Regularity of the Circuit Lattice of Oriented Matroids
- extended abstract -

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1 Overview

Let $\mathcal{O}$ be an oriented matroid on a ground set E with n elements and $\mathcal{C}$ its set of signed circuits. We identify a circuit $C \in \mathcal{C}$ with its signed characteristic vector $\chi_C \in \{0, \pm 1\}^n$. In this paper we will discuss some questions concerning the integer lattice of $\mathcal{C}$ denoted by $\mathcal{L}(\mathcal{C})$ which are quite natural in the context of integer lattices:

1. What is the dimension of $\mathcal{L}(\mathcal{C})$?
2. Is there a discrepancy between $\mathcal{L}(\mathcal{C})$ and the integral points of the linear hull of $\mathcal{C}$, i. e. is $\mathcal{L}(\mathcal{C})$ regular and therefore $\mathcal{L}(\mathcal{C}) = \operatorname{lin} \mathcal{C} \cap \mathbb{Z}^n$?
3. Has $\mathcal{L}(\mathcal{C})$ a short characterization?
4. Does $\mathcal{C}$ contain a basis of $\mathcal{L}(\mathcal{C})$?

For the signed circuits of a regular oriented matroid (and more particular of a digraph) the above questions have been studied very well in past. From graph theory it is known that the dimension of the circuit space of a connected digraph is $|E| - |V| + 1$, that the circuit space $\mathcal{L}(\mathcal{C})$ is regular and that the elementary circuits $\{C(B, e)\}_{e \in E \setminus B}$ form a basis of $\mathcal{L}(\mathcal{C})$ for any basis B of $\mathcal{O}$. For general oriented matroids question 1 is a known problem (see Björner et al. [1, 4.45(d)]) but the other problems have not been considered in the literature yet. Furthermore, we extend the set of questions by a fifth, coming from the theory of nowhere-zero flows in a digraph:

5. What is the smallest number k, so that there is a vector $x \in \mathcal{L}(\mathcal{C})$ satisfying $0 < |x_i| < k$ for all $i = 1, \ldots, n$?

We completely analyze the circuit lattice for uniform oriented matroids and general oriented matroids of rank 3. It will turn out that there is a gap in the dimension between regular and non-regular oriented matroids of the considered types, e. g. there is no simple and co-simple rank 3 oriented matroid $\mathcal{O}$ with $|E| - 3 < \dim \mathcal{F}(\mathcal{O}) < |E| - 1$ while the values $|E| - 1$ and $|E|$ are obtained. Furthermore, there is only one case beyond the considered types where a circuit lattice is not regular (which exactly holds for non-neighborly uniform oriented matroids of odd rank).

Table 1 shows the results presented in this paper. The case of a non-regular circuit lattice is marked with a box.

Table 1. Results of the paper concerning the circuit lattice of a connected simple and co-simple oriented matroid $\mathcal{O}$ with more than $r + 3$ elements

	non-regular				regular
	non-uniform	uniform			
	rank 3	r even	r odd		
$\dim \mathcal{F}_\mathcal{O}$	$\lvert E\rvert$	$\lvert E\rvert$	$\lvert E\rvert - 1$	$\lvert E\rvert$	$\lvert E\rvert - r$
$\mathcal{F}_\mathcal{O}$	$\mathbb{Z}^{\lvert E\rvert}$	$\mathbb{Z}^{\lvert E\rvert}$	$\{v\}^\perp \cap \mathbb{Z}^{\lvert E\rvert}$ $v \in \{1, -1\}^{\lvert E\rvert}$	$\boxed{\{x^T \mathbb{1} \text{ even}\}}$	regular
$\Phi_\mathcal{L}$	2	2	2 if $\lvert E\rvert$ even 3 if $\lvert E\rvert$ odd	2 if $\lvert E\rvert$ even 3 if $\lvert E\rvert$ odd	$\chi(\mathcal{O}^*)$

Section 2 provides the necessary terminology and Section 3 summarizes the current situation for regular matroids. In Sections 4 and 5 we analyze the circuit lattice for uniform and rank 3 oriented matroids by answering the raised questions. The last section concludes with possible future research on this topic.

2 Notation

Regular Lattices

Let $\mathcal{C} \subset \mathbb{Z}^n$ be a finite set of integral vectors. We define by

$$\mathcal{L}(\mathcal{C}) := \{\sum_{x \in \mathcal{C}} \lambda_x \cdot x : \lambda_x \in \mathbb{Z}\}$$

the *integer lattice* of $\mathcal{C}$. The *dimension* of $\mathcal{L}(\mathcal{C})$ is the rank of the matrix containing the elements of $\mathcal{C}$ as rows and is denoted by $\dim \mathcal{L}(\mathcal{C})$. A lattice $\mathcal{L}(\mathcal{C})$ is called *regular* if for each $x \in \mathcal{L}(\mathcal{C})$ there are $x_1, \ldots, x_m \in \{0, \pm 1\}^n$ such that $\operatorname{supp}(x_i) \subseteq \operatorname{supp}(x)$ and $x = \sum_i x_i$. Vectors of minimal support in $\mathcal{L}(\mathcal{C})$ are called *elementary* and vectors in $\mathcal{L}(\mathcal{C}) \cap \{0, \pm 1\}^n$ are called *primitive.*

Oriented Matroids

We use standard notation for oriented matroids as in Björner et al. [1]. Let $\mathcal{C}$ be the set of signed characteristic vectors of circuits of an oriented matroid $\mathcal{O}$. We denote by $\mathcal{F}(\mathcal{O}) := \mathcal{L}(\mathcal{C})$ the *circuit lattice* of $\mathcal{O}$. An oriented matroid is called *regular* if it can be represented by a totally unimodular matrix. It can be checked easily that the primitive vectors of a lattice $\mathcal{L}$ satisfy the circuit axioms of an oriented matroid which we will denote by $\mathcal{O}(\mathcal{L})$ (see Tutte [7]).

Proposition 1. *The following conditions are known to be equivalent:*

· $\mathcal{L}$ *is regular.*
· $\mathcal{O}(\mathcal{L})$ *is regular.*
· $\mathcal{L} = \mathcal{F}(\mathcal{O}(\mathcal{L}))$.
· $\mathcal{L} = \operatorname{lin} \mathcal{L} \cap \mathbb{Z}^n$.

Flows

If $\mathcal{O}$ is a graphic oriented matroid of a digraph $D(V, E)$ the vectors in $\mathcal{F}(\mathcal{O})$ correspond to circular flows in D. The flow number $\varphi(D)$ of D is defined as the smallest k such that there exists a nowhere-zero k-flow (note that $\varphi(D)$ only depends on the underlying graph). For a survey on nowhere-zero flows in a graph see Seymour [6].

This approach to define a flow number of a digraph recently has been generalized to oriented matroids by Hochstättler and Nešetřil [3]. They defined the *flow number* $\Phi_{\mathcal{L}}(\mathcal{O})$ of an oriented matroid $\mathcal{O}$ as the minimum k such that there is a vector $x \in \mathcal{F}(\mathcal{O})$ satisfying $0 < |x_e| < k$ for all $e \in E$. Although the computation of $\varphi(D)$ is known to be an $\mathcal{NP}$-hard problem the determination of the generalized flow number turns out to be trivial for uniform oriented matroids and matroids of rank 3.

3 Regular Oriented Matroids

Regular oriented matroids have been studied a lot in the literature so that the dimension of the circuit lattice is well-known:

$$\mathcal{O} \text{ regular} \iff \dim \mathcal{F}(\mathcal{O}) = n - r.$$

Concerning the structure of the lattice it is known that $\mathcal{F}(\mathcal{O})$ is regular. A basis of signed circuits can be constructed easily as the set of elementary circuits $\{C(B, e) \mid e \in E\}$ of a basis B of $\mathcal{O}$. The problem of computing the flow number $\Phi_{\mathcal{L}}(\mathcal{O})$ is known to be $\mathcal{NP}$-hard.

4 Uniform Oriented Matroids

In the recent work of Hochstättler and Nešetřil [3] which introduces the flow number of an oriented matroid the flow lattice turns out to be trivial (i. e. $\mathcal{F}(\mathcal{O}) = \mathbb{Z}^n$) for the case of even rank. The case of odd rank requires the consideration of balanced circuits (i. e. $|C^+| = |C^-|$) and leads to an exception concerning neighborly matroid polytopes. For this we call a uniform oriented matroid of odd rank *neighborly* if it is reorientation equivalent to a neighborly matroid polytope.

Theorem 1. *Let $\mathcal{O}$ be a uniform oriented matroid with $n \geq r+2$ elements. Then $\dim \mathcal{F}(\mathcal{O}) < n$ if and only if there is a reorientation ${}_I\mathcal{O}$ of $\mathcal{O}$ such that all circuits of ${}_I\mathcal{O}$ are balanced. In particular we then have $\mathcal{F}({}_I\mathcal{O}) = \{\mathbb{1}\}^T \cap \mathbb{Z}^n$ and $\dim \mathcal{F}(\mathcal{O}) = n-1$.*

The proof of this theorem proceeds by induction on the number of elements of $\mathcal{O}$ starting from the unique uniform oriented matroid with $r+2$ elements.

We remark that the class of uniform oriented matroids with circuit lattice dimension of $n-1$ is exactly the class of neighborly oriented matroids of odd rank.

The above statements characterize the circuit lattice of uniform oriented matroids almost completely:

Theorem 2. *Let $\mathcal{O}$ be a uniform oriented matroid on n elements. Then*

$$\mathcal{F}(\mathcal{O}) = \begin{cases} \mathbb{Z}^n & \text{if } r \text{ is even} \\ \{v\}^\perp \cap \mathbb{Z}^n \text{ for some } v \in \{1,-1\}^n & \text{if } r \text{ is odd and } \dim \mathcal{F}_{\mathcal{O}} = n-1 \\ \{x \in \mathbb{Z}^n : x^T\mathbb{1} \text{ is even}\} & \text{if } r \text{ is odd and } \dim \mathcal{F}_{\mathcal{O}} = n. \end{cases}$$

In particular, the circuit lattice of $\mathcal{O}$ is not regular if and only if $\mathcal{O}$ has odd rank and is not neighborly.

For the case of even rank and full dimension we investigate that if $\mathcal{O}$ is not neighborly then there is an element $i \in E$ such that $\mathcal{O} \setminus i$ is not neighborly or $\mathcal{O}$ has $r+3$ elements. Therefore, we can proceed by induction and show that

$$\begin{aligned} \mathcal{F}(\mathcal{O}) &= \mathcal{L}(\{e_i \pm e_j \mid i \neq j \in E\}) \\ &= \{x \in \mathbb{Z}^n : x^T\mathbb{1} \text{ is even}\}. \end{aligned}$$

From the structure we easily conclude with the known result ([3]) that

$$\Phi_{\mathcal{L}}(\mathcal{O}) = \begin{cases} 2 & \text{if } nr \text{ is even} \\ 3 & \text{if } nr \text{ is odd} \end{cases}$$

The construction of a basis proceeds as follows: We start with a basis of an orientation of $U_{r,r+2}$ which is a restriction of any uniform oriented matroid having more than $r+2$ elements. This restriction $\mathcal{O}_{r+2}$ is neighborly and unique up to reorientation. If $\mathcal{O}$ has odd rank and is neighborly we can extend $\mathcal{O}_{r+2}$ by successively adding elements and the basis of $\mathcal{O}_{r+2}$ is extended by adding corresponding new circuits. We proceed similarly if $\mathcal{O}$ has even rank. If $\mathcal{O}$ has odd rank and is not neighborly we extend a basis of $\mathcal{O}_{r+2}$ by two new circuits so that the dimension increases by 2. These two circuits can be selected by considering the dual of the extension which has rank 3 and is therefore representable as a pseudo-line arrangement. This extension of $\mathcal{O}_{r+2}$ now has full flow lattice dimension and a basis can be extended to a basis of $\mathcal{O}$ as in the other cases.

For further details on the flow lattice of uniform oriented matroids see [4].

5 Non-uniform Oriented Matroids with Rank 3

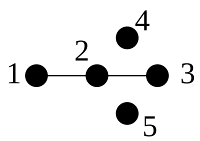
While in the last section $\mathcal{O}$ was assumed to have a special structure we now turn to general oriented matroids but limit the rank to be 3. Note that this case contains regular and uniform oriented matroids as well. Precisely, we will show among other things that any rank-preserving non-regular simple and co-simple extension of a regular oriented matroid increases the dimension of the lattice by 4 (instead of at most two in the uniform case).

While in the uniform case we did not have to deal with loops, co-loops, parallels or co-parallels we now have to consider these cases separately.

Proposition 2. *Let $\mathcal{O}$ be an oriented matroid on the ground set E.*

- $\dim \mathcal{F}(\mathcal{O}) = \dim \mathcal{F}(\mathcal{O} \setminus e) + 1$ *for a loop* $e \in E$
- $\dim \mathcal{F}(\mathcal{O}) = \dim \mathcal{F}(\mathcal{O} \setminus e) + 1$ *if* e *is parallel to some* $f \in E$
- $\dim \mathcal{F}(\mathcal{O}) = \dim \mathcal{F}(\mathcal{O}/e)$ *for a co-loop* $e \in E$
- $\dim \mathcal{F}(\mathcal{O}) = \dim \mathcal{F}(\mathcal{O}/e)$ *if* e *is co-parallel to some* $f \in E$

The following oriented matroid (which we call $\mathcal{O}_5$) is contained in almost any non-regular and non-uniform oriented matroid of rank 3:



+−+00
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$\mathcal{O}_5$, being a coextension of the 4-point line by a coparallel, has a circuit lattice dimension of 4 and

$$\mathcal{F}(\mathcal{O}_5) = \mathcal{L}(\{e_1, e_2, e_3, e_4 - e_5\}).$$

At first, we point out that the flow lattice is trivial for any $U_{2,4}$-free restriction of $\mathcal{O}_5$.

Lemma 1. *Any connected single element extension of $\mathcal{O}_5$ which does not contain a 4-point line has trivial flow lattice.*

Proof. Consider a single element extension $\mathcal{O}_6$ of $\mathcal{O}_5$ which does not contain a 4-point line. Hence, 4 and 5 are not coparallel. We show that the circuits of $\mathcal{O}_6$ generate $\{e_i\}_{i=1,\dots,6}$. Obviously this holds for $e_1, \dots, e_3$. Therefore, we can skip the first three coordinates from our considerations. Choose a reorientation so that

$$\begin{array}{cccccc} * & * & * & + & + & 0 \\ * & * & * & \alpha & 0 & + \\ * & * & * & \beta & \gamma & + \end{array}$$

are circuits of $\mathcal{O}_6$ where $\alpha, \beta, \gamma \in \{+, -\}$. Note that $\mathcal{O}_6$ must contain the third circuit because 4 and 5 do not form a three-element basis with one of the first three elements. For any choice of α, β and γ which does not violate the circuit axioms the rows of

$$\begin{pmatrix} + & + & 0 \\ \alpha & 0 & + \\ \beta & \gamma & + \end{pmatrix}$$

span $\mathbb{Z}^3$. Consequently, any one-element extension of $\mathcal{O}_5$ has trivial circuit lattice. On the other hand, any orientation of P_6, R_6, Q_6 or $\mathcal{W}^3$ contains $\mathcal{O}_5$ as a minor. $\square$

By considering the enumeration of oriented matroids of Finschi and Fukuda [2] we remark that any 4-line-free connected single element extension of $\mathcal{O}_5$ is an orientation of one of P_6, R_6, Q_6 or $\mathcal{W}^3$ and on the other hand, any orientation of P_6, R_6, Q_6 or $\mathcal{W}^3$ contains a reorientation of $\mathcal{O}_5$ as a minor (see Oxley [5]).

Lemma 2. *Let $\mathcal{O}$ be a non-uniform oriented matroid on more than 5 elements which does not contain the deletion minors P_6, R_6, Q_6 or $\mathcal{W}^3$. Then $\mathcal{O}$ must be either an orientation of $\mathcal{M}(K_4)$ or contains an $(n-2)$-point line.*

Proof. Assume that $U_{2,n-2}$ is not a deletion minor and $\underline{\mathcal{O}} \ncong \mathcal{M}(K_4)$. Then we can find a deletion minor $\mathcal{O}_6$ of $\mathcal{O}$ with 6 elements which contains a 3-point line but not a 4-point line and is not isomorphic to $\mathcal{M}(K_4)$ (Note that only the non-orientable Fano plane has no other minor than $\mathcal{M}(K_4)$). This matroid must be an orientation of P_6, R_6, Q_6 or $\mathcal{W}^3$. $\square$

Theorem 3. *Let $\mathcal{O}$ be a simple non-uniform oriented matroid of rank 3 on a ground set E with $n > 6$ elements. Then $\mathcal{F}_{\mathcal{O}} = \mathbb{Z}^n$ if and only if there is a deletion minor $\mathcal{O} \backslash X$ isomorphic to P_6, R_6, Q_6 or $\mathcal{W}^3$. Furthermore, the circuit lattice of any simple and co-simple non-uniform oriented matroid of rank 3 is regular.*

Proof. The "if" case is Lemma 1 together with the fact that any extension of a trivial flow lattice again must be trivial. The "only if" case is Lemma 2. Obviously, trivial lattices are regular, as well as circuit lattices of regular oriented matroids. Any other rank 3 non-uniform oriented matroid must contain an $(n-2)$-line and therefore either has a co-loop or a co-parallel. $\square$

As an obvious consequence the flow number of non-regular non-uniform simple and co-simple matroids of rank three is 2. The construction of a basis is done by starting from a basis of P_6, R_6, Q_6 or $\mathcal{W}^3$ and extending the basis by a circuit containing only some of the current elements plus one new element.

6 Open Questions

Is there a possibility to determine the circuit lattice dimension of an arbitrary oriented matroid (this question is raised in Björner et al. [1, 4.45(d)]) by decomposing the matroid into direct sums and 2-sums of 3-connected oriented matroids? While for a direct sum of two oriented matroids the dimensions are simply added we conjecture the dimension to be $d_1 + d_2 - \{1, 2\}$ for 2-sums.

Another interesting question involves the gap in the flow lattice dimension between regular and non-regular oriented matroids: Does the dimension always increase rapidly when extending a regular oriented matroid non-regularly? To be more offensive, is the flow lattice of a non-regular extension of a (maximum) regular oriented matroid always trivial?

As we could see in the considered cases, the flow number is invariant with respect to the reorientation class of an oriented matroid, i. e. it is a matroid invariant for regular, rank 3, and uniform oriented matroids. Hochstättler and Nešetřil [3] ask whether this holds in general even if it was shown above that different orientations of the same underlying matroid can lead to a significantly different flow lattice structure. Furthermore, we look for other cases where the dimension differs between reorientation classes of the same underlying matroid as found for uniform oriented matroids of odd rank.

By combining the last two open questions we could also ask whether the flow number is a matroid invariant only because the flow lattice is trivial for almost any connected non-regular oriented matroid.

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