is frequently rather useful for analytical purposes, to consider the problem from a more theoretical point of view. Namely, the exponential function can be generalized to a mapping defined on the set of square matrices. One recalls that the set of linear operators $\mathcal{L}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is an n^2-dimensional Banach space equipped with the operator norm

$$\|A\| = \sup_{|v|=1} |Av| .$$

Now, one defines the exponential map $\exp\colon \mathcal{L}(\mathbb{R}^n) \to \mathcal{L}(\mathbb{R}^n)$ with

$$\exp\colon A \mapsto \mathcal{I} + \sum_{k=1}^{\infty} \frac{1}{k!} A^k,$$

and sets $e^A := \exp(A)$. Now, if $A \in \mathcal{L}(\mathbb{R}^n)$, then the infinite series

$$\exp(A) := \mathcal{I} + \sum_{k=1}^{\infty} \frac{1}{k!} A^k$$

is absolutely convergent (the proof can be found in [13, Proposition 2.31] for example). The considerations just above, motivate the following (see [48, Definition 3.3.1]) function $z \mapsto e^{Az}$ defined through

$$e^{Az} = \sum_{k=0}^{N} \frac{A^k z^k}{k!}, \quad z \in \mathbb{C}.$$

It is called the **matrix exponential** for the matrix A, and has the decisive property

$$\begin{aligned}
\frac{d}{dz} \sum_{k=0}^{N} \frac{A^k z^k}{k!} &= \sum_{k=1}^{N} \frac{A^k z^{k-1}}{(k-1)!} \\
&= A \sum_{k=0}^{N-1} \frac{A^k z^k}{k!} \\
&\xrightarrow[N \to \infty]{} A e^{zA},
\end{aligned}$$

i.e. the uniform convergence on compact sets of $\mathbb{C}$ with regards to z. It follows from the property just above, that

$$x(t) = x_0 e^{(t-t_0)A}, \tag{11}$$

with $x_0 \in \mathbb{R}^n$, is the unique solution of the system (10). Now, one can observe that for an eigenvector v from A,

$$\begin{aligned}
e^{tA} v &= \left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \right) v \\
&= \sum_{k=0}^{\infty} \frac{A^k v t^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k v t^k}{k!} \\
&\underset{Av = \lambda v}{=} \left(\sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} \right) v \\
&= e^{\lambda t} v.
\end{aligned}$$

It follows that $x(t) := e^{\lambda t} v$ is a solution for the linear system defined in (10). In summary: the eigenvalues and eigenvectors of A are closely related to the solutions of $\dot{x} = Ax$. For the sake of convenience, it shall be assumed that for $k = 1, \dots, n$, the eigenvalues λ_k of A are distinct, with their corresponding eigenvectors v_k. In this situation, the system (10) possesses n independent solutions, namely

$$e^{\lambda_1 t} v_1, \dots, e^{\lambda_n t} v_n, \tag{12}$$

which are gathered as columns in a matrix

$$\Phi := (x_1(t) \dots x_n(t)),$$

in the literature sometimes referred to as the fundamental matrix. Now, every solution of the IVP (10) can be put in the form

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0.$$

It follows from Equation (11) that $t \mapsto e^{At}$ is the fundamental matrix for the system (10). One therefore obtains

$$\Phi(t) = e^{tA} \text{ and consequently } \Phi^{-1}(t) = e^{-tA}.$$

Further details of the theory of solutions of linear systems, especially when λ_k is an eigenvalue with multiplicity n, and the dimension of its eigenspace is smaller than n, can be found in [3, Section 25] and [1, Section 12]. As one just saw, the eigenvalues play a central role in the solutions of (10). It is well-known from the qualitative theory of differential equations, that the solutions of (10) form an n-dimensional space $\mathcal{L} \subset C^1(I; \mathbb{R}^n)$, with $I \subset \mathbb{R}$. Thus, considering the linear combinations of the solutions in (12), leads naturally to the following result.

Proposition 2.14. *Let $A \in \mathbb{R}^{n\times n}$. The system $\dot{x} = Ax$ has:*

- *Solutions converging to 0 as $t \to +\infty$ if and only if $Re(\lambda_i) < 0, i = 1, \dots, n$;*
- *Bounded Solutions in $[0, \infty)$, if and only if $Re(\lambda_i) \leq 0, i = 1, \dots, n$. In the case of $\lambda_m = 0$, for $1 \leq m \leq n$, the algebraic and geometric multiplicities of λ_m concur.*

Proof. The proof and details can be found in [11, Section 2.3]. □

Proposition 2.14 is an essential connection between the behaviour of solutions for the system (10), and the eigenvalues for its defining matrix A. Furthermore, it builds a bridge for stability considerations of linear systems in Section 4.1.1 later on.

2.2.3 Continuous Dependence on the Initial Conditions

Let $x(t)$ be the solution of (2) on its maximal interval of existence (t_-, t_+). $J = [a, b]$ a compact interval of (t_-, t_+), $t_0 \in [a, b]$ and define $\mathrm{graph}_J(x) := \{(t, x(t)|t \in J)\} \subset G$. This leads to the following (see [48, Definition 4.1.1]) definition.

Definition 2.15. *A solution $x(t)$ is called continuously dependent on the data (t_0, x_0, f), if for every compact interval of $J \subset (t_-, t_+)$, there is a compact neighbourhood $K \subset G$ of $graph_J(x)$, such that: For every $\epsilon > 0$ there is a $\delta > 0$ such that the solution of the IVP $\dot{y} = g(t, y), y(\tau_0) = y_0$ exists for all $t \in [a, b]$ and the inequality*

$$|x(t) - y(t)| \leq \epsilon, \qquad \textit{for all } t \in [a, b]$$

is satisfied, assuming that $g: \mathbb{R} \times G \to \mathbb{R}^n$ is continuous and locally Lipschitz in x. Furthermore,

$$|\tau_0 - t_0| \leq \delta, |x_0 - y_0| \leq \delta \textit{ and } \sup_{(s,z)\in K} |f(s, z) - g(s, z) \leq \delta|$$

must hold.

This definition leads to an important result.

Theorem 2.16. Let $G \subset \mathbb{R}^{n+1}$ be an open set with $(t_0, x_0) \in G$, $f : G \to \mathbb{R}^n$ continuous and locally Lipschitz in x. Then, the solution $x(t)$ of (2) is continuously dependent on the data (t_0, x_0, f).

Proof. The proof can be found in [48, Satz 4.1.2]. □

An important property of the solutions for autonomous ordinary differential equations of the form (3) is the following (see [15, Lemma 2.2]) result.

Lemma 2.17 (Translation invariance)**.** *Let $\Omega_0 \subset \mathbb{R}^n$ and $J \subset \mathbb{R}$. One considers the mapping $f : \Omega_0 \to \mathbb{R}^n$, $x_0 \in \Omega_0$. For every solution $x \in C^1(J; R^n), t_0 \in J$, of the autonomous initial value problem* (3)*, time translation by $t \in \mathbb{R}$ produces a solution $x(t - \tau) \in C^1(J; R^n)$ of the IVP*

$$\begin{cases} \dot{x} = f(x) \\ x(t_0 + \tau) = x_0. \end{cases} \tag{13}$$

Proof. One has $\dot{x}(t) = f(x(t))$, for all $t \in J$, and it follows

$$\begin{aligned} \frac{d}{dt}(x(t - \tau)) &= \frac{\partial x}{\partial t}(t - \tau)\frac{d}{dt}(t - \tau) \\ &= \dot{x}(t - \tau) \end{aligned}$$

$$= f(x(t-\tau)).$$

Finally, taking $t = t_0 + \tau$ and plugging it into $x(t-\tau)$ yields $x(t_0 + \tau - \tau) = x(t_0) = x_0$ and the proof is concluded. □

It follows from Lemma 2.17 that in autonomous cases the starting time t_0 is not of any importance. A translation in time maps solutions for distinct starting times, but equal starting values, one into another. Thus, t_0 can be freely set.

Lemma 2.18. *Consider the autonomous system* (3)*, and f shall be locally Lipschitz in an open set $G \subset \mathbb{R}^n$. If its solution $x(t)$ exists for all $t \geq t_0$ and if $x_\infty = \lim_{t\to\infty} x(t)$ exists and belongs to $G \subset \mathbb{R}^n$, then x_∞ is a critical point; i.e. $f(x_\infty) = 0$.*

Proof. The proof can be found in [60, Chapter 10], for example. □

The following definitions (see [48, Definition 1.4.1, 60, Chapters X and XI]) are useful for this thesis:

Definition 2.19 (Isocline, nullcline)**.** *A curve having the equation $f(x) = k$ for some constant $k \in \mathbb{R}$ is called an **isocline** for the differential equation $\dot{x} = f(x)$. The special case of $f(x) = 0$ is called a **nullcline**.*

Definition 2.20 (Equilibrium point)**.** *A point $\tilde{a} \in G \subset \mathbb{R}^n$ is called an equilibrium point (also a stationary point or critical point) of f, if $f(\tilde{a}) = 0$. If $\tilde{a}$ is a critical point of f, then $x(t) = \tilde{a}$ is a (constant) solution.*

2.3 Dynamical Systems

This section gives a brief introduction to the main ideas of dynamical systems and is mainly based on [27, Chapter 1], [48, Section 4.4] and [29, Chapter 1]. Roughly speaking, three things mainly characterize dynamical systems:

1. The **phase space** Ω, whose elements stand for possible states of the system.
2. The **time** t, which can be discrete or continuous (in this thesis only the latter). If t can be extended to the past, one has a reversible process, otherwise irreversible. In the case of a continuous-time process, t is represented by the set $\mathbb{R}$ if the process is reversible and by $\mathbb{R}^+$ if otherwise.
3. The **time-evolution law**. Roughly speaking, it allows one to determine the state of the system at each t from its states at previous times.

Moreover, characteristic in the dynamical theory is the main focus on the asymptotic behaviour, i.e. one is interested in the state of the system after a very long time (typically

$t \to \infty$). The notion of a dynamical system has its origins (see [33, Section 1.1]) in the qualitative theory of differential equations, developed in the last two decades of the 19th century by Poincaré and Lyapunov. After more than 50 years of searching, the following abstract notion (central for the scope of this thesis) arose.

Definition 2.21 (Dynamical System). *Let $M \subset \mathbb{R}^n$ and (M, d) its corresponding metric space. The mapping $\phi\colon \mathbb{R} \times M \to M$, $(t, x) \mapsto \phi(t, x)$ is called a **dynamical system (flow)** if the following conditions are satisfied:*

$$\phi(0, x) = x \quad \forall x \in M, \tag{DS1}$$

$$\phi(t + s, x) = \phi(t, \phi(s, x)) \quad \forall t, s \in \mathbb{R}, x \in M \text{ and} \tag{DS2}$$

$$\phi \text{ is continuous in } (t, x) \in \mathbb{R} \times M. \tag{DS3}$$

Condition (DS2) is called the group property and states that the dynamical system can be reinitialized at a point $x(s)$ along its trajectory to obtain the same result $x(t + s)$, starting at $x(0)$ and travelling for some time $t + s$. If only $\mathbb{R}_+$ is being considered in the definition, one speaks of a semi-dynamical system on M (semi-flow), and in this case, condition (DS2) is named semi-group property. The dynamical system is called discrete if $\mathbb{R}$ is replaced by $\mathbb{Z}$. The mapping Φ describes the dynamic of the system. Define the system to be at the time $t = 0$ in x, then at a time $\tilde{t}$ it will be at $\phi(\tilde{t}, x)$. The set M is called the **phase space** of the dynamical system. The **phase portrait** of a dynamical system is the set of all its solution curves in the phase space M.

As already mentioned above, Definition 2.21 is rather abstract. However, this thesis mainly focuses on dynamical systems defined through solutions of autonomous differential equations leading to a very important result.

Theorem 2.22. [48, Satz 4.4.2] Let $G \subset \mathbb{R}^n$ be an open set, $f\colon G \to \mathbb{R}^n$ locally Lipschitz and for every initial value $y \in G$, the solution $x(t, y)$ of the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = y \end{cases} \tag{1}$$

shall exist globally, i.e. $\forall t \in \mathbb{R}$. Then, the function

$$\phi(t, y) := x(t, y), y \in G, t \in \mathbb{R}$$

defines a dynamical system. If the solutions exist at least globally to the right, then one obtains a semi-flow.

Proof. To prove this claim, one has to verify that the three conditions in Definition 2.21 are fulfilled. One has $y = x(0) = x(0, y) = \phi(0, y)$ so (DS1) holds. Now, $\phi(t+s, y) = x(t+s, y)$ is, due to the autonomy of the system and Lemma 2.17, a solution of (1). Same goes for $\phi(t, \phi(s, y)) = x(t, x(s, y))$. Setting $t = 0$, one has $x(0 + s, y) = x(s, y) = x(0.x(s, y))$. Because uniqueness of the solution was assumed, it follows $x(t + s, y) = x(t, x(s, y))$, and this fulfils condition (DS2). Finally, $x(t, y) = \phi(t, y)$ continuously depends on y because of Theorem 2.16. Therefore, (DS3) also holds and the proof is complete. □

Theorem 2.22 can be nicely illustrated with the following example.

Example 2.23. ([46, Section 1.1]) Consider the (uncoupled) linear system

$$\dot{x} = Ax \text{ with } A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},$$

with its solutions $x_1(t) = y_1 \mathrm{e}^{-t}$ and $x_2(t) = y_2 \mathrm{e}^{2t}$, leading to

$$x(t, y) = y \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{2t} \end{pmatrix}.$$

The premises of Theorem 2.22 are fulfilled, leading to the dynamical system defined by the linear system above:

$$\phi(t, y) := y \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{2t} \end{pmatrix}.$$

The importance of Theorem 2.22 lies in the fact, that if the autonomous IVP (3) has a unique global solution, then it always defines a dynamical system. If the solution exists globally to the right, one still has a semi-dynamical system. It has to be noted that the autonomy of (3) is essential to obtain a dynamical system. One considers, for instance the IVP $\dot{x} = tx, x(0) = y$ with solution $x(t) = y\mathrm{e}^{\frac{1}{2}t^2}$. Now, it holds

$$x(t + s, y) = y\mathrm{e}^{t+s} = y\mathrm{e}^{s}\mathrm{e}^{t} \text{ and } \dot{x}(t + s, y) = y\mathrm{e}^{t+s} \neq ty\mathrm{e}^{t+s} = f(t, x)$$

and therefore $x(t + s, y)$ is not a solution of (1).

Remark 2. One can reformulate Lemma 2.18 in the language of dynamical systems. Let $\phi(t, x)$ be a semi-flow, and the limit value $x_\infty := \lim_{t\to\infty} \phi(t, x_0)$ shall exist. One has

$$\phi(t, x_\infty) = \phi(t, \lim_{s\to\infty} \phi(s, x_0)) \underset{\text{(DS3)}}{=} \lim_{s\to\infty} \phi(t, \phi(s, x_0)) \underset{\text{(DS2)}}{=} \lim_{s\to\infty} \phi(t + s, x_0) = x_\infty.$$

It follows, that limit points (i.e. for $t \to \infty$) from solutions of (3) defined for all $t \geq 0$, are always stationary points of the equation.

There is however a significant downside in Theorem 2.22, namely to prove the (global) existence of the solution for the IVP, which can be a tedious task, especially when a closed-form of the solution is not available.

3 Invariance

Let $G \subset \mathbb{R}^n$ be an open set, $f : \mathbb{R} \times G \to \mathbb{R}^n$ continuous. One recalls the initial value problem

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{2}$$

with $x_0 \in G, t_0 \in \mathbb{R}$, defined in Section 2.2. Let x_1 be the solution of (2) on the interval $[t_0, t_1]$. The following useful notion shall be reminded: Let $t_+(t_0, x_0) \in \mathbb{R}$ be defined as $t_+ = t_+(t_0, x_0) := \sup\{t_1 \geq t_0 : \text{a solution } x_1 \text{ exists for (2) on the closed interval } [t_0, t_1]\}$. The interval $[t_0, t_+)$ shall then be called the maximal interval of existence of the solution to the right. The maximal solution of (2) is defined as $x(t) = x_1(t)$ for all $t \in [t_0, t_1]$. Moreover, a solution of (2) shall be designated with $x(t; t_0, x_0)$ to emphasize the initial value $x(t_0) = x_0$.

3.1 Invariant Sets

For the rest of this section, one assumes uniqueness of solutions to the right. Let $x(t; t_0, x_0)$ be a solution of (2) on the maximal interval $J_+(t_0, x_0) := [t_0, t_+(t_0, x_0)]$. One of the essential concepts in the qualitative theory of ordinary differential equations is the notion of an invariant set (see [48, Definition 7.1.1]).

Definition 3.1. *Let $D \subset G$. The set D is called positively invariant for* (2)*, if the solution satisfies $x(t; t_0, x_0) \in D$ for all $t \in J_+(t_0, x_0)$, assuming $x_0 \in D$. Negative invariance is defined accordingly, assuming uniqueness of solutions of* (2) *to the left. D is finally called invariant if it is both positively and negatively invariant.*

The invariance of a set D for (2) can be summarized as follows: **If the initial value lies in D, the solution of (2) will remain in D for all $t \in \mathbb{R}$.**

Invariant sets of dynamic systems play an essential part in various situations, in which the behaviour of the solution is restrained. The existence of an invariant set in the state space of a system entails bounds for the solution's behaviour. The validation of a priori specified constraints (physical constraints, the positivity of solutions) can be confirmed with the help of invariant sets.

Before looking at examples of invariant sets, a few things can already be noted:

- $\emptyset$ and G are trivial examples of positively invariant sets.
- One can see, that unions (same goes for intersections) of (positively) invariant sets are (positively) invariant.

- The smallest (biggest) invariant subset of D, is the intersection (union) of all the invariant subsets of D.

Definition 3.2 (Orbit). *Let $y \in G \subset \mathbb{R}^n$ be fixed. The set*

$$\gamma(y) := \{\phi(t, y) | t \in \mathbb{R}\} \tag{3}$$

*is called the **orbit** (or **trajectory**) through the point y. Correspondingly, $\gamma_+(y) = \phi(\mathbb{R}_+, y)$ is called the positive semi-orbit of y.*

Lemma 3.3.

1. *If D is (positively) invariant, then this is also true for $\overline{D}$.*

2. *If D_1 and D_2 are invariant, then so is the complement $D_1 \setminus D_2$.*

Proof. Let D be a (positively) invariant set, $x \in \overline{D}$ and a fixed t in the maximal interval of existence $(t_-(t_0, x), t_+(t_0, x))$. If x is in the closure of D, it implies that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $\lim\limits_{n \to \infty} x_n = x$. Now, for an n_0 chosen large enough, one has for $t \in (t_-(t_0, x_n), t_+(t_0, x_n))$ and $n \geq n_0$

$$\lim_{n \to \infty} \phi(t, x_n) = \phi(t, x) \in \overline{D}.$$

In summary, one has derived that if $x \in \overline{D}$ then $\phi(t, x) \in \overline{D}$, hence the closure of D is also (positively) invariant.
Now, let $d_1 \in D_1 \setminus D_2$. If the intersection of the orbit passing through d_1 with D_2 contains a point d_2, then it follows the equality $\gamma(d_1) = \gamma(d_2)$, and both trajectories belong to D_2, in contradiction to the assumption $d_1 \notin D_2$. It follows $\gamma(d_1) \subseteq D_1 \setminus D_2$, and thus $D_1 \setminus D_2$ is invariant. □

For the sake of getting a first idea of what invariant sets looks like, examples are useful.

Example 3.4. An equilibrium point $\{x_0\}$ (see Definition 2.20), with $f(t, x_0) = 0$ for all $t \in \mathbb{R}$, would be the simplest invariant set one can think of, considering that a solution starting at this point would remain there forever.

Example 3.5. A rather simple invariant set is a periodic orbit Γ (see, e.g. [46, Section 3.3]), determined by the initial condition x_0 and a period T, defined as the shortest time $T > 0$ for which $\phi(t + T, x_p) = \phi(t, x_p)$ holds.

After a few examples, it is now necessary to find conditions for the positive invariance of D. The following considerations are mainly based on [48, Section 7.1]. Let $x_0 \in D$ be arbitrary, D be positively invariant , $h > 0$ sufficiently small and set $\tilde{x} = x_0 + hf(t_0, x_0)$.

- If $\tilde{x} \in int(D)$, then it follows $0 = dist(\tilde{x}, D) \leq |\tilde{x} - x(t_0 + h; x_0)|_2$.
- Else, the distance from $\tilde{x}$ to D will be less or equal than the Euclidean distance between $\tilde{x}$ and $x(t_0 + h; t_0, x_0)$.

It follows the inequality

$$dist(\tilde{x}, D) \leq |\tilde{x} - x(t_0 + h; x_0)|_2 . \tag{4}$$

By the fundamental theorem of calculus, $x(t_0 + h; x_0)$ can be written as

$$x(t_0 + h; x_0) = x_0 + \int_{t_0}^{t_0+h} \dot{x}(s; t_0, x_0)\,\mathrm{d}s.$$

Plugged in the inequality (4), it follows with $\dot{x}(t_0; t_0, x_0) = f(t_0, x_0)$

$$dist(\tilde{x}, D) \leq \left| h\dot{x}(t_0) - \int_{t_0}^{t_0+h} \dot{x}(s; t_0, x_0)\,\mathrm{d}s \right|_2 ,$$

and further

$$\begin{aligned}\left| h\dot{x}(t_0) - \int_{t_0}^{t_0+h} \dot{x}(s; t_0, x_0)\,\mathrm{d}s \right|_2 &= |h\dot{x}(t_0) - (x(t_0 + h) - x(t_0))|_2 \\ &= h \left| \dot{x}(t_0) - \frac{(x(t_0 + h) - x(t_0))}{h} \right|_2 .\end{aligned}$$

Now, if $h \to 0$, then $\lim\limits_{h \to 0} \frac{x(t_0+h)-x(t_0)}{h} = \dot{x}(t_0)$. It means that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$dist(x_0 + hf(t_0, x_0), D) \leq \epsilon h$$

is fulfilled, if $h \in (0, \delta]$. One has derived an essential result with these considerations.

Proposition 3.6 (The subtangential condition (SC))**.** *Let x be an element of an invariant subset D, then*

$$\lim_{h \to 0_+} \frac{1}{h} dist(x + hf(t, x), D) = 0$$

holds, for all $t \in \mathbb{R}$.

Proposition 3.6 leads to a result of significant importance. But before the next step, the theorem of Arzelà-Ascoli shall be reminded (a proof can be found in [52, Theorem 11.28]):

Theorem 3.7 (Arzelà-Ascoli)**.** Let F be a pointwise bounded equicontinuous collection of complex functions on a metric space X, and that X contains a countable dense subset E. Every sequence $\{f_n\}$ in F possesses then a subsequence, that converges uniformly on every compact subset of X.

Proposition 3.8. *Let $G \subset \mathbb{R}^n$ be an open set, $f : \mathbb{R} \times G \to \mathbb{R}^n$ continuous and $D \subset G$ closed. Then the following assertions are equivalent:*

a) D is positively invariant for (2)*,*
b) f and D satisfy the SC.

Proof. The implication $a \Rightarrow b$ has already been derived at the beginning of this section (see Proposition 3.6). It remains to prove the existence of a solution for (2), such that $x(t) \in D$ for all $t \in [t_0, t_0 + a]$ ($a \in \mathbb{R}$), assuming that the SC is fulfilled. The proof for the local solution is sufficient since uniqueness of a solution for the initial value problem (2) would imply, that the maximal solution in D to the right would remain in D. The existence of a solution $x(t) \in D$ for (2) on the interval $[t_0, t_0 + a]$ is shown with a sequence of points (t_j, x_j) that approximate its graph. The proof is rather lengthy, so only the main ideas are being presented here.

- Let $M := \max\{|f(t, x)| : t \in [t_0, t_0 + 1], x \in B_r(x_0) \cap D\}$, $a := \min\{1, \frac{r}{M+1}\}$ and $\epsilon \in (0, 1)$ arbitrary. Now, let (t_j, x_j) be already constructed. Because one assumes that the SC is fulfilled in this point, one can find $h_{j+1} > 0$ and $x_{j+1} \in D$ (more precisely $x_{j+1} \in \partial D$) such that the following holds:

$$dist(x_j + h_{j+1} f(t_j, x_j), D) = |x_j + h_{j+1} f(t_j, x_j) - x_{j+1}| \leq \epsilon h_{j+1}, \quad \epsilon \in (0, 1).$$

- One then shows that the sequence aborts when $t_{j+1} \geq t_0 + a$ occurs (there is only a finite number of $t_j \leq t_0 + a$).

- The linear spline passing through the points (t_j, x_j) and (t_{j+1}, x_{j+1}) has the form

$$x_\epsilon(t) = \frac{t - t_j}{h_{j+1}} x_{j+1} + \frac{t_{j+1} - t}{h_{j+1}} x_j, \quad t_j \leq t < t_{j+1},$$

  (for simplicity, $x(t) \in \mathbb{R}$). One then has the inequality

$$\left| x_\epsilon(t) - x_0 - \int_{t_0}^{t} f(s_j, x_j)\, \mathrm{d}s \right| \leq \epsilon a.$$

- Now, if one takes for ϵ the values $\epsilon_k = 1/k$, the corresponding splines $x_{\epsilon_k}(t)$ are uniformly bounded and Lipschitz continuous with Lipschitz constant $L = M + 1$ on the interval $[t_0, t_0 + a]$, thus they are equicontinuous. The existence of a solution follows with Theorem 3.7 from Arzelà-Ascoli: x_{ϵ_k} possesses a uniformly convergent, D-valued subsequence $x_{\epsilon_{k_m}} \to x$.

More details of the proof (especially for $b \Rightarrow a$) can be found in [48, Satz 7.1.4]. □

In the case that the solutions for (2) are not unique, if at least one of them exists in D, then D is called **positively weakly invariant**. Historically, the concept of the subtangential condition can be traced back to Bouligand's (as cited in [6, Definition 4.6]) definition of the "tangent cone" in the 1930s.

Definition 3.9. *(Tangent cone) Let S be a closed set, the tangent cone to S at x is then being defined as*

$$T_D(x) = \left\{ z \in \mathbb{R}^n : \liminf_{h \to 0_+} \frac{dist(x + hz, D)}{h} = 0 \right\}. \tag{5}$$

It should be noted, that even if $dist(x+hz, D)$ depends on the concrete norm, the set $T_D(x)$ doesn't. Two more similar definitions of the tangent cone exist. One was made by Bony [8, Definition 2.2], although he did not specifically use this term. He used the notion of a vector field $X(x)$ tangent to a closed set F ("Un champ de vecteurs $X(x)$ est dit tangent au fermé F si, ..."). The other definition traces back to Clarke [14, Section 1.2, p11]. Nevertheless, in the case of convex sets, the three definitions are equivalent. Furthermore, if D is a convex set, so is $T_D(x)$ and therefore, lim inf can be replaced by lim in (5). With the notion of the tangent cone, one can reformulate Proposition 3.8 (in the autonomous case) as follows:

Theorem 3.10 (The SC revisited)**.** One considers the system $\dot{x}(t) = f(x(t))$. The existence of its (not necessarily unique) solution for all $t \geq 0$ shall be assumed, for an initial value $x(0)$ in an open set O. Let $D \subset O$ be a closed set. Then, D is weakly positively invariant for the system if and only if Nagumo's condition $f(x) \in T_D(x)$ for all $x \in D$, is satisfied. One obtains positive invariance if one assumes the uniqueness of the solution.

Proof. The proof dates back to Nagumo [45] in 1942. In the literature, this result is sometimes referred to as the Nagumo theorem. □

The subtangential condition is equal to the fact, that the vector field $f(x)$ **points inward** or **lies below the tangent plane** along ∂D. There lies the origin of the term "subtangential condition", which was first formulated by Nagumo. Interestingly, he does not use the word positive invariance in his paper; he used the notion of "majorant to the right in D" instead ("nach rechts majorant in D"). Nevertheless, his findings did not get much

attention until the late sixties, when the invariance problem was brought to life again.
A drawback from Proposition 3.8, is the prerequisite of the presumed invariant set D. Its application, therefore, depends on an educated guess, or mathematical intuition. A possibility to circumvent this hurdle is the following important result from Wilke and Prüss (see [48, Satz 7.2.1]):

Theorem 3.11. Let $\Phi \in C^1(\mathbb{R}^n;\mathbb{R})$ and let $a \in \mathbb{R}$ be a regular value for Φ, i.e. $\nabla\Phi(x) \neq 0$ for all $x \in \Phi^{-1}(a)$. Then, the following assertions are equivalent:

1. $D = \Phi^{-1}((-\infty, a])$ is positively invariant for (2).

2. $(f(t,x)|\nabla\Phi(x)) \leq 0$ for all $t \in \mathbb{R}, x \in \Phi^{-1}(a) = \partial D$.

Proof. Assume that 1. is valid. Set $t \in \mathbb{R}, x \in \partial D$ and define $y = f(t,x)$. According to Proposition 3.8, f and D satisfy the SC, entailing the existence of a function $g \in D$ so that for every $\epsilon \in (0,1)$, there is a $\delta > 0$ with

$$\|x + hy - g(h)\|_2 \leq \epsilon h,$$

for every $h \in (0, \delta]$. Considering that Φ is continuously differentiable, there exits a $\zeta > 0$ with

$$|\Phi(g(h)) - \Phi(x) - (\nabla\Phi(x)|g(h) - x)| \leq \epsilon\|g(h) - x\|_2,$$

if $\|g(h) - x\|_2 \leq \zeta$ is fulfilled. One can see that $\|g(h) - x\|_2 \leq h(1 + \|y\|_2)$ holds, and therefore $0 < h \leq \min\{\delta, \frac{\zeta}{(1+\|y\|_2)}\}$. Because of the definition of D and x, one has $\Phi(x) = a$ and $\Phi(g(h)) \leq a$. Thus (one remembers the bilinearity of a scalar product),

$$\begin{aligned}
(y|\nabla\Phi(x)) &= \big(y - h^{-1}(g(h) - x)|\nabla\Phi(x)\big) + \big(h^{-1}(g(h) - x)|\nabla\Phi(x)\big) \\
&\leq \|\nabla\Phi(x)\|_2\epsilon + h^{-1}\left|\Phi(g(h)) - \Phi(x) - (\nabla\Phi(x)|g(h) - x)\right| \\
&\leq \|\nabla\Phi(x)\|_2\epsilon(1 + \|y\|_2) \\
&\leq k\epsilon
\end{aligned}$$

with $k \in \mathbb{R}$. Considering that ϵ is arbitrary, the claim follows.
Now, one assumes that 2. holds, and sets $t_0 \in \mathbb{R}, x_0 \in D$. The following ODE

$$\dot{x} = f(t,x) - \epsilon\nabla\Phi(x)$$

shall be considered. Theorem 2.1 (Peano's existence theorem) guarantees the existence of a solution $x_\epsilon(t)$ with initial value $x_\epsilon(t_0) = x_0$. One defines $\phi(t) = \Phi(x_\epsilon(t))$. One can see that $\phi(t_0) \leq a$, and also (with the chain rule and bilinearity of the scalar product once

more)
$$\dot{\phi}(t) = (\dot{x}_\epsilon(t)|\nabla\Phi(x_\epsilon(t))) = (f(t, x_\epsilon(t))|\nabla\Phi(x_\epsilon(t))) - \epsilon\,|\nabla\Phi(x_\epsilon(t))|^2$$

hold. Now, if one assumes $x_\epsilon(t) \notin D$ for some $t > t_0$, then there exists a $t_1 \geq t_0$ with $x_\epsilon(t_1) \in \partial D$ and $x_\epsilon(t) \in D$ if $t \leq t_1$. One then obtains

$$\dot{\phi}(t_1) \leq -\epsilon\,|\nabla\Phi(x_\epsilon(t_1))|^2 < 0,$$

since a is a regular value. On the other hand, one has with the formal definition of the derivative
$$\dot{\phi}(t_1) = \lim_{h\to 0_+} \frac{\phi(t_1) - \phi(t_1 - h)}{h} = \lim_{h\to 0_+} \frac{a - \phi(t_1 - h)}{h} \geq 0,$$

and therefore a contradiction to $\dot{\phi}(t_1) < 0$. It follows, that the solution $x_\epsilon(t)$ stays in D, on its maximal interval of existence. Finally, one can see that $x_\epsilon(t)$ converges uniformly to $x(t)$ (the solution of (2)) on compact intervals for $\epsilon \to 0$, and it follows the positive invariance of D. □

Example 3.12. One considers the system of ODEs

$$\begin{aligned}\dot{x} &= y^2 - x\\ \dot{y} &= -xy,\end{aligned}$$

and the right-hand side shall be designated with $f(x, y)$. f is continuously differentiable, guaranteeing a unique local solution of the system. Moreover, one has

$$(f(x, y)|\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)) = xy^2 - x^2 - xy^2 = -x^2 \leq 0$$

and it follows with Proposition 2.11, that all the solutions of the system exist globally to the right. Let $0 \neq a \in \mathbb{R}$ be arbitrary but fixed and one defines $\Phi(x, y) = \frac{1}{2}(x^2 + y^2)$. One recognizes $\Phi \in C^1(\mathbb{R}^2; \mathbb{R})$ with $\nabla\Phi(x, y) = (x, y)^T$. Now, $\nabla\Phi(x) = 0$ if and only if $(x, y)^T = (0, 0)$. Considering the fact that $\Phi(0, 0) = 0 \neq a$, one has $(0, 0)^T \notin \Phi^{-1}(a)$, and therefore a is a regular value for Φ. It holds

$$(f(t, x)|\nabla\Phi(x)) = (y^2 - x, -xy)^T \begin{pmatrix}x\\y\end{pmatrix} = y^2x - x^2 - xy^2 = -x^2 \leq 0$$

for all $t \in \mathbb{R}, x \in \Phi^{-1}(a)$. According to Theorem 3.11, $D = \Phi^{-1}((-\infty, a])$ is positively invariant for the system of ODEs, D being the circle with centre $(0, 0)$ and radius $\sqrt{2a}$.

It has to be said, that in Proposition 3.8, the difficulty of guessing a presumed invariant set D, has been replaced with the task of finding a function Φ with the desired properties stated in Theorem 3.11. However, this result will be improved in Section 4.2, especially with Corollary 5.11.

3.2 Invariant Convex Sets

Before exploring further the geometric meaning of the SC, the notions of the outer normal (see [49, Appendix, Section B]) and convex sets shall be reminded.

Definition 3.13 (The outer normal)**.** *Let $D \subset \mathbb{R}^n$ and $x \in \partial D$. The vector $\nu \neq 0 \in \mathbb{R}^n$ is called **outer normal** of D in x, if $B_{|\nu|}(x+\nu) \cap D = \emptyset$. The set of outer normals in $x \in \partial D$ should be named $N_D(x)$.*

One can say, that ν is an outer normal to D at $x \in \partial D$ if the open ball B has no shared point with D. From a geometric point of view, B touches D at $x \in \partial D \cap \bar{B}$. It can further be seen from Definition 3.13, that an outer normal must not always exist. As an example, one can take a concave polygon, which possesses at least one "jumping in" corner. This unpleasant property will be getting ridden off in this section, when convexity of the set is assumed.

Definition 3.14 (Convex sets)**.** *A set $K \subset \mathbb{R}^n$ is called convex if it satisfies:*

$$x, y \in K \text{ then } \lambda x + (1-\lambda)y \in K \text{ for all } 0 \leq \lambda \leq 1.$$

It can be formulated in the following way: For two distinct points in K, their connecting line entirely lies in K.

Some essential properties of convex sets shall be reminded:

- The intersection of convex sets is convex.
- The union of convex sets is usually non-convex (two intersecting circles for example).
- The closure of a convex set yields a convex set.

The definitions and proofs for the properties of convex sets can be found in [61, Section 1.22]. With the concept of the outer normal, it will now be possible to formulate a criterion to characterize invariant convex sets of (2). But before that, one needs the following (see [48, Lemma 7.3.3]) result.

Lemma 3.15. *Let $D \subset \mathbb{R}^n$ be a convex closed set, $x \in \partial D$ and $z \in \mathbb{R}^n$. The SC*

$$\lim_{h \to 0_+} \frac{1}{h} dist(x + hz, D) = 0$$

is then equivalent to

$$(z|y) \leq 0 \text{ for all } y \in N(x).$$

Proof. The proof for the implication $\lim_{h\to 0_+} \frac{1}{h} dist(x + hz, D) = 0 \Rightarrow (z|y) \leq 0$ for all $y \in N_D(x)$ can be found in [48, Lemma 7.1.3]. Now, assume that $(z|y) \leq 0$ for all $y \in N_D(x)$ holds, but that the SC is wrong. Thus, there exists an $\epsilon > 0$ and a sequence $h_n \to 0_+$ with

$$\epsilon h_n \leq dist(x + h_n z, D) \leq h_n \|z\|_2.$$

It follows (one remembers the projection P)

$$\epsilon \leq \left| h_n^{-1}(x - P(x + h_n z)) + z \right| \leq \|z\|_2.$$

It follows with Bolzano-Weierstrass that there exists a subsequence $\tilde{h}_n$ with $\tilde{h}_n \to 0_+$ and a sequence

$$y_n := \tilde{h}_n^{-1}(x - P(x + \tilde{h}_n z)) + z$$

that converges to a $y \neq 0$. It can be shown (see [48, Lemma 7.3.1] for a proof), that for every $u \in \mathbb{R}^n$, there is precisely a $Pu \in D$ satisfying

$$(u - Pu|v - Pu) \leq 0 \tag{6}$$

for all $v \in D$. Now, it follows for a $v \in D$ with (6)

$$0 \leq (y_n|P(x + \tilde{h}_n z) - v) \to (y|x - v)$$

and thus the positivity of $(y|x - v)$. One further recognizes

$$\|x + y - v\|_2^2 = \|x - v\|_2^2 + 2(x - v|y) + \|y\|_2^2 \geq \|y\|_2^2$$

for all $v \in D$. This entails $B_{\|y\|_2}(x + y) \cap D = \emptyset$, hence y is an outer normal on D in the point x. Further, one obtains again with (6) by setting $v = x$ and $u = x + \tilde{h}_n z$ the inequality

$$(x + \tilde{h}_n z - P(x + \tilde{h}_n z)|x - P(x + \tilde{h}_n z)) \leq 0.$$

Taking the limit $n \to \infty$, one obtains

$$(z|y) \geq \|y\|_2^2 > 0,$$

a contradiction to $(z|y) \leq 0$ and the claim is proven. $\square$

To grasp a geometric interpretation of Lemma 3.15, one needs the well-known relation of the angle Θ between two non-zero vectors and their scalar product $\cos(\Theta) = \frac{v \cdot w}{||v||||w||}$. Now, if $(z|y) \leq 0$, it entails $\cos(\Theta) \leq 0$, considering the positivity of a norm. It follows that Θ is an angle between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, thus an obtuse angle. With the help of Lemma 3.15, the SC can be interpreted as follows: The angle Θ between $\dot{x}(t)$ and every outer normal in $x(t) \in \partial D$ is always higher than $\frac{\pi}{2}$. Geometrically speaking, if a solution of (2) reaches the boundary of D, then the SC forces it to turn back. It can be visualized graphically, that a convex set D is positively invariant if a trajectory through each boundary point "moves inward".

Theorem 3.16. [48, Satz 7.3.4] Let G be an open set of $\mathbb{R}^n$ and $D \subset G$ a closed convex set. Then the following assertions are equivalent:

i) D is positively invariant for (2);

ii) $(f(t,x)|y) \leq 0$ for all $t \in \mathbb{R}, x \in \partial D, y \in N_D(x)$.

Proof. According to Proposition 3.8, it follows from the positive invariance of D for (2), that $\lim\limits_{h \to 0_+} \frac{1}{h} dist(x + hf(t,x), D) = 0$. Combining it with Lemma 3.15, one obtains the equivalence $(f(t,x)|y) \leq 0$ for all $y \in N_D(x)$. □

A particular case of Theorem 3.16 is the positive set

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \colon x_k \geq 0, k = 1, \ldots, n\}.$$

To this end, one considers the set

$$H_k := \{x \in \mathbb{R}^n \colon x_k \geq 0\}.$$

Let $\alpha > 0$ and e_k be the k-th unit vector in $\mathbb{R}^n$. One can see that the outer normal in $x \in \partial H_k$ is given by $y = -\alpha e_k$. Theorem 3.16 yields positive invariance for H_k if

$$x_k = 0 \Rightarrow f_k(x) \geq 0$$

is satisfied. One further obtains for $x \in \partial \mathbb{R}^n_+$

$$N_{\mathbb{R}^n_+}(x) = \{y \in -\mathbb{R}^n_+ \colon x_k y_k = 0 \text{ for all } k\}.$$

One has just proven the following positivity criterium (see [49, Appendix]):

Lemma 3.17. *One considers the positive set* $\mathbb{R}^n_+$*. The following are then equivalent:*

1. $\mathbb{R}^n_+$ *is positively invariant for* (3)*;*

2. *For* $x \in \mathbb{R}_+^n, x_k = 0 \Rightarrow f_k(x) \geq 0$ *for all* k.

In the particular case of $f(x) = Ax$ with a matrix $A \in \mathbb{R}^{n \times n}$, property 2 in Lemma 3.17 is satisfied if all the non-diagonal entries of A are non-negative. These matrices are called quasi-positive. For this reason, a function f satisfying property 2 in Lemma 3.17 shall also be called **quasi-positive**.

The following example from Walter (see [60, Chapter 10]) is a well-studied model originating in the life sciences. Despite being rather simple in its construction, it is nevertheless fascinating for visualization purposes of the SC.

Example 3.18. (Competing species) One considers the autonomous system

$$\dot{u} = u(3 - u - 2v), \quad \dot{v} = v(4 - 3u - v), \tag{7}$$

which describes the numbers of animals in two different populations ($u(t)$ and $v(t)$), feeding on the same limited food source, with positive initial values u_0, v_0. As usual, t represents the time variable and let $f_1(u, v) := u(3 - u - 2v)$ and $f_2(u, v) := v(4 - 3u - v)$. One has $f_1(0, v) = f_2(u, 0) = 0$, and it follows with Lemma 3.17, that the solutions of the system are positive for $t \geq 0$, if the initial values u_0 and v_0 are positive. Therefore, one can restrain seeking positively invariant sets on $Q := [0, \infty] \times [0, \infty]$. Now, one can see, that the right-hand side $f(u, v)$ is polynomial, and in particular continuously differentiable, which guarantees a unique solution of the system in an open interval containing $t_0 = 0$. Let $I = [0, t_+]$ be the maximal interval of existence. One has

$$\dot{u} = u(3 - u - 2v) \leq 3u, \quad \dot{v} = v(4 - 3u - v) \leq 4v$$

and therefore, $0 \leq u(t) \leq e^{3t}u_0$ and $0 \leq v(t) \leq e^{4t}v_0$. It follows, that the solutions $u(t), v(t)$ are bounded on every finite interval, and thus, $t_+ = \infty$ and the solutions exist globally for $t \geq 0$.

Setting the right-hand side of the system to zero provides (after a short calculation) the equilibria $(0, 0), (0, 4)$ and $(3, 0)$. As already pointed out in Remark 2, limit points from solutions of (3) defined for all $t \geq 0$, are always stationary points of the equation. Therefore, a solution of the system will eventually converge to one of these points. A repeatedly encountered difficulty in the search for invariant sets is its localization: One needs an educated guess or a conjecture of where the invariant sets could be. In this example, one can analyse the nullclines, as a starting point. Setting $f_1(u, v)$ and $f_2(u, v)$

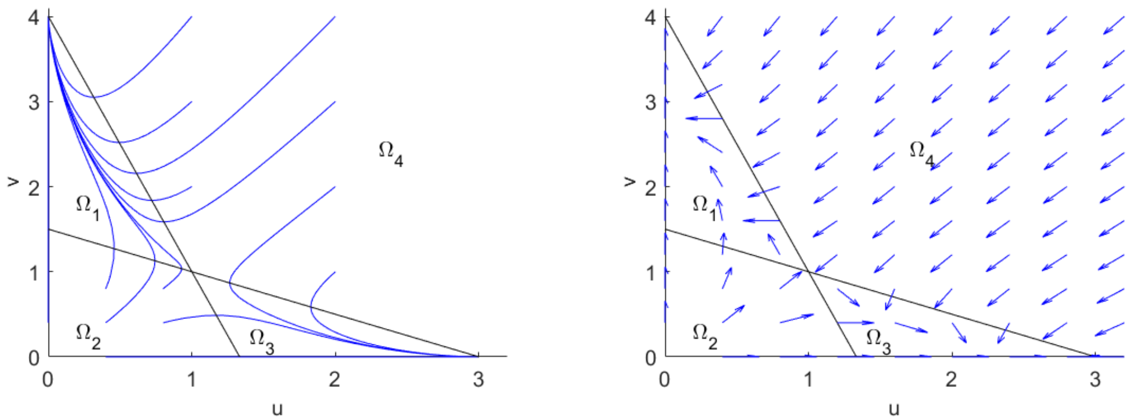


Figure 1: Phase portrait (left) and vector field (right) for a competing species model.

to zero yields the nullclines

$$u = 0, v = 0, v = \frac{3-u}{2} \text{ and } v = 4 - 3u,$$

depicted in Figure 1 (right).

Remark 3. Setting the nullclines to equality, leads to $f_1(u,v) = f_2(u,v) = 0$, and thus $\dot{u} = \dot{v} = 0$. It follows that each point of intersection of the nullclines is an equilibrium point.

Now, the (convex) set Ω_1 (depicted in Figure 1 on the left) shall be considered.

- On the left boundary, one has $n_l := (-1,0)^T$ as outer normal, and $u = 0$. Thus, $(f(u,v)|n_l) = (0, v(4-v))^T \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right) = 0$.
- On the upper boundary, one has $n_u := (3,1)^T$ as outer normal, $u \in [0,1]$ and $v = 4 - 3u$. One obtains $(f(u,v)|n_l) = (u(3-u-2(4-3u)),0)^T \left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right) = 0 = 15(u-1) \leq 0$, if $u \in [0,1]$.
- On the lower boundary, one has $n_d := (-1,-2)^T$ as outer normal, $u \in [0,1]$ and $v = \frac{3-u}{2}$. Thus $(0, \frac{3-u}{2}(4-3u-\frac{3-u}{2}))^T \left(\begin{smallmatrix} -1 \\ -2 \end{smallmatrix}\right) = -2 \underbrace{(5-5u)\frac{3-u}{4}}_{\geq 0 \text{ if } u \in [0,1]} \leq 0.$

It follows with Theorem 3.16 that Ω_1 is a positively invariant set for the system (7). One can show with analogous calculations, that Ω_3 is also a positively invariant set for the competing species system.

The second example, for visualization purposes of the *SC* among other things, from Prüss, Zacher, and Schnaubelt [49, Chapter 8] is by now famous in the world of dynamical systems.

Example 3.19. (Mathematical Modelling of Infectious Diseases) Many models that describe the spreading of an epidemic work similarly: one partitions the host population into categories. In the **SIR model** of Kermack and McKendrick, there are three different dividing classes:

1. Susceptible (**S**): The persons who are not immune to the infectious organism, and therefore could contract the infection in case of an exposition.

2. Infectious (**I**): The persons who are at present infected, and therefore can transmit the disease to responsive subjects they are in contact with

3. Removed (**R**): The individuals who are immune to the disease, and for this reason do not have any influence on the transmission of the infectious agent in case of contact with other individuals

The SIR model's main goal is to describe the changes in the sizes of the three compartments over a specific period. Diseases that lead to a possible immunity have a different class structure from diseases which do not lead to immunity. In the above SIR model, the disease confers immunity against re-infection (in the case of measles, for example). It should be assumed that the infection of a non-infected person with an infected one, happens with a fixed relative frequency. Therefore, $\dot{S}$ is proportional to S and I with some proportionality constant $r > 0$, called **infection rate**. One further assumes that a formerly infected person has gained immunity after overcoming the disease, i.e. the healing process occurs randomly with a fixed rate $a > 0$. The above discussion entails that individuals pass through the following scheme

$$S \underbrace{\hookrightarrow}_{rSI} I \underbrace{\hookrightarrow}_{aI} R,$$

leading to the system of ODEs

$$\begin{aligned}
\dot{S} &= -rSI := f_1(S, I, R), \\
\dot{I} &= rSI - aI := f_2(S, I, R), \\
\dot{R} &= aI := f_3(S, I, R),
\end{aligned}$$

with given constants $a, r > 0$. Furthermore, due to the fact that S, I and R represent the size of a portion of the population N, the initial conditions $S(0) = S_0$, $I(0) = I_0$ and $R(0) = R_0$ are positive, and one defines the (initial) total population $N = S_0 + I_0 + R_0$. One further defines the reproduction rate of the infection $\mathcal{R} = \frac{rN}{a}$, describing how many infections an individual causes in a middle period of infection, presupposing that the number of individuals which could be infected is constant. First of all, the right-hand side of the system (i.e. $f(S, I, R)$) is obviously continuously differentiable, and thus locally Lipschitz with respect to S, I, R. It follows with Theorem 2.4 that the system possesses a unique solution (S, I, R) for given positive initial values. Now, if one assumes $S, I, R \geq 0$, then it follows

$$f_1(0, I, R) = f_2(S, 0, R) = 0 \text{ and } f_3(S, I, 0) = aI \geq 0.$$

It follows with Lemma 3.17 that the solution of the SIR model is positive for all $t \geq 0$. Furthermore, one has $\dot{S} + \dot{I} + \dot{R} = 0$, and thus the total host population size $N = S(t) + I(t) + R(t)$ is constant on the interval of existence of the solution. Thus, $S(t), I(t), R(t) \leq N$, due to their just proven positivity. Hence, the solution exists for all $t \geq 0$ following the extension theorem (see Theorem 2.8), and furthermore defines a dynamical system. Now, it can be beneficial to "normalize" the system, utilizing the new variables

$$u(at) = \frac{S(t)}{N}, v(at) = \frac{I(t)}{N}, w(at) = \frac{R(t)}{N}, u_0 = \frac{S_0}{N}, v_0 = \frac{I_0}{N}, w_0 = \frac{R_0}{N}.$$

One can see from these definitions, that

$$u(t) + v(t) + w(t) = \frac{S(t) + I(t) + R(t)}{N} = 1 = \frac{S_0 + I_0 + R_0}{N} = u_0 + v_0 + w_0. \tag{8}$$

The idea behind normalization is to resize the variables on values between 0 and 1. Now, one has

$$\frac{du(at)}{dt} = a\dot{u} = \frac{\dot{S}}{N} = \frac{-rSI}{N} = \frac{-rN}{a}\frac{a}{N}\frac{SI}{N} = -\mathcal{R}uva,$$

i.e. one obtains $\dot{u} = -\mathcal{R}uv$. Analogous computations lead to the "normalized" system of ODEs

$$\dot{u} = -\mathcal{R}uv \tag{9}$$

$$\dot{v} = \mathcal{R}uv - v, \tag{10}$$

$$\dot{w} = v, \tag{11}$$

for all $t \geq 0$, with initial values $u(0) = u_0, v(0) = v_0, w(0) = w_0$. One can see, that the variable w only appears in Equation (11), and therefore, one can restrict the analysis of the "reduced" system (9)-(10). From Equation (8), one readily see, that $u + v \leq 1$, and therefore, the solutions of (9)-(10) lie in the triangle

$$T := \{(u, v) \in \mathbb{R}^2 : u + v \leq 1 \text{ with } u, v \geq 0\}.$$

A quick calculation yields the solutions for Equation (11)

$$w(t) = w_0 + \int_0^t v(\xi)\,\mathrm{d}\xi$$

for $t \geq 0$. Now, on the (convex) triangle T one obtains:

- On the left boundary, one has $n_l := (-1,0)^T$ as outer normal, and $u = 0$. Thus, $(f(u,v)|n_l) = (0,-v)^T \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right) = 0$.
- On the upper boundary, one has $n_u := (1,1)^T$ as outer normal, $u \in [0,1]$ and $v = 1-u$. One obtains $(f(u,v)|n_u) = (-\mathcal{R}u(1-u), \mathcal{R}u(1-u)-1+u)^T \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) = -1+u \leq 0$, if $u \in [0,1]$.
- On the lower boundary, one has $n_d := (0,-1)^T$ as outer normal, $u \in [0,1]$ and $v = 0$. Thus $(f(u,v)|n_d) = (0,0)^T \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right) = 0$.

It follows with Theorem 3.16 that the triangle T is a positively invariant set for the system (9)-(10).

3.3 Further Examples of Invariant Sets

Before considering furthermore complicated examples of invariant sets, some first basic ideas on the notion of **stability** shall be introduced. The concept of stability is an essential and extensively studied property of invariant sets (equilibria in particular), and can roughly be summarized as follows:

- An invariant set (equilibrium) is called stable if orbits starting close to it, stay close to it;
- Unstable if they do not;
- Asymptotically stable if the invariant set (equilibrium) is stable, and additionally orbits starting close to it converge to it if $t \to \infty$.

A rigorous concept of stability is being formally defined and studied in Chapter 4. This section mainly follows the ideas of Teschl ([57, Section 6.3]) , Wilke/Prüss ([48, Section 8.4]) and Meiss ([40, Section 4.9]). The considerations are restricted to autonomous initial value problems, i.e. on ODE systems of the form

$$\dot{x} = f(x) \tag{12}$$

with $t_0 \geq 0$ and $x(0) = x_0$. As usual, $f\colon G \to \mathbb{R}^n$ shall be continuous, with $G \subset \mathbb{R}^n$ being an open set. The uniqueness of solutions shall not necessarily be assumed. As a reminder, a neighbourhood of a set A is an open set U with $A \subset U$. An interesting observation is the following: if $q \in \gamma(p)$ then $q = \phi(t', p)$ for some $t' \in (a, b)$. It follows

$$\phi(t, q) = \phi(t, \phi(t', p)) = \phi(t + t', p),$$

again with the group property of a dynamical system, and one obtains

$$\gamma(q) = \gamma(p),$$

i.e. orbits either concur or do not cross. It has to be emphasized, that this property is inherent for autonomous systems, and is not necessarily true for non-autonomous systems, as the following example from Lebovitz [34, Example 7.1.1] demonstrates.

Example 3.20. One considers the non-autonomous system (with $k \in \mathbb{R}$)

$$\begin{aligned} \dot{x} &= -x + k(\cos(t) - \sin(t)) \\ \dot{y} &= y + k(\cos(t) + \sin(t)) \end{aligned}$$

with solutions

$$x(t) = ae^{-t} + k\cos(t), \quad y(t) = be^{t} + k\sin(t),$$

for $a, b \in \mathbb{R}$. The choice $a = b = 0$ leads to an orbit described by a circle with centre $(0, 0)$ and radius k. Now, taking as initial values $(0, \frac{k}{2})$ lead to the fact, that the solutions in this case

$$x(t) = k(\cos(t) - e^{-t}), \quad y(t) = k(\frac{1}{2}e^{t} + \sin(t)),$$

leave the circle of radius k, and thus intersect it at a certain time $t > 0$.

3.3.1 Limit Sets

Let $x\colon \mathbb{R}_+ \to \mathbb{R}^n$ be a continuous solution of the system (12).

Definition 3.21. *The set*

$$\begin{aligned} \omega_+(x) :=& \{y \in \mathbb{R}^n |\ \text{there exists a sequence } (t_k) \in \mathbb{R} \text{ with } t_k \to \infty \text{ and } x(t_k) \to y\} \\ =& \{y \in \mathbb{R}^n |\ \text{there is a sequence } (t_k) \in \mathbb{R} \text{ with } t_k \to \infty \text{ and } \phi(t_k, x) \to y\} \end{aligned}$$

is called the (positive) limit set of x. The negative limit set ω_- is defined accordingly.

Definition 3.21 can be extended to some set $X \subseteq M \subset \mathbb{R}^n$. The positive limit set of X

is the set $\omega_+(X)$ of all points $y \in M$, for which there exists sequences $t_n \to +\infty$ and $x_n \in X$ with $\phi(t_n, x_n) \to y$. It should be noticed that it follows from this definition, that the inclusion

$$\bigcup_{x \in X} \omega_+(x) \subseteq \omega_+(X)$$

is valid, equality being rarely the case (for further details, see [57, Section 8.1]).

Roughly speaking, limit sets are states, that a dynamical system can attain when time goes to infinity, either by passing forward or backwards in time. Limit sets are crucial since they can be exploited to understand the long term behaviour of a dynamical system. The first type that would come to mind for a limit set would be an asymptotically stable equilibrium (see Definition 4.3). Limit sets can be further divided into the following categories:

- A periodic orbit (see Example 3.5);
- An attractor (see Section 3.3.3);
- A limit cycle (introduced in Section 3.3.2).

One can see from the definition of a limit set, that $w_+(x)$ is empty if a solution passing through x is not defined for all $t \geq 0$, or unbounded as $t \to \infty$. The following example from Knauf (see [30, 5.1 Beispiel]) illustrates that further premises are needed to ensure the non-emptiness of the limit set.

Example 3.22. Let $a \in \mathbb{R}$, and one defines the ODE $\dot{x} = ax$ with initial value $x(0) = x_0$. Its flow then reads $\phi(t, x_0) = e^{at}x_0$. One obtains the following cases:

- If $a < 0$, then $\omega_+(x_0) = \{0\}$;
- If $a = 0$, then $\omega_+(x_0) = \{x_0\}$;
- If $a > 0$, then $\omega_+(x_0) = \emptyset$.

If one assumes, that the positive orbit is contained in a **compact** set C, then the solution $x(t)$ is included in C, as long as it exists for $t > 0$. According to the theorem of Bolzano-Weierstrass, every sequence $x(t_k)$ has a convergent subsequence. One has derived a first significant result concerning limit sets.

Proposition 3.23. *Let the forward orbit of x be contained in a* ***compact*** *set. The limit set $\omega(x)$ is then non-empty, compact, and connected. One further has for $t \to \infty$*

$$\phi(t, x) \to \omega_+(x).$$

Proof. The proof can be found in [40, Lemma 4.42]. □

Very interesting for the scope of this thesis is the following result:

Proposition 3.24. *The limit set $\omega_+(x)$ is a closed,* ***invariant*** *set.*

Proof. The empty set is trivially closed and invariant. It can, therefore, be assumed, that $w_+(s)$ is non-empty for some $s \in \mathbb{R}^n$.
Let (s_n) be a sequence converging to s. One knows from the definition of the limit set, that there exists a sequence $(t_{n,k})$, with $t_{n,k} \to \infty$, with $\phi(t_{n,k}, x) \to s_n$ as $k \to \infty$. Now, for each n, one can choose an $L(n)$ satisfying $\|\phi(t_{n,k}, x) - s_n\| < \frac{1}{n}, \forall k > L(n)$. Now, for any given $\epsilon > 0$, one can set an N such that $\|s - s_n\| < \frac{\epsilon}{2}$, if $n > N$. One then obtains

$$\begin{aligned}\|\phi(t_{n,L(n)}, x) - s\| &\leq \|\phi(t_{n,L(n)}, x) - s_n\| + \|s - s_n\| \\ &< \frac{1}{n} + \frac{\epsilon}{2} \\ &< \epsilon\end{aligned}$$

if $n > \max\{N, \frac{2}{\epsilon}\}$ is satisfied. It follows that s lies in $\omega_+(x)$, and therefore, $\omega_+(x)$ is a closed set. Now, let $x \in \omega_+(s)$, and one considers $\phi(T, x)$ for a fixed $t \in \mathbb{R}$. If $\phi(t_n, x) \to y$, then it follows (once again) from the group property and by the continuity of the flow

$$\phi(t_n + t, x) = \phi(t, \phi(t_n, x)) \underbrace{\to}_{n \to \infty} \phi(t, y)$$

for all t on the maximal interval of existence of the solution. It follows

$$\phi(t, x) \in \omega_+(s)$$

proving the invariance of $\omega_+(s)$. □

It follows from Proposition 3.24, that limit sets are another important class of invariant sets, that occur in systems of dimension two or higher. It has to be emphasized, that Proposition 3.24 is not necessarily true in the non-autonomous case.

Example 3.25. One considers the IVP $\dot{x} = \frac{1}{1+t^2}$, with $x(t_0) = x_0$. One can see that the solution takes the form (keeping in mind that $\lim\limits_{t \to \pm\infty} \arctan = \pm\frac{\pi}{2}$)

$$x(t) = x_0 + \frac{\pi}{2} - \arctan(t_0).$$

One recognizes, that the limit set depends on the initial value, and would therefore not be inevitably invariant.

3.3.2 Limit Cycles

Before diving in the concept of limit cycles, a quick reminder of the notion of a closed curve (see, e.g. [19, Definition 2.5.4]) is necessary:

Definition 3.26. *The curve* Γ *is a closed curve contained in* $\mathbb{R}^n$*, if there exists a continuous mapping* $\gamma : [0,1] \to \mathbb{R}^n$ *satisfying* $\gamma(0) = \gamma(1), \Gamma = \{\gamma(t) : t \in [0,1]\},$ *with* $\Gamma \neq \gamma(0)$*.*

Roughly speaking, a limit cycle is a closed trajectory in the phase space, possessing the property that at least another path spirals into, or away from it. More precisely, this leads naturally (see, e.g. [19, Definition 2.54]) to the following notion.

Definition 3.27. *One considers the system* (12)*. A* ***limit cycle*** *of* (12) *is a closed curve* $\Gamma \subset \mathbb{R}^n$*, such that* Γ *is the positive limit set of the positive orbit* $\gamma_+(x)$ *of* (12)*, or the negative limit set of the negative orbit* γ_- *of* (12)*, presupposing that* $x \notin \Gamma$*.*

Definition 3.27 entails that a cycle of (12) is a closed solution curve (not an equilibrium point) of the system (12). Therefore, periodic orbits are made of closed curves in the phase space (see [5, Section 2.1]).

In general, the task of finding cycles is not an easy one. In the preceding examples in two dimensions, the solutions behaved in a rather smooth way. Most of the time, they would either converge to an equilibrium point or a periodic orbit. The reason for this distinctive behaviour in two dimensions (in contrast to $\mathbb{R}^n, n \geq 3$), is the validity of the **Jordan curve theorem**:

Theorem 3.28. Let C be a connected simple plane curve that is a closed subset of $\mathbb{R}^2$. Then, C separates $\mathbb{R}^2$ into two connected regions, i.e. $\mathbb{R}^2 \setminus C$ has exactly two attached components with a common boundary C.

Proof. An astonishing fact about the Jordan curve theorem is the "simplicity" of its content, and the surprising difficulties arising, when one tries to prove it. A proof can be found in [42, Theorem 9.14] for example. □

After a substantial amount of definitions, the following modified example from Richards (see [51, Section 17.3.1]) is useful to get more clarity:

Example 3.29. One considers the system

$$\dot{x} = y - x(x^2 + y^2 - 1) \tag{13}$$

$$\dot{y} = -x - y(x^2 + y^2 - 1). \tag{14}$$

The right-hand side is polynomial, and therefore continuously differentiable, securing the uniqueness of the solutions of the system in an open interval containing $t_0 \in \mathbb{R}$.

Furthermore, one has

$$\begin{aligned}(f(x,y)|(x,y)) &= xy - x^2(x^2+y^2-1) - xy - y^2(x^2+y^2-1)\\ &= -(x^2+y^2-1)(x^2+y^2)\\ &\leq \|(x,y)\|_2^2,\end{aligned}$$

and it follows with Proposition 2.11, that the solutions of (13)-(14) exist globally to the right. Let (x,y) be a solution of this system with corresponding polar coordinates $r \geq 0$ and ϕ, i.e. one has $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$ and therefore $r^2(t) = x^2(t)+y^2(t)$. Differentiating on both sides yields $r'(t)2r(t) = 2x(t)x'(t) + 2y(t)y'(t)$, and it follows

$$\begin{aligned}r'(t)r(t) &= x(t)x'(t) + y(t)y'(t)\\ &= x(t)y(t) - x^2(t)(x^2(t)+y^2(t)-1) - x(t)y(t) - y^2(t)(x^2(t)+y^2(t)-1)\\ &= -(x^2(t)+y^2(t)-1)(x^2(t)+y^2(t)).\end{aligned}$$

Substituting back yields, that $r(t)$ solves the ODE $\dot{r} = -(r^2(t)-1)r(t)$. One further derives from $\tan\phi(t) = \frac{y(t)}{x(t)}$

$$\begin{aligned}\frac{\phi'(t)}{\cos^2(t)} =& \frac{y'(t)x(t) - y(t)x'(t)}{x^2(t)}\\ =& \frac{-x^2(t) - x(t)y(t)(x^2(t)+y^2(t)-1) - y^2(t) + x(t)y(t)(x^2(t)+y^2(t)-1))}{x^2(t)}\\ =& -\frac{r^2(t)}{r^2(t)\cos^2(t)}.\end{aligned}$$

The system is therefore equivalent to the polar coordinate system

$$\begin{aligned}\dot{r}(t) &= -(r^2(t)-1)r(t)\\ \dot{\phi}(t) &= -1.\end{aligned}$$

One can see, that $(0,0)$ is an equilibrium of this system, and there exists a periodic orbit for $r=1$. Now, one has

$$\dot{r}|_{r=\frac{1}{2}} = -(\frac{1}{4}-1)\frac{1}{2} = \frac{3}{8} > 0,$$

implying that all orbits crossing the circle with radius $r = \frac{1}{2}$ and centre $(0,0)$, enter the exterior of this circle when t increases. One further has

$$\dot{r}|_{r=\frac{6}{5}} = -((\frac{6}{5})-1)\frac{6}{5} = -\frac{66}{125} < 0,$$

implying this time, that orbits intersecting the circle with radius $r = \frac{6}{5}$ and centre $(0, 0)$, enter the interior of this circle when t increases. It follows from these two observations, that the annulus region

$$D := \{(x, y) : \frac{1}{2} < r < \frac{6}{5}\}$$

is positively invariant for this system. Thus, if the initial values (x_0, y_0) are non zero, then one has

$$\omega_+(x_0, y_0) = \partial B_1(0)$$

as a limit cycle.

3.3.3 Dynamical Systems in Higher Dimensions

In most applications, the principal goal is to analyse the long-time behaviour of the flow of a dynamical system (especially generated by a differential equation in this thesis). Thus, it is crucial to understand how solutions starting in a given set D will behave in the future. To this purpose, in this section based on [57, Sections 8.1-2], some additional definitions are necessary.

Definition 3.30 (Stable and unstable sets)**.** *Let D be an invariant set $D \subset M \subset \mathbb{R}^n$. The sets*

$$W^+(D) = \{x \in M : \lim_{t \to +\infty} dist(\phi(t, x), D) = 0\} \textit{ and}$$

$$W^-(D) = \{x \in M : \lim_{t \to -\infty} dist(\phi(t, x), D) = 0\}$$

*are called the **stable** and **unstable sets** of D respectively.*

This leads to the following notion:

Definition 3.31. *An invariant set D is named **attracting** if $W^+(D)$ is a neighbourhood of D. The set $W^+(D)$ is then referred to as **domain** or **basin of attraction** for D.*

Furthermore, for any positively invariant neighbourhood U for D, one has

$$W^+(D) = \bigcup_{t<0} \phi(t, U).$$

As already mentioned in Section 3.1, the union and the complement of invariant sets are also invariant. It follows, that if additionally, one chooses U as an open set, then it follows that the basin of attraction of $W^+(D)$ is an invariant and open set. Furthermore, the boundary $\partial W_+(D) = \overline{W_+(D)} \setminus W_+(D)$ is also invariant. A difficulty that now arises is how to determine an attracting set. Fortunately, there is a course of action with the definitions from above. An open, connected set E possessing a compact closure is called a **trapping region** for the flow, if $\phi(t, E) \subset E$ is satisfied for all $t > 0$. In most of the situations, a trapping region can be found by seeking areas bounded by some surface (see Section 5.1), such that the SC is satisfied, i.e. the vector field points inwards on that surface (see Chapter 3 as a reminder).

Lemma 3.32. *Consider a trapping region E. One then has, that the set*

$$D = \omega_+(E) = \bigcap_{t\geq 0} \phi(t, E)$$

is a nonempty, compact, invariant, and connected attracting set.

Proof. One has

$$\bigcap_{t\geq 0} \phi(t, E) = \bigcap_{t\geq 0} \phi(t, \overline{E}) = \bigcap_{t\geq 0} \phi(t, \overline{\gamma_+(E)}) = \omega_+(E).$$

Moreover, this concludes the proof of the first part with Propositions 3.23 and 3.24. It remains to show that D is an attracting set. One shall assume, that there are an $e \in E$ and a sequence $t_n \to \infty$ with $dist(\phi(t_n, e), D) \geq \epsilon > 0$. Considering that E is a trapping region, it would follow that $\phi(t_n, e)$ stays in the compact set $\overline{E}$, and thus $\phi(t_n, e) \to \tilde{e}$ (with a subsequence by Bolzano-Weierstrass). Nevertheless, one has $\tilde{e} \in \omega_+(e) \subseteq \omega_+(E)$, a contradiction. □

Regrettably, the notion of an attracting set is not sufficient in some cases, considering the fact, that it always contains the unstable sets of all its equilibria (see [57, Lemma 8.6]). To eliminate potentially unpleasant situations, one has to guarantee that an attracting set cannot be divided into smaller invariant sets. A possible loophole is to define an **attractor**, which is an attracting set that cannot be split into smaller attracting sets.

Example 3.33. In Example 3.29, the attractor with respect to $M = \mathbb{R}^2 \setminus \{0\}$ is the unit circle $\partial B_1(0)$.

Example 3.34. One considers the system of ODEs

$$\begin{aligned} \dot{x} &= x(1 - x^2) \\ \dot{y} &= -y, \end{aligned}$$

with initial value (x_0, y_0). The right-hand side is Lipschitz in (x, y), and one has

$$(f(x, y)|(x, y)) = x^2(1 - x^2) - y^2 \leq x^2 + y^2 = \|(x, y)\|_2^2.$$

It follows with Proposition 2.9 that the solutions exist (at least) globally to the right. Setting $f(x, y)$ to zero yields the equilibria $(\pm 1, 0)$ and $(0, 0)$. Now, let $0 < \epsilon < 1, \epsilon \in \mathbb{R}$.

A vertical segment through the origin, i.e. the set $\{(x, y) \in \mathbb{R}^2 : x = 0, -\epsilon \leq y \leq \epsilon\}$ is obviously a (closed) convex set. One obtains for $x = 0$ and the normal vector $n = (\pm 1, 0)^T$

$$(f(0, y)|n) = (0, -y)^T \left(\begin{smallmatrix} \pm 1 \\ 0 \end{smallmatrix} \right) = 0.$$

Moreover, it follows with Theorem 3.16, that the segment is a positively invariant set for the system. Furthermore, one considers the squares with side lengths 2ϵ S_ϵ^+ and S_ϵ^- with centres $(1, 0)$ and $(-1, 0)$, respectively. Applying Theorem 3.16 again, it can be shown that S_ϵ^+ and S_ϵ^- are trapping regions for $(1, 0)$ and $(-1, 0)$ respectively, for any ϵ defined just above, particularly when $\epsilon \to 0$. It follows from these considerations, that the basins of attraction for the first two equilibria are $W^+(\{(\pm 1, 0)\}) = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : x = 0\}$. Considering the fact, that only $(-1, 0)$ and $(1, 0)$ are attracting fixed points (see Section 4.1.1 for further details), it follows that $\{(-1, 0)\}$ and $\{(1, 0)\}$ are **attractors** for this system.

The preceding examples may have given the impression, that limit sets and attractors always possess a simple structure, and that the number of limit sets can still be calculated. Nevertheless, one could not be further from the truth. It becomes evident when one studies models of a dynamical system for $\mathbb{R}^n, n \geq 3$. Even worse: one considers the case of planar polynomial systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

where P and Q are polynomials in the variables x and y. The challenge is to evaluate the maximal number and relative positions of limit cycles in this system (also known in the mathematical world as "**The Second Part of Hilbert's Sixteenth Problem**"). Despite being easily stated (one remembers the Jordan curve theorem), the second part of Hilbert's sixteenth problem remains mostly completely unresolved (see [36, Section 17.1] for further details). These considerations lead to a famous system in the world of dynamical systems (see, e.g. [57, Section 8.2]):

Example 3.35 (The Lorenz equations)**.** One of the most renowned dynamical systems exhibiting **chaotic** behaviour is the **Lorenz equation**, a system of ODEs defined as

$$\dot{x} = -\sigma(x - y), \tag{15}$$
$$\dot{y} = rx - y - xz, \tag{16}$$
$$\dot{z} = xy - bz, \tag{17}$$

with σ, r and b being positive constants. Lorenz derived these equations by modelling a fluid cell in two dimensions, between two parallel plates being at different temperatures.

A first possible solution of the system can be obtained by setting $x(t) = 0, y(t) = 0$, yielding $\dot{z} = -bz$, and thus with initial value $z(t_0) = z_0$, one obtains $z(t) = z_0 e^{-bt}$. It follows, that the $z-$axis is an invariant set for the system in this case.

Equilibria of the Lorenz system:

Setting Equation (15) to zero yields $x = y$. Inserted in (17), one obtains $x^2 - bz = 0$ and hence $\frac{x^2}{b} = z$. Putting z in Equation (16) yields $rx - x - \frac{x^3}{b} = 0$ and thus $x(r - 1 - \frac{x^2}{b}) = 0$. These calculations yield the possible first equilibrium $x^* = (0, 0, 0)^T$. For further equilibria, one has to consider the equation

$$r - 1 - \frac{x^2}{b} = 0 \Leftrightarrow b(r-1) = x^2. \tag{18}$$

One can recognize from equation (18), that if $r \leq 1$, then $b(r-1) \leq 0$, and thus x^* is the only equilibrium. More importantly, one has the following result:

Lemma 3.36. *If $r \leq 1$, then the Lorenz equation has only the equilibrium x^*, and every solution converges to the origin if $t \to \infty$.*

Proof. This lemma can be proven with the concept of Lyapunov functions, introduced in Section 4.2. The proof can be found in [57, Lemma 8.7] for example. □

If on the other hand, one considers $r > 1$, then there are two equilibria in this case, namely

$$x_1 := \begin{pmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{pmatrix}, x_2 := \begin{pmatrix} -\sqrt{b(r-1)} \\ -\sqrt{b(r-1)} \\ r-1 \end{pmatrix}.$$

Now, this is where the situation becomes interesting. To simplify the analysis of the system, one considers the case $\sigma = 10$ and $b = \frac{8}{3}$. If one sets $r_1 \cong 1.3456, r_2 \cong 24.737$, Boyce, DiPrima, and Meade [9, Section 9.8] show that:

- For $1 < r < r_1$ or $r_1 < r < r_2$, the equilibria x_1 and x_2 are asymptotically stable, and x^* is unstable.
- For $r > r_2$, all the critical points are unstable. Most solutions close to x_1 or x_2 spiral away from the equilibrium point.

One would assume in the case of $r > r_2$, that since none of the equilibrium points is stable, most orbits should diverge and tend to infinity. Surprisingly, this is not the case: it can be proven that every solution remains bounded, as $t \to \infty$.

Invariant sets of the Lorenz system:

As a starting point, one considers the function

$$\Phi(x,y,z) := rx^2 + \sigma y^2 + \sigma(z-2r)^2.$$

Now, one has

$$\begin{aligned}\dot{\Phi}(x,y,z) :=& (f(x,y,z)|\nabla\Phi(x,y,z))\\ =&(-\sigma(x-y), rx-y-xz, xy-bz)^T \begin{pmatrix} 2rx \\ 2\sigma y \\ 2\sigma(z-2r)\end{pmatrix}\\ =& -2\sigma(rx(x-y)-y(rx-y-xz)-(z-2r)(xy-bz))\\ =& -2\sigma(rx^2+y^2+b(z-r^2)-br^2).\end{aligned}$$

One defines the set $E = \{(x,y,z) : \dot{\Phi}(x,y,z) \geq 0\}$. Because σ is presupposed positive, it follows that $\dot{\Phi} \geq 0$ only holds, if $rx^2+y^2+b(z-r^2)-br^2 \leq 0$, i.e. the set E defines an ellipsoid. Thus, E is a bounded set. Considering, that $\dot{\Phi}(x,y,z)$ is a continuous function, it follows that E is also a closed set, hence E is a compact set, entailing that $\Phi(x,y,z)$ possesses a maximum $M := \max\limits_{(x,y,z)\in E} \Phi(x,y,z)$. Now, one defines a further ellipsoid

$$E_M = \{(x,y,z) : \Phi(x,y,z) < M+1\}.$$

Because of this definition, any point outside of E_M is also outside of E, and it follows that $\dot{\Phi}(x,y,z) < 0$ for these points. Nevertheless, this entails, that for all $(x,y,z) \in \mathbb{R}^3 \setminus E_M$, Φ strictly decreases in value along the trajectory of the system. Thus, it has to enter the ellipsoid E_M at a certain point in time. Furthermore, E_M is a trapping region for the Lorenz equations, and the related attracting set $D = \omega_+(E_M)$ is called the **attractor of the Lorenz equations**. It follows that the solutions of the system exist for all $t \geq 0$ (and therefore the Lorenz equations define a dynamical system). It can be recognized in Figure 2, that the attracting set D possesses a quite complicated nature. This is the reason why it is sometimes called the **strange attractor** of the Lorenz equations (for further details, see, e.g. [57, Chapter 11, 55, Chapter 9]). The example shows again the very complicated behaviour, even for assumed simple systems in $\mathbb{R}^3$. Such strange behaviour is ruled out in $\mathbb{R}^2$.

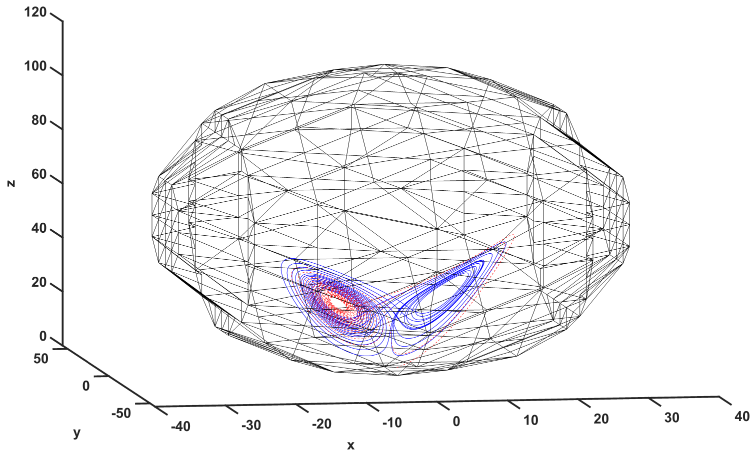


Figure 2: Solution for the Lorenz equation with the initial condition $(1, 1, 1)$ and parameters $\sigma = 10, b = 8/3, r = 28$. The red dotted line and the blue line represent the evolution of the solution on the time segments $[0, 20]$ and $[20.01, 40]$, respectively.

4 The Concept of Stability

As mentioned earlier in this thesis, one of the main questions is the **long-time behaviour of a dynamical system**. Notably, one often wants to know whether the solution is stable or not. As one recalls Definition 2.20, an equilibrium point (in the autonomous case) of f is a point $\tilde{x} \in G \subset \mathbb{R}^n$ satisfying $f(\tilde{x}) = 0$. Usually, one considers a fixed point and wants to know the future behaviour of a solution if the starting point is close to it. Taking into account that equilibria are valuable invariant sets, the notion of stability shall be defined and considered on them in this section. These definitions can be extended to more general types of invariant sets. Before diving into the concept of stability, there is an interesting link between a convex, compact and positively invariant set, and an equilibrium (see [35, Theorem 4.45]).

Proposition 4.1. *Let $D \subset G \subset \mathbb{R}^n$ be a non-empty, convex and compact set. If D is positively invariant under the local flow $\phi(t,x)$, then D contains an equilibrium.*

Useful for the proof of Proposition 4.1 is an essential theorem from Brouwer (a proof can be found in [63, Section 1.14.4]).

Theorem 4.2 (Brouwer's fixed-point theorem)**.** Let M be a non-empty, convex and compact set in a finite-dimensional normed space over a field K, and a continuous operator $A : M \to M$. Then, A has at least one fixed-point, i.e. there exists at least one point $x \in M$ satisfying $A(x) = x$.

Proof of Proposition 4.1. Because D is assumed being positively invariant, one considers the sequence (t_n) with values in $(0, \infty)$ and converging to 0, i.e. $\lim_{n\to\infty} t_n \to 0$. Further, because of the invariance of D, it follows $\phi(t_n, x) \in D$ for all $x \in D, n \in \mathbb{N}$. Now, one defines the continuous map

$$f_n(x) = \phi(t_n, x)$$

for all $x \in D$. The prerequisites of Brouwer's fixed-point Theorem 4.2 are satisfied, and therefore there exists a sequence $x_n \in D$ with

$$x_n = f_n(x_n) = \phi(t_n, x_n)$$

for all $n \in \mathbb{N}$. Now, using the group property DS2 of a dynamical system, one has

$$\begin{aligned}\phi(t_n + t_n, x_n) &= \phi(t_n, \phi(t_n, x_n)) \\ &= \phi(t_n, x_n)\end{aligned}$$

entailing that x_n is a periodic point for each $n \in \mathbb{N}$, and thus the maximum interval of existence of x_n is $\mathbb{R}$. Now, considering that D is compact, there exists an $x \in D$ with $\lim\limits_{n\to\infty} x_n \to x$. It remains to show, that x is a fixed point, i.e. $\phi(t,x) = x$ for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ be arbitrary, one has

$$\|\phi(t,x) - x\| = \underbrace{\|\phi(t,x) - \phi(t,x_n)\|}_{\to 0,} + \underbrace{\|\phi(t,x_n) - \phi(t_n,x_n)\|}_{(*)} + \underbrace{\|\phi(t_n,x_n) - x\|}_{=\|x_n - x\| \to 0}$$

because of $x_n \to x$ and continuity of ϕ, as $n \to \infty$. It remains to consider the term $(*)$. As already shown above, x_n is t_n-periodic, so it holds

$$\phi(kt_n, x_n) = x_n \text{ for all } k \in \mathbb{Z}, n \in \mathbb{N}.$$

However, this means that for all $n \in \mathbb{N}$, there is a $k_n \in \mathbb{Z}$ satisfying $k_n t_n \leq t < (k_n + 1)t_n$ and thus, there is a $q_n \in [0,1)$ with $t = k_n t_n + q_n t_n$. It follows

$$\begin{aligned}\phi(t,x_n) &= \phi(k_n t_n + q_n t_n, x_n) \\ &\underbrace{=}_{\substack{\text{group property} \\ DS2}} \phi(q_n t_n, x_n) \text{ for all } n \in \mathbb{N},\end{aligned}$$

and thus

$$\begin{aligned}\|\phi(t,x_n) - \phi(t_n,x_n)\| &= \|\phi(q_n t_n, x_n) - x_n\| \\ &\to \|\phi(0,x) - x\| \\ &= \|x - x\| \\ &= 0 \quad \text{ if } t \to \infty.\end{aligned}$$

□

The study of an equilibrium $\tilde{x}$ can be reduced to the consideration of $\tilde{x} = 0$ with the following trick (see [48, Section 5.1]): Let $x_1(t), t \geq t_0$ be a designated solution for (2), with initial value $x_1(t_0) = x_{1,0}$. Now, if one defines $y(t) = x(t) - x_1(t), t \geq t_0$, then one has

$$\dot{y}(t) = \dot{x}(t) - \dot{x_1}(t) = f(t,x(t)) - f(t,x_1(t)) = f(t,y(t) + x_1(t)) - f(t,x_1(t)).$$

Because $x_1(t)$ is a fixated solution, the last term only depends on $y(t)$, and thus one can define $g(t,y(t)) = f(t,y(t) + x_1(t)) - f(t,x_1(t))$. This leads to the new ODE

$$\dot{y}(t) = g(t,y(t))$$

which is locally Lipschitz in y and continuous in t. Furthermore, one has

$$g(t,0) = f(t, 0 + x_1(t)) - f(t, x_1(t)) = 0,$$

entailing that the trivial solution $y(t) = 0$ solves $\dot{y} = g(t, y)$, with initial value $y(t_0) = 0$. For this reason, the sole consideration of the case $f(t, 0) = 0$ with $t \geq t_0$ is sufficient.

For the study of the stability of equilibria, two methods have been proven particularly fruitful:

1. The principle of linearized stability;

2. The Lyapunov stability with the help of Lyapunov functions.

These two methods will be described in Sections 4.1 and 4.2, respectively.

4.1 The Principle of Linearized Stability

One considers the autonomous ODE

$$\dot{x} = f(x) \tag{19}$$

with $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, i.e. f is continuously differentiable on $\mathbb{R}^n$, and thus locally Lipschitz due to Proposition 2.5. Now, let $\tilde{x}$ be an equilibrium of (19), and $x(t)$ one of its solutions. Setting $y(t) := x(t) - \tilde{x}$ leads to

$$\begin{aligned}
\dot{y}(t) &= \dot{x}(t) - \underbrace{\frac{d}{dt}\tilde{x}}_{=0} \\
&= \dot{x}(t) - f(\tilde{x}), \quad \tilde{x} \text{ being an equilibrium point, i.e. } f(\tilde{x}) = 0 \\
&= f(y(t) + \tilde{x}) - f(\tilde{x}), \quad \text{ due to the definition of } y(t).
\end{aligned}$$

Now, applying Taylor's theorem for multivariate functions (see, e.g. [16, Corollar 1]) on $f(y(t) + \tilde{x})$, one obtains

$$f(\tilde{x} + y(t)) = f(\tilde{x}) + Df(\tilde{x})y(t) + o(\|y\|),$$

if $\|y\| \to 0$, with

$$Df(x) := \frac{\partial f}{\partial x}(x) \in \mathbb{R}^{n \times n}$$

being the well known Jacobi matrix. One further defines the Landau symbol o as follows: Let ζ be a function defined in a neighbourhood of $0 \in \mathbb{R}^n$ with values in $\mathbb{R}^n$. Then,

$$\lim_{x \to 0} \frac{\zeta(x)}{\|x\|} = 0,$$

can be written as

$$\zeta(x) = 0(\|x\|).$$

It follows

$$\begin{aligned}\dot{y}(t) &= f(\tilde{x}) + Df(\tilde{x})y(t) + o(\|y\|) - f(\tilde{x}) \\ &= Df(y)y(t) + o(\|y\|).\end{aligned}$$

If one sets $r(y) := o(\|y\|)$ for $\|y\| \to 0$, and $A := Df(\tilde{x})$, one obtains the equivalent semi-linear system to (19):

$$\dot{y} = Ay + r(y), \tag{20}$$

which will be compared to the linear system

$$\dot{y} = Ay. \tag{21}$$

The system $\dot{x} = Ax$ is called the **linearization** of $\dot{x} = f(x)$ at the point $x = 0$, and gives pieces of information about the behaviour of the non-linear system, in a neighbourhood of the zero point. Thus, it is of great importance to consider linear systems in general [10, Section 1.4.2, 48, Section 5.4]. As mentioned just above, the systems (19) and (21) shall be compared, and to this end, the main goal of this section will lie on the study of autonomous linear systems. The considerations in the two following sections are mainly based on [48, Sections 5.2-5.3], [10, Section 1.4.3] and [57, Section 3.2], and will show how closely the eigenvalues of A and the stability properties of $\tilde{x} = (0, \ldots, 0)^T$ are linked.

4.1.1 Stability of Linear Systems

Definition 4.3. *[48, Definition 5.1.1] Let $f\colon \mathbb{R} \times G \to \mathbb{R}^n$ be continuous and locally Lipschitz in x, and further satisfy $f(t,0) = 0$, and let $x(t, x_0), t \geq t_0$ be the solution for the IVP* (2)*.*

1. *The trivial solution $\tilde{x} = 0$ is called **stable**, if for every $\epsilon > 0$, there is a $\delta > 0$ with $\overline{B}_\delta(0) \subset G$, and the solution $x(t, x_0)$ with $x_0 \in \overline{B}_\delta(0)$ exists for all $t \geq t_0$ and*

$$\|x(t, x_0)\| \leq \epsilon, \text{ for all } \|x_0\| \leq \delta, \text{ and } t \geq t_0 \tag{22}$$

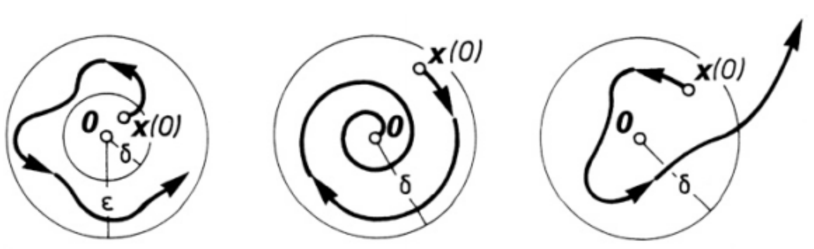


Figure 3: Different stability situations for the equilibrium $x(t) = 0$: stable (left), asymptotically stable (middle) and unstable (right). Image from [10, page 123].

is satisfied.

2. *The solution $\tilde{x}$ is called* ***unstable*** *if x_* is not stable.*

3. *The solution $\tilde{x}$ is called* ***attractive****, if a $\delta_0 > 0$ exists, such that $\overline{B}_{\delta_0} \subset G$, the solution $x(t, x_0)$ with $x_0 \in \overline{B}_{\delta_0}(0)$ exists for all $t \geq t_0$ and*

$$\lim_{t\to\infty} \|x(t, x_0)\| = 0 \textit{ for all } x_0 \in \overline{B}_{\delta_0}(0) \tag{23}$$

holds.

4. *The solution $\tilde{x} = 0$ is called* ***asymptotically stable****, if $x_* = 0$ is* ***stable*** *and* ***attractive****.*

One can see from Definition 4.3, that an asymptotically stable equilibrium is an **attractor** of (19). Central for the stability of $\tilde{x} = 0$ for linear systems of the form (21) is the following result.

Theorem 4.4 (Stability of linear systems)**.** The equilibrium $\tilde{x} = 0$ of the linear system $\dot{x} = Ax$ is in its stability behaviour determined by the eigenvalues of A. Namely, 0 is:

- **Asymptotically stable**, if $Re\lambda_i < 0$ for all $i = 1, \dots n$;
- **Stable**, if $Re\lambda_i \leq 0$, and if for each eigenvalue λ with $Re\lambda = 0$, the algebraic and geometric multiplicities concur;
- **Unstable**, if there is at least a λ with $Re\lambda > 0$.

Proof. The proof can be found in [10, Section 1.4.3]. □

For more clarity, planar linear systems are being explicitly considered, i.e. autonomous linear ODEs of the form:

$$\dot{x} = Ax, \quad \text{with } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}. \tag{24}$$

Known from the linear algebra, to determine the eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$, one has to consider the characteristic polynomial $p_A(\lambda)$ from A:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I - A) \\ &= (\lambda - a_{11})(\lambda - a_{22}) - a_{21}a_{12} \\ &= \lambda^2 - \lambda \underbrace{a_{11} + a_{22}}_{=: \text{ trace } A} + \underbrace{a_{11}a_{22} - a_{21}a_{12}}_{= \det A} \\ &= \lambda^2 - p\lambda + q, \end{aligned}$$

with $p =$ trace A(written trA) and $q = \det A$. Setting $p_A(\lambda) = 0$, one gets with the quadratic formula

$$\lambda_{1,2} = \frac{trA}{2} \pm \sqrt{\frac{(trA)^2}{4} - \det A}. \tag{25}$$

There are many different phase portraits (14 in total), depending on the constellation of the eigenvalues $\lambda_{1,2}$, which can be classified as follows:

a. $\lambda_{1,2} \in \mathbb{R}, \lambda_1 \neq \lambda_2$;

b. $\lambda_1 = \lambda_2$ with $\lambda_{1,2} \in \mathbb{R}$;

c. $\lambda_1 = a + ib, \lambda_2 = a - ib, a, b \in \mathbb{R}, b \neq 0$.

This leads to the following notions (see [10, Definition 1.12]):

Definition 4.5.

1. *If λ_1 and $\lambda_2 \in \mathbb{R}$ have the same sign, the equilibrium 0 is called:*

   a) A ***stable knot****, if $\lambda_1 < \lambda_2 < 0$;*

   b) An ***unstable knot*** *if $0 < \lambda_1 < \lambda_2$.*

2. *If $\lambda_1 = \lambda_2 \in \mathbb{R}$, 0 is called a* ***degenerate knot****.*

3. *If $\lambda_1, \lambda_2 \in \mathbb{R}$ differ in their signs, 0 is called a* ***saddle point****.*

- If both eigenvalues $\lambda_{1,2}$ are positive, then all solutions grow exponentially as $t \to \infty$ and decay as $t \to -\infty$. In this situation, the origin is called a **source**, as can be recognized in Figure 4 (right). Similarly, if both eigenvalues $\lambda_{1,2}$ are negative, the situation can be mirrored with the previous one by exchanging t by $-t$. In this case, the phase portrait remains the same, except that solution curves, are being travelled in the reversed direction, as can be seen in Figure 4 (left). One calls the origin a **sink** in this case.

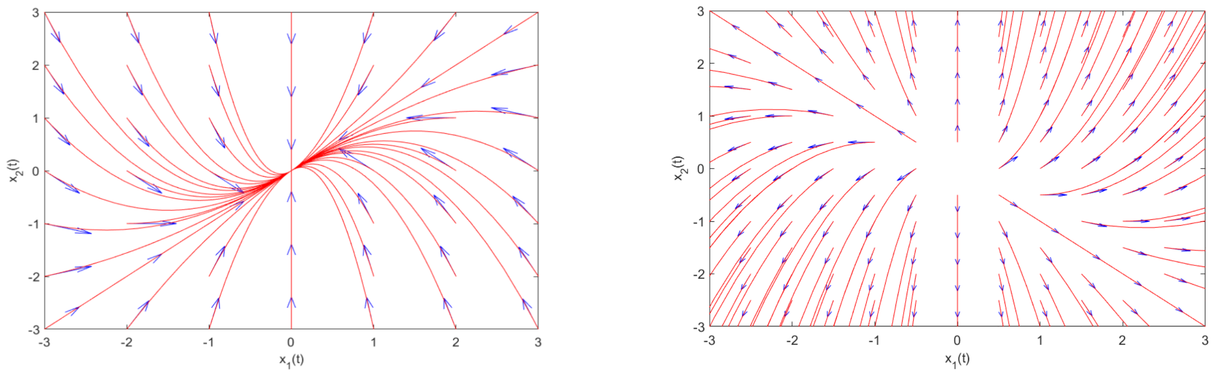


Figure 4: Phase portraits for a stable knot with $\lambda_1 = -2, \lambda_2 = -1$(left), and an unstable knot with $\lambda_1 = 1, \lambda_2 = 2$ (right).

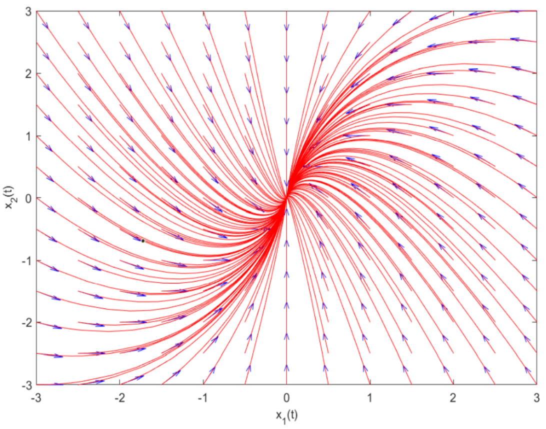
- In the case of Definition 4.5.2, for $\lambda := \lambda_{1,2} < 0$ or $\lambda > 0$, stable or unstable knots, respectively, appear, as can be seen in Figure 5 (left). In the presence of a saddle point, the orbits are given by the eigenvectors of the negative eigenvalue and eventually converge to the critical point. The orbits that are given by the eigenvectors of the positive eigenvalue move in exactly the opposite direction. Saddle points are always unstable, as can be seen in Figure 5 (right).

It can be recognized from (25), that:

- If $\det A > 0, tr A < 0$, then $\tilde{x}$ is asymptotically stable;
- If $\det A < 0$, then $\tilde{x}$ is saddle point;
- If $\det A > 0$, then $\tilde{x}$ is either a knot or a spiral.

It is worth noting that in the case of linear systems, the basin of attraction of an asymptotically stable equilibrium is the whole state space. Every initial condition leads to an orbit approaching the origin. The considerations in Section 4.1.1 show, that boundedness of solutions of $\dot{x} = Ax$ for $t \in [0, \infty)$ and stability of its equilibria lead to algebraic questions. However, already for $n \geq 4$, the calculation of the eigenvalues by seeking the roots of $p_A(\lambda)$ can be a tedious task. Thus, criteria securing that the roots of the characteristic polynomial of A have the desired properties from Theorem 4.4 are of great interest. A very well-known criterion is due to Routh and Hurwitz, given here without a proof (for further details, see, e.g. [11, Section 2.4]).

Proposition 4.6. *(Routh-Hurwitz criterion) Let* $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ *be a*



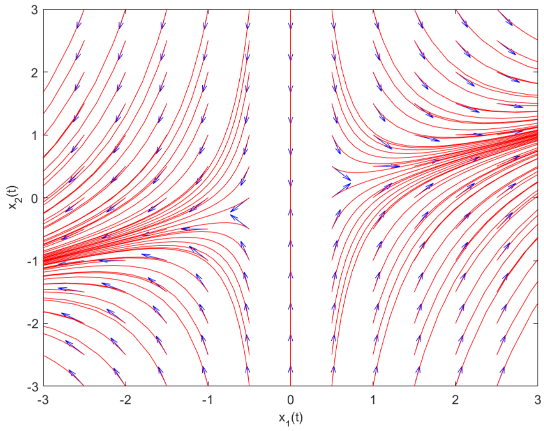


Figure 5: Phase portraits for an unstable knot with $\lambda_1 = \lambda_2 = -1$ (left) and a saddle point with $\lambda_1 = -2, \lambda_2 = 1$,(right).

polynomial with real coefficients, and define $D_1 = a_1$, and

$$D_k = \det \begin{pmatrix} a_1 & a_3 & a_5 & \dots & a_{2k-1} \\ 1 & a_2 & a_4 & \dots & a_{2k-2} \\ 0 & a_1 & a_3 & \dots & a_{2k-3} \\ 0 & 1 & a_2 & \dots & a_{2k-4} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_k \end{pmatrix}, \quad k = 2, 3, \dots n,$$

with $a_i = 0$ if $i > n$. If for all determinants $D_k > 0, k = 1, 2, \dots, n$ holds, then all the roots of $p(z)$ have negative real parts.

Example 4.7. (Generalized competing species model) Example 3.18 can be generalized to a frequently used model to describe the dynamics of n competing species: the quadratically non-linear Gauss-Lotka-Volterra equations, described by May and Leonard [38], which this example is based on. One defines the non-linear system

$$\frac{dN_i(t)}{dt} = r_i N_i(t) \left(1 - \sum_{j=1}^{n} \alpha_{ij} N_j(t)\right), \tag{26}$$

with parameters

- $N_i(t)$: the number of individuals in the ith population at time t;
- r_i: the intrinsic growth rate of the ith population;
- α_{ij}: competition coefficients that assess the amount, to which the jth species alters the growth rate of the ith.

In the case of three competing species (i.e. $n = 3$) which will be called 1, 2 and 3 respectively, the number of parameters can be reduced with the following assumptions:

- Symmetry, i.e. $r_1 = r_2 = r_3 = r$;
- Concerning competition, 2 affects 1, as 3 affects 2, as 1 affects 3, i.e. $\alpha_{12} = \alpha_{23} = \alpha_{31} = \alpha$, and by the same argument, $\alpha_{21} = \alpha_{32} = \alpha_{13} = \beta$;
- The number of individuals N_i can be rescaled to obtain $\alpha_{ii} = 1$, and t can be rescaled to attain $r = 1$.

Assuming $\alpha, \beta > 0$ leads to a competitive system. These presumptions lead to the simplified system:

$$\frac{dN_1}{dt} = N_1(1 - N_1 - \alpha N_2 - \beta N_3) =: f_1(N_1, N_2, N_3), \tag{27}$$
$$\frac{dN_2}{dt} = N_2(1 - \beta N_1 - N_2 - \alpha N_3) =: f_2(N_1, N_2, N_3), \tag{28}$$
$$\frac{dN_3}{dt} = N_3(1 - \alpha N_1 - \beta N_2 - N_3) =: f_3(N_1, N_2, N_3) \tag{29}$$

and its Jacobian matrix

$$\begin{pmatrix} 1 - 2N_1 - \alpha N_2 - \beta N_3 & -\alpha N_1 & -\beta N_1 \\ -\beta N_2 & 1 - \beta N_1 - 2N_2 - \alpha N_3 & -\alpha N_2 \\ -\alpha N_3 & -\beta N_3 & 1 - \alpha N_1 - \beta N_2 - 2N_3 \end{pmatrix}.$$

The right-hand side of the system (27)-(29) shall be designated by $f(N)$. One can see, that for $(N_1, N_2, N_3) \in \mathbb{R}^3_+$ and $N_k = 0$, one has $f_k(N_1, N_2, N_3) = 0$ for all $k = 1, 2, 3$. It follows with Lemma 3.17, that $\mathbb{R}^3_+$ is positively invariant for (27)-(29). Moreover, one has

$$\begin{aligned} (f(N)|N) &= N_1^2(1 - N_1 - \alpha N_2 - \beta N_3) + N_2^2(1 - \beta N_1 - N_2 - \alpha N_3) \\ &\quad + N_3^2(1 - \alpha N_1 - \beta N_2 - N_3) \\ &\leq N_1^2 + N_2^2 + N_3^2 \\ &= \|N\|_2^2, \end{aligned}$$

due to the positiveness of N_1, N_2, N_3, α and β. It follows with Proposition 2.11 that all the solutions of (27)-(29) exist globally to the right. Now, setting the right-hand side of the system (27)-(29) to zero yields eight equilibria.

- The non-existence of specimens at all $(0, 0, 0)$ with triple eigenvalue $\lambda = 1$. One has with Proposition 4.4 that the origin is an unstable knot.

- Only one species exists, i.e. one has the three equilibria $(1,0,0),(0,1,0)$ and $(0,0,1)$, with corresponding eigenvalues $(-1,1-\alpha,1-\beta)$ in each case. Using Proposition 4.4 again, it follows that these equilibria are stable if $1-\alpha\le 0$ and $1-\beta\le 0$ hold.

- The coexistence of two species with equilibria

$$\frac{1}{1-\alpha\beta}\begin{pmatrix}1-\alpha\\1-\beta\\0\end{pmatrix},\frac{1}{1-\alpha\beta}\begin{pmatrix}0\\1-\alpha\\1-\beta\end{pmatrix},\frac{1}{1-\alpha\beta}\begin{pmatrix}1-\beta\\0\\1-\alpha\end{pmatrix}$$

which shall not be further investigated.

- Probably the most compelling case is the three-species equilibrium $E_\infty := \frac{1}{1+\alpha+\beta}\begin{pmatrix}1\\1\\1\end{pmatrix}$.

The corresponding Jacobian matrix reads

$$\frac{1}{1+\alpha+\beta}\underbrace{\begin{pmatrix}-1&-\alpha&-\beta\\-\beta&-1&-\alpha\\-\alpha&-\beta&-1\end{pmatrix}}_{=:A_{E_\infty}}.$$

Furthermore, it follows with Proposition 4.4 once more, that the three-species equilibrium is stable if all eigenvalues of the matrix A_{E_∞} have negative real parts (α,β are greater than 0 by assumption). Now, one has as characteristic polynomial of A_{E_∞}

$$\lambda^3+3\lambda^2+(3-3\alpha\beta)\lambda+\alpha^3+\beta^3-3\alpha\beta+1.$$

Setting $D_1 := 3$,

$$D_2 := \det\begin{pmatrix}3&\alpha^3+\beta^3-3\alpha\beta+1\\1&3-3\alpha\beta\end{pmatrix}\text{ and}$$

$$D_3 := \det\begin{pmatrix}3&\alpha^3+\beta^3-3\alpha\beta+1&0\\1&3-3\alpha\beta&0\\0&3&\alpha^3+\beta^3-3\alpha\beta+1\end{pmatrix},$$

leads to the inequalities $8-6\alpha\beta-\alpha^3-\beta^3>0$ and $\alpha^3+\beta^3-3\alpha\beta+1>0$, derived from $D_2>0$ and $D_3>0$, respectively. With the Routh-Hurwitz criterion stated in Proposition 4.6, it follows that the three-species equilibrium is stable if the inequalities just above are satisfied.

4.2 Lyapunov Stability

One of the most common issues in the study of dynamical systems is its stability. It is characterized by considering the response of a dynamical system to minor perturbations. More precisely, an equilibrium of a dynamical system is called stable if, for "small" values of initial disturbances, the perturbed orbit stays in a randomly prescribed "small" area of the state space (see Section 3.3). Furthermore, stability is equal to the continuity of solutions as a function of the initial values of a system in a neighbourhood of the fixed-point uniformly in time. If additionally, all orbits of the dynamical system come close to the equilibrium for $t \to \infty$, then the equilibrium point is designated as asymptotically stable. In this section based on Sections 5.5 and 8.1 from [48], Section 7.1 from [41] and Appendix C from [49], the concepts of stability shall further be investigated. At the beginning of the qualitative analysis of dynamical systems, problems originating in mechanics lead to the studies of their stability properties and equilibria. The most exhaustive contribution to the stability analysis of non-linear dynamical systems was introduced at the end of the nineteenth century by A. M. Lyapunov, who provided a mighty framework for analysing the stability of non-linear dynamical systems (see [19, Section 3.1] for further details).

4.2.1 Lyapunov Functions

One considers the autonomous ODE

$$\dot{x} = f(x), \tag{30}$$

$f\colon G \to \mathbb{R}^n$ being locally Lipschitz, $G \subset \mathbb{R}^n$ open. Central in the qualitative theory of ODEs is the following notion:

Definition 4.8. *A function $V \in C(G;\mathbb{R})$ is called a **Lyapunov function** for* (30) *if V is decreasing alongside the solutions of* (30)*. It means that the function $\phi(t) := (V \circ x)(t)$ is decreasing with respect to t for every solution $x(t)$ of* (30)*. If $\phi(t)$ is strictly decreasing for every nonconstant solution of* (30)*, then V is called a **strict Lyapunov function**.*

Now, let $x(t)$ be a solution of (30) and $V \in C^1(G;\mathbb{R})$. If one applies the chain rule on $\phi(t)$, one has

$$\dot{\phi}(t) = (\nabla V(x(t))|\dot{x}(t)) = (\nabla V(x(t))|f(x(t))).$$

From elementary analysis, it is well known that the function $\phi(t)$ is decreasing if $\dot{\phi}(t) \leq 0$. It follows that V is a Lyapunov function, if and only if

$$(\nabla V(x(t))|f(x(t))) \leq 0 \quad \text{for all } x \in G \tag{31}$$

holds. Similarly, V is a strict Lyapunov function, if and only if

$$(\nabla V(x(t))|f(x(t))) < 0 \quad \text{for all } x \in G \setminus \mathcal{E}, \tag{32}$$

with $\mathcal{E} = \{x \in G | f(x) = 0\}$ being the set of equilibria of (30).

To extend the concept of the Lyapunov function to the non-autonomous case, one considers the IVP (2), with the interval of existence $J(x)$, and defines $J_+(x) = J(x) \cap [0, \infty)$.

Definition 4.9. *[48, Definition 8.1.1] Let $V: \mathbb{R}_+ \times G \to \mathbb{R}$ be continuous, $G \subset \mathbb{R}^n$. $V(t, x)$ is a Lyapunov function for* (2)*, if the function*

$$\phi(t) = V(t, x(t)), \quad t \in J_+(x),$$

is decreasing for every solution $x(t)$ of (2)*.*

If $V \in C^1(\mathbb{R}_+ \times G)$, then so is $\phi(t)$ and the chain rule yields

$$\begin{aligned} \dot{\phi}(t) &= \frac{dV(t, x(t))}{dt} + \left(\nabla_x V(t, x(t)|\dot{x(t)})\right) \\ &= \frac{dV(t, x(t))}{dt} + (\nabla_x V(t, x(t)|f(t, x(t))) . \end{aligned}$$

One can see that the right-hand side of the last equation only depends on t and x, and therefore it is reasonable to define

$$\dot{V}(t, x) = \frac{dV(t, x(t))}{dt} + (\nabla_x V(t, x(t)|f(t, x(t))) , \ t \in \mathbb{R}_+, x \in G.$$

$\dot{V}$ is called the **orbital derivative** of V alongside the solutions of (2) [43, Definition 2.18]. In the case of autonomous systems and that V does not explicitly depend on t, one has

$$\dot{V}(x) = (\nabla_x V(x(t))|f(x(t))) , \quad x \in G.$$

Example 4.10. (Virus dynamics) Bonhoeffer et al. [7] suggest a basic model for viral dynamics, consisting of the following components:

1. The uninfected cells x produced at a constant rate λ and meeting their demise at a rate mx.
2. The infected cells y, dying at a rate ay and producing new virus agents at a rate ky.
3. The free virus agents v dying at a rate uv. They infect not infected cells to produce

infected cells y at a rate βxv.

It follows from the above definitions that $1/m, 1/a,$ and $1/u$ give the average life-spans of uninfected cells, infected cells, and free virus agents, respectively. The average number of virus agents generated over the lifespan of a single infected cell (the burst size) is given by k/a. The premises and definitions finally lead to the system:

$$\dot{x} = \lambda - mx - \beta xv, \tag{33}$$

$$\dot{y} = \beta xv - ay, \tag{34}$$

$$\dot{v} = ky - uv. \tag{35}$$

The system is continuously differentiable, guaranteeing a unique local solution for the system. The right-hand side of the system shall be by designated by $f(x, y, v)$.

Definition 4.11. *(Basic reproductive ratio) One defines the basic reproductive ratio R_0, as the average number of newly infected cells which arise from a single infected cell when almost all cells are uninfected.*

In the above virus model, one obtains $R_0 = \frac{\beta\lambda k}{amu}$. Now, one can recognize, that a small initial amount of virus agents v_0, can only spread itself in an organism if the number of newly infected cells through a single infected cell is greater than 1, i.e. $R_0 > 1$. In the absence of an infection, i.e. $y = v = 0$, one can readily see that the equilibrium is $(\lambda/m, 0, 0)$. Therefore, one assumes at least that v_0 is strictly greater than zero. Now, setting (34) and (35) to 0, one obtains

$$x^* = \frac{au}{\beta k} = \frac{\lambda}{mR_0}. \tag{36}$$

If x^* is plugged into Equation (33) and set to 0, one obtains

$$v^* = \frac{\lambda - mx}{\beta x} = \frac{m}{\beta}(R_0 - 1). \tag{37}$$

Finally, putting v^* in (35) and setting it to 0 yields

$$y^* = \frac{um(R_0 - 1)}{\beta k} = \frac{u}{k}v^*. \tag{38}$$

Bonhoeffer et al. [7] show that the system converges in damped oscillations to the equilibrium (x^*, y^*, v^*) if the basic reproduction ratio is greater than 1. Furthermore, Korobeinikov shows that the Lyapunov function for the system (33)-(35) is built accordingly to R_0 (see [32, 31]).

Case $R_0 > 1$:

One defines the function

$$V(x,y,v) = x^*(\frac{x}{x^*} - \ln\frac{x}{x^*}) + y^*(\frac{y}{y^*} - \ln\frac{y}{y^*}) + \frac{a}{k}v^*(\frac{v}{v^*} - \ln\frac{v}{v^*}).$$

It follows

$$(\nabla V(x,y,v)|f(x,y,v)) = \begin{pmatrix}1 - \frac{x^*}{x} 1 - \frac{y^*}{y}\frac{a}{k}(1-\frac{v^*}{v})\end{pmatrix}^T \begin{pmatrix}\lambda - mx - \beta xv \\ \beta xv - ay \\ ky - uv\end{pmatrix}$$

$$= \lambda - mx - \beta xv - \lambda\frac{x^*}{x} + mx^* + \beta x^* v + \beta xv - ay - \beta xv\frac{y^*}{y} + ay^* + ay - \frac{a}{k}uv - ay\frac{v^*}{v} + \frac{a}{k}uv^*.$$

Using the equalities derived for x^*, y^* and v^* in (36),(37) and (38), respectively, one obtains

$$(\nabla V(x,y,v)|f(x,y,v)) = mx^*(2 - \frac{x}{x^*} - \frac{x^*}{x}) + ay^*(3 - \frac{x^*}{x} - \frac{xvy^*}{x^*v^*y} - \frac{yv^*}{y^*v}).$$

The following well known inequality shall be reminded: $\sqrt[n]{a_1 \dots a_n} \leq \frac{a_1 + \dots a_n}{n}$, with equality holding if and only if all a_i's are equal, $a_1, \dots a_n \in \mathbb{R}$ being positive numbers. Using this inequality on x, x^* yields

$$\sqrt[2]{xx^*} \leq \frac{x + x^*}{2} \Longleftrightarrow xx^* \leq \frac{x^2 + (x^*)^2 + 2xx^*}{4} \Longleftrightarrow 0 \leq \frac{x}{x^*} + \frac{x^*}{x} - 2$$

and in a similar manner, $\frac{x^*}{x} + \frac{xvy^*}{x^*v^*y} + \frac{yv^*}{y^*v} - 3 \geq 0$ if $x, y, v > 0$ is assumed. It follows

$$(\nabla V(x,y,v)|f(x,y,v)) \leq 0 \text{ for all } x,y,v > 0, \text{ if } x^*, y^*, v^* \geq 0.$$

Case $R_0 \leq 1$:

As already mentioned above, in the case of $R_0 \leq 1$, no infection of the organism is possible, and therefore $Q_0 = (\frac{\lambda}{m}, 0, 0)$ is the only possible equilibrium. Let $x_0 = \frac{\lambda}{m}$ and one defines the function $U(x,y,v) = x_0(\frac{x}{x_0} - \ln\frac{x}{x_0}) + y + \frac{a}{k}v$. It holds:

$$\dot{U}(x,y,z) = \begin{pmatrix}1 - \frac{x_0}{x}, 1, \frac{a}{k}\end{pmatrix}^T \begin{pmatrix}\lambda - x(m + \beta v) \\ \beta xv - ay \\ ky - uv\end{pmatrix} = \lambda(2 - \frac{x}{x_0} - \frac{x_0}{x}) + \frac{au}{k}(R_0 - 1)v.$$

Using the inequalities $2-\frac{x}{x_0}-\frac{x_0}{x} \leq 0$ and $R_0-1 \leq 0$, it follows $(\nabla U(x,y,v)|f(x,y,v)) \leq 0$ for all $x, v > 0$. Interestingly, the variable y has no influence in this case, whether $U(x,y,v)$ is a Lyapunov function for the system or not.

The application of Lyapunov's direct method just above (also named the second method of Lyapunov) makes it possible to determine the stability of a system, without explicitly solving the non-linear system of differential equations. The main idea can be summarized as follows: The considered trajectories in the phase plane are characterized by decreasing values of a non-negative function, the **Lyapunov function**. The trajectory and its corresponding Lyapunov function will both decrease until the zero value has been attained. Thus, a root of the Lyapunov function (i.e. $V(x) = 0$) characterizes an equilibrium point (see [21, Section 9.4.3] for further details).

As promised at the end of Section 2.2.1, a method to prove the global existence to the right for an autonomous system (30) should be caught up, with the concept of the Lyapunov function.

Proposition 4.12. *([48, Proposition 5.5.2]) Let $G \subset \mathbb{R}^n$ be open, $f\colon G \to \mathbb{R}^n$ locally Lipschitz, $V \in C(G;\mathbb{R})$ a Lyapunov function for* (30) *and*

1. $\lim\limits_{|x|\to\infty} V(x) = \infty$, *if G is boundless (V is called coercive)*
2. $\lim\limits_{x\to\partial G} V(x) = \infty$

shall hold. Then, every solution of (30) *exits globally to the right. Furthermore,*

$$\sup_{t\geq 0} |x(t)| < \infty \text{ and } \inf_{t\geq 0} dist(x(t), \partial G) > 0$$

are fulfilled.

Proof. Let $x(t)$ solve the system (30) on its maximal interval of existence $[0, t_+)$. Following the property of a Lyapunov function, $V(x(t))$ is monotonically decreasing for all $t \in [0, t_+)$. The proof shall be carried out by contradiction.

- Firstly, it shall be assumed that $x(t)$ is unbounded. Then, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ that converges to t_+, satisfying

$$\lim_{n\to\infty} |x(t_n)| \to \infty.$$

  Now, because $V(x)$ is continuous on $x \in G$, one obtains with the sequential criterion for the continuity of a function

$$V(\lim_{n\to\infty} x(t_n)) = \lim_{n\to\infty} V(x(t_n)) \underbrace{\to}_{\substack{\text{coercivity}\\ \text{of } V(x)}} \infty,$$

which contradicts the monotony of $V(x)$.

- Secondly, $\lim\limits_{t_n\to t_+} dist(x(t_n), \partial G) \to 0$ shall be assumed. From the second premiss, it follows $V(x(t_n)) \to \infty$ if $n \to \infty$, which is again a contradiction. According to Theorem 2.8, it follows the existence of a global solution to the right, i.e. $t_+ = \infty$.

□

Example 4.13. The Lyapunov functions for the virus dynamics model (see Example 4.10) are

$$V(x,y,v) = x^*(\frac{x}{x^*} - \ln\frac{x}{x^*}) + y^*(\frac{y}{y^*} - \ln\frac{y}{y^*}) + \frac{a}{k}v^*(\frac{v}{v^*} - \ln\frac{v}{v^*})$$

and

$$U(x,y,v) = x_0(\frac{x}{x_0} - \ln\frac{x}{x_0}) + y + \frac{a}{k}v,$$

if $R_0 > 1$ and $R_0 \le 1$ respectively. Let $G = (0,\infty)^2$. The prerequisites 1. and 2. in Proposition 4.12 are satisfied in both cases, and therefore, every solution exits globally to the right.

As stated at the beginning of this section, the stability properties of equilibria can be studied employing Lyapunov functions.

Theorem 4.14. Let $\tilde{x}$ be an equilibrium point for $\dot{x} = f(x)$, with $f\colon \mathbb{R}^n \to \mathbb{R}^n$. Let $V\colon U \to \mathbb{R}$, be a $C^1(U)$ (Lyapunov) function on a neighbourhood U of $\tilde{x}$, satisfying

i) $V(\tilde{x}) = 0$ and $V(x) > 0$ for $x \in U \setminus \{\tilde{x}\}$,

ii) $\dot{V}(x) \le 0$ in $U \setminus \{\tilde{x}\}$.
Then $\tilde{x}$ **is stable**. Additionally, if

iii) $\dot{V}(x) < 0$ in $U \setminus \{\tilde{x}\}$,
then $\tilde{x}$ is **asymptotically stable**.

Proof. The proof can be found in [12, Theorem 2.6, 48, Satz 5.5.4]. □

Example 4.15. (**Linear Systems** [59, Chapter 2])

One considers the linear unforced system $\dot{x}(t) = Ax(t), x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$. It is reasonable to seek a Lyapunov function of the form $V(x) = x^T Px, P$ being a positive definite symmetric matrix. The goal is to build the matrix P, so that the conditions for asymptotic stability in Theorem 4.14 are satisfied. One has:

$$\begin{aligned}\dot{V}(x) =& \dot{x}^T P x + x^T P \dot{x}\\ =& x^T A^T P x + x^T P A x\\ =& x^T \underbrace{(A^T P + PA)}_{=:Q} x.\end{aligned}$$

Requiring the orbital derivative of V to be a negative definite function leads to

$$\dot{V}(x) = x^T(A^T P + PA)x = -x^T Q x < 0.$$

Thus, $\dot{V}(x) < 0$ is satisfied, if and only if Q is a positive definite symmetric matrix.

One obtains the following result for a linear unforced system: The equilibrium $\tilde{x} = 0$ is asymptotically stable in the Lyapunov sense, if

$$A^T P + PA = -Q \tag{39}$$

holds, P and Q being positive definite symmetric matrices. In the literature, Equation (39) is sometimes referred to as the **Lyapunov equation**. In practice, the analysis of the stability of a linear system is carried out in the following way: usually, one chooses the identity matrix for Q. The symmetric matrix P is then determined through the Lyapunov Equation (39). If P is positive definite, then the linear system is asymptotically stable in the Lyapunov sense.

Remark 4. If one considers closely Theorem 4.14, one observes that $V(x) = x^T P x > 0$ for all $x \neq 0$ is equivalent to $P \succ 0$. Therefore, 4.14 ii) can be reformulated as

$$\begin{aligned}0 \geq \dot{V}(x) &= (\nabla V(x)|\dot{x})\\ &= 2\,(Px|Ax)\\ &= 2x^T P A x \quad (\text{ due to } P^T = P)\\ &= x^T(A^T P + PA)x,\end{aligned}$$

and one obtains

$$A^T P + PA \preceq 0 \quad \text{for } x \in U \setminus \{\tilde{x}\}.$$

If one considers the Lyapunov functions from the virus dynamics model in Example 4.10, one has

$$V(x^*, y^*, z^*) \neq 0 \text{ and } U(x_0, 0, 0) \neq 0 \text{ if } R_0 > 1 \text{ and } R_0 \leq 1, \text{ respectively,}$$

and therefore, the prerequisites of Theorem 4.14 are not fulfilled. Thus, these Lyapunov functions are not suited for the study of the stability of their corresponding equilibrium.

Two main features of Lyapunov's second method crystallised out in this section (for further details, see [50, Section 2]):

1. No precise insight is needed concerning the considered non-linear system. Only the existence of a decreasing function alongside the trajectories of the system's solutions is required to identify the long-term behaviour. Therefore, the solving of the equations is not necessary.

2. There is however a severe drawback: no systematic approach is (until now) known, to determine if a dynamical system admits a Lyapunov function or not. Worse, in the case where the mere existence would be proven for a system, there still would be the (mostly) difficult task to find such a function.

5 Construction of Invariant Sets

In Chapter 3, the concept of an invariant set was introduced. As one recalls, a set that possibly fulfils the conditions of invariance was already given. A question that naturally arises is **how to find or construct such a set**. In this thesis, the following two methods are presented in detail:

1. **Construction with primary geometric regions.**
   If there is no possibility to postulate a rather primary function that yields a closed formula for the boundary of a positively invariant set, then one can try to localize its bounds as a curve (or a surface if one works in higher dimensions). The approach would consist of putting the boundary as several simple pieces together, i.e. straight segments, or parts of an ellipsoid.

2. **Construction based on Lyapunov functions.**

5.1 Construction with Basic Geometric Regions

As one can easily guess, one of the most straightforward positively invariant set would be a rectangle (or a rectangular box in the case of higher dimensions) with parallel sides to the coordinate axes.

Example 5.1. [36, Section 11.2, Example 2] The system of ODEs

$$\dot{x} = y - 8x^3, \quad \dot{y} = 2y - 4x - 2y^3 \tag{40}$$

is being considered, let $\tilde{x} = (x, y)^T$. As a starting point, one can suspect the positive invariance of a rectangle R with corners at $(\pm 1, \pm 2)$. Now, the outer normal vectors on the boundaries (normalized to unit magnitude) are $n_r = (1, 0)$ on the right side, $n_l = (-1, 0)$ on the left side, $n_u = (0, 1)$ on the upside and $n_d = (0, -1)$ on the downside. Therefore $N_D(\tilde{x}) = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. One has

- $(f(\tilde{x})|n_r) = y - 8x^3 = y - 8 \leq 0,$ if $x = 1$ and $y \in [-2, 2]$,
- $(f(\tilde{x})|n_l) = -y + 8x^3 \leq 0,$ if $x = -1$ and $y \in [-2, 2]$,
- $(f(\tilde{x})|n_u) = 2y - 4x - 2y^3 = -4x - 12 \leq 0,$ if $y = 2$ and $x \in [-1, 1]$ and
- $(f(\tilde{x})|n_d) = -2y + 4x + 2y^3 = -20 + 4x \leq 0,$ if $y = -2$ and $x \in [-1, 1]$.

It follows with Theorem 3.16 that the rectangle R is positively invariant for (40).

In the rest of this section based on [23] and [6, Section 4], continuous linear dynamical systems described by the (linear) equation

$$\dot{x}(t) = Ax(t), \tag{41}$$

with a constant real matrix $A \in \mathbb{R}^{n\times n}$, are being considered. As usual, let $x(t) \in \mathbb{R}^n$, $x(t_0) = x_0$ the initial value and $t \in \mathbb{R}$. One may assume, without loss of generality, that A is not the zero matrix. **Ellipsoids and polyhedral sets** have been, without a doubt, the most successful categories of eligible invariant sets. For this reason, it is legitimate to consider cases, in which these types of invariant sets appear.

5.1.1 Polyhedral Sets

Definition 5.2. *A polyhedron $P \subset \mathbb{R}^n$ can be defined as the intersection of a finite number of half-spaces $P = \{x \in \mathbb{R}^n : Gx \leq b\}$, with $G \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.*

Lemma 5.3. *([23, Lemma 3.5]) Let P be a polyhedron as in Definition 5.2 and the non-zero set of indices I_x for all $x \in P$. One has that P is an invariant set for the continuous system* (41) *if and only if for every $x \in \partial P$ (i.e. $G_i^T x = b_i$) the following inequality holds:*

$$G_i^T Ax \leq 0 \textit{ for all } i \in I_x. \tag{42}$$

Proof. To apply Theorem 3.10, the tangent cone for the closed convex polyhedron of Definition 5.2 is needed. Let x be an element on the boundary of P. Now, one has for the tangent cone (see [22, Examples 5.2.6 (c)])

$$T_P(x) = \{x \in \mathbb{R}^n | (g_i, d) \leq 0 \text{ for } i \in I_x\} = \{x \in \mathbb{R}^n | G_i^T x \leq 0 \text{ for } i \in I_x\}.$$

The proof concludes straightforward with Nagumo's Theorem 3.10 choosing $f(x) = Ax$. □

Example 5.4. [23, Example 4.1] Let the polyhedron be defined by

$$P = \{(x, y) | x + y \leq 1, -x + y \leq 1, x - y \leq 1, -x - y \leq 1\}$$

i.e. one has the form in Definition 5.2 with

$$G = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

and the system of ODEs

$$\dot{x} = -x, \dot{y} = -y \quad (\text{ i.e. } A = -I_2) \tag{43}$$

with solutions $x(t) = x_0 \exp(-t)$ and $y(t) = y_0 \exp(-t)$. Now, one has on the first boundary the relation $G_1^T \begin{pmatrix} x \\ y \end{pmatrix} = x + y = 1$. It follows

$$G_1^T A \begin{pmatrix} x \\ 1-x \end{pmatrix} = -x - 1 + x \leq 0.$$

The other inequalities are checked in the same manner, and one obtains the inequality $G_i^T Ax \leq 0$ for every $i \in I_x$ and $\forall x \in \partial P$. Thus, with Lemma 5.3, the polyhedron P is an invariant set for the system of ODEs, assuming that $(x_0, y_0) \in P$.

5.1.2 Ellipsoids

Definition 5.5. *An ellipsoid $E \subset \mathbb{R}^n$ centred at the origin is defined as $E = \{x \in \mathbb{R}^n \colon x^T Q x \leq 1\}$ with a symmetric matrix $Q \in \mathbb{R}^{n \times n}$ satisfying $Q \succ 0$. An ellipsoid which is not centred at the origin can be transformed into an ellipsoid centred at the origin.*

In the case of an ellipsoid, its invariance can be proven with the following result:

Theorem 5.6. An ellipsoid E given as in Definition 5.5 is an invariant set for the continuous system (41) if and only if the following relation is satisfied:

$$A^T Q + QA \preceq 0. \tag{44}$$

Proof. Define the ellipsoid $E = \{x \in \mathbb{R}^n : x^T Q x \leq 1\}$ with boundary $\partial E = \{x \in \mathbb{R}^n : x^T Q x = 1\}$ and set $\phi(x_1, \ldots, x_n) = x^T Q x$. The vector $\nabla \phi$ is normal to the surface $\phi(x_1, \ldots, x_n) = 1$, and considering the symmetry of Q, one obtains $\nabla \phi = 2Qx$. More details of the derivation of $\nabla \phi$ can be found in [17, Section 4.6]. Thus, the outer normal vector of the ellipsoid E at an x on the boundary is Qx. Stern and Wolkowicz have shown ([54, Section 2]) that the tangent cone at $x \in \partial E$ has the form $\mathcal{T} = \{y \in \mathbb{R}^n | y^T Q x \leq 0\}$. With Nagumo's Theorem 3.10, one obtains that E is an invariant set for the linear system (41), if and only if

$$(Ax)^T Q x \leq 0 \tag{45}$$

holds, for all x on the boundary. Now, inequality (45) can be rewritten as

$$x^T (A^T Q + QA) x \leq 0 \quad \text{ for all } x \in \partial E. \tag{46}$$

Moreover, one can see that (46) follows directly from (44). If inequality (46) is being assumed, then for all $0 \neq y \in \mathbb{R}^n$, one can find an x on the boundary of E and an $\alpha \in \mathbb{R}$, satisfying $y = \alpha x$, and one obtains

$$y^T(A^TQ + QA)y = \frac{1}{\alpha^2}x^T(A^TQ + QA)x \leq 0,$$

which is equivalent to (44). $\square$

Amazingly, the inequality (44) is identical to the one already encountered in Remark 4. In summary: The study of the stability of an equilibrium point $\tilde{x}$ for an autonomous system of ODEs employing a Lyapunov function defined by $V(x) = x^TPx, P$ being a positive definite symmetric matrix, is directly related to **finding an invariant set** for a linear system. This leads to the illustrating modified example from Horváth, Song, and Terlaky [23, Example 4.3]:

Example 5.7. Let $E = \{x \in \mathbb{R}^n : x^TQx \leq 1\}$ be the ellipsoid with $Q = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ and the system of ODEs with $A = \begin{pmatrix} -9 & 0 \\ 4 & -4 \end{pmatrix}$, initial values $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now, one has

$$A^TQ + QA = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

with its corresponding eigenvalues$\{-3, -1\}$. It follows $A^TQ + QA \preceq 0$, and one obtains with Theorem 5.6, that E is an invariant set for the system of ODEs defined by A.

5.1.3 Lorenz Cones

Definition 5.8. *A Lorenz cone, $C_L \subset \mathbb{R}^n$ with a vertex at the origin is defined as $C_L = \{x \in \mathbb{R}^n : x^TQx \leq 0, x^Tu_n \geq 0\}$ where $Q \in \mathbb{R}^{n\times n}$ is a symmetric nonsingular matrix with one negative eigenvalue λ_n and corresponding eigenvector u_n. As already seen in Section 5.1.2, any Lorenz cone with non-zero vertex can also be transformed to a Lorenz cone with a vertex at the origin.*

In the case of a Lorenz cone, Horváth, Song, and Terlaky [23, Theorems 3.34 and 3.35] show that its invariance can be proven with the following result:

Theorem 5.9. A Lorenz cone C_L is an invariant set for a system of ODEs (41) if and only if

$$\exists \eta \in \mathbb{R} \text{ satisfying } A^TQ + QA - \eta Q \preceq 0.$$

Theorem 5.9 can be nicely visualized with the following example from Horváth, Song, and Terlaky [23, Example 4.4]:

Example 5.10. Let $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ with corresponding sole negative eigenvalue -1 and its eigenvector $(0,0,1)^T$. Obviously, Q is symmetric and nonsingular. Thus, one obtains the Lorenz cone $C_L = \{x^2 + y^2 \leq z^2, z \geq 0\}$. Now, let the system of ODEs be $\dot{x} = x - y, \dot{y} = x + y$ and $\dot{z} = z$, i.e. $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with initial values $(0,0,0.3)$. One obtains as solutions for the system $x(t) = y(t) = 0$ and $z(t) = \frac{1}{3}\mathrm{e}^t$. It follows

$$A^TQ + QA + \eta Q = \begin{pmatrix} \eta + 2 & 0 & 0 \\ 0 & \eta + 2 & 0 \\ 0 & 0 & -\eta - 2 \end{pmatrix}.$$

Setting $\eta = -2$, one sees that the conditions in Theorem 5.9 are satisfied, and therefore the Lorenz cone C_L is an invariant set of the system of ODEs defined by A.

Let $f\colon G \to \mathbb{R}^n$ be continuous with $G \subset \mathbb{R}^n$ open. If not stated otherwise, the IVP

$$\dot{x} = f(x), t \geq 0, \text{ with } x(0) = t_0, \tag{47}$$

shall be considered for the remainder of this chapter.

5.2 Construction based on Lyapunov Functions

Another possibility to build (positively) invariant sets lies in the interesting (at first sight surprising) feature of Lyapunov functions (see, e.g. [48, Korollar 8.1.3]).

Corollary 5.11. Let V be an autonomous Lyapunov function (which does not explicitly depend on t) for the system (2), and $\alpha \in \mathbb{R}$. Then, the sets

$$D = V^{-1}((-\infty, \alpha]) \subset G$$

are positively invariant for (2).

Proof. Since only positive invariance is considered, let $t \in \mathbb{R}_+$ and $t_0 \geq 0$. The definition of D leads to $V(x_0) \leq 0$, and because the Lyapunov function for the system (2) is decreasing in time alongside its solutions, it holds

$$V(x(t)) \leq V(x_0) \leq \alpha$$

for all $t \geq t_0$. $\square$

Example 5.12. (Pendulum [18, Section 8.2]) The pendulum is a classical example originating from mechanics. A mass m is hung up at the end of a rigid (massless) rod of length R. Let α be the angle between the pendulum and the vertical. The force acting on the pendulum is then $-mg\sin(\alpha)$ in the direction of its movement ($g = 9,81m/s^2$ being the constant earth acceleration). Now, with Newton's famous second law $F = ma$, one obtains the differential equation

$$-mg\sin(\alpha) = m\ddot{\alpha}. \tag{48}$$

As one can see, this equation is of second order, thus not in the desired form (2). This issue is easily solvable with a well known-trick: one defines the vector $x = (x_1, x_2)^T := (\alpha, \dot{\alpha}) \in \mathbb{R}^2$, then the system of ODEs

$$\dot{x}(t) = \begin{pmatrix} x_2(t) \\ -g\sin(x_1(t)) \end{pmatrix} \tag{49}$$

is equivalent to the differential equation (48). The system can be extended by a linear friction term:

$$\dot{x}(t) = \begin{pmatrix} x_2(t) \\ -g\sin(x_1(t)) - kx_2(t) \end{pmatrix} \tag{50}$$

with $k > 0$ being the friction coefficient. Now, though it is a tedious task, Lyapunov functions can be constructed through the help of a physical thought process. The energy of the pendulum is the sum of its kinetic energy $E_{kin} := \frac{\dot{x}_1^2}{2} = \frac{x_2^2}{2}$, and its potential energy $E_{pot} = \int\limits_0^{x1} g\sin(\alpha)\,\mathrm{d}\alpha = g(1-\cos(x_1))$ (the integral over the force necessary to shift the pendulum from position 0 to the position x_1). As an ansatz for the Lyapunov function, one can try the total energy of the pendulum, i.e.

$$V(x_1, x_2) = \frac{x_2^2}{2} + g(1-\cos(x_1)), \text{ with } V(x_1, x_2) \in C^1(\mathbb{R}^2; \mathbb{R}).$$

It follows

$$(\nabla V(x_1, x_2)|f(x_1, x_2)) = \begin{pmatrix} g\sin(x_1) & x_2 \end{pmatrix}^T \begin{pmatrix} x_2(t) \\ -g\sin(x1) - kx_2 \end{pmatrix}$$

$$
\begin{aligned}
&= g\sin(x_1)x_2 - x_2 g\sin(x_1) - kx_2^2 \\
&= -kx_2^2 \\
&\le 0, \quad \text{if } k > 0 \text{ for all } x \in \mathbb{R}^2.
\end{aligned}
$$

Hence, one sees from (31), that V is a Lyapunov function for the system (50). If $x_2 \neq 0$, $(\nabla V(x_1, x_2)|f(x_1, x_2)) < 0$ holds, and thus $V(x)$ is even a strict Lyapunov function for the pendulum system in this case. The equilibria of the system are $(0,0)$ and $(2k\pi, 0)$ with $0 \neq k \in \mathbb{Z}$. Now, one has $V(0,0) = V(2k\pi, 0) = 0$, and $V(x_1, x_2) > 0$ for $(x_1, x_2)^T \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $\mathbb{R}^2 \setminus \{(2k\pi, 0)\}$, respectively. As shown just above, one has $\dot{V}(x_1, x_2) \le 0$, if $k > 0$ for all $x \in \mathbb{R}^2$. Furthermore, with Theorem 4.14, the equilibria $(2k\pi, 0)$ are stable. If one assumes small amplitude oscillations, then the only equilibrium $(0,0)$ is asymptotically stable. Because of $\dot{V}(x_1, x_2) = -kx_2^2$, the orbital derivative is only zero and stays zero, if x_2 is zero. This entails $\dot{x_2} = 0$, and consequently $x_1 = 0$. It shows, that $\dot{V}(x_1, x_2) = 0$, only when $x_1 = x_2 = 0$, proving the asymptotic stability of $(0,0)$ (see [28, Example 2.2.2] for further details). Another approach is to modify the Lyapunov function so that Theorem 4.14 iii) is applicable. One defines the modified Lyapunov function

$$V_\alpha(x_1, x_2) = \frac{x_2^2}{2} + g(1 - \cos(x_1)) + \alpha x_2 \sin(x_1),$$

with a parameter $\alpha > 0$. For $x_1 \in [-\pi, \pi]$ and $0 < \alpha < \min\{k, \frac{4gk}{4g+k^2}\}$, there exists a constant $c > 0$ with

$$\dot{V}_\alpha(x_1, x_2) \le -c(\sin^2(x_1) + x_2^2),$$

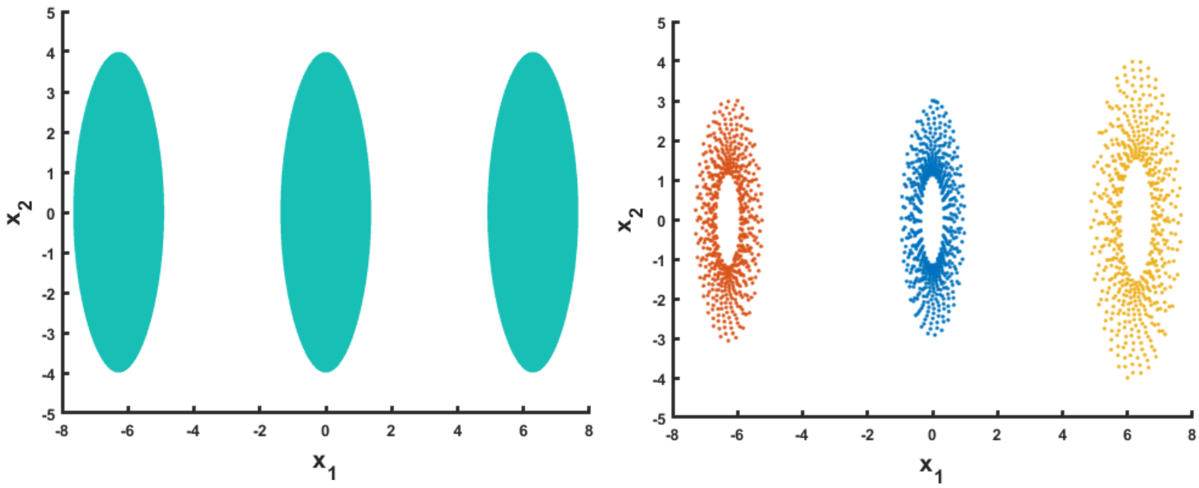
and it follows that the prerequisites of Theorem 4.14 iii) are satisfied, proving the asymptotic stability of $(0,0)$. More details can be found in [18, Section 8.2]. An application of Corollary 5.11 with $\alpha = 8$ yields the family of sets depicted in Figure 6a.

The competing species example shall be considered again in regard to a Lyapunov function.

Example 5.13. (Competing species revisited) Example 3.18 is a two-dimensional competitive Lotka-Volterra system

$$\dot{u} = u(b_1 - u - \alpha v) \tag{51}$$
$$\dot{v} = v(b_2 - \beta u - v) \tag{52}$$

with $b_1 = 3, \alpha = 2, b_2 = 4$ and $\beta = 3$. An application of Lemma 3.17 confirms that the solutions $u(t), v(t)$ are positive for $t \ge 0$ if the initial values are positive (which they are again because one considers the number of animals). Tang, Yuan, and Ma [56] show the



(a) Representation of the invariant sets $V(x_1, x_2) \leq 8$, with $V(x_1, x_2) = \frac{x_2^2}{2} + g(1 - \cos(x_1))$.

(b) Solutions for the pendulum system with initial values $(-6, 3)$ (red), $(0, 3)$ (blue) and $(6, -4)$ (yellow).

Figure 6: Visualization and confirmation of Corollary 5.11 for the pendulum system (50).

existence of a Lyapunov function for the system (51)-(52) in the form of

$$V(u, v) = \frac{\beta}{2}u^2 + \frac{\alpha}{2}v^2 - \beta b_1 u - \alpha b_2 v + \alpha\beta uv, \quad \alpha, \beta \in \mathbb{R}.$$

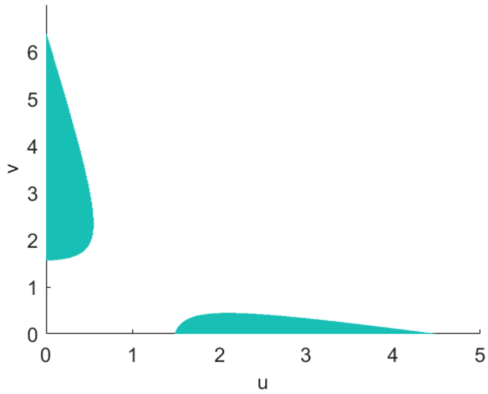
Applied on Example 3.18, one obtains the function $V(u, v) = \frac{3}{2}u^2 + v^2 - 9u - 8v + 6uv$. It follows

$$\begin{aligned}
(\nabla V(u, v)|f(u, v)) &= \Big(3u - 9 + 6v, \quad 2v - 8 + 6u\Big)^T \begin{pmatrix} u(3 - u - 2v) \\ v(4 - 3u - v) \end{pmatrix} \\
&= \Big(-3(-u + 3 - 2v), \quad -2(-v + 4 - 3u)\Big)^T \begin{pmatrix} u(3 - u - 2v) \\ v(4 - 3u - v) \end{pmatrix} \\
&= -3u(3 - u - 2v)^2 - 2v(4 - 3u - v)^2 \leq 0
\end{aligned}$$

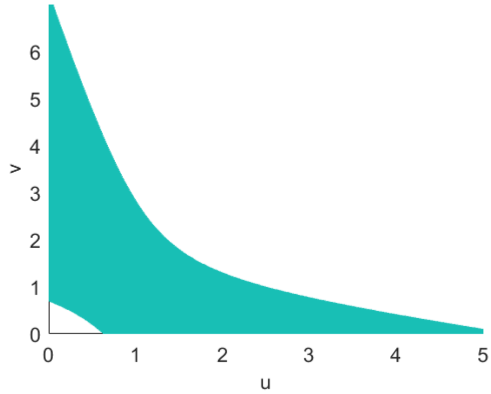
due to the positivity of $u(t), v(t)$. Therefore, the function V defined above is indeed a Lyapunov function for the considered system, in Example 3.18. The application of Corollary 5.11 with $\alpha = -10$ and $\alpha = -5$ yields the family of sets depicted in Figures 7a and 7b, respectively.

Example 5.14 (The discrete p-Laplacian revisited)**.** Let $\mathcal{G} = (V, E)$ be an undirected graph with $n = |V|$ vertices, $m = |E|$ edges, and its corresponding incidence matrix $\mathcal{I}$. One further defines the Laplacian $\Delta = \mathcal{I}\mathcal{I}^T$.

Lemma 5.15. *The discrete Laplacian Δ is positive semi-definite and singular.*



(a) Representation of the invariant sets $\{(u,v) \in \mathbb{R}^2 : \frac{3}{2}u^2 + v^2 - 9u - 8v + 6uv \leq -10\}$.



(b) Representation of the invariant set $\{(u,v) \in \mathbb{R}^2 : \frac{3}{2}u^2 + v^2 - 9u - 8v + 6uv \leq -5\}$.

Figure 7: Visualization and confirmation of Corollary 5.11 for the competing species system.

Proof. Let λ be an eigenvalue of Δ with corresponding eigenvector v, with $\|v\| = 1$. One has

$$\lambda = v^T \Delta v = (v^t \mathcal{I})(\mathcal{I}^T v) = (\mathcal{I}^T v)^T (\mathcal{I}^T v) \geq 0.$$

Moreover, considering that the sum of the elements of every column yields zero, it follows that Δ is singular. □

One can see that $\Delta^T = (\mathcal{I}\mathcal{I}^T)^T = \mathcal{I}\mathcal{I}^T = \Delta$, and it follows that Δ is a symmetric matrix. It is well-known from the linear algebra, that symmetric matrices possess only real eigenvalues. It follows that the eigenvalues of the Laplacian are non-negative real numbers. Moreover, zero is always an eigenvalue of Δ whose multiplicity agrees with the number of connected components of $\mathcal{G}$ (see [44, Section 1]). The eigenvalues of Δ can, therefore, be ordered as follows:

$$0 = \lambda_1 \leq \ldots \leq \lambda_n.$$

For the sake of simplicity, $\mathcal{G}$ shall be connected, because in this case, one obtains the following result:

Lemma 5.16. *Let $\mathcal{G}$ be connected, then the null space of the Laplacian Δ is of dimension one and generated by the vector* $(1, \ldots, 1)^T$.

Proof. Let x be an element of the null space of Δ. One has

$$x^T \Delta x = \sum_{(i,j) \in E} (x_i - x_j)^2 = 0.$$

It follows that $x_i = x_j$ for every edge $(i,j) \in E$. Considering that $\mathcal{G}$ is connected, one obtains that all x_i's are equal. $\square$

Let $\Delta_p : H \to \mathbb{R}^n$, $H \subset \mathbb{R}^n$ open, with $\Delta_p x = \mathcal{I}(\left|\mathcal{I}^T x\right|^{p-2} \mathcal{I}^T x)$. The initial value problem defined in Example 2.12

$$\frac{dx}{dt} = -\Delta_p x(t), \tag{53}$$

with $x(t_0) = x_0$ shall be considered again. The right-hand side of (2.12) shall be denoted by $f(x)$. It was shown in Example 2.12, that the IVP (53) possesses a unique solution, which exists globally to the right, and therefore defines a dynamical system. Setting $f(x) = 0$ yields the equilibria of the system (53). One recognizes that the set of the fixed points consists of the elements of the null space of Δ. Now, it follows with Lemma 5.15, that the system has an infinite number of equilibrium points (Δ is singular). Furthermore, Lemma 5.16 yields the set of equilibria $\mathcal{E} := \{x_\lambda := (\lambda, \ldots, \lambda) : \lambda \in \mathbb{R}\}$. If one defines the vector

$$x - \lambda \mathbf{1} = \begin{pmatrix} x_1 - \lambda \\ \vdots \\ x_n - \lambda \end{pmatrix}$$

and the function $V \colon H \to \mathbb{R}$ defined by

$$V(x) = \frac{1}{2}(x_1 - \lambda, \ldots, x_n - \lambda)^T (x_1 - \lambda, \ldots, x_n - \lambda) = \frac{1}{2}(x - \lambda\mathbf{1})^T (x - \lambda\mathbf{1}), \tag{54}$$

one obtains with Lemma 5.15 (Δ is positive semi-definite)

$$\begin{aligned} (\nabla V(x) | f(x)) &= -(x - \lambda\mathbf{1})^T \mathcal{I}(\left|\mathcal{I}^T (x - \lambda\mathbf{1})\right|^{p-2} \mathcal{I}^T (x - \lambda\mathbf{1})) \\ &= -\left|\mathcal{I}^T (x - \lambda\mathbf{1})\right|^{p-2} (x - \lambda\mathbf{1})^T \mathcal{I}\mathcal{I}^T (x - \lambda\mathbf{1}) \\ &\leq 0. \end{aligned}$$

It follows that the function V, defined in (54), is indeed a Lyapunov function for the considered system. Now, one sees that $V(x_\lambda) = 0$ and $V(x_\lambda) > 0$ for $x \in H \setminus \{x_\lambda\}$. Furthermore, if $x \notin \mathcal{E}$, then $x - \lambda\mathbf{1} \notin \mathcal{E}$ and one obtains $(x - \lambda\mathbf{1})^T \mathcal{I}\mathcal{I}^T (x - \lambda\mathbf{1}) > 0$. One recognizes $\dot{V}(x) < 0$ in $H \setminus \{x_\lambda\}$, yielding with Theorem 4.14, that every equilibrium x_λ is asymptotically stable. It follows with Corollary 5.11, that the sets $\{\frac{1}{2}(x - \lambda\mathbf{1})^T (x - \lambda\mathbf{1}) \leq c : c, \lambda \in \mathbb{R}, \}$ are positively invariant for the system (53).

Before moving on to the next example, one needs the notion of a maximal weak-invariant subset (see [48, Section 8.4]). Let $D \subset G \subset \mathbb{R}^n$; if one considers the orbits $\gamma(x) = x(\mathbb{R}) \subset D$, then the **maximal weak-invariant subset** M_D of D is defined as the union of these orbits. The concept of an attractor was defined in Section 3.3.3, but no method to find one has been presented so far. This gap shall be closed with a result from Prüss and Wilke [48, Satz 8.4.6].

Proposition 5.17. *Let $G \subset \mathbb{R}^n$ be open, $V\colon G \to \mathbb{R}$ a locally Lipschitz Lyapunov function satisfying*

$$V(x) \to \infty \quad \text{for } dist(x, \partial G) \to 0 \text{ or } \|x\| \to \infty.$$

Moreover, for an $\alpha \in \mathbb{R}$,

$$\dot{V}(x) < 0 \quad \text{for all } x \in G \text{ with } V(x) > \alpha$$

shall hold. Let $D = \{x \in G : V(x) \leq \alpha\}$ and M the maximal weak-invariant subset of D. Then, all starting solutions in G exist globally in G. The set $D \subset G$ is compact and positively invariant. Furthermore, M is a ***global attractor*** *for* (47) *in G.*

The considerations just above lead to a well-studied system (see [48, Sections 8.4, 14.1]).

Example 5.18 (The FitzHugh-Nagumo system)**.** One considers the system originating in the electro-physiology

$$\dot{x} = g(x) - y, \tag{55}$$

$$\dot{y} = \sigma x - \gamma y. \tag{56}$$

$\sigma, \gamma > 0$ are constants, $g\colon \mathbb{R} \to \mathbb{R}$ is continuous with $g(0) = 0$. This system was proposed by FitzHugh, to simplify the complex Hodgkin-Huxley model, which describes excitement conduct in the nerve tracts. Moreover, the "simplified" model should retain the qualitative properties of the original system. Usually, one takes for g a cubic function possessing three roots $\{0, a, b\}$ with $0 < a < b$. For instance, one has $g(x) = -x(x-a)(x-b)$. It should further be assumed, that there exists an $r > 0$ satisfying

$$xg(x) < 0, \quad \text{for } |x| \geq r. \tag{57}$$

The right-hand side (denominated by $f(x, y)$) is continuous, and it follows with Peano, that the system possesses a local solution (uniqueness is not required).

Equilibria:

One can see that $\tilde{x}_0 := (0, 0)$ is always an equilibrium of the system, for any value of the parameters. Now, setting $f(x, y) = 0$, one obtains $g(x) = y$ from (55). Inserting $g(x)$ in (56) leads to the consideration of the solutions from the following equation (with $\delta = \sigma/\gamma$):

$$p(x) := \sigma x - \gamma g(x) = x(\delta + (x-a)(x-b)) = 0.$$

One obtains the solutions ($x = 0$ excluded)

$$\tilde{x}_{\pm} = \frac{a+b}{2} \pm \frac{1}{2}\sqrt{(b-a)^2 - 4\delta}.$$

Moreover, one recognizes that $\tilde{x}_- \neq \tilde{x}_+$ holds, if $\delta < \delta_0 := (b-a)^2/4$ is satisfied. One obtains with $y_{\pm} = \delta x_{\pm}$ the equilibria

$$\tilde{x}_{\pm} := (x_{\pm}, y_{\pm}).$$

Stability:

One has

$$A := f'(x,y) = \begin{pmatrix} g'(x) & -1 \\ \sigma & -y \end{pmatrix},$$

and obtains

$$trA = -\gamma + g'(x) = -h'(x), \quad h(x) := \gamma x - g(x),$$

and

$$\det A = -\gamma g'(x) + \sigma = \gamma p'(x).$$

It follows with the principle of linearized stability (see Section 4.1), that $\tilde{x}_0$ is asymptotically stable for any value of the parameters, due to $0 < h'(0)$ and $0 < p'(0)$. One can see that $p'(\tilde{x}_-) < 0$ and $p'(x_+) > 0$ and it follows that $\tilde{x}_-$ and $\tilde{x}_+$ are a saddle point and a knot (or a spiral), respectively. Furthermore, if $-h'(\tilde{x}_+) > 0$, i.e. $\gamma < g'(\tilde{x}_+) < \delta$, then $\tilde{x}_+$ is unstable.

Lyapunov function:

Now, if one sets $G = \mathbb{R}^2$ and defines the function $V(x,y) = \frac{x^2}{2} + \frac{y^2}{2\sigma}$, then one obtains

$$\dot{V}(x,y) = (\nabla V(x,y)|f(x,y)) = (x, \frac{y}{\sigma})^T \begin{pmatrix} g(x) - y \\ \sigma x - \gamma y \end{pmatrix} = xg(x) - \frac{\gamma}{\sigma}y^2.$$

It follows $\dot{V}(x,y) < 0$ if (57) is satisfied. Now, if one considers the set $V^{-1}(\alpha)$, then one has for $(x,y) \in V^{-1}(\alpha)$

$$\frac{x^2}{2} + \frac{y^2}{2\sigma} \leq \alpha \Leftrightarrow y^2 \leq 2\alpha\sigma - \sigma x^2.$$

If one defines $\alpha_0 = \max\{xg(x) + \gamma x^2 : |x| \leq r\}/2\gamma$, then it follows

$$\dot{V}(x,y) = xg(x) - 2\alpha\gamma + \gamma x^2 < 0,$$

if $\alpha > \alpha_0$ is satisfied. With Proposition 5.17, one obtains global existence of the solution to the right, and the existence of a global attractor $M \subset D := V^{-1}(-\infty, \alpha_0]$. One sees from these considerations that the whole dynamics of the FitzHugh-Nagumo system occur inside the ellipse $E := \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{2} + \frac{y^2}{2\sigma} \leq \alpha_0\}$.

Last but not least, the behaviour of systems of chemical and biological oscillators shall be studied, based on the considerations of Jadbabaie, Motee, and Barahona [24].

Example 5.19 (The Kuramoto Model of Coupled Non-linear Oscillators)**.** Significant attention has been dedicated to the question of the interconnected motion of multiple self-governing entities. Numerous disciplines such as ecology, the social sciences, or computer graphics are developing an apprehension of how a class of moving objects can attain an agreement and move in an organized way without centralized coordination. These models can especially be studied through the lens of network dynamics, i.e. the relationship between a graph structure and the dynamical behaviour of large networks. The standard **Kuramoto model** describes the dynamics of a set of N phase oscillators θ_i with natural frequencies ω_i. This model is one of the most popular dynamical systems which can be studied on networks (see [47, Section 3.6]). The time evolution of an *ith* oscillator is described through the following ODE:

$$\theta_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \tag{58}$$

for every node i with its associated phase $\theta_i(t) \in [0, 2\pi)$, K is the coupling strength. Kuramoto himself analysed the system (58) based on the order parameter $\tilde{r} \exp(\psi) = \frac{1}{N} \sum_{j=1}^{N} \exp(i\theta_j)$ as a measure of synchronization (ψ is called the average phase). One can see from his definition, that if all the ω_i 's are identical then $\tilde{r} = 1$. If all the oscillators are spaced equally on the unit circle, then $\tilde{r} = 0$. An in-depth analysis of the equilibria of (58) would go beyond the scope of this thesis, and shall, therefore, be omitted (Mehta et al. [39] point out that even for humble sizes $N \in [10, 20]$, the number of equilibria rises above 100000). Now, for an oriented graph $\mathcal{G}$ with N vertices and e edges, one considers the $N \times e$ incidence matrix B and the $N \times N$ symmetric Laplacian $L = BB^T$ (see Example 5.14). In the framework of graph theory, the Kuramoto model (58) can be rewritten for an unweighted graph, with the $N \times 1$ vectors θ and ω as:

$$\dot{\theta} = \omega - \frac{K}{N} B \sin(B^T \theta). \tag{59}$$

The order parameter $\tilde{r}$ can be "generalized" to (i is the imaginary unit):

$$r^2 = 1 - \frac{(\exp(i\theta))^H L(\exp(i\theta))}{N}, \tag{60}$$

with $(\exp(i\theta)) := (\exp(i\theta_1), \dots \exp(i\theta_N)))^T$ and H denoting the complex conjugate transpose. This representation yields an interesting physical interpretation: each oscillator i can be viewed as a rotor travelling on a circle with unit radius, and velocity vector $v_j = \exp(i\theta_j)$. Now, the Kuramoto model (59) shall be considered in its "unperturbed" version, i.e. all the frequencies ω_i's are identical, over an arbitrary connected graph. Furthermore, it can be assumed that the ω_i's are without loss of generality equal to zero. One, therefore, studies the system

$$\dot{\theta} = -\frac{K}{N} B \sin(B^T \theta). \tag{61}$$

If one considers the function

$$V_1(\theta) := \frac{4 \left| \sin(B^T\theta/2) \right|_2^2}{N^2},$$

then one has

$$\nabla V_1 = \frac{4}{N^2} B \sin\left(\frac{B^T\theta}{2}\right) \cos\left(\frac{B^T\theta}{2}\right) = \frac{2}{N^2} B \frac{\sin(B^T\theta)}{2} = -\frac{2}{NK}\dot{\theta},$$

using the identity $2\sin(x)\cos(x) = \sin(2x)$. One then obtains

$$\dot{V}_1(\theta) = (\nabla V(\theta)|\dot{\theta}) = -\frac{2}{NK}\dot{\theta}^T\dot{\theta} \leq 0,$$

and it follows that V_1 is a Lyapunov function for the system (61). With the help of the function V_1, Jadbabaie, Motee, and Barahona [24] show that all trajectories converge to the set of equilibria solutions, for any value of K, and that the synchronized state is locally asymptotically stable. They further point out, that one could use alternatively $V_2 := \frac{1}{2}\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}(\theta_i - \theta_j)^2 = \theta^T L_c \theta$, as a candidate for a Lyapunov function, where $L_c = NI - \mathbf{1}\mathbf{1}^T$ is the Laplacian of a complete graph (I being the identity matrix). Finally, it follows with Corollary 5.11, that the sublevel sets of V_1 and V_2 which are contained inside $|\theta_i| < \frac{\pi}{2}$, with $i = 1, \dots, e$, are positively invariant.

A notable aspect of Corollary 5.11, is its similarity with Theorem 3.11, which delivered a positively invariant set employing a function $\Phi(x) \in C^1(\mathbb{R}^n;\mathbb{R})$. In contrast, Corollary 5.11 gives a **whole family of positively invariant sets**. Furthermore, every solution starting in D will remain in D. It follows that D is **not only** weakly positively invariant, but also positively invariant.

6 Conclusion and Further Prospects

The main goal of this thesis was to introduce the reader to the concept of invariance in dynamical systems. To this end, the appropriate mathematical framework had to be established first, especially the theory of differential equations and dynamical systems. Several important notions and theorems were stated from the literature. Some criteria for showing or disproving the invariance of a given set $D \subset \mathbb{R}^n$ were given, based mainly on [48, Section 7], and then used in the final chapter for the construction of invariant sets. A system of non-linear ODEs possesses in its essence many (positively) invariant sets; the whole domain is especially always (a trivial) one. Therefore, it is of great interest to narrow the invariant sets down as much as possible. One tries to show more precisely where orbits (or solutions) of the system tend to when t approaches the upper bound of the maximal time interval.

Stability is a frequently studied property of invariant sets, especially from fixed points of a system. Therefore, the stability definitions were stated formally for equilibria, and two methods to determine their stability were introduced. It turned out that Lyapunov functions were beneficial for finding invariant sets. However, it can be seen from the definition that stability describes a local property. If one is interested in the global behaviour, an essential (and complicated) question is which orbits will be attracted by a stable equilibrium, and which will deviate from it. Unfortunately, results on this very matter are rarely treated in the literature: invariance and stability theory are almost always considered separately. Furthermore, research papers on this matter mostly address specific examples and do not seek overall results in the qualitative theory of non-linear differential equations.

Considering the results from this thesis as a starting point, a possibility for further research in this area could be seeking the existence and construction of Lyapunov functions, leading to geometrically more complicated structures of invariant sets. One can think of continuous piecewise-defined Lyapunov functions, for example. This is especially important in the case of structurally complex orbits, as was seen in the Lorenz equation.

A still open research item beyond the scope of a master's thesis is the evaluation of the number of limit cycles in a planar system. Furthermore, it should be investigated in-depth if the methods in this thesis for finding invariant sets, could be extended to partial differential equations, or ODEs defined on graph structures.

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Erklärung

Ich erkläre, dass ich die Masterarbeit selbstständig und ohne unzulässige Inanspruchnahme Dritter verfasst habe. Ich habe dabei nur die angegebenen Quellen und Hilfsmittel verwendet und die aus diesen wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht. Die Versicherung selbstständiger Arbeit gilt auch für enthaltene Zeichnungen, Skizzen oder graphische Darstellungen. Die Arbeit wurde bisher in gleicher oder ähnlicher Form weder derselben noch einer anderen Prüfungsbehörde vorgelegt und auch nicht veröffentlicht. Mit der Abgabe der elektronischen Fassung der endgültigen Version der Arbeit nehme ich zur Kenntnis, dass diese mit Hilfe eines Plagiatserkennungsdienstes auf enthaltene Plagiate geprüft werden kann und ausschließlich für Prüfungszwecke gespeichert wird.

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