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Problems on Bispanning Graphs and Block Matroids (Extended Abstract)

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1 Definition

A graph $G = (V, E)$ is a *bispanning graph* if its edge set admits a partition $E = E_1 \dot\cup E_2$ into two spanning trees. A matroid M on a finite set E is a *block matroid* if E is the disjoint union of two bases of M. Bispanning graphs and block matroids come into play whenever one is interested in connectivity properties of exchange graphs on spanning trees or bases.

The following characterization is easily derived from matroid intersection duality:

Theorem 1 (implicitely in [7]:) *A matroid of rank r on a finite set r with $|E| = 2r$ is a block matroid if and only if*

$$\forall A \subseteq E : 2r(A) \geq |A|.$$

The recognition of bispanning graphs is particularly easy:

Proposition 1 *A bridgeless simple graph $G = (V, E)$ with $2|V| - 2$ edges is bispanning if and only if sucessively removing vertices of degree 2 does not produce a bridge.*

Example 1 *A matroid of rank 5 with 5 points on a line and 5 points in general position shows that this is not suffient in the matroid case.*

2 Cyclic base order

The following was brought to my attention by Jan van den Heuvel [6].

Conjecture 1 ([5, 12]) *Let M be a block matroid of rank r with base partition (B_1, B_2). Then the bases can be reordered to $\tilde{B}_1, \tilde{B}_2$ such that an r consecutive elements in the cyclic order $(\tilde{B}_1, \tilde{B}_2)$ form a basis.*

For graphs this is a theorem due to Farber, Richter and Shank [4]. The proof should generalize to regular matroids.

The *graph $\tau_2(M)$ of a block matroid M* has as vertices all pairs (B_1, B_2) of disjoint bases $E = B_1 \dot\cup B_2$. We have an edge $((B_1, B_2), (B'_1, B'_2))$ if and only if (B'_1, B'_2) arises from (B_1, B_2) by a *symmetric swap*, i.e.

$$B'_1 = (B_1 \cup \{e\}) \setminus \{f\} \text{ and } B'_2 = (B_2 \setminus \{e\}) \cup \{f\}.$$

The cyclic base order conjecture asserts, that every vertex (B_1, B_2) lies on a circuit of length $2r(M)$, where antipodal vertices (i.e. vertices at distance $r(M)$), consist of the same pair of bases but interchanged. It seems to be unknown whether $\tau_2(M)$ is connected in general [3] (true for regular matroids).

3 k-th minimum spanning tree

The *graph* $\tau_1(M)$ *of a matroid* M has as vertices its bases. We have an edge (B_1, B_1') if and only if $B_1' = (B_1 \cup \{e\}) \setminus \{f\}$. Note that identifying B_1 with $(B_1, E \setminus B_1)$, $\tau_2(M)$ maybe considered as induced subgraph of $\tau_1(M)$.

Let $w : E \to \mathbb{R}$ be a weight function and B_0 be a minimum weight basis. Consider the set of all possible weights of bases

$$\{w(B) \mid B \in \mathcal{B}(M)\} = \{w_0, w_1, \ldots, w_k\}.$$

We say a basis is of *order* i if $w(B) = w_i$. In particular a basis is of minimum weight if and only if it has order 0.

Kano [10] conjectured that in a graphic matroid G for $0 \le i \le k$ there is always a *basis of order* i *at distance at most* i from B_0 in $\tau(G)$. This was proven by Mayr and Plaxton [9] and generalized to matroids by Lemos [8].

Mayr and Plaxton used bispanning graphs in their proof and showed

Theorem 2 ([9]) *If* $(G, B_1 \dot\cup B_2, w)$ *is a weighted bispanning graph, where* B_1 *is a minimum spanning tree and* B_2 *is the only basis of weight* $w(B_2)$*, then* B_2 *has order at least* $|B_2|$*.*

Furthermore they conjectured

Conjecture 2 ([9]) *If* $(G, B_1 \dot\cup B_2, w)$ *is a weighted bispanning graph, where the order of* B_1 *is strictly smaller than the order of* B_2 *and* B_2 *is the only basis of weight* $w(B_2)$*, then* B_2 *has order at least* $|B_2|$*.*

Matthias Baumgart [2] verified this for weight functions where B_1 is the only basis of weight $w(B_1)$, and for series-parallel graphs.

4 Unique Exchange Properties

The *graph* $\tau_4(M)$ *of left unique exchanges* has as vertices all pairs (B_1, B_2) of disjoint bases $E = B_1 \dot\cup B_2$. We have an edge $((B_1, B_2), (B_1', B_2'))$ if and only if there exist $e \in B_2$, $f \in B_1$ such that

$$B_1' = (B_1 \cup \{e\}) \setminus \{f\} \text{ and } B_2' = (B_2 \setminus \{e\}) \cup \{f\}$$

and if $C(B_1, e)$ denotes the fundamental circuit and $D(B_2, e)$ the fundamental cut, then

$$C(B_1, e) \cap D(B_2, e) = \{e, f\}. \tag{1}$$

The definition of the *graph* $\tau_3(M)$ *of unique exchanges* is the same except that (1) is replaced by

$$C(B_1, e) \cap D(B_2, e) = \{e, f\} \text{ or } D(B_1, f) \cap C(B_2, f) = \{e, f\}. \tag{2}$$

Theorem 3 ([1]) *$\tau_4(M)$ is disconnected for the cycle matroid of K_4 but connected for all larger wheels.*

Problem 1 *Can you characterize the graphs (or block matroids) where $\tau_4(M)$ is connected?*

We have $$E(\tau_4(M)) \subseteq E(\tau_3(M)) \subseteq E(\tau_2(M)) = E(\tau_1(M)[V(\tau_2(M))]).$$

Conjecture 3 ([11]) *If M is regular, then $\tau_3(M)$ is connected.*

We do not even know whether $\tau_3(M)$ is connected for bispanning graphs. Recall that $\tau_2(M)$ is known to be connected for regular matroids and conjectured to be connected in general, while $\tau_1(M)$ is known to be connected.

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