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The game chromatic number and the game colouring number of classes of oriented cactuses

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Abstract

We prove that the game chromatic and the game colouring number of the class of orientations of cactuses with girth of 2 or 3 is 4. We improve this bound for the class of orientations of certain forest-like cactuses to the value of 3. These results generalise theorems on the game colouring number of undirected forests [3] resp. orientations of forests [1]. For certain undirected cactuses with girth 4 we also obtain the tight bound 4, thus improving a result of Sidorowicz [7].

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1 Introduction

In 2007, Sidorowicz [7] determined the game chromatic and game colouring number of the class of undirected cactuses. Here, we will extend her examinations to several classes of oriented and undirected cactuses. We start by fixing some definitions.

An *orientation* of a graph $G = (V, E)$ is a digraph $D = (V, E')$ in which each edge $vw \in E$ is replaced by exactly one of the arcs (v, w) or (w, v) in E'. A *semiorientation* of a graph $G = (V, E)$ is a digraph $D = (V, E')$ in which each edge $vw \in E$ is replaced by at least one of the arcs (v, w) or (w, v) in E', possibly both. An *oriented cycle* is any orientation of a cycle with $n \geq 3$ vertices. A *directed cycle* is a connected digraph in which every vertex has in-degree and out-degree 1. An *edge* consists of two vertices v and w and the two arcs (v, w) and (w, v). A *semicycle* is the semiorientation of a cycle with $n \geq 3$ vertices.

An *(undirected) cactus* is a graph in which every pair of cycles has at most one common vertex. An *oriented cactus* is a semiorientation of a cactus with the property that every semicycle is an oriented cycle. An oriented cactus is called *directed cactus* if every oriented cycle is a directed cycle.

An undirected or oriented cactus is *n-fat* for $n \geq 1$ if it has girth at least n. So n-fat oriented cactuses do neither contain edges nor loops for $n \geq 3$, but in 2-fat oriented cactuses edges are allowed. A 3-fat oriented cactus is simply an orientation of a cactus. We denote the class of n-fat oriented cactuses by $\vec{\mathcal{C}}_n$ and the class of n-fat directed cactuses by $\vec{\mathcal{C}}_n^{dir}$.

The maker-breaker games which give us the parameters we will consider are generalisations for digraphs of two games known as Bodlaender's graph colouring game [2] and Zhu's graph marking game [8]. In the last years these games have arisen some attention to graph theorists. Their generalisations for digraphs were introduced in [1].

The digraph colouring game is played by two players, Alice and Bob, on a given, initially uncoloured, digraph D. There is a fixed set C of k colours. The players alternately colour uncoloured vertices of D with colours from C. Alice has the first move. The players have to respect the rule that an uncoloured vertex may receive only a colour different from the colours of its in-neighbours. If this is not possible any more or all vertices are coloured, the game is over. If all vertices are coloured at the end, Alice wins, otherwise Bob. In order to simplify thinking about the game we can say that, whenever a vertex is coloured, all its in-arcs are deleted, since they do not play any role for further moves. The smallest number k such that Alice has a winning strategy for the game played on D is called *game chromatic number* $\vec{\chi}_g(D)$. We further define for a class $\mathcal{C}$ of digraphs

$$\vec{\chi}_g(\mathcal{C}) = \sup_{D \in \mathcal{C}} \vec{\chi}_g(D).$$

The digraph marking game is played by Alice and Bob on a given, initially unmarked, digraph D. There is a fixed score of k. The players move alternately. Alice begins. A move consists in marking an unmarked vertex in such a way that the number of its marked in-neighbours is at most $k-1$. Again we can think of deleting all in-arcs whenever a vertex is marked. The game ends when no such move is possible any more. Alice wins if all vertices are marked at the end, otherwise Bob wins. The smallest number k such that Alice has a winning strategy for the game played on D is called *game colouring number* $\vec{\mathrm{col}}_g(D)$. We further define for a class $\mathcal{C}$ of digraphs

$$\vec{\mathrm{col}}_g(\mathcal{C}) = \sup_{D \in \mathcal{C}} \vec{\mathrm{col}}_g(D).$$

Obviously, for a digraph D, we always have

$$\vec{\chi}_g(D) \leq \vec{\mathrm{col}}_g(D). \tag{1}$$

We remark that our game chromatic resp. game colouring number, when applied to graphs (which in our case are digraphs in which each arc has an antiparallel arc) gives us Bodlaender's game chromatic number χ_g resp. Zhu's game colouring number col_g. Therefore in the following we will omit the vector $\vec{}$ in $\vec{\chi}_g$ and $\vec{\mathrm{col}}_g$.

The study of game chromatic and game colouring numbers of certain classes of undirected graphs concentrated on forests [3], planar graphs [9], outerplanar graphs [4] and graphs embeddable in a surface [6]. Some more specific classes of graphs were considered, too (see e.g. [5]). In this paper we mainly generalise the result from Sidorowicz [7] and consider cactuses. In [7] it is proven that the game chromatic and game colouring number of the class of (3-fat) undirected cactuses is 5. We prove a similar result for the class of 2- and 3-fat oriented cactuses and refine this result in a special case. The upper bounds will be given in Section 2, the lower bounds in Section 3. In Section 4 we refine the result of Sidorowicz in a special case.

2 Upper bounds

It was observed by Sidorowicz [7] that any cactus has an edge partition into a forest and a matching. So any 3-fat oriented cactus C has an arc partition into the orientation C_1 of a forest and the orientation C_2 of a matching. By the formula (see [1, 4, 8])

$$\mathrm{col}_g(C) \leq \mathrm{col}_g(C_1) + \Delta^-(C_2), \tag{2}$$

which holds for every arc partition $C_1|C_2$ of a digraph C, and by the result

Theorem 1 (Andres [1]). $\mathrm{col}_g(C_1) \leq 3$ *for any orientation* C_1 *of a forest.*

we conclude that $\mathrm{col}_g(\vec{\mathcal{C}}_3) \leq 4$. Here we will prove more, namely that this bound already holds for $\vec{\mathcal{C}}_2$.

We introduce a notion we need. An oriented cycle or an edge P is called *pendant* if it contains a vertex v, called the *neck* of P, whose removal from P leaves a component, called the *head* H_P of P, which is an oriented tree and contains $P - v$. In particular H_P contains no cycles and no edges.

Sidorowicz [7] introduced a similar notion for undirected cactuses and proved the following

Lemma 2 (Sidorowicz [7]). *If* K *is a cactus with at least one cycle then* K *contains a pendant cycle.*

By carefully reading the proof of this lemma it can be observed that there is no need to make the precondition that K is a simple graph (and thus has girth at least 3), but the proof still works when 2-cycles (or even loops) are allowed. Moreover, by orienting such cactuses, we conclude

Lemma 3. *Every 2-fat oriented cactus has at least one pendant oriented cycle or one pendent edge or no cycles and edges at all.*

Lemma 4. *Let* $C \in \vec{\mathcal{C}}_2$. *Then* C *has an arc partition* $C_1|C_2$, *where* C_1 *is an oriented forest, and* C_2 *is an oriented forest with maximum in-degree* $\Delta^-(C_2) \leq 1$.

Proof. Let k be the number of cycles and edges of C. We prove the lemma by induction on k. If C does not contain cycles or edges, we are done by $C_1 = C$ and $C_2 = \emptyset$. Otherwise, let P be a pendant oriented cycle or a pendant edge, which exists by Lemma 3. Let v be the neck of P and H_P be the head. By induction, since $C - H_P$ has $k-1$ oriented cycles and edges, there is an arc partition $C_1'|C_2'$ of $C - H_P$, so that C_1' and C_2' are oriented forests and $\Delta^-(C_2) \leq 1$.

If P is an oriented cycle, then P contains an arc (x, y) with $x \neq v$ and $y \neq v$. Let $A := P \cup H_P \setminus \{(x, y)\}$. A is the arc set of an oriented tree. Define $C_1 := C_1' \cup A$ and $C_2 := C_2' \cup \{(x, y)\}$. Note that the intersection of the induced vertex sets of C_1' and A is $\{v\}$. Hence $C_1' \cup A$ is a forest as well. $C_1|C_2$ is the arc partition we wanted to obtain.

If P is an edge, then P contains a unique vertex w different from the neck v. Let B be the arcs of P' together with (w, v). Then we define $C_1 := C_1' \cup B$ and $C_2 := C_2' \cup \{(v, w)\}$. Again, C_1 and C_2 are forests. Since (v, w) is directed towards the exterior, the in-degree is increased at w only. But in C_2' w was an isolated vertex, so in C_2 the degree of w is 1. This concludes the proof. □

Our main Theorem 5 follows immediately from Lemma 4, Theorem 1 and (2):

Theorem 5. $\mathrm{col}_g(\vec{\mathcal{C}}_2) \leq 4$.

Corollary 6 (Faigle et al. [3])**.** *Let F be an undirected forest. Then*

$$\mathrm{col}_g(F) \leq 4.$$

Now we restrict ourselves from arbitrary oriented to directed cactuses and consider a special kind of tree-like directed cactuses. We call a vertex in a directed cactus *thin* if it has in- and out-degree 1, otherwise thick. A *directed cactus tree* is a directed cactus where each cycle has at most two thick vertices. Note that a directed cactus tree is not necessarily connected. Note further that the class of directed cactus trees may as well be defined as follows. Replace all arcs in a forest either by an arc or by a directed n-cycle, $n \geq 3$.

The following theorem is a generalization of Theorem 1.

Theorem 7. *Let C be a directed cactus tree. Then* $\mathrm{col}_g(C) \leq 3$.

Proof. We have to show that Alice has a winning strategy for the directed marking game ensuring that each marked vertex has at most 2 marked in-neighbours at the time it is marked.

Recall that, by a rule of the game, each time a vertex is marked we remove all its in-arcs. Thus when the players mark vertices, the directed cactus tree is decomposed into more and more components. If we remove all marked vertices the (actual) directed cactus tree decomposes into several even smaller (open) components. Such a component together with its marked neighbours and their arcs leading into it is called a *closed component*. Note that two closed components are not necessarily disjoint (but have at most one vertex in common). Call a marked vertex m in a closed component Γ *easy* if its — necessarily unique — neighbour in Γ is a thin vertex, otherwise m is *critical*.

Alice's winning strategy is to guarantee that after each of her moves every closed component contains at most one critical vertex.

If Bob, during his move, creates an independent component with two marked critical vertices v and w — and he can create only one such independent component — Alice considers the shortest path from v to w. Since the first and the last arc of the path are directed towards the interior of the closed component by the rules of the game, there is a vertex z which has two incoming arcs in the path and hence z is thick. Alice marks z. This is feasible because the thick vertex z may have only critical marked neighbours.

Now we claim that Alice has reinstalled her invariant. By marking z, Alice has separated the closed component of v and w into several components, and v and w lie in different components now. If z does not lie in any of the closed components containing v and w, we are done. Otherwise, let Θ be a closed component containing z and $u \neq z$ a marked vertex in Θ which is critical. Note that $u \in \{v, w\}$ since z was in an open component before Alice's move. We claim that z is easy. If z were critical, then it had a thick out-neighbour x. Since the first arc $\vec{a}$ on the shortest z-u-path P_1 was an in-arc of z, it has been deleted when z was marked. So P_1 does not exist any more after Alice's move, but since u and z belong to the same closed component there must be another z-u-path P_2 containing the arc (z, x). So there is a thick vertex x', so that x' is the first inner vertex where P_1 and P_2 meet, and before Alice's move there has been a directed cycle X containing the thick vertices x' and z. Since x is a thick vertex on this cycle, and $x \neq z$, and every directed cycle has at most two thick vertices, we have $x = x'$. Let P_i' be the subpath of P_i from z to x, $i = 1, 2$. So P_2' consists only of the arc (z, x). Since P_1 was a shortest z-u-path, P_1' is a shortest z-x-path and hence consists of the single arc $\vec{a}$. So X is a 2-cycle contradicting that C has girth at least 3. Hence Alice has successfully reinstalled her invariant.

Consider the case that after Bob's move Alice's invariant is still satisfied. If no closed component contains a marked as well as an unmarked thick vertex, Alice marks an arbitrary vertex, preferably a thick one. This way she does not create any new critical vertex. Let Θ denote a closed component that contains at least one marked thick vertex m and one unmarked thick vertex u. Recall that C may be considered as an oriented tree T, where some arcs have been replaced by directed cycles. Clearly, this tree contains u and m. Alice now marks the first thick vertex v on the m-u-path in T. The only case where it is not immediate that after such a move u and m do not belong to the same closed component any more is when they form a pair of thick vertices of an oriented cycle X. But since X is directed, both of its u-m-paths have been destroyed by marking m and u. Therefore, in every case Alice's invariant holds after her move. $\square$

3 Lower bounds

In this section we prove that the bound from Theorem 5 is tight.

Theorem 8. *$\chi_g(\vec{\mathcal{C}}_3^{dir}) \geq 4$.*

Proof. Consider the digraph D depicted in Fig. 1. We construct a digraph D' by taking two identical copies of D glued together at the copies of vertex u_4. The digraph D'' is formed by two disjoint copies of D'. We will prove that Bob has a winning strategy for the directed colouring game played on D'' with 3 colours.

By the construction, after both players have moved twice, Bob can create a situation in which there is a copy of D where u_1 and u_4 are coloured with colour 1 (by Bob) and all other vertices of D are uncoloured. (W.l.o.g. we may assume that Alice plays in some other copies of D in her first two moves.) Now Alice's third move will bring the decision. We distinguish 5 cases:

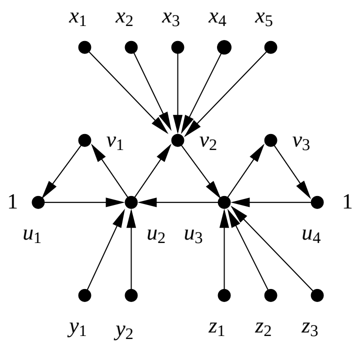

Figure 1: The digraph D

1. **Alice colours u_3 (with colour 2).** Then Bob colours y_1 with colour 3, and u_2 cannot be coloured any more.

2. **Alice colours v_2 with colour 2.** Then Bob colours z_1 with colour 3 and u_3 cannot be coloured any more.

3. **Alice colours v_2 with colour 1.** Then Bob colours z_1 with colour 3. In order to prevent Bob from colouring z_2 or z_3 with colour 2 and leaving u_3 uncoloured, Alice has to colour u_3 with colour 2. But then we are in the same situation as in Case 1 and Bob has a winning strategy.

4. **Alice colours u_2 (with colour 2).** Then Bob colours x_1 with colour 1. In order to prevent Bob from colouring x_2 or x_3 with colour 3 and leaving vertex v_2 uncoloured, Alice has to colour v_2 with colour 3. Now Bob can colour z_1 with colour 2 in order to win since u_3 cannot be coloured any more.

5. **Alice colours a vertex different from u_2, v_2 and u_3.** In this case Bob colours one of the vertices x_j with a first colour not used so far for the vertices x_j. Alice is stuck now. If she colours u_2 (resp. v_2, resp. u_3) with colour 2, then Bob colours a vertex x_j (resp. z_j, resp. y_j) with colour 3 and wins. Otherwise Bob colours another vertex x_j with a second colour not used so far for the vertices x_j. Either this results in a win for Bob since v_2 has three distinctly coloured in-neighbours x_j or again, Alice may not colour u_2, v_2 or u_3. No matter what she does, Bob colours another vertex x_j with the last colour not used so far for the vertices x_j and wins. Note that in the latter situation the fifth vertex x_j is still uncoloured before Bob's move since Alice may have coloured at most two vertices x_j.

So in any case, Bob wins. □

Corollary 9. *For $n \in \{2, 3\}$ we have*

$$\chi_g(\vec{\mathcal{C}}_n) = \mathrm{col}_g(\vec{\mathcal{C}}_n) = 4$$
$$\chi_g(\vec{\mathcal{C}}_n^{dir}) = \mathrm{col}_g(\vec{\mathcal{C}}_n^{dir}) = 4$$

Proof. This follows from Theorems 5 and 8 by (1) and by the definition of n-fat. □

4 Undirected forests with thin 4-cycles

In this section we consider a special type of undirected cactuses. A *forest with thin 4-cycles* is an undirected cactus in which each cycle is a 4-cycle of the form $abcd$, where b and d are non-adjacent vertices of degree 2, the *thin* vertices of the cycle. Again, the vertices of this cactus which are not thin are called *thick*. Note that thick vertices may be the vertices a or c of a 4-cycle, or any terminal vertex of an edge. The class of forests with thin 4-cycles is denoted by $\mathcal{C}_4^{th}$.

The following theorem generalises the result of Faigle et al. [3] concerning the game colouring number of forests to forests with thin 4-cycles, improving the result of Sidorowicz [7] concerning the game colouring number of cactuses in this special case.

Theorem 10. $\mathrm{col}_g(\mathcal{C}_4^{th}) = 4$.

Proof. By a result of Bodlaender [2] there is a tree T with $\chi_g(T) = 4$. Therefore

$$\mathrm{col}_g(\mathcal{C}_4^{th}) \geq \mathrm{col}_g(T) \geq \chi_g(T) \geq 4.$$

For the inverse estimation let $C \in \mathcal{C}_4^{th}$. We have to prove that Alice has a winning strategy for C, so that every unmarked vertex is adjacent to at most 3 marked vertices. We consider *unmarked components*, which are the dynamic components of the graph induced by the unmarked vertices. Note that each unmarked vertex is contained in a unique unmarked component. Alice's winning strategy will be that after each of her moves every unmarked component U is of one of the following types:

(a) U is adjacent to at most one marked vertex.

(b) U is adjacent to exactly two marked vertices at least one of which is a thick vertex or U is adjacent to exactly two marked vertices which are thin vertices of the same 4-cycle.

(c) U is adjacent to exactly three marked vertices, two thick vertices and one thin vertex which is adjacent to one of the marked thick vertices.

Bob can destroy Alice's invariant by his move in several ways. First, he can mark a thin vertex in a component of type (a) with a marked thin vertex. Since we assume that Bob has destroyed Alice's invariant, the two marked thin vertices belong to different cycles, otherwise we would have two components of type (b). In this case Alice marks a thick neighbour of one of the marked vertices in order to split the component in two components of type (b) and maybe some components of type (a). See Fig. 2 (a).

Second, Bob can mark a vertex in a component of type (b). After his move there is at most one component that does not satisfy Alice's invariant. Let U_0 be such a component. If all three marked vertices v_1, v_2, v_3 adjacent to U_0 lie on a path, then the middle vertex v_2 must be a thin vertex with two unmarked neighbours. Note that in case the marked vertices lie on a path, v_1 and v_3 are not thin vertices of the same cycle, since if the only thick vertex on one of the paths from v_1 to v_3 were marked, every component would satisfy Alice's invariant. So we may assume w.l.o.g. that v_1 is a thick vertex. Then Alice marks the thick vertex adjacent to v_2 which is nearer to v_3. By this move, U_0 is

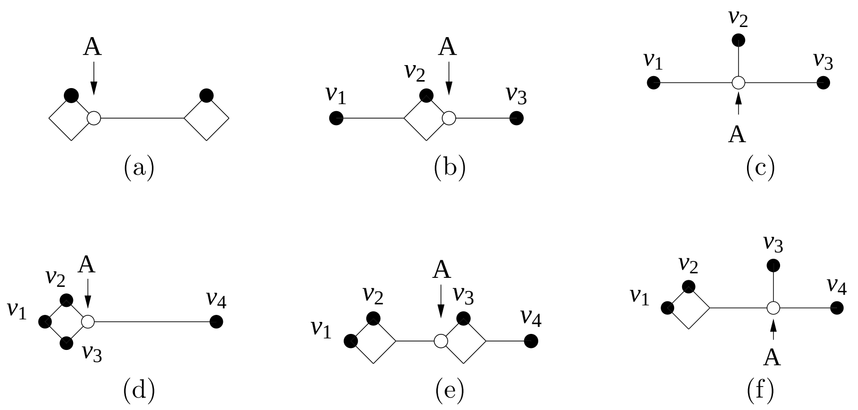

Figure 2: Alice's answer in the different cases

split into a component of type (c) adjacent to v_1, v_2, w and a component of type (b) adjacent to w, v_3 and maybe some components of type (a). See Fig. 2 (b).

If the three marked vertices v_1, v_2, v_3 adjacent to U_0 do not lie on a path, then there is a unique (thick) vertex w in the intersection of all paths between pairs of these three marked vertices. Alice marks w and splits U_0 into at most three components of type (b) and maybe some components of type (a). See Fig. 2 (c).

The third possibility is that Bob marks a vertex in a component of type (c). Assume that Bob destroys Alice's invariant. Then there is a (unique) component U_1 adjacent to at least three of four marked vertices v_1, v_2, v_3, v_4, where v_1 and v_4 are thick vertices, v_2 is the thin vertex adjacent to v_1, and v_3 is the vertex marked by Bob. If v_3 is the other thin vertex of the cycle containing v_1 and v_2, then Alice marks the other thick vertex of this cycle (which has at most three marked neighbours, namely v_2, v_3, v_4), obtaining at most one component of type (b) and maybe some components of type (a). See Fig. 2 (d).

If the vertices v_1, v_3, v_4 lie on a path, then v_3 is a thin vertex, and Alice marks the — necessarily unmarked — thick neighbour of v_3 in direction towards v_1. This vertex has at most two neighbours, namely v_2, v_3. So Alice splits U_1 into two components of the types (b) or (c) and maybe some components of type (a). See Fig. 2 (e).

If the vertices v_1, v_3, v_4 do not lie on a path, there is a unique (thick) vertex w on all paths between pairs of these vertices. Alice marks w and splits U_1 into one component of type (b) or (c) and at most two further components of type (b) and maybe some components of type (a). Note that w is not adjacent to v_1, therefore adjacent to at most three marked vertices. See Fig. 2 (f).

We are left with the case that Bob does not destroy Alice's invariant. If there is a component with no marked thick vertex and at most one marked thin vertex, Alice marks a thick vertex in this component, preferably a neighbour of the marked thin vertex. Otherwise, Alice marks a thick neighbour of a marked thick vertex or of two marked thin vertices belonging to the same cycle. If there is no such vertex, she marks a thick vertex at distance two of a marked thick vertex. If there is no unmarked thick vertex at all, she marks an arbitrary thin vertex. Note that in all cases she does not destroy her invariant. In particular, she will mark thin vertices only at the end of the game. Since thin vertices have at most two neighbours, this will be no problem. By induction, Alice wins. □

5 Open questions

Our considerations and those of Sidorowicz [7] arise some interesting questions:

Problem 11. *Determine the game chromatic and game colouring number of the class of n-fat oriented (resp. directed resp. undirected) cactuses for $n \geq 4$.*

The answers to Problem 11 may be 3 or 4 in the case of oriented or directed cactuses, the answers in the case of undirected cactuses may be 4 or 5.

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