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A Hadwiger Conjecture for Hyperplane Arrangements

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1 Introduction

Hadwiger [7] called a *Streckenkomplex* a $K(k)$ if it has a simplex of order k, but none of order $k+1$ as a minor and conjectured that a $K(k)$ is always k-colorable:

> Verschiedene Feststellungen stützen nämlich die Vermutung, dass die chromatische Zahl eines $K(k)$ nicht grösser als k ausfällt.

In today's notation this is known as the following parametrized conjecture:

Conjecture 1 (H(k)[7])**.** *If a graph is not k-colorable, then it must have a K_{k+1}-minor.*

While H(1) and H(2) are trivial, Hadwiger proved his conjecture for $k = 3$ and pointed out that Klaus Wagner proved that H(4) is equivalent to the Four Color Theorem [22, 1, 15]. Robertson, Seymour and Thomas [16] reduced H(5) to the Four Color Theorem. The conjecture remains open for $k \geq 6$.

The purpose of this note is to point out a generalization of this conjecture to (projective) hyperplane arrangements or more general oriented matroids.

The paper is organized as follows. In the next section we review the famous flow conjectures of Tutte, generalizations to regular matroids and discuss their relation to Hadwiger's conjecture. Then we will sketch a possible generalization to oriented matroids. In the last section we will give a detailed interpretation of $H(3)$ in affine hyperplane arrangements which we pose as an open problem. We assume familiarity with the basics of graph theory, matroid theory and oriented matroids. Standard references are [5, 14, 2].

2 Tutte's flow Conjectures

Let $A \in \{0, +1, -1\}^{r \times E}$ be a totally unimodular matrix representing a regular matroid M of rank r on a finite set E and G an Abelian group. A *G-NZ-flow* in M is a vector $f \in (G \setminus \{0_G\})^E$ such that $Af = 0$. If $G = \mathbb{Z}$ and $0 < |f(e)| < k$ we call f a NZ-k-flow.

First we consider the case that A is the incidence matrix of a directed graph $D = (V, E)$. Tutte [18] pointed out that the Four Color Theorem is equivalent to the statement that every planar graph admits an NZ-4-flow. Generalizing this to arbitrary graphs he conjectured that

Conjecture 2 (Tutte's Flow Conjecture [18])**.** *There is a finite number $k \in \mathbb{N}$ such that every graph admits a NZ-k-flow.*

And moreover that

Conjecture 3 (Tutte's Five Flow Conjecture [18])**.** *Every graph admits a NZ-5-flow.*

Note that the latter is best possible as the Petersen graph does not admit a NZ-4-flow. Conjecture 2 has been proven independently by Kilpatrick [12] and Jaeger [11] with $k = 8$ and improved to $k = 6$ by Seymour [17].

Conjecture 3 has a sibling which is a more direct generalization of the Four Color Theorem.

Conjecture 4 (Tutte's Four Flow Conjecture [20, 21])**.** *Every graph without a Petersen-minor admits a NZ-4-flow.*

In [20, 21] Tutte cited Hadwiger's conjecture as a motivating theme and pointed out that while

> "Hadwiger's conjecture asserts that the only irreducible chain-group which is graphic is the coboundary group of the complete 5-graph"

Conjecture 4 means that

> "the only irreducible chain-group which is cographic is the cycle group of the Petersen graph."

The first statement refers to the case where the rows of A consist of a basis of signed characteristic vectors of cycles of a digraph.

Combining these we derive the following formulation in terms of regular matroids which can be seen to be equivalent to Conjecture 4. First let us call any integer combination of the rows of A a *coflow*. Clearly, by duality resp. orthogonality, flows and coflows yield the same concept in regular matroids. Note that the existence of a NZ-k-coflow in a graph is equivalent to k-colorability [20].

Conjecture 5 (Tutte's Four Flow Conjecture, matroid version)**.** *A regular matroid that does not admit a NZ-4-flow has a the cographic matroid of the K_5 or the graphic matroid of the Petersen graph as a minor or, equivalently, a regular matroid that is not 4-colorable, i.e. that does not admit a NZ-4-coflow has a K_5 or a Petersen-dual as a minor.*

Using the Four Color Theorem Lai, Li and Poon have proven that

Theorem 1 ([13])**.** *A regular matroid that is not 4-colorable has a K_5 or a K_5-dual as a minor.*

Tutte's Five Flow Conjecture now suggests the following matroid version of Hadwiger's conjecture:

Conjecture 6. *If a regular matroid is not k-colorable for $k \geq 5$, then it must have a K_{k+1}-minor.*

It is well known that

Theorem 2 ([4, 19])**.** *a) . The number of G-NZ-coflows in a regular matroid depends only on the order of the group, not on its structure.*

b) . The existence of a G-NZ-coflow is equivalent to the existence of a NZ-$|G|$-coflow.

While in the case of the Four Flow Conjecture the simple structure of the additive group of $GF(4)$ allows to combine 4-flows or 4-coflows along 2-sums and 3-sums and prove that Conjectures 4 and 5 are equivalent the author has no general argument why the same should be possible for k-flows and $k \geq 5$. While Conjecture 6 implies Tutte's Five Flow Conjecture as well as H(k) for $k \geq 5$ it is not clear whether the converse holds.

Problem 1. *Is Conjecture 6 equivalent to $H(k)$ and Conjecture 3 for $k \geq 5$?*

We conclude this section considering the remaining cases of k. Again the cases $k = 1$ and $k = 2$ are trivial.

Theorem 3 (H(3) for regular matroids)**.** *If a regular matroid M is not 3-colorable, then it has a K_4-minor.*

Proof. If M has no K_4-minor it is the matroid of some series-parallel network (see [14] Corollary 11.2.15). Hence, the assertion follows from H(3) for graphs. □

3 Oriented Matroids

In the sixties of the last century the term orientable matroid was used for what is now known as a regular matroid. A matroid is *regular*, if there is an orientation of its circuits and cocircuits such that for all circuits C and all cocircuits D

$$|C^+ \cap D^+| + |C^- \cap D^-| = k \iff |C^+ \cap D^-| + |C^- \cap D^+| = k. \tag{1}$$

This has changed with the appearance of oriented matroids [6, 3]. We call a matroid *orientable* if there is an orientation of its circuits and cocircuits such that for all circuits C and all cocircuits D

$$|C^+ \cap D^+| + |C^- \cap D^-| > 0 \iff |C^+ \cap D^-| + |C^- \cap D^+| > 0. \tag{2}$$

In general, this definition will destroy orthogonality between directed circuits and cocircuits and the *chain group* ([19]) generated by the signed characteristic vectors of cocircuits will no longer coincide with the integer kernel of the matrix the rows of which are the signed characteristic vectors of circuits. Even worse, the latter is frequently trivial. Thus, to define flows or coflows we go back to Tutte's original definition of chain groups which we prefer to call *integer lattices.*

Definition 1. *Let $\mathcal{D}$ denote the set of signed cocircuits of an oriented matroid $\mathcal{O}$ on a finite set E. For each $D \in \mathcal{D}$ we define its* signed characteristic vector *$\vec{\chi}_D \in \{\pm 1, 0\}^E$ as*

$$\vec{\chi}_D(e) = \begin{cases} 1 & \textit{if} \quad e \in D^+ \\ -1 & \textit{if} \quad e \in D^- \\ 0 & \textit{if} \quad e \notin D. \end{cases} \tag{3}$$

The lattice of coflows *$\mathcal{F}^*(\mathcal{O})$ is defined as*

$$\mathcal{F}^*(\mathcal{O}) := \left\{ \sum_{D \in \mathcal{D}} \lambda_D \vec{\chi}_D \mid \lambda_D \in \mathbb{Z} \right\}. \tag{4}$$

We say that an oriented matroid $\mathcal{O}$ is k-colorable*, if there exists a coflow $f^* \in \mathcal{F}^*(\mathcal{O})$ such that*

$$\forall e \in E : 0 < |f(e)| < k.$$

The chromatic number $\chi(\mathcal{O})$ *of an oriented matroid is the smallest k such that $\mathcal{O}$ is k-colorable.*

Note that this definition is compatible with the case of regular matroids and graphs. The flow lattice of an oriented matroid has been introduced in [8]. The following theorem and numerical results from [9] suggest that the graphic case should be the worst case for the chromatic number:

Theorem 4 ([10])**.** *If $\mathcal{O}$ is an oriented matroid of rank k then $\chi(\mathcal{O}) \leq k+1$. Furthermore, equality holds if, and only if $\mathcal{O}$ is the oriented matroid of an orientation of K_{k+1}.*

This tempts us to replace the term "regular" in Conjectures 5 and 6 by "orientable". Explicitely:

Problem 2. H(4) for oriented matroids: *Does an oriented matroid that is not 4-colorable, necessarily have an orientation of K_5 or of the Petersen-dual as a minor?*

H(k) for oriented matroids and $k \geq 5$: *Does an oriented matroid that is not k-colorable for $k \geq 5$, necessarily have an orientation of K_{k+1} as a minor?*

The next section will be devoted to a generalization of $H(3)$.

While two orientations of a regular matroid differ only by reorientation, in general this is not true. E.g. there are orientations $\mathcal{O}_1, \mathcal{O}_2$ of U_3^6, the uniform matroid of rank 3 on 6 elements, such that $\dim \mathcal{F}(\mathcal{O}_1) = 6$ while $\dim \mathcal{F}(\mathcal{O}_2) = 5$. Nevertheless, we do not know a matroid where the chromatic numbers differ for different orientations.

Very little is known about algebraic coflows in oriented matroids. The example $\mathcal{O}_1$ just mentioned together with $\mathbb{Z}_4$ and the additive group of $GF(4)$ form a counterexample to a generalization of Theorem 2 a) , while we do not have an immediate counterexample to an oriented matroid version of Theorem 2 b) .

4 H(3) for hyperplane arrangements

By the topological representation theorem for oriented matroids ([6], see also [2] 5.2.1) every oriented matroid can be represented as an arrangement of pseudohyperspheres on the sphere. To simplify the discussion we will consider only the linear case, i.e. hyperplane arrangements, here.

Thus, let $\mathcal{H} = (H_e)_{e \in E}$ be an arrangement of affine hyperplanes in $\mathbb{R}^d$. A *plane* of $\mathcal{H}$ is any non empty intersection of elements of $\mathcal{H}$. Planes of dimension zero or one are called *vertices* resp. *lines*. In order to avoid working in projective spaces we call an arrangement *proper* if no two lines of $\mathcal{H}$ are parallel. If $S, T \subseteq E$ are two disjoint subsets of E such that $P = \bigcap_{e \in S} H_e$ is a plane which is contained in no hyperplane H_t with $t \in T$ then the hyperplane arrangement defined on P by $(P \cap H_t)_{t \in T}$ is called a *minor* of $\mathcal{H}$.

The arrangement of the K_4 is derived from the hyperplanes defined by $x_i = 0$ and $x_i - x_j = 0$ for $1 \leq i < j \leq 3$ and $x_i = 0$ in $\mathbb{R}^4$ by dehomogenization and is depicted in Figure 1.

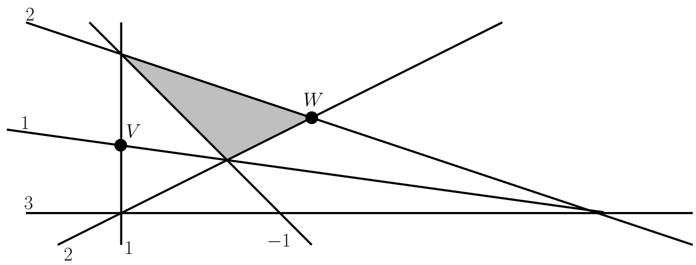

Figure 1: The arrangement of the K_4

Let $\mathcal{V}$ denote the set of vertices of a proper arrangement $\mathcal{H}$. A *coflow in* $\mathcal{H}$ consists of a maximal cell X and a map $z : \mathcal{V} \to \mathbb{Z}$. We say that a vertex $V \in \mathcal{V}$ is *positive* with respect to a hyperplane H_e, if $V \notin H_e$ and V lies on the same side of H_e as X, and *negative* if $V \notin H_e$ and V and X are on different sides of H_e. The *value* $f_z(e)$ *of a hyperplane* H_e *in a coflow* is defined as

$$f_z(e) = \sum_{V \text{ is positive wrt.} H_e} z(V) - \sum_{V \text{ is negative wrt.} H_e} z(V). \tag{5}$$

A NZ-k-coflow is a coflow such that $0 < |f_z(e)| < k$. The choice of X corresponds to the choice of an acyclic orientation in a graph.

If we choose the shaded region in Figure 1 as X, set $z(V) = 2, z(W) = 1$ and $z(U) = 0$ for all other vertices U, we yield the indicated NZ-4-coflow. This is best possible, since $\chi(K_4) = 4$. Theorem 4 asserts that all (pseudo)-line arrangements, which are not projectively equivalent to Figure 1 admit a NZ-3-coflow.

Hadwiger's Theorem H(3) then becomes

Conjecture 7. *If an arrangement does not admit a NZ-3-coflow it must have a configuration projectively equivalent to Figure 1 as a minor.*

References

[1] Kenneth I. Appel and Wolfgang Haken, *Every planar map is four colorable*, Bull. Amer. Math. Soc. **82** (1976), no. 5, 711–712.

[2] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler, *Oriented matroids*, 2nd ed., Cambridge University Press, Cambridge, 1999.

[3] Robert G. Bland and Michel Las Vergnas, *Orientability of matroids*, Journal of Combinatorial Theory Series B **23** (1978), 94–123.

[4] Henry H. Crapo, *The tutte polynomial*, Aequationes Mathematicae **3** (1969), no. 3, 314.

[5] Reinhard Diestel, *Graph theory*, 3rd ed., Springer, February 2006.

[6] Jon Folkman and Jim Lawrence, *Oriented matroids*, Journal of Combinatorial Theory. Series B **25** (1978), no. 2, 199–236.

[7] Hugo Hadwiger, *Über eine Klassifikation der Streckenkomplexe*, Vierteljahresschrift der Naturforschenden Gesellschaft in Zürich **88** (1943), 133–142.

[8] Winfried Hochstättler and Jaroslav Nešetřil, *Antisymmetric flows in matroids*, Eur. J. Comb. **27** (2006), no. 7, 1129–1134.

[9] Winfried Hochstättler and Robert Nickel, *The flow lattice of oriented matroids*, Contributions to Discrete Mathematics **2** (2007), no. 1, 68–86.

[10] Winfried Hochstättler and Robert Nickel, *On the chromatic number of an oriented matroid*, J. Comb. Theory, Ser. B **98** (2008), no. 4, 698–706.

[11] F. Jaeger, *Flows and generalized coloring theorems in graphs*, Journal of Combinatorial Theory, Series B **26** (1979), no. 2, 205 – 216.

[12] Peter Allan Kilpatrick, *Tutte's first colour-cycle conjecture.*, Master's thesis, University of Cape Town, 1975.

[13] Hong-Jian Lai, Xiangwen Li, and Hoifung Poon, *Nowhere zero 4-flow in regular matroids*, J. Graph Theory **49** (2005), no. 3, 196–204.

[14] James G. Oxley, *Matroid theory*, The Clarendon Press Oxford University Press, New York, 1992.

[15] Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas, *The Four-Colour theorem*, Journal of Combinatorial Theory, Series B **70** (1997), no. 1, 2–44.

[16] Neil Robertson, Paul Seymour, and Robin Thomas, *Hadwiger's conjecture for k_6-free graphs*, Combinatorica **13** (1993), 279–361.

[17] P. D. Seymour, *Nowhere-zero* 6*-flows*, J. Combin. Theory Ser. B **30** (1981), no. 2, 130–135. MR MR615308 (82j:05079)

[18] W.T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. **6** (1954), 80–91.

[19] ———, *A class of abelian groups*, Canad. J. Math. **8** (1956), 13–28.

[20] ———, *On the algebraic theory of graph colorings*, Journal of Combinatorial Theory **1** (1966), no. 1, 15 – 50.

[21] ———, *A geometrical version of the four color problem*, Combinatorial Math. and Its Applications (R. C. Bose and T. A. Dowling, eds.), Chapel Hill, NC: University of North Carolina Press, 1967.

[22] K. Wagner, *Über eine Eigenschaft der ebenen Komplexe*, Mathematische Annalen **114** (1937), no. 1, 570–590.