Graph Coloring

Application of the Ellipsoid Method in Combinatorial Optimization

Robert Nickel

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Abstract

The following article is the working out of a talk in a seminar about the ellipsoid method and it's consequences in combinatorial optimization at Brandenburg Technical University Cottbus. It deals with the ellipsoid method used as a tool for proving polynomial time solvability of combinatorial optimization problems. In 1970 SHOR ([24],[23]) gave a first outline of this method and it first explicitly appears in 1977 in one of his papers about convex nondifferential programming ([25]). Two years later L.G. KHACHIYAN ([17]) showed that the ellipsoid method is a polynomial algorithm to solve linear optimization problems (in theory) and gave a proof in 1980 ([18]). In the following years many new algorithms appeared which solve linear optimization problems in time polynomially bounded but that require the complete knowledge of the constraint system in opposite to the ellipsoid method which is therefore applicable to those problems of combinatorial optimization which lead to a linear programming description with a potentionally exponential number of constraints. This applicability was discovered by KARP and PAPADIMITRIOU ([16]-1980), PADBERG and RAO ([21]-1981) and GRï¿½SCHEL, LOVï¿½Z and SCHRIJVER ([11]-1981) who showed that many combinatorial problems can be formulated as an optimization of a linear functional over a polytope.

I decided to focus on graph coloring as a very famous problem of graph theory since it has many practical and academic applications. It will be shown that minimum graph coloring is polynomially solvable for a subclass of undirected graphs by applying the ellipsoid method.

1 Motivation

A minimum coloring of a graph G with vertices V and edges E is a labeling of the vertices in a way that no connected two vertices have the same label such that a minimum number of labels is used. The minimum coloring problem (even calculating the minimum number of colors needed) for general undirected graphs is $\mathcal{NP}$-hard.

The most popular application of graph coloring is the coloring of a map. Here each country is given a color such that no neighbouring contries have the same color (to make the borders recognizable) and a minimum number of colors is used. Identifying each country with a vertex and connecting two neighbouring countries with an edge leads to a vertex coloring problem. Another academic example of graph coloring is the problem to accommodate a number of children in a minimum number of rooms such that no room contains two children that dislike each other. Here the children are the vertices and they are connected with an edge whenever they dislike each other.

But also in real world applications graph coloring is of big relevance:

- **Time tabling and scheduling** In scheduling problems one often has a set of pairwise restrictions on which jobs can be done simultaneously. For instance scheduling classes at a university such that no two courses involving the same teacher or a similar group of students are scheduled for the same time period. The problem of finding the minimum number of time periods needed subject to these restrictions is a graph coloring problem.

- **Frequency assignment** A number of mobile radio transmitters have to be assigned a frequency such that two transmitters that are close to each other are assigned different frequencies and a minimum number of frequencies is used.

- **Register allocation** The problem is to assign variables to a limited number of hardware registers during a program execution. Since there are typically far more variables in a program than registers it is necessary to assign multiple variables to registers such that they do not conflict with each other.

For an extensive introduction into the theory of graph coloring and existing coloring algorithms see [15] and [10].

I will first give an outline of the ellipsoid method to show that it can be used to optimize a linear functional over a bounded convex set with nonempty interior whenever a polynomial algorithm is known to solve the separation problem. After that a section about graph theory and perfect graphs follows in which the coloring problem is formulated in terms of independent sets. Then the ellipsoid method will be used to show the polynomial solvability of a semidefinite optimization problem which leads to the Lovï¿½$\frac{1}{2}$z number (or sandwich number) that is equal to the chromatic number (minimum number of colors needed to color a graph) for perfect graphs. From the calculation of the chromatic number follows an algorithm that evaluates a minimum covering of the perfect graph with disjoint independent sets which is exactly a minimum coloring. The proofs given for several theorems are often taken from papers or books listed as references at the end of the paper and those citations are always explicitly pointed out. I tried to choose only those proofs that are of a special need for understanding the conclusions (e.g. it might be useful to point out exactly why the ellipsoid method is applicable to graph coloring). On the other hand it is often necessary to look into several areas of optimization which are of discrete, linear and semidefinite kind. Some knowledge (particular the duality theorems of linear and semidefinite optimization) is required or at least useful. Especially the section about semidefinite programming requires some background information but it seemed useful to me not to leave LOVï¿½$\frac{1}{2}$Z' sandwich theorem unproved. For an introduction into semidefinite programming with applications in combinatorial optimization see [14].

2 The ellipsoid method

Consider the following two simple problems for a convex set $B \subseteq \mathbf{R}^n$:

- **Optimization**: Given a vector $c \in \mathbf{R}^n$. Find a vector $x \in B$ such that $c^T x \geq \max\{c^T x | x \in B\} - \varepsilon$ for a given $\varepsilon > 0$.
- **Separation**: Given a vector $x \in \mathbf{R}^n$. Verify that $x \in B$ or find a vector $c \in \mathbf{R}^n$ such that $c^T x > \max\{c^T b | b \in B\}$ which means that c describes a separating hyperplane.

Definition 1. *Let $\mathfrak{B}$ a class of convex sets. $\mathfrak{B}$ is called solvable if there exists an algorithm that solves the optimization problem for each $c \in \mathbf{R}^n$ over any $B \in \mathfrak{B}$ in time polynomially bounded.*

In that case the algorithm is ment to be polynomial in the input data used to describe the convex set and in the encoding length of the vector c. We will use graphs to describe those convex sets which means that the algorithm must be polynomial in the number of vertices of the graph.

Now I want to show that with the ellipsoid method a polynomial algorithm for the optimization problem follows directly from an algorithm solving the separation problem in polynomial time over B whenever B contains an interior point $a_0 \in \mathbf{Q}^n$ and there exist numbers r and R such that $S(a_0, r) \subseteq B \subseteq S(a_0, R)$ where $S(a_0, \rho)$ denotes the sphere centered at a_0 with radius ρ. Furthermore the inner and outer radii r and R must be explicitly known and neither dominator nor denominator occuring in a_0, $r \in \mathbf{Q}$ and $R \in \mathbf{Q}$ should be larger than a constant $T \in \mathbf{N}$.

The ellipsoid method now produces a sequence of ellipsoids $\{A_k\}$ such that the sequence of their centers $\{a_k\}$ converges to an optimal solution of $\min c^T x, x \in B$:

- Let $A_0 = S(a_0, R), k := 0$
- **repeat**
  - **if** $a_k \in B$ construct a smaller ellipsoid A_{k+1} that contains the level set $L(a_k) = \{x \in B | c^T a_k > c^T x\}$
  - **if** $a_k \notin B$ construct a separating hyperplane b and a smaller ellipsoid A_{k+1} that contains $A_k \cap \{x \in \mathbf{R}^n | b^T x < 0\}$
  - Let $a_{k+1} = \mathit{center}(A_{k+1})$
- **until** stopping criterion is met.

In [11] it is shown that the ellipsoids A_k can be chosen in a way such that the following theorem holds:

Theorem 2. *Given an algorithm for the separation problem the ellipsoid method (if applicable) solves the optimization problem for a given $\varepsilon > 0$ in $N = 4n^2 \left\lceil \log \frac{2R^2\|c\|}{r\varepsilon} \right\rceil$ iterations.*

Corollary 3. *If a polynomial separation algorithm for a class $\mathfrak{B}$ exists then $\mathfrak{B}$ is solvable.*

3 Graph theory

Let $V = \{1, ..., n\}$ a finite set and $E \subseteq \{\{u, v\}|u, v \in V\}$. Then $G = G(V, E)$ is called an *undirected graph with vertices* V *and edges* E (or simply *graph*). The *complementary graph* is denoted by $\bar{G} = G(V, \bar{E})$ where $\bar{E} = \{\{u, v\}|\{u, v\} \notin E, u, v \in V\}$. $G' = G(V', E')$ with $V' \subset V$ and $E' \subset E$ such that $(\{u, v\} \in E') \Leftrightarrow (\{u, v\} \in E)$ and $u, v \in V'$ is called an *induced subgraph of* G. A *clique* $C \subseteq V$ is a vertex set such that $\{u, v\} \in E\ \forall u, v \in C$ and an *independent set* (or *stable set*) $S \subseteq V$ is a vertex subset such that $\{u, v\} \notin E\ \forall u, v \in S$. It follows directly that the cliques of G are exactly the independent sets of $\bar{G}$.

Now a coloring can be defined in terms of graph theory:

Definition 4. *A coloring is a covering of* V *with disjoint independent sets* $S_1, ..., S_k$ *of* $G = G(V, E)$*, i.e.* $\bigcup_{i=1}^{k} S_i = V$.

This leads to the following problems:

Problem 5. *Find the minimum number of colors needed for a coloring of* G *that means computing* $\chi(G) = \min\{k|S_1, ..., S_k$ *is a coloring*$\}$. $\chi(G)$ *is called the chromatic number of* G.

Problem 6. *Find a minimum coloring* $S_1, ..., S_{\chi(G)}$.

A simple lower bound for the chromatic number $\chi(G)$ is the cardinality of the maximum clique in G denoted by $\omega(G)$ because each vertex in such a maximum clique has to be assigned a different color.

4 Perfect graphs

Definition 7. *A graph* $G = G(V, E)$ *is perfect* $\Leftrightarrow \omega(G') = \chi(G')$ *for any induced subgraph* G' *of* G.

The idea of perfect graphs appeared first in an article of T. GALLAI ([7]-1958/[8]-1959) who proved (in a different terminology) that the complement of a bipartite graph is perfect again. Other early results where obtained by HAJNAL and SURï¿½YI ([13]-1958), BERGE ([1]-1960/[2]-1961), DIRAC ([6]-1961) and GALLAI ([9]-1962). Anyway the first definition of perfect graphs was given by BERGE ([3]-1963, [4]-1966) whose articles are generally mentioned as the official sources for perfect graphs. In [4] he made the following conjectures:

Theorem 8 (Perfect Graph Theorem)**.** G *is perfect* $\Leftrightarrow$ $\bar{G}$ *is perfect.*

Theorem 9 (Strong Perfect Graph Theorem)**.** G *and respectively* $\bar{G}$ *is perfect if and only if it does not contain an odd hole or antihole with at least five vertices as an induced subgraph.*

The perfect graph theorem was proved by LOVï¿½Z ([19]-1972) and the strong perfect graph theorem was just proved by M.CHUDNOVSKY and P. SEYMOUR building on an earlier work with N. ROBERTSON and R. THOMAS ([5]). But although BERGE's conjectures have been proved now it is still open, whether perfect graphs can be recognized in polynomial time even if it is easy to prove that G is not perfect (by guessing an odd hole / antihole with at least five vertices). BFS can shurely find a minimum odd circle in a graph but if that circle is not an induced subgraph of G it might be necessary to look at any circle in the graph which leads to a non-polynomial algorithm.

Now semidefinite optimization can be used to calculate $\omega(G)$ respectively $\chi(G)$ for perfect graphs.

5 Semidefinite optimization

The existence of a number ϑ such that $\omega(G) \leq \vartheta \leq \chi(G)$ that can be computed in polynomial time would directly lead to an algorithm for the chromatic number of perfect graphs. Such a *sandwich number* was introduced by LOVï¿½Z in 1979 ([20]) as the solution of the following semidefinite optimization problem[1]:

$$
\begin{aligned}
\vartheta(G) := \max \sum_{i,j=1}^{n} b_{ij} &= \max e^T B e \\
\text{s.t.} \quad b_{ij} &= 0 \quad \text{if } \{i,j\} \notin E,\ i \neq j \\
Tr(B) &\leq 1 \\
B &\succeq 0 \quad \text{(symmetric and positiv semidefinite)}
\end{aligned} \tag{1}
$$

For an optimum solution B^* of (1) the equality $Tr(B^*) = 1$ must hold (otherwise multiplying B^* with $\frac{1}{TrB^*}$ would lead to a better solution. Taking the equality trace condition as a constraint the semidefinite dual of (1) has the following form (see [22] for further details):

$$
\begin{aligned}
&\min \lambda \\
\text{s.t.} \quad &\lambda I - Y - ee^T \succeq 0 \\
&y_{ij} = 0 \quad \text{if } \{i,j\} \in E \text{ or } i = j
\end{aligned} \tag{2}
$$

Now all preparations are done to prove the so-called *sandwich theorem* introduced by LOVï¿½Z ([20]-1979). The proof is taken from [22]:

Theorem 10 (The Lovï¿½z sandwich theorem)**.**

$$\omega(G) \leq \vartheta(G) \leq \chi(G)$$

Proof. I will show both inequalities seperately:

1. Left inequality: Let $V = \{1, ..., n\}$ such that $C = \{1, ..., k\}$ is a clique (without loss of generality because vertices can be enumerated arbitrarily). Set $x_C = (1, ..., 1, 0, ..., 0)$ with k ones and $n - k$ zeros. Then

$$B_k := \frac{1}{k} x_C x_C^T$$

[1] $\vartheta(G)$ is defined in many different ways. I follow the definition of [22]. For a variety number of different definitions see [12] 286-290 or [14]

is primal feasible for (1) with $e^T B_k e = k$. The clique was arbitrarily chosen. Chosing a maximum clique ($k = \omega(G)$) leads to

$$\omega(G) \leq \vartheta(G).$$

2. Right inequality: Let $(S_1, ..., S_k)$ a coloring of G with k colors and $\gamma_i := |S_i|$. For $i = 1, ..., k$ define

$$M_i := k\,(J_{\gamma_i} - I_{\gamma_i})$$

where J_{γ_i} denotes the all one matrix of size $\gamma_i \times \gamma_i$ and I_{γ_i} the identity matrix of the same size.

Then the block matrix

$$Y_k := \begin{pmatrix} -M_1 & 0 & \cdots & 0 \\ 0 & -M_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & -M_k \end{pmatrix}$$

is dual feasible with $\lambda = k$. Chosing $S_1, ..., S_k$ such that $k = \chi(G)$ leads to

$$\vartheta(G) \leq \chi(G).$$

$\square$

6 Application of the ellipsoid method

Now I will give a proof that the ellipsoid method is applicable to the class of feasible sets of (1) induced by arbitrary undirected graphs.

Let $\mathfrak{B}_{\vartheta(G)}$ the feasible region of (1).

Theorem 11. *$\mathfrak{B}_{\vartheta(G)}$ is convex and bounded with nonempty interior which means that there are $\left(\frac{n^2+n}{2} - |\bar{E}|\right)$-dimensional spheres with radii r and R and an interior point B_0 such that $S(B_0, r) \subseteq \mathfrak{B}_{\vartheta(G)} \subseteq S(B_0, R)$.*[2]

Proof. Let $B_1, B_2 \in \mathfrak{B}_{\vartheta(G)}$ and $B_3 = tB_1 + (1-t)B_2$ for some $t \in [0, 1]$. B_3 is symmetric and positive semidefinite. It's trace is less or equal to one and each zero components of B_1 and B_2 remain zero.

$$\Rightarrow \mathfrak{B}_{\vartheta(G)} \text{ is convex}$$

$B_0 := \frac{1}{2n} I$ is an interior point of $\mathfrak{B}_{\vartheta(G)}$ and it can be verified easily that $\mathfrak{B}_{\vartheta(G)}$ contains the sphere $S(B_0, r)$ with $r := \frac{1}{2n^2+1}$.

[2] Construct such a sphere as follows: Ignore all coordinates for $\{i, j\} \in \bar{E}$ and all coordinates below the main diagonal of the matrix B_0. The remaining coordinates build a vector in the $\left(\frac{n^2+n}{2} - |\bar{E}|\right)$-dimensional euklidean space.

An outer radius can be constructed as follows: Let $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$, $\|x\|_\infty := \max_{i=1,\dots,n} |x_i|$ and $\|B\|_p$ and $\|B\|_\infty$ the induced matrix norms. By definition it follows that

$$\|x\|_\infty \leq \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{n \max_{i=1,\dots,n} |x_i|^2} = \sqrt{n}\|x\|_\infty$$

which means that these norms are equivalent. Now the constants for the equivalence of the corresponding matrix norms can be given:

$$||B||_2 = \max_{x\in\mathbf{R}^n} \frac{\|Bx\|_2}{\|x\|_2} \leq \max_{x\in\mathbf{R}^n} \frac{\sqrt{n}\|Bx\|_\infty}{\|x\|_\infty} = \sqrt{n}\|B\|_\infty$$
$$||B||_\infty = \max_{x\in\mathbf{R}^n} \frac{\|Bx\|_\infty}{\|x\|_\infty} \leq \max_{x\in\mathbf{R}^n} \frac{\sqrt{n}\|Bx\|_2}{\|x\|_2} = \sqrt{n}\|B\|_2$$

$$\Rightarrow \frac{1}{\sqrt{n}}\|B\|_\infty \leq \|B\|_2 \leq \sqrt{n}\|B\|_\infty$$

It follows from $Tr(B) = \sum_{i=1}^n \lambda_i \leq 1$ and $B \succeq 0$ that $0 \leq \lambda_{min} \leq \lambda_{max} \leq 1$, where λ_i denotes the i^{th} eigenvalue of B and $\lambda_{min} = \min \lambda_i$, $\lambda_{max} = \max \lambda_i$.

Hereby one can easily find that

$$\max_{i,j} |b_{ij}| \leq \|B\|_\infty \leq \sqrt{n}\|B\|_2 = \sqrt{n}\lambda_{max} \leq \sqrt{n}$$

because $\|\cdot\|_\infty$ is the matrix norm of row sums and $\|\cdot\|_2$ is the spectral norm. This means that $R = 2\sqrt{n}$ is a candidate for an outer radius and

$$\mathfrak{B}_{\vartheta(G)} \subseteq S(B_0, R).$$

□

The following theorem points out the applicability of the ellipsoid method and proves that the class of feasible reagions of (1) is solvable for any graph G (see [11] for further details).

Theorem 12. *For any undirected graph G the separation problem is solvable on $\mathfrak{B}_{\vartheta(G)}$.*

Proof. Given a symmetric $n \times n$-matrix B with $b_{ij} = 0$ if $\{i,j\} \notin E$ (otherwise we would have to operate in an $\frac{n^2+n}{2}$-dimensional space while we have shown the applicability of the ellipsoid method for $\frac{n^2+n}{2} - |E|$ dimensions) we have to verify in polynomial time whether $B \in \mathfrak{B}_{\vartheta(G)}$. If not we have to give a separating hyperplane C. C separates B from $\mathfrak{B}_{\vartheta(G)}$ iff

$$C \bullet B = \sum_{i,j=1}^n c_{ij} b_{ij} > \max\{C \bullet \tilde{B} | \tilde{B} \in \mathfrak{B}_{\vartheta(G)}\}.$$

1. $Tr(B) \leq 1$ is verified in polynomial time. If $Tr(B) > 1$ then $C := \frac{1}{n} I$ is a separating hyperplane.

2. Now the positive semidefiniteness of B is to be verified and a separating hyperplane is to be constructed if B is not positive semidefinite.

   Let $rg(B) = m$ and find m linear independent column vectors of B. Without loss of generallity let these m vectors be the first m columns of B.[3] Let furthermore $B_t := (b_{ij})_{i,j=1}^n$ and compute $\det B_t$ for $t = 1, ..., m$ which can be done in polynomial time by the gaussian algorithm. If $\det B_t \geq 0\ \forall t \in \{1, ..., m\}$ then $B \in \mathfrak{B}_{\vartheta(G)}$. Otherwise one can construct a separating hyperplane:

   Let t the first index with $\det B_t < 0$ and

   $$\phi_i := \begin{cases} (-1)^{i+t} \det B_{i,t} & i = 1, ..., t \\ 0 & \text{otherwise} \end{cases}$$

   where $B_{i,t}$ denotes the matrix B_t without the i^{th} row and t^{th} column.

   Then $\det B_t = \sum_{i=1}^t b_{it}\phi_i$ and $\phi_t = \det B_{t-1}$.

   Since any $\tilde{B} \in \mathfrak{B}_{\vartheta(G)}$ is positive semidefinite the inequality $\phi^T \tilde{B} \phi \geq 0$ must hold. But for B we can verify that

   $$\phi\phi^T \bullet B = \phi^T B \phi = \sum_{i=1}^{n} \sum_{i=1}^{n} \phi_i \phi_j b_{ij} = \sum_{i=1}^{t-1} \phi_i \sum_{j=1}^{t} \phi_j b_{ij} + \det B_{t-1} \det B_t.$$

   Now $\sum_{j=1}^t \phi_j b_{ij}$ is the determinant of the matrix $(B_t^1, B_t^2, ..., B_t^i, ..., B_t^{t-1}, B_t^i)^T$ where B_t^k denotes the k^{th} row of B_t.

   Since this determinant is zero it is shown that

   $$\phi\phi^T \bullet B = \det B_{t-1} \det B_t < 0$$

   and the matrix $C := -\phi\phi^T$ is a separating hyperplane.

□

Now the ellipsoid method can be applied and it is possible to approximate the value $\vartheta(G)$ with accuracy ε in time polynomially bounded by $|V|$ and $|\log \varepsilon|$. If G is known to be perfect then $\omega(G) = \vartheta(G) = \chi(G)$ and ε can be chosen $\frac{1}{2}$ because $\vartheta(G) \in \mathbf{N}$.

7 Constructing a coloring

With the described method we can compute $\omega(G)$ in polynomial time for perfect graphs without evaluating a maximum clique. But knowing $\omega(G)$ this can be done as follows:

1. Let $i := 0$, $V = 1, ..., n$

2. Compute $\omega(G)$

[3] Otherwise swapping columns and corresponding rows which is the same as enumerating the vertices in a different order leads to such a matrix. Even the positive (in)definiteness of B remains.

3. Compute $\omega(G')$ where G' arises from G by removing vertex i and the corresponding edges

4. If $\omega(G) = \omega(G')$ set $G := G'$

5. $i := i + 1$, If $i \leq n$ goto 2

To make this algorithm faster one can remove all adjacent vertices of i if $\omega(G) = \omega(G')$ and all non-adjacent vertices of i otherwise because these cannot belong to a maximum clique. After less than n iterations the algorithm terminates with a maximum complete subgraph of $G(V, E)$. The knowledge of such a maximum clique can be used to construct a minimum covering of G with independent sets:

A vector $x \in \mathbf{R}^n$ with $x_i = \begin{cases} 1 & i \in V' \subset V \\ 0 & \text{otherwise} \end{cases}$ is called a *characteristic vector* of the vertex set $V' \subset V$. Now let A the matrix which has as rows the characteristic vectors of all independent sets of G and consider the following linear programming problem:

$$\begin{aligned} \max\ & e^T x \\ s.t.\ & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{aligned} \tag{3}$$

If the components of x were additionaly restricted to $\{0, 1\}$ the solution of the problem would exactly be a maximum clique with $\omega(G)$ as maximum value since $Ax \leq \mathbf{1}$ ensures that the vertex set represented by x intersects at most one independent set and $\max e^T x$ maximizes the cardinality of the clique.

$$\Rightarrow \omega(G) \leq \max e^T x \qquad Ax \leq \mathbf{1},\ x \geq \mathbf{0}$$

The linear dual problem of (3) can be written down and interpreted, too:

$$\begin{aligned} \min\ & e^T y \\ s.t.\ & A^T y \geq \mathbf{1} \\ & y \geq \mathbf{0} \end{aligned} \tag{4}$$

If again the components of y are additionally restricted to $\{0, 1\}$ y can be interpreted as a characteristic vector of a subset of all independent sets of G. Furthermore y chooses a minimum covering of G with independent sets while $A^T y \geq \mathbf{1}$ ensures that each vertex is covered by at least one independent set and $\min e^T y$ minimizes the number of sets used. Whenever the resulting sets are not disjoint, i.e. $S_k \cap S_l \neq \emptyset$ this can be corrected by removing $S_k \cap S_l$ from S_l. Hereby it follows that $\chi(G) \geq \min e^T y, \quad A^T y \geq \mathbf{1},\ y \geq 0$.

If now G is perfect we can get a maximum clique of G and by linear programming duality a covering of G with independent sets which is the coloring we are looking for.

8 Conclusion and final remarks

If we can consider a graph G to be perfect it is possible to compute the maximum clique number $\omega(G)$ by applying semidefinite programming and the ellipsoid method. Afterwards it is easy to determine a maximum clique and by linear programming duality we also get a solution of the dual problem of (3) which is a coloring if we make the covering of the graph by independent sets disjoint as described above. This means that for perfect graphs a coloring can be computed in time polynomially bounded.

But it is still unknown if it is possible to recognize whether a given graph is perfect or not although one can prove the imperfectness of a graph.

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