Flows in Oriented Matroids

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Outline

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Oriented Matroids.
Some Terminology

- Let $E = \{1, \ldots, n\}$, the ground set.
- A signed subset of $C \subseteq E$ is a partition $C = (C^+, C^-)$.
- A family $\mathcal{C}$ of signed subsets is the set of signed circuits of an oriented matroid $\mathcal{O}$ if it satisfies the circuit axioms.
- By forgetting the signs we get the underlying matroid.

Circuit Axioms

(C1) $\mathcal{C} = -\mathcal{C}$

(C2) $C_1 \subseteq C_2 \Rightarrow C_1 = \pm C_2$

(C3) $e \in C_1{}^+ \cap C_2{}^- \Rightarrow \exists Z \in \mathcal{C}:$
$Z^+ \subseteq (C_1{}^+ \cup C_2{}^+) \setminus e$
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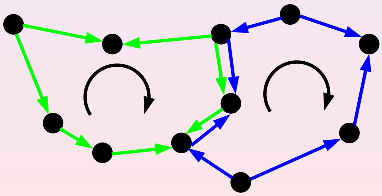

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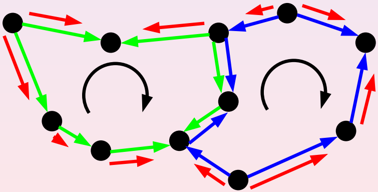

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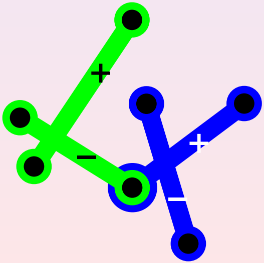

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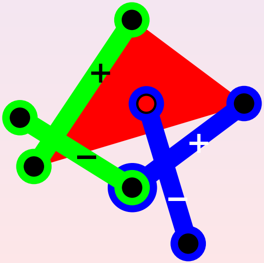

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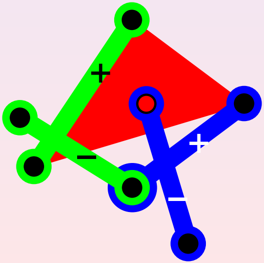

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- Reorienting $e \in E$ means changing the sign of e in every circuit
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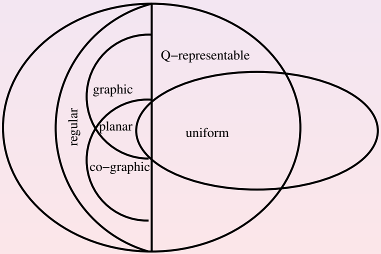

Outline

The Flow Lattice of an Oriented Matroid.

Circuits and Flows:

- A signed circuit $C = (C^+, C^-) \in \mathcal{C}$ yields characteristic vector (e. g. $\chi_C = (+1, -1, 0, 0, -1, -1, +1)$)
- We define the Flow Lattice of $\mathcal{O}$ as

$$\mathcal{F}_{\mathcal{O}} := \{x = \sum_{C \in \mathcal{C}} \lambda_C \chi_C \mid \lambda_c \in \mathbb{Z}\}$$

- and the Flow Number as

$$\phi(\mathcal{O}) := \min_{x \in \mathcal{F}_{\mathcal{O}}} \{k : 0 < |x_i| < k\}$$

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Why Does It Make Sense?

Flows in Digraphs

- If $\mathcal{O}$ is graphic, $\mathcal{F}_{\mathcal{O}}$ consists of all circular flows
- $\phi(\mathcal{O})$ is the known flow number of the corresponding digraph G
- If $\mathcal{O}$ is co-graphic, $\phi(\mathcal{O}^*)$ is the chromatic number of $G(\mathcal{O}^*)$

Flow Lattice Structure of a Digraph

- $\dim \mathcal{F}_G = |E| - |V| + \mathrm{comp}(G)$
- Characterization of $\mathcal{F}_G$ uses vertices (Kirchhoff's law)
- Determination of $\phi(G)$ is $\mathcal{NP}$-hard.
- $\phi(G)$ does not depend on the orientation (is graph invariant)
- Elementary circuits of a spanning tree form a basis of $\mathcal{F}_G$

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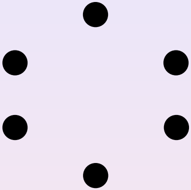
What We Want to Know.

Questions:

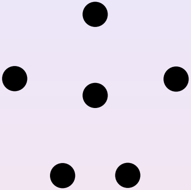
- Determine $\dim \mathcal{F}_{\mathcal{O}}$
- Is there a simple characterization of $\mathcal{F}_{\mathcal{O}}$?
- Can $\phi(\mathcal{O})$ be determined for other classes?
- Is $\phi(\mathcal{O})$ a matroid invariant?
- Is there a basis of $\mathcal{F}_{\mathcal{O}}$ containing circuits only?

Outline

Some Examples.



$\dim \mathcal{F}_{\mathcal{O}} = n - 1$



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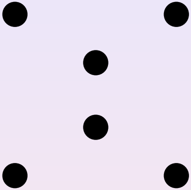

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- The first example is a neighborly polytope
  In $\mathbb{R}^d$: Every set of at most $d/2$ vertices forms a facet.
- The third example is a reorientation of a the 6-gon
- Only neighborly matroid polytopes have balanced circuits only

The Structure of $\mathcal{F}_{\mathcal{O}}$ for Uniform Oriented Matroids.

Let $\mathcal{O}$ be a uniform rank r oriented matroid on a ground set $E = \{1, \ldots, n\}$.

Theorem

$\mathcal{O}$ has a reorientation such that

$$\mathcal{F}_{\mathcal{O}} = \begin{cases} \mathbb{Z}^n & \text{if } r \text{ is even} \\ \{\mathbf{1}\}^{\perp} \cap \mathbb{Z}^n & \text{if } r \text{ is odd and } \mathcal{O} \text{ is neighborly} \\ \{x^T\mathbf{1} \text{ is even}\} & \text{otherwise.} \end{cases}$$

Answers for Uniform Oriented Matroids.

Our Results:

- The co-dimension is either 0 or 1
- $\mathcal{F}_{\mathcal{O}}$ is trivial or can be characterized by an orthogonality condition or a modular equation
- $\phi(\mathcal{O})$ is either 2 or 3 but matroid invariant
- $\dim \mathcal{F}_{\mathcal{O}}$ is not matroid invariant
- A basis $B \subset \mathcal{C}$ of $\mathcal{F}_{\mathcal{O}}$ can be constructed

Outline

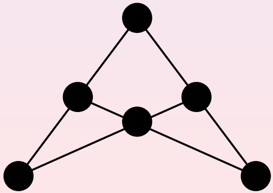
The Flow Lattice of Rank 3 Oriented Matroids.

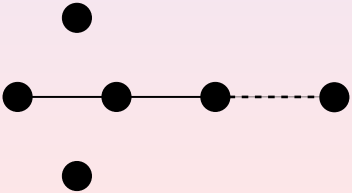
...is trivial

- Consists of regular and uniform OMs as well
- $\mathcal{O}(K_4)$ is maximum regular (has 6 elements)

Theorem

Let $\mathcal{O}$ a simple and co-simple non-uniform rank 3 oriented matroid over $E = \{1, \dots, n\}$ with $n > 6$. Then $\mathcal{F}_{\mathcal{O}} = \mathbb{Z}^n$.





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Proof.

- $\mathcal{O}_5$ is contained in any non-regular, non-uniform rank 3 oriented matroid with more than 5 elements.
- Any co-simple extension of $\mathcal{O}_5$ yields a trivial flow lattice

$\square$

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+ - + 0 0
+ + 0 - -
+ 0 + - -
0 + - - -
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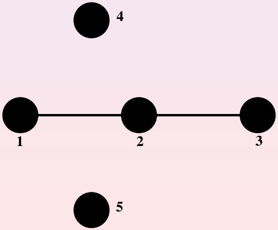

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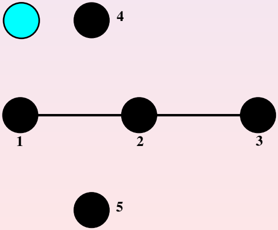

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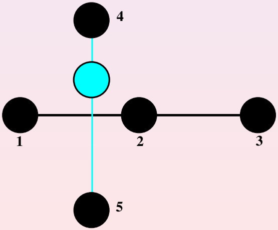

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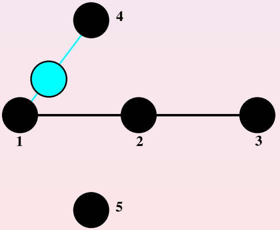

Answers for Rank 3 Oriented Matroids.

Our Results:

- The co-dimension of $\mathcal{F}_{\mathcal{O}}$ is 0
- $\mathcal{F}_{\mathcal{O}}$ is trivial whenever $\mathcal{O}$ is simple, co-simple, non-regular and non-uniform with more than 4 elements
- $\phi(\mathcal{O}) = 2$ and therefore matroid invariant
- A basis $B \subset \mathcal{C}$ of $\mathcal{F}_{\mathcal{O}}$ can be constructed

Outline

The General Case. Direct Sum and 2-Sum.

- Any co-dimension of $\mathcal{F}_{\mathcal{O}}$ can be constructed.
- Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be circuits of two oriented matroids $\mathcal{O}_1$ and $\mathcal{O}_2$ on $E_1 \cap E_2 = \emptyset$.

Direct Sum

Let $\mathcal{C}_\oplus := \mathcal{C}_1 \cup \mathcal{C}_2$. Then $\mathcal{C}_\oplus$ is the set of signed circuits of an oriented matroid $\mathcal{O}_\oplus$.

- $\dim \mathcal{F}_{\mathcal{O}} = \dim \mathcal{F}_{\mathcal{O}_1} + \dim \mathcal{F}_{\mathcal{O}_2}$
- Problem: The resulting oriented matroid is not connected
- Define a 2-sum similar to the 2-sum of graphs

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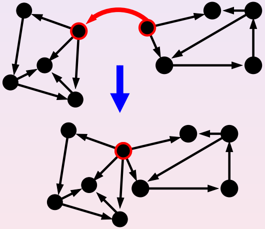

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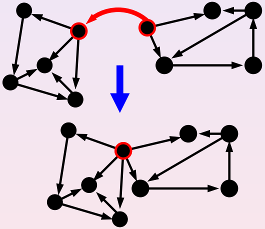

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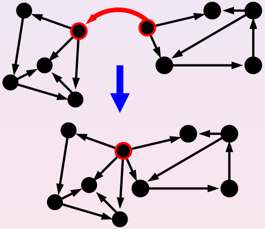
The General Case. Direct Sum and 2-Sum.

- Any co-dimension of $\mathcal{F}_{\mathcal{O}}$ can be constructed.
- Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be circuits of two oriented matroids $\mathcal{O}_1$ and $\mathcal{O}_2$ on $E_1 \cap E_2 = \emptyset$.

Direct Sum

Let $\mathcal{C}_{\oplus} := \mathcal{C}_1 \cup \mathcal{C}_2$. Then $\mathcal{C}_{\oplus}$ is the set of signed circuits of an oriented matroid $\mathcal{O}_{\oplus}$.



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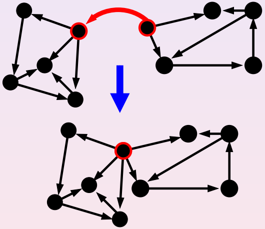

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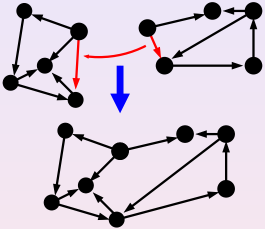

A 2-Sum

Let $\mathcal{C}_{\oplus_2} := \mathcal{C}(\mathcal{O}_1 \setminus f) \cup \mathcal{C}(\mathcal{O}_2 \setminus f)$ $\cup\{(C_1 \cup C_2) \setminus f : C_i \in \mathcal{C}_i, f \in C_1^+ \cap C_2^-\}$. Then $\mathcal{C}_{\oplus_2}$ is the set of signed circuits of an oriented matroid $\mathcal{O}_{\oplus_2}$.

Theorem

$$\dim \mathcal{F}_{\mathcal{O}_{\oplus_2}} = \dim \mathcal{F}_{\mathcal{O}_1} + \dim \mathcal{F}_{\mathcal{O}_2} - \begin{cases} 2 \textit{ if } e_f \in \operatorname{lin} \mathcal{C}_1 \textit{ and } e_f \in \operatorname{lin} \mathcal{C}_2 \\ 1 \textit{ otherwise} \end{cases}$$

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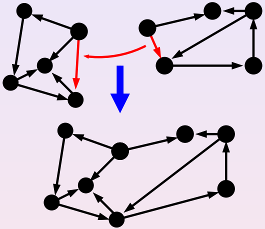

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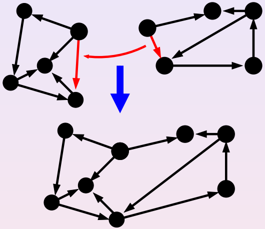

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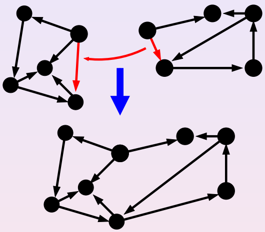

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Outline

A Decomposition Theorem for Oriented Matroids.

Theorem

Let $\mathcal{O}$ be an oriented matroid. Then $\mathcal{O}$ can be decomposed into direct sums and 2-sums of 3-connected oriented matroids.

- It suffices to consider
  - 3-connected
  - simple
  - co-simple
  - non-regular
  - non-uniform

  oriented matroids

Numerical Results.

Questions:

Under the above assumptions:

1. Is the co-dimension always 0 or 1 ?
2. Is $\mathcal{F}_{\mathcal{O}}$ always trivial or has a characterization by an orthogonality condition or an integral modular equation (mod 2 ?) ?
3. Is $\phi(\mathcal{O}) \leq 3$?
4. Is $\phi(\mathcal{O})$ matroid invariant?

Numerical Results.

Our Test Sets:

We evaluated $\mathcal{F}_{\mathcal{O}}$ for the following test sets of oriented matroids:

- The entire catalogue of small OMs from Lukas Finschi `http://www.om.math.ethz.ch`
- All orientations of the examples of James G. Oxley ("Matroid Theory")
- All projective incidence structures from Jürgen Richter-Gebert's Darmstadt-dissertation (On the Realizability Problem of Combinatorial Geometries)

Some Answers and Corrections.

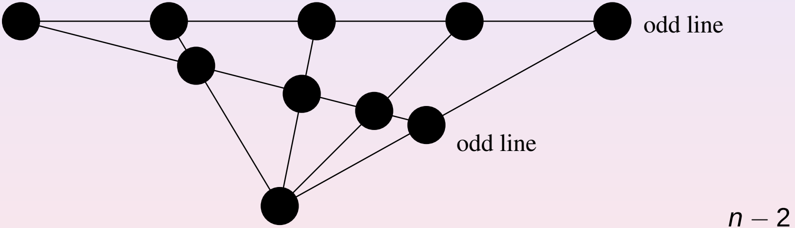
Question

Is the co-dimension always 0 or 1 ?

Some Answers and Corrections.

Question

Is the co-dimension always 0 or 1 ? NO



Correction

Can the co-dimension of a 3-connected non-regular OM become arbitrary large?

Some Answers and Corrections.

Question

Is $\mathcal{F}_{\mathcal{O}}$ always trivial or has a characterization by an orthogonality condition or an integral modular equation (mod 2 ?) ?

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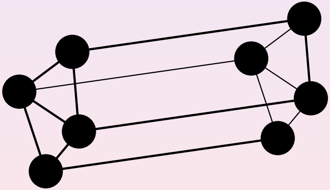
Is $\mathcal{F}_{\mathcal{O}}$ always trivial or has a characterization by an orthogonality condition or an integral modular equation (~~mod 2 ?~~) ?

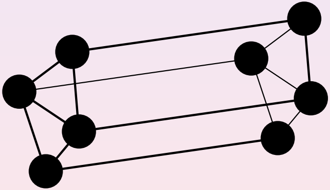


$$\mathcal{F}_{\mathcal{O}_{Vamos}} = \{3 | (2,2,2,2,1,1,2,1)^T x\}$$

Some Answers and Corrections.

Question

Is $\mathcal{F}_{\mathcal{O}}$ always trivial or has a characterization by an orthogonality condition or an integral modular equation (~~mod 2 ?~~) OR BOTH?



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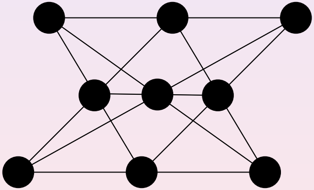

$$\mathcal{F}_{\mathcal{O}_{Pappus^*}} = \left\{ \begin{array}{l} 2 | (1,1,1,0,0,0,1,0,1,1)^T x \\ (0,0,1,1,1,1,1,1,0)^T x = 0 \end{array} \right\}$$

Summary

The current state:

- The flow lattice and the flow number are known for uniform and rank 3 OMs
- The co-dimension of $\mathcal{F}_{\mathcal{O}}$ is "actually" 0 or 1 (the single counterexample could not be generalized)
- $\phi(\mathcal{O}) \leq 3$ and remains matroid invariant for the considered classes
- $\mathcal{F}_{\mathcal{O}}$ has a simple characterization