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BICIRCULAR MATROIDS ARE 3-COLORABLE

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Abstract. Hugo Hadwiger proved that a graph which is not 3-colorable must have a K_4-minor and conjectured that a graph which is not k-colorable must have a K_{k+1}-minor. Meanwhile the class of graphs without K_4-minor has been identified as the class of series-parallel networks.

[Hochstättler, Nešetřil 2006] extended the theory of colorings and nowhere-zero-flows to oriented matroids.

We generalize the notion of being series-parallel to oriented matroids and show that generalized series-parallel oriented matroids are 3-colorable. Finally, we prove that every orientation of a bicircular matroid is generalized series-parallel.

1. Introduction

Hadwiger's conjecture, that every graph which is not k-colorable must have a K_{k+1}-minor, is "one of the deepest unsolved problems in graph theory"[5]. While the first non-trivial case, $k = 3$, was proved by Hadwiger, he pointed out that Klaus Wagner had shown that $k = 4$ is equivalent to the Four Color Theorem [22, 2, 16]. Robertson, Seymour and Thomas [17] reduced the case $k = 5$ to the Four Color Theorem. The conjecture remains open for $k \geq 6$.

Recently, Goddyn and Hochstättler [11] observed that a proper generalization of Hadwiger's conjecture to regular matroids includes Tutte's flow conjectures for the cases $k \in \{4, 5\}$ [19, 20, 21] while the case $k = 3$ remains the only non-trivial settled case. Let N be a matroid. We say that an oriented matroid $\mathcal{O}$ is N-*free* if no minor of its underlying matroid $\underline{\mathcal{O}}$ is isomorphic to N. Hadwiger's conjecture for regular matroids asserts that an oriented matroid $\mathcal{O}$ that is $M(K_{k+1})$-free is either k-colorable or $k = 4$ and $\underline{\mathcal{O}}$ has the cographic matroid of Petersen's graph as a minor.

Hochstättler and Nešetřil [12] generalized the theory of nowhere-zero flows to the class of oriented matroids. Hochstättler and Nickel [13, 14] extended this work and defined the chromatic number of an oriented matroid. They also showed that the chromatic number of a simple oriented matroid of rank r is at most $r + 1$, with equality if and only if the matroid is an orientation of $M(K_{r+1})$. This suggests that, in some sense, complete graphs are necessary to construct oriented matroids with large chromatic number. With their definition of chromatic number, Hadwiger's conjecture may, to our knowledge, hold true for the class of oriented matroids. But, in this setting, the case $k = 3$ remains open: Are $M(K_4)$-free oriented matroids always 3-colorable?

Jim Geelen asked [8] for some characterization of matroids without an $M(K_4)$-minor. Here we define the class of generalized series-parallel oriented matroids (Definition 2), which might provide some progress towards an answer to Geelen's question in the oriented case. While it is clear by definition that every generalized series-parallel matroid is $M(K_4)$-free, the converse statement, if true, would verify Hadwiger's conjecture for oriented matroids in the case $k = 3$.

We assume some familiarity with graph theory, matroid theory and oriented matroids; standard references are [23, 15, 4]. The paper is organized as follows. In the next section we review the definition of the chromatic number of an oriented matroid. In Section 3 we define generalized series-parallel oriented matroids and prove that they are 3-colorable. In Section 4 we prove that every orientation of a bicircular matroid is generalized series-parallel, and is therefore 3-colorable. We conclude with several open problems.

2. Nowhere-Zero Coflows in Oriented Matroids

A *tension* in a digraph $D = (V, A)$ is a map $f^* : A \to \mathbb{R}$ such that

$$\sum_{a \in C^+} f^*(a) - \sum_{a \in C^-} f^*(a) = 0$$

for all cycles C in the underlying graph of D, with forward arcs C^+ and backward arcs C^-. We say that f^* is a *nowhere-zero-k tension (NZ-k tension)* if $f^* : A \to \{\pm 1, \pm 2 \ldots, \pm(k-1)\}$.

Let $c : V \to \{0, 1, \ldots, k-1\}$ be a proper k-coloring of a connected graph $G = (V, E)$. Let $D = (V, A)$ be some orientation of G and define $f^* : A \to \{\pm 1, \ldots, \pm(k-1)\}$ by $f^*((u, v)) = c(v) - c(u)$. Then f^* is a NZ-k tension. The Kirchhoff Voltage Law implies every NZ-k tension arises from a proper k-coloring this way (see for example [19, 9]).

The notion of a NZ-k tension can be generalized to a regular matroid M (essentially done by Arrowsmith and Jaeger [3]) by referring to the integer combinations of rows in a totally unimodular matrix representation of M. If one attempts to further generalize NZ-k tensions to the class of oriented matroids one faces the difficulty that, for example, the four point line does not admit a nontrivial tension. The following definitions from [12, 13, 14] avoid this difficulty.

The *coflow lattice* of an oriented matroid $\mathcal{O}$ on a finite set E, denoted as $\mathcal{F}_{\mathcal{O}^*}$, is the integer lattice generated by the signed characteristic vectors of the signed cocircuits $\mathcal{D}$ of $\mathcal{O}$, where the *signed characteristic vector* of a signed cocircuit $D = (D^+, D^-)$ is defined by

$$\vec{D}(e) := \begin{cases} 1 & \text{if } e \in D^+ \\ -1 & \text{if } e \in D^- \\ 0 & \text{otherwise.} \end{cases}$$

That is

$$\mathcal{F}_{\mathcal{O}^*} = \left\{ \sum_{D \in \mathcal{D}} \lambda_D \vec{D} \mid \lambda_i \in \mathbb{Z} \right\} \subseteq \mathbb{Z}^{|E|}.$$

We call any $x \in \mathcal{F}_{\mathcal{O}^*}$ a *coflow* of $\mathcal{O}$. Such an x is a *nowhere-zero-k coflow (NZ-k coflow)* if $0 < |x(e)| < k$ holds for $e \in E$. The *chromatic number* $\chi(\mathcal{O})$ is the minimum k such that $\mathcal{O}$ has a NZ-k coflow. It is easy to see that $\chi(\mathcal{O})$ has a finite value if and only if $\mathcal{O}$ has no loops. Since reorienting an element of e in the groundset of $\mathcal{O}$ corresponds to negating $x(e)$ in every coflow x, the chromatic number is reorientation invariant. It is shown [13] that $\chi(\mathcal{O})$ is a matroid invariant for uniform or corank 3 oriented matroids. It is unknown whether this is the case in general, but it is known [13] that the dimension of $\mathcal{F}_{\mathcal{O}^*}$ in general is not a matroid invariant. Although there are alternative ways to define the chromatic number of an oriented matroid [10, 6], the following theorem supports the suitability of our definition of $\chi(\mathcal{O})$ with regard to Hadwiger's conjecture.

Theorem 1 ([14]). *Let $\mathcal{O}$ be a simple oriented matroid of rank $r \geq 3$. Then $\chi(\mathcal{O}) \leq r + 1$. Moreover, $\chi(\mathcal{O}) = r + 1$ if and only if $\mathcal{O}$ is an orientation of $M(K_{r+1})$.*

3. Generalized Series-parallel Oriented Matroids

Recall that a *parallel extension* of a matroid M is an extension by an element e' which is parallel to some element e of M while a *series extension* is a coextension by an element e' which is coparallel to some element. A matroid is called *series-parallel* [1] if it can be obtained from a one element matroid by a sequence of series and parallel extensions.

It is well known that series-parallel matroids are exactly the graphic matroids of series-parallel networks, which are the graphs without a K_4-minor. The latter makes an inductive proof of Hadwiger's conjecture for the case $k = 3$ almost trivial: if G arises by a parallel extension from G', any proper 3-coloring of G' remains a proper 3-coloring of G whereas if G arises by subdividing an edge e into e and e', we have a free color for the new vertex. Translating this proof into a proof for the resulting NZ-3 coflow in an orientation of G, the key point arises in the case $G' = G/e$ (series extension) where the induction step relies on the existence of a $\{0, \pm 1\}$-valued coflow in $\vec{G}$, say g^*, whose support is $\{e, e'\}$. It has been shown in [13, Theorem 3.8] that such a coflow g^* exists in the coflow lattice of any orientation of a uniform oriented matroid, proving that this class of matroids is 3-colorable. This motivates the following definition.

Definition 2. *Let $\mathcal{O}$ be an oriented matroid. We say that $\mathcal{O}$ is* generalized series-parallel (GSP) *if every simple minor of $\mathcal{O}$ has a $\{0, \pm 1\}$-valued coflow which has exactly one or two nonzero entries.*

By its definition, the class of GSP oriented matroids is closed under minors. Note that an orientation of a regular matroid is GSP if and only if it is an orientation of a series-parallel matroid.

THEOREM 3. *Let $\mathcal{O}$ be a GSP oriented matroid on a finite set E without loops. Then $\mathcal{O}$ has a NZ-3 coflow.*

Proof. We proceed by induction on $|E|$. If $\mathcal{O}$ is not simple, then it has two parallel or antiparallel elements e and e'. Since $\mathcal{O} \setminus \{e'\}$ is generalized series-parallel without loops, by induction it admits a NZ-3 coflow which extends to a NZ-3 coflow in $\mathcal{O}$ because every cocircuit contains both or none of e and e' and either always with the same or always with opposite sign. Thus, we may assume $\mathcal{O}$ is simple and that $\mathcal{O}$ has a $\{0, \pm 1\}$-valued coflow, say g^*, whose support is $\{e, e'\}$ (possibly with $e = e'$). By reorienting e' if necessary, we may assume that $g^*(e') = 1$. Let $\mathcal{O}' = \mathcal{O}/e$. Then $\mathcal{O}'$ is a GSP oriented matroid without loops and hence there exists a NZ-3 coflow f'^* in $\mathcal{O}'$, which extends to a coflow f^* in $\mathcal{O}$ which is 0 on e and has entries from $\{\pm 1, \pm 2\}$ elsewhere. If $f^*(e') \in \{-2, 0, 1\}$, then $f^* + g^*$ is a NZ-3 coflow in $\mathcal{O}$. Otherwise we have $f^*(e') \in \{-1, 2\}$, so $f^* - g^*$ is a NZ-3 coflow in $\mathcal{O}$. □

An practical way to prove that every orientation of a matroid from a certain minor closed class is GSP is to show the existence of *positive colines* for that class.

DEFINITION 4. *Let M be a matroid. A* copoint *of M is a flat of codimension 1, i.e., a hyperplane. A* coline *is a flat of codimension 2. If L is a coline of M and $L \subseteq H$ a copoint, then we say H is a* copoint on L. *The* copoint *is* simple *if $|H \setminus L| = 1$, otherwise it is* fat. *A coline L is* positive *if there are more simple than fat copoints on L.*

LEMMA 5. *If an orientable matroid M has a positive coline, then every orientation $\mathcal{O}$ of M has a $\{0, \pm 1\}$-valued coflow which has exactly two nonzero entries.*

Proof. Let L be a positive coline of M and $\mathcal{O}$ some orientation of M. By [4] Proposition 4.1.17 the interval $[L, \hat{1}]$ in the big face lattice has the structure of the big face lattice of an even cycle. Here, each antipodal pair of vertices of the even cycle corresponds to a copoint of L. Neighboring copoints in this cycle are conformal vectors (see [4] 3.7). Since L is positive, by the pidgeonhole principle there must exist two neighboring simple copoints $D_1, D_2 \in \mathcal{D}$ in the even cycle. Since both copoints are simple, $(D_1 \cup D_2) \setminus L = \{e_1, e_2\}$ and since they are conformal, $\vec{D}_1 - \vec{D}_2$ is a vector with exactly two non-zeroes which are $+1$ or -1, namely on e_1 and e_2. □

4. Bicircular Matroids

In this section we will show that orientations of bicircular matroids are GSP and hence 3-colorable by Theorem 3.

DEFINITION 6. [18] *Let $G = (V, E)$ be a not necessarily simple graph. The* bicircular matroid $B(G)$ represented *by G has as its independent sets the subsets of E that contain the edge set of at most one cycle of G in each of its connected components. A matroid is* bicircular *if deleting its loops results in $B(G)$ for some graph G.*

Note that the original definition of a bicircular matroid did not allow loops. We conclude easily from the definition:

PROPOSITION 7. (1) *Bicircular matroids are closed under taking minors.*
(2) *For any graph G, the edge set of every forest in G defines a closed set, i.e., a flat of $B(G)$.*

The following result suffices for our purpose.

THEOREM 8. *Every simple bicircular matroid M of rank ≥ 2 has a positive coline.*

Proof. If M has two coloops, then deleting both of them results in a coline of M which has exactly 2 simple copoints on it, and no other copoints. If M has exactly one coloop, say e, then we must have $\text{rank}(M) \geq 3$. Applying induction on $|M|$, we find that M/e has a positive coline which, together with e, gives a positive coline of M. Similarly, if a connected component of M has a positive coline, then adding to it every other component of M results in a positive coline of M. Thus we may assume M is connected, simple, with rank ≥ 3, and without coloops. Therefore $M = B(G)$ for some connected graph G with $|V(G)| = \text{rank}(M) \geq 3$. We select G to have the fewest possible number of loops. Let T_1 be a spanning tree of G, and let v be a leaf of T_1. Let $\{e_1, e_2, \ldots, e_k\}$ be the set of non-loop edges of G which are incident with v, for some $k \geq 1$. Since M is simple, v is incident to at most one loop edge of G; we denote this loop edge by ℓ if it exists in G. If

$k = 1$, then ℓ must exist, for otherwise e_1 is a coloop of M. But now we have that $B(G) = B(G')$ where G' is the graph obtained from G by repositioning ℓ so that e_1 and ℓ are parallel edges in G'. This contradicts the choice of G. So we may assume $k \geq 2$.

Exactly one edge in $\{e_1, e_2, \ldots, e_k\}$ belongs to T_1, say $e_1 \in E(T_1)$. Let $F = T_1 - e_1$ and let $T_i = F + e_i$, for $1 \leq i \leq k$. Then T_i is a spanning tree of G and v is a leaf of T_i, for $1 \leq i \leq k$. By Proposition 7, the set $L := E(F)$ is a flat of M. Since it is also independent in M and $|L| = |V(G)| - 2$, it is a coline of M. Also, for $1 \leq i \leq k$, the set $L \cup \{e_i\} = E(T_i)$ is a simple copoint on L. If ℓ exists then $L \cup \{\ell\}$ is also a simple copoint on L. Every remaining edge $e \in E(G) - L - \{e_1, e_2, \ldots, e_k, \ell\}$ has both of its endpoints in the connected subgraph $F - v$, so the closure of $L \cup \{e\}$ in M is the copoint $E - \{e_1, e_2, \ldots, e_k, \ell\}$. There are no other copoints on L, and at most one of them is fat. Since $k \geq 2$, we have shown that L is a positive coline of M. □

Theorem 9. *Every orientation of a bicircular matroid is GSP.*

Proof. The statement trivially holds for bicircular matroids of rank ≤ 1. The result now follows from Lemma 5, Theorem 8 and Proposition 7. □

Corollary 10. *Bicircular matroids are 3-colorable.*

5. Open Problems and Conjectures

The definition of an oriented matroid being GSP does not seem to be related to duality.

Question 11. *Are GSP oriented matroids closed under duality?*

Since $M(K_4)$ is not series-parallel, every GSP oriented matroid is $M(K_4)$-free. Probably it would be too much to hope for the other inclusion:

Question 12. *Does there exist an $M(K_4)$-free oriented matroid which is not GSP?*

The previous question begs the following, more fundamental one.

Question 13. *Does there exist a $M(K_4)$-free matroid which is not orientable?*

Not every $M(K_4)$-free matroid has a positive coline, an example being P_7. We checked with Lukas Finschi's database [7] that P_7 has a unique reorientation class. Although there is no positive coline, this class still is GSP. Every bicircular matroid is a gammoid. Since P_7 is not a gammoid, the following might be a possible extension of Theorem 9 to the class of gammoids, which includes the class of transversal matroids.

Conjecture 14. *Every simple gammoid of rank at least two has a positive coline.*

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