Flows in Oriented Matroids

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Abstract

Recently Hochstättler and Nešetřil [3] introduced the flow lattice of an oriented matroid as generalization of the lattice of all integer flows of a digraph or more general a regular matroid. This lattice is defined as the integer hull of the characteristic vectors of signed circuits.

We describe the structure and the dimension of the flow lattice for uniform and rank 3 oriented matroids and construct a basis of signed circuits. Furthermore, we analyze the behaviour of the dimension of the flow lattice under the 2-sum of oriented matroids and present some quesions based on computational results on catalogs of oriented matroids.

Key words: oriented matroids, flow number, integer lattices

1 Overview

Let $\mathcal{O}$ be an oriented matroid on a ground set E with n elements and $\mathcal{C}$ its set of signed circuits. We identify a circuit $C \in \mathcal{C}$ with its signed characteristic vector $\chi_C \in \{0, \pm 1\}^n$. We use standard notation for oriented matroids as in

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Björner et al. [1], denote by

$$\mathcal{F}_{\mathcal{O}} := \left\{ \sum_{C \in \mathcal{C}} \lambda_C \chi_C \in \mathbb{Z}^E : \lambda_C \in \mathbb{Z} \right\}$$

the integer lattice of signed circuits of $\mathcal{O}$ and by $\Phi_{\mathcal{L}}(\mathcal{O})$ the *flow number* of $\mathcal{O}$, i. e. the smallest k so that there is an $x \in \mathcal{F}_{\mathcal{O}}$ satisfying $0 < |x_e| < k$ for all $e \in E$. $A(\mathcal{C})$ is the matrix containing the signed characteristic vectors of $\mathcal{C}$ as rows.

If $\mathcal{O}$ is a graphic oriented matroid of a digraph $D(V,E)$ the vectors in $\mathcal{F}(\mathcal{O})$ correspond to circular flows in D. The flow number $\varphi(D)$ of D is then equal to $\Phi_{\mathcal{L}}(\mathcal{O})$. Note that $\varphi(D)$ only depends on the underlying graph.

In this paper we will discuss some questions concerning the integer lattice of $\mathcal{C}$ (i. e. the *flow lattice* $\mathcal{F}_{\mathcal{O}}$):

(1) What is the dimension of $\mathcal{F}_{\mathcal{O}}$?

(2) Has $\mathcal{F}_{\mathcal{O}}$ a short characterization?

(3) Does $\mathcal{C}$ contain a basis of $\mathcal{F}_{\mathcal{O}}$?

(4) What is the smallest number k, so that there is a vector $x \in \mathcal{F}_{\mathcal{O}}$ satisfying $0 < |x_i| < k$ for all $i = 1, \ldots, n$?

For the signed circuits of a regular oriented matroid (and more particular of a digraph) the above questions have been studied very well in past. From graph theory it is known that the dimension of the circuit space of a connected digraph is $|E| - |V| + 1$ and more general $|E| - \operatorname{rank}(\mathcal{O})$ for a regular matroid. The elementary circuits $\{C(B,e)\}_{e \in E \setminus B}$ form a basis of $\mathcal{F}_{\mathcal{O}}$ for any basis B of $\mathcal{O}$. Concerning the last question it is known that $k \leq 6$ and conjectured that $k \leq 5$ for every bridge-less digraph (Tutte's 5-flow conjecture) but the computation of $\Phi_{\mathcal{L}}$ is an $\mathcal{NP}$-complete problem for regular oriented matroids, as the flow number of a cographic matroid is the chromatic number of the corresponding graph. For general oriented matroids Question 1 was stated in a slightly different context as research problem in Björner et al. [1, 4.45(d)]. The other problems seem not to have been considered in the literature yet.

We completely analyze the flow lattice for uniform oriented matroids and general oriented matroids of rank 3. It will turn out that there is a surprising gap in the dimension between regular and non-regular oriented matroids of the considered types, e. g. there is no simple and co-simple rank 3 oriented matroid $\mathcal{O}$ with $|E| - 3 < \dim \mathcal{F}(\mathcal{O}) < |E| - 1$ while the values $|E| - 1$ and $|E|$ are obtained.

In Sections 2 and 3 we analyze the flow lattice for uniform and rank 3 oriented matroids by answering the raised questions. In the last section we define the 2-sum of oriented matroids which, surprisingly, we could not find in the literature. Using this 2-sum we can construct for an arbitrary gap $g \in \mathbb{N}$ a connected oriented matroid that satisfies $\dim \mathcal{F}_{\mathcal{O}} = |E| - r + g$. Finally, we briefly report on and draw some conclusions from computational results on the flow lattices of catalogs of oriented matroids.

2 Uniform Oriented Matroids

In the recent work of Hochstättler and Nešetřil [3] which introduces the flow number of an oriented matroid the flow lattice turns out to be trivial (i. e. $\mathcal{F}(\mathcal{O}) = \mathbb{Z}^n$) for the case of even rank. It was also observed that the component sum for odd rank always must be even. Our analysis of the dimension in this case requires the consideration of balanced circuits (i. e. $|C^+| = |C^-|$). It turns out that $\mathcal{F}(\mathcal{O})$ does not have full dimension if and only if $\mathcal{O}$ is reorientation equivalent to a neighborly matroid polytope. We call such a uniform oriented matroid of odd rank *neighborly*. This leads to a complete characterization:

Theorem 1 *Let $\mathcal{O}$ be a uniform oriented matroid on n elements. Then*

$$\mathcal{F}(\mathcal{O}) = \begin{cases} \mathbb{Z}^n & \textit{if } r \textit{ is even} \\ \{v\}^\perp \cap \mathbb{Z}^n \textit{ for some } v \in \{1, -1\}^n & \textit{if } r \textit{ is odd and } \mathcal{O} \textit{ is neighborly} \\ \{x \in \mathbb{Z}^n : 2 | \mathbb{1}^T x\} & \textit{otherwise.} \end{cases}$$

This easily implies the result of [3]

$$\Phi_{\mathcal{L}}(\mathcal{O}) = \begin{cases} 2 & \text{if } nr \text{ is even} \\ 3 & \text{if } nr \text{ is odd.} \end{cases}$$

Theorem 2 *If $\mathcal{C}$ is the set of signed circuits of a uniform oriented matroid $\mathcal{O}$ then $\mathcal{C}$ contains a basis of $\mathcal{F}_{\mathcal{O}}$.*

We construct a basis of circuits as follows: We start with a basis of an orientation of $U_{r,r+2}$ which is a restriction of any uniform oriented matroid having more than $r + 2$ elements. This restriction $\mathcal{O}_{r+2}$ is neighborly and unique up to reorientation. If $\mathcal{O}$ has odd rank and is neighborly we can extend $\mathcal{O}_{r+2}$ by successively adding elements and the basis of $\mathcal{O}_{r+2}$ is extended by adding corresponding new circuits. We proceed similarly if $\mathcal{O}$ has even rank. If $\mathcal{O}$ has odd rank and is not neighborly we can show that $\mathcal{O}$ has a restriction minor

$\mathcal{O}_{r+3}$ with $r+3$ elements that is not neighborly. We extend a basis of $\mathcal{O}_{r+2}$ by two new circuits so that the dimension increases by two to a basis of $\mathcal{F}(\mathcal{O}_{r+3})$. We have to take some care selecting these two circuits. In order to find them it turns out to be helpful to consider the dual $\mathcal{O}^*_{r+3}$ which is of rank 3 and therefore representable as a pseudo-line arrangement. The basis of $\mathcal{F}(\mathcal{O}_{r+3})$ now can be extended to a basis of $\mathcal{O}$ as in the other cases.

3 Non-uniform Oriented Matroids with Rank 3

While in the last section $\mathcal{O}$ was assumed to be uniform we now turn to general oriented matroids but limit the rank to be 3. Note that this case contains regular and uniform oriented matroids as well. Precisely, we will show that $\mathcal{F}_{\mathcal{O}}$ is trivial (up to co-loops and co-parallels) whenever $\mathcal{O}$ has at least 6 elements and is neither $\mathcal{M}(K_4)$ nor uniform. Consequently, any non-regular simple and co-simple extension of a regular rank 3 oriented matroid increases the dimension of the lattice by 4 (instead of at most two in the uniform case).

Theorem 3 *Let $\mathcal{O}$ be a simple co-simple non-uniform oriented matroid of rank 3 on a ground set E with $n \geq 6$ elements. Then $\mathcal{F}_{\mathcal{O}} = \mathbb{Z}^n$ if and only if $\underline{\mathcal{O}} \ncong \mathcal{M}(K_4)$.*

To prove this theorem we consider the single element extensions of an oriented matroid that is contained in almost any non-uniform oriented matroid of rank 3 as a deletion minor (a co-extension of the 4-point line) and show that all of its co-simple extensions have trivial flow lattice.

Corollary 4 *Let $\mathcal{O}$ be a non-regular non-uniform simple and co-simple oriented matroid of rank 3. Then $\Phi_{\mathcal{L}}(\mathcal{O}) = 2$.*

The construction of a basis is done by starting from a basis of P_6, R_6, Q_6 resp. $\mathcal{W}^3$ and extending the basis by a circuit containing only some of the current elements plus one new element.

4 General Oriented Matroids

It suffices to study 3-connected oriented matroids which is a consequence of an analogue to a decomposition theorem from matroid theory. We introduce the 2-sum of oriented matroids $\mathcal{O}_1(E_1, \mathcal{C}_1)$ and $\mathcal{O}_2(E_2, \mathcal{C}_2)$ $(E_1 \cap E_2 = \{f\})$ as $\mathcal{O}((E_1 \cup E_2) \setminus f, \mathcal{C}_{\oplus_2})$ with the circuit set

$$\mathcal{C}_{\oplus_2} = \mathcal{C}_1^{f^0} \cup \mathcal{C}_2^{f^0} \cup \{(C_1 \cup C_2) \setminus f : C_i(f) = -C_2(f) \neq 0\}.$$

$\mathcal{C}_{\oplus_2}$ can be shown to satisfy the circuit axioms of oriented matroid theory. By the definition of this 2-sum and the direct sum we derive

Lemma 5 *Every oriented matroid $\mathcal{O}$ can be decomposed into direct sums and 2-sums of 3-connected oriented matroids.*

Aiming to determine the dimension of $\mathcal{F}_{\mathcal{O}}$ for general oriented matroids we analyze the flow lattice dimension of 1- and 2-sums. While the dimension of $\mathcal{F}_{\mathcal{O}_1 \oplus \mathcal{O}_2}$ does not depend on the structure of $\mathcal{O}_1$ and $\mathcal{O}_2$ (i. e. $\dim \mathcal{F}_{\mathcal{O}_1 \oplus \mathcal{O}_2} = \dim \mathcal{F}_{\mathcal{O}_1} + \dim \mathcal{F}_{\mathcal{O}_2}$) there are two possibilities for $\dim \mathcal{F}_{\mathcal{O}_1 \oplus_2 \mathcal{O}_2}$:

Lemma 6 *If i (resp. j) is the column index in $A(\mathcal{C}_1)$ (resp. $A(\mathcal{C}_2)$) corresponding to f, then*

$$\dim \mathcal{F}_{\mathcal{O}_1 \oplus_2 \mathcal{O}_2} = \dim \mathcal{F}_{\mathcal{O}_1} + \dim \mathcal{F}_{\mathcal{O}_2} - \begin{cases} 2 & \text{for } e_i \in \operatorname{lin}(\mathcal{C}_1) \text{ and } e_j \in \operatorname{lin}(\mathcal{C}_2) \\ 1 & \text{otherwise.} \end{cases}$$

As a consequence we get for the flow number of the 2-sum

Corollary 7 $\Phi_{\mathcal{L}}(\mathcal{O}_1 \oplus_2 \mathcal{O}_2) \leq \max\{\Phi_{\mathcal{L}}(\mathcal{O}_1), \Phi_{\mathcal{L}}(\mathcal{O}_2)\}$.

To get an idea of the structure of the flow lattice of general oriented matroids we determined $\mathcal{F}_{\mathcal{O}}$ for small oriented matroids from Finschi [2] and Oxley [4]. Based on these results we raise the following questions:

(1) Is $\dim \mathcal{F}_{\mathcal{O}} \in \{n-1, n\}$ for any simple co-simple non-regular 3-connected oriented matroid?
(2) Can $\mathcal{F}_{\mathcal{O}}$ be characterized (similar to the uniform case) by either an orthogonality condition, an integral linear modular equation, or both? The most complicated lattice we encountered was that of an orientation of the dual of the Pappus matroid with the flow lattice

$$\mathcal{F}_{\mathcal{O}} = \left\{ x \in \mathbb{Z}^n : 2 | (1,1,1,2,2,2,1,2,1,1)^T x \text{ and } \sum_{i=3}^{8} x_i = 0 \right\}.$$

(3) Does $\mathcal{C}$ always contain a basis of $\mathcal{F}_{\mathcal{O}}$?
(4) Is $\mathcal{F}_{\mathcal{O}}$ trivial if $\mathcal{O}$ is a single element extension of a maximum regular oriented matroid?

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