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STICKY MATROIDS AND KANTOR'S CONJECTURE

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Dedicated to Achim Bachem on the occasion of his 70th birthday.

Abstract. We prove the equivalence of Kantor's Conjecture and the Sticky Matroid Conjecture due to Poljak und Turzík.

1. Introduction

The purpose of this paper is to prove the equivalence of two classical conjectures from combinatorial geometry. Kantor's Conjecture [5] adresses the problem whether a combinatorial geometry can be embedded into a modular geometry, i.e., a direct product of projective spaces. He conjectured that for finite geometries this is always possible if all pairs of hyperplanes are modular.

The other conjecture, the Sticky Matroid Conjecture (SMC) due to Poljak and Turzík [8] concerns the question whether it is possible to glue two matroids together along a common part. They conjecture that a "common part" for which this is always possible, a sticky matroid, must be modular. It is well-known (see eg. [7]) that modular matroids are sticky and easy to see [8] that modularity is necessary for ranks up to three. Bachem and Kern [1] proved that a rank-4 matroid that has two hyperplanes intersecting in a point is not sticky. They also stated that a matroid is not sticky if for each of its non-modular pairs there exists an extension decreasing its modular defect. The proof of this statement had a flaw which was fixed by Bonin [2]. Using a result of Wille [9] and Kantor [5] this implies that the sticky matroid conjecture is true if and only if it holds in the rank-4 case. Bonin [2] also showed that a matroid of rank $r \geq 3$ with two disjoint hyperplanes is not sticky and that non-stickiness is also implied by the existence of a hyperplane and a line that do not intersect but can be made modular in an extension.

We generalize Bonin's result and show that a matroid is not sticky if it has a non-modular pair that admits an extension decreasing its modular defect. Moreover by showing the existence of the proper amalgam of two arbitrary extensions of the matroid we prove that in the rank-4 case this condition is also necessary for a matroid not to be sticky. As a consequence from every counterexample to Kantor's conjecture arises a matroid that can be extended in finite steps to a counterexample of the (SMC), implying the equivalence of the two conjectures. A further consequence of our results is the equivalence of both conjectures to the following:

Conjecture 1. *In every finite non-modular matroid there exists a non-modular pair and a single-element extension decreasing its modular defect.*

Finally, we present an example proving that the (SMC), like Kantor's Conjecture fails in the infinite case.

We assume familiarity with matroid theory. The standard reference is [7].

2. Our results

Let M be a matroid with groundset E and rank function r. We define the *modular defect* $\delta(X,Y)$ of a pair of subsets $X,Y \subseteq E$ as

$$\delta(X,Y) = \mathrm{r}(X) + \mathrm{r}(Y) - \mathrm{r}(X \cup Y) - \mathrm{r}(X \cap Y).$$

By submodularity of the rank function, the modular defect is always non-negative. If it equals zero, we call (X,Y) a *modular pair*. A matroid is called *modular* if all pairs of flats form a modular pair.

An *extension* of a matroid M on a groundset E is a matroid N on a groundset $F \supseteq E$ such that $M = N|E$. If N_1, N_2 are extensions of a common matroid M with groundsets F_1, F_2, E resp. such that $F_1 \cap F_2 = E$, then a matroid $A(N_1,N_2)$ with groundset $F_1 \cup F_2$ is called an *amalgam* of N_1 and N_2 if $A(N_1,N_2)|F_i = N_i$ for $i = 1,2$.

Theorem 1 (Ingleton see [7] 11.4.10 (ii))**.** *If M is a modular matroid then for any pair (N_1,N_2) of extensions of M an amalgam exists.*

We found a proof of this result only for finite matroids (see eg. [7]). We will show that it also holds for infinite matroids of finite rank.

Conjecture 2 (Sticky Matroid Conjecture (SMC) [8])**.** *If M is a matroid such that for all pairs (N_1,N_2) of extensions of M an amalgam exists, then M is modular.*

The following preliminary results concerning the (SMC) are known:

Theorem 2 ([8, 1, 2])**.** *Let M be a matroid.*

- *(i) If $\mathrm{r}(M) \leq 3$ then the (SMC) holds for M.*
- *(ii) If the (SMC) holds for all rank-4 matroids, then it is true in all ranks.*
- *(iii) Let l be a line and H a hyperplane in M such that $l \cap H = \emptyset$. If M has an extension M' such that $\mathrm{r}_{M'}(\mathrm{cl}_{M'}(l) \cap \mathrm{cl}_{M'}(H)) = 1$, then M is not sticky.*
- *(iv) If M has two disjoint hyperplanes then is not sticky.*

We will generalize the last two assertions and prove:

Theorem 3. *Let M be a matroid, X and Y two flats such that $\delta(X,Y) > 0$. If M has an extension M' such that $\delta_{M'}(\mathrm{cl}_{M'}(X), \mathrm{cl}_{M'}(Y)) < \delta(X,Y)$ then M is not sticky.*

We postpone the proof of Theorem 3 to Section 3.

We call a matroid *hypermodular* if each pair of hyperplanes forms a modular pair. With this notion we can rephrase Kantor's Conjecture.

Conjecture 3 (Kantor [5], page 192)**.** *Every finite hypermodular matroid embeds into a modular matroid.*

Like the (SMC) Kantor's Conjecture can be reduced to the rank-4 case (see Corollary 3, Section 5).

Next, we consider the correspondence between single-element extensions of matroids and modular cuts.

Definition 1. *A set $\mathcal{M}$ of flats of a matroid M is called a* modular cut *of M if the following holds:*

- *(i) If $F \in \mathcal{M}$ and F' is a flat in M with $F' \supseteq F$, then $F' \in \mathcal{M}$.*
- *(ii) If $F_1, F_2 \in \mathcal{M}$ and (F_1,F_2) is a modular pair, then $F_1 \cap F_2 \in \mathcal{M}$.*

Theorem 4 (Crapo 1965 [3])**.** *There is a one-to-one-correspondence between the single-element extensions $M +_{\mathcal{M}} p$ of a matroid M and the modular cuts $\mathcal{M}$ of M. $\mathcal{M}$ consists precisely of the set of flats of M containing the new point p in $M +_{\mathcal{M}} p$.*

The set of all flats of a matroid M is a modular cut, the *trivial modular cut*, corresponding to an extension with a loop. The empty set is a modular cut corresponding to an extension with a coloop, the only single-element extension increasing the rank of M. For a flat F of M, the set $\mathcal{M}_F = \{G \mid G \text{ is a flat of } M \text{ and } G \supseteq F\}$ is a modular cut of M. We call it a *principal modular cut*. We say that in the corresponding extension the new point is *freely added* to F. A *modular cut $\mathcal{M}_\mathcal{A}$ generated by a set of flats $\mathcal{A}$* is the smallest modular cut containing $\mathcal{A}$.

The following is immediate from Theorem 7.2.3 of [7].

Proposition 1. *If (X, Y) is a non-modular pair of flats of a matroid M, then there exists an extension decreasing its modular defect (we call the pair* intersectable*) if and only if the modular cut generated by X and Y is not the principal modular cut $\mathcal{M}_{X\cap Y}$.*

We call a matroid *OTE (only trivially extendable)* if all of its modular cuts different from the empty modular cut are principal.

Most of this paper will be devoted to the proof of the following theorem.

Theorem 5. *If M is a rank-4 matroid that is OTE, then M is sticky.*

As we will prove with Theorem 9, Theorem 3 implies that a matroid that is not OTE is not sticky. Hence Theorem 5 implies that for rank-4 matroids being sticky is equivalent to being OTE. Since the (SMC) is reducible to the rank-4 case, it is equivalent to the conjecture that every rank-4 matroid that is OTE is already modular. For finite matroids, this is our Conjecture 1, which is also reducible to the rank-4 case (see the remark after the proof of Corollary 3).

Like Kantor's Conjecture our Conjecture 1 is no longer true in the infinite case. This will be a consequence of the following theorem, proven in Section 5.

Theorem 6. *Every finite matroid can be extended to a (not necessarily finite) matroid of the same rank that is OTE.*

Starting from, say, the Vámos matroid this yields an infinite rank-4 non-modular matroid that is OTE, hence a counterexample to the (SMC) in the infinite case.

Finally, Theorem 11 will imply that any finite counterexample to Kantor's Conjecture can be embedded into a finite non-modular matroid that is OTE. In the rank-4 case any counterexample to Kantor's Conjecture this way yields a finite counterexample to the (SMC). We wil show in Corollary 3 that Kantor's Conjecture is reducible to the rank-4 case, hence the (SMC) implies Kantor's Conjecture. It had already been observed by Faigle (see [1]) and was explicitely mentioned by Bonin in [2] that Kantor's Conjecture implies the (SMC). The latter is now immediate from Theorem 3 and the former establishes the equivalence of the two conjectures.

Corollary 1. *Kantor's Conjecture holds true if and only if the Sticky Matroid Conjecture holds true.*

3. Proof of Theorem 3

We start with a proposition that states that the so called Escher matroid ([7] Fig. 1.9) is not a matroid.

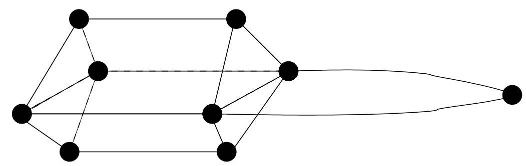

FIGURE 1. This is not a matroid

Proposition 2. *Let l_1, l_2, l_3 be three lines in a matroid that are pairwise coplanar but not all lying in a plane. If l_1 and l_2 intersect in a point p, then p must also be contained in l_3.*

Proof. By submodularity of the rank function we have

$$r\left((l_1 \vee l_3) \wedge (l_2 \vee l_3)\right) \leq \mathrm{r}(l_1 \vee l_3) + (l_2 \vee l_3) - \mathrm{r}(l_1 \vee l_2 \vee l_3) = 3 + 3 - 4 = 2.$$

Now $l_3 \vee p \leq (l_1 \vee l_3) \wedge (l_2 \vee l_3)$ and hence p must lie on l_3. □

Probably the easiest way to prove that the (SMC) holds for rank 3 is to proceed as follows. If a rank-3 matroid M is not modular, then it has a pair of disjoint lines. We consider two extensions N_1 and N_2 of M such that N_1 adds to the two lines a point of intersection and N_2 erects a Vámos-cube (V_8 in [7]) using the disjoint lines as base points. By Proposition 2 the amalgam of N_1 and N_2 cannot exist (see Figure 1).

Bonin [2] generalized this idea to the situation of a disjoint line-hyperplane pair in matroids of arbitrary rank. We further generalize this to a non-modular pair of a hyperplane H and a flat F that can be made modular by a proper extension. Our first aim is to show that such a pair exists in any matroid that is not OTE. Again, the following is immediate:

Proposition 3. *Let M be a matroid, M' an extension of M and (X, Y) a modular pair of flats in M. Then $(\mathrm{cl}_{M'}(X), \mathrm{cl}_{M'}(Y))$ is a modular pair in M'. Moreover*

$$\mathrm{cl}_{M'}(X) \cap \mathrm{cl}_{M'}(Y) = \mathrm{cl}_{M'}(X \cap Y).$$

Proposition 4. *Let M be a matroid, $\mathcal{M}$ a modular cut in M and $M' = M +_{\mathcal{M}} p$ the corresponding single-element extension. If M' does not contain a modular pair of flats $X' = X \cup p, Y' = Y \cup p$ such that X and Y are a non-modular pair in M, then*

$$\mathcal{M}' := \{\mathrm{cl}_{M'}(F) \mid F \in \mathcal{M}\}$$

is a modular cut in M'.

Lemma 1. *Let M_0 be a matroid that is not OTE and (X, Y) be a non-modular pair of smallest modular defect $\delta := \delta(X, Y)$ such that there is a single-element extension decreasing their modular defect. Then there exists a sequence $M_1, \ldots, M_\delta$ of matroids such that M_i is a single-element extension of M_{i-1} for $i = 1, \ldots, \delta$ and $\delta_{M_i}(\mathrm{cl}_{M_i}(X), \mathrm{cl}_{M_i}(Y)) = \delta - i$. In particular $(\mathrm{cl}_{M_\delta}(X), \mathrm{cl}_{M_\delta}(Y))$ are a modular pair in M_δ.*

Proof. Let $\mathcal{M}$ denote the modular cut generated by X and Y in M_0. Inductively we conclude, that by the choice of X and Y

$$\mathcal{M}_i := \{\mathrm{cl}_{M_i}(F) \mid F \in \mathcal{M}\}$$

is a modular cut in M_i for $i = 1, \dots, \delta - 1$ implying the assertion. □

Lemma 2. *Let M be a matroid that is not OTE. Then there exists an intersectable non-modular pair (F, H) of smallest modular defect, where F is a minimal element in the modular cut $\mathcal{M}_{F,H}$ generated by H and F, and H is a hyperplane of M.*

Proof. Since M is not OTE, it is not modular and hence of rank at least three. Every non-modular pair of flats in a rank-3 matroid clearly satisfies the assertion. Hence we may assume $\mathrm{r}(M) \geq 4$. Let (X, Y) be a non-modular intersectable pair of flats in M of smallest modular defect δ_{min} and chosen such that, first, X is of minimal and, second, Y of maximal rank. We claim that $F = X$ and $H = Y$ are as required. Let $\mathcal{M}_{X,Y}$ be the modular cut generated by these two flats.

Assume, contrary to the first assertion, that there exists an $F \in \mathcal{M}_{X,Y}$ with $F \subsetneq X$. Since the principal modular cut $\mathcal{M}_{X\cap Y}$ contains X and Y, it is a superset of the modular cut $\mathcal{M}_{X,Y}$. Hence we obtain $X \cap Y \subseteq F$. Since $\mathcal{M}_{X,Y}$ contains F and Y but not $X \cap Y = F \cap Y$, the pair (F, Y) is non-modular and intersectable in M (according to Proposition 4). Due to submodularity of r we have $\mathrm{r}(X) + \mathrm{r}(F \cup Y) \geq \mathrm{r}(X \cup Y) + \mathrm{r}(F)$ and hence:

$$\begin{aligned}\delta(F, Y) &= \mathrm{r}(F) + \mathrm{r}(Y) - \mathrm{r}(F \cup Y) - \mathrm{r}(F \cap Y) \\ &\leq \mathrm{r}(X) + \mathrm{r}(Y) - \mathrm{r}(X \cup Y) - \mathrm{r}(X \cap Y) = \delta(X, Y) = \delta_{min},\end{aligned}$$

contradicting the choice of X. Next we show that $\mathrm{cl}(X \cup Y) = E(M)$. Assume to the contrary that there exists $p \in E(M) \setminus \mathrm{cl}(X \cup Y)$ and let $Y_1 = \mathrm{cl}(Y \cup p)$. Then $X \cap Y = X \cap Y_1$ and hence $\delta(X, Y_1) = \delta(X, Y)$. Since $\mathcal{M}_{X,Y_1} \subseteq \mathcal{M}_{X,Y}$, the pair (X, Y_1) remains intersectable, contradicting the choice of Y, and hence verifying $\mathrm{cl}(X \cup Y) = E(M)$. Finally, assume Y is not a hyperplane. Let $Y' = \mathrm{cl}(Y \cup p)$ with $p \in X \setminus Y$. Then

$$\begin{aligned}\delta(X, Y') &= \mathrm{r}(X) + \mathrm{r}(Y') - \mathrm{r}(X \cup Y') - \mathrm{r}(X \cap Y') \\ &= \mathrm{r}(X) + \mathrm{r}(Y) + 1 - \mathrm{r}(X \cup Y) - \mathrm{r}(X \cap Y) - 1 = \delta(X, Y).\end{aligned}$$

Since Y is not a hyperplane and $\mathrm{cl}(X \cup Y) = E(M)$, we must have $X \cap Y' \subsetneq X$, and X being minimal in $\mathcal{M}_{X,Y}$ implies $X \cap Y' \notin \mathcal{M}_{X,Y}$. Now $\mathcal{M}_{X,Y'} \subseteq \mathcal{M}_{X,Y}$ yields that $X \cap Y' \notin \mathcal{M}_{X,Y'}$ and thus by Proposition 4 the pair (X, Y') is intersectable with $\delta(X, Y') = \delta(X, Y) = \delta_{min}$, contradicting the choice of Y. □

Lemmas 1 and 2 now imply the following:

Theorem 7. *Let M be a matroid that is not OTE. Then there exist*

(i) a non-modular pair (F, H) where H is a hyperplane of M and
(ii) an extension N of M such that $(\mathrm{cl}_N(F), \mathrm{cl}_N(H))$ is a modular pair in N.

On the other hand we also have:

Theorem 8. *Let M be a matroid and (F, H) a non-modular pair of disjoint flats, where H is a hyperplane of M. Then there exists an extension N of M such that for every extension N' of N, $(\mathrm{cl}_{N'}(F), \mathrm{cl}_{N'}(H))$ is not a modular pair in N'.*

Proof. We follow the idea from [1] and Bonin's proof [2] and erect a Vámos-type matroid above F and H. Clearly, $r := \mathrm{r}_M(M) \geq 3$ and $2 \leq \mathrm{r}_M(F) \leq r-1$. We extend M by first adding a set A of $r-1-\mathrm{r}_M(F)$ elements freely to H. Next, we add, first, a coloop e, and then an element f freely to the resulting matroid, yielding an extension N_0 with groundset $E(M) \cup A \cup \{e,f\}$ and of rank $r+1$. Note, that $\mathrm{cl}_{N_0}(H) = H \cup A$. We consider the following sets:

- $T_1 = F \cup A \cup e$
- $T_2 = H \cup A \cup e$
- $B_1 = F \cup A \cup f$
- $B_2 = H \cup A \cup f$

Note that (T_1, T_2), (B_1, B_2) are non-modular pairs of hyperplanes of rank r in N_0 with the same modular defect

$$\delta(T_1, T_2) = 2r - (r+1) - (r - \mathrm{r}_M(F)) = \mathrm{r}_M(F) - 1 = \delta(B_1, B_2).$$

Any non-modular pair of hyperplanes in a matroid is intersectable because the modular cut generated by the two hyperplanes contains additionally only the groundset of the matroid and hence is non-principal (see Proposition 1). In the corresponding single-element extension the modular defect of the hyperplane-pair decreases by one. If this defect is still non-zero these two hyperplanes remain intersectable. Repeating this process until they become a modular pair, the modular defect of other hyperplane-pairs stays unaffected in these extensions. This way, we obtain an extension N of the matroid N_0 of rank $r+1$ with groundset $E(N_0) \cup P \cup Q$ where P and Q are independent sets of size $\mathrm{r}_M(F) - 1$ such that $(\mathrm{cl}_N(T_1), \mathrm{cl}_N(T_2))$ and $(\mathrm{cl}_N(B_1), \mathrm{cl}_N(B_2))$ are modular pairs in N and $P \subseteq \mathrm{cl}_N(T_1) \cap \mathrm{cl}_N(T_2)$ resp. $Q \subseteq \mathrm{cl}_N(B_1) \cap \mathrm{cl}_N(B_2)$. We will show now that the matroid N is as required.

Assume to the contrary that there exists an extension N' of N such that $(\mathrm{cl}_{N'}(F), \mathrm{cl}_{N'}(H))$ is a modular pair. As $A \subseteq \mathrm{cl}_{N'}(H)$ and $A \cap \mathrm{cl}_{N'}(F) = \emptyset$ we compute

$$\begin{aligned}
&\mathrm{r}_{N'}((\mathrm{cl}_{N'}(F) \cap \mathrm{cl}_{N'}(H)) \cup A) = \mathrm{r}_{N'}(\mathrm{cl}_{N'}(F) \cap \mathrm{cl}_{N'}(H))) + |A| \\
&= \mathrm{r}_{N'}(\mathrm{cl}_{N'}(F)) + |A| + \mathrm{r}_{N'}(\mathrm{cl}_{N'}(H)) - \mathrm{r}_{N'}(\mathrm{cl}_{N'}(F) \cup \mathrm{cl}_{N'}(H))) \\
&= \mathrm{r}_{N'}(\mathrm{cl}_{N'}(F \cup A)) + \mathrm{r}_{N'}(\mathrm{cl}_{N'}(H)) - \mathrm{r}_{N'}(\mathrm{cl}_{N'}(F \cup H)) \\
&= (r-1) + (r-1) - r = r-2. \qquad (1)
\end{aligned}$$

Let $D_1 = \mathrm{cl}_{N'}(A \cup P \cup e)$ and $D_2 = \mathrm{cl}_{N'}(A \cup Q \cup f)$. Proposition 3 yields $\mathrm{cl}_{N'}(\mathrm{cl}_N(T_1)) \cap \mathrm{cl}_{N'}(\mathrm{cl}_N(T_2)) = \mathrm{cl}_{N'}(\mathrm{cl}_N(T_1) \cap \mathrm{cl}_N(T_2))$ and it holds $\mathrm{r}_{N'}(D_1) = \mathrm{r}_{N'}(D_2) = r-1$. We obtain

$$\begin{aligned}
&\mathrm{r}_{N'}((\mathrm{cl}_{N'}(F) \cap \mathrm{cl}_{N'}(H)) \cup D_1) \leq \mathrm{r}_{N'}((\mathrm{cl}_{N'}(F \cup D_1) \cap \mathrm{cl}_{N'}((H \cup D_1)) \\
&= \mathrm{r}_{N'}(\mathrm{cl}_{N'}(T_1) \cap \mathrm{cl}_{N'}(T_2)) = \mathrm{r}_{N'}(\mathrm{cl}_N(T_1) \cap \mathrm{cl}_N(T_2)) = r-1 = \mathrm{r}_{N'}(D_1). \qquad (2)
\end{aligned}$$

This implies $\mathrm{cl}_{N'}(F) \cap \mathrm{cl}_{N'}(H) \subseteq D_1$. Similarly, using B_1 and B_2 instead of T_1 and T_2, we get $\mathrm{cl}_{N'}(F) \cap \mathrm{cl}_{N'}(H) \subseteq D_2$ and conclude $(\mathrm{cl}_{N'}(F) \cap \mathrm{cl}_{N'}(H)) \cup A \subseteq D_1 \cap D_2$. This yields

$$\mathrm{r}_{N'}(D_1 \cap D_2) \geq \mathrm{r}_{N'}((\mathrm{cl}_{N'}(F) \cap \mathrm{cl}_{N'}(H)) \cup A) \overset{(1)}{=} r-2. \qquad (3)$$

From $\mathrm{r}_{N'}(D_1 \cup D_2) = \mathrm{r}_{N'}(A \cup P \cup Q \cup e \cup f)) = r+1$ we finally obtain

$$\mathrm{r}_{N'}(D_1) + \mathrm{r}_{N'}(D_2) \overset{(2)}{=} 2r-2 < (r+1) + (r-2) \overset{(3)}{\leq} \mathrm{r}_{N'}(D_1 \cup D_2) + \mathrm{r}_{N'}(D_1 \cap D_2)$$

contradicting submodularity. □

Summarizing the two previous theorems yields the final result of this section:

Theorem 9. *Let M be a matroid that is not OTE. Then M is not sticky.*

Proof. By Theorem 7, M has a non-modular intersectable pair of flats (F, H) such that H is a hyperplane, and there exists an extension N_1 of M such that $(\mathrm{cl}_{N_1}(F), \mathrm{cl}_{N_1}(H))$ is a modular pair. Possibly contracting $(F \cap H)$, and referring to Lemma 7 of [1], we may assume that F and H are disjoint. Thus, by Theorem 8, there also exists an extension N_2 of M such that in every extension N of N_2 the pair $(\mathrm{cl}_N(F), \mathrm{cl}_N(H))$ is not modular. Hence M is not sticky. □

4. Hypermodularity and OTE matroids

We collect some facts about hypermodular matroids and OTE matroids that we need for the proof of Theorem 5 and the embedding theorems in the next section. Recall that a matroid is hypermodular if any pair of hyperplanes intersects in a coline. Modular matroids are hypermodular and hypermodular matroids of rank at most 3 must be modular. Thus, a contraction of a hypermodular matroid of rank n by a flat of rank $n-3$ is a modular matroid of rank 3. Every projective geometry $P(n, q)$ is hypermodular and remains hypermodular if we delete up to $q-3$ of its points. In the following we will focus on the case of hypermodular matroids of rank 4.

Proposition 5. *Let M be a hypermodular rank-4 matroid. If M contains a disjoint line and hyperplane, then M also contains two disjoint coplanar lines. The same holds for a modular cut in M.*

Proof. Let (l_1, e_1) be a disjoint line-plane pair in M. Take a point p in e_1. Because of hypermodularity, the plane $l_1 \vee p$ intersects the plane e_1 in a line l_2 in M. The lines l_1 and l_2 are coplanar and disjoint. If now l_1 and e_1 are elements of a modular cut $\mathcal{M}$ in M then it holds also $l_2 \in \mathcal{M}$. □

The next results are matroidal versions of similar results of Klaus Metsch (see [6]) for linear spaces.

Lemma 3. *Let M be a hypermodular matroid of rank 4 on a groundset E. Let l_1, l_2 be two disjoint coplanar lines. Then E can be partitioned into l_1, l_2 and lines that are coplanar with l_1 and with l_2. The modular cut $\mathcal{M}$ generated by l_1 and l_2 always contains such a line-partition of E.*

Proof. We set $e = \mathrm{cl}(l_1 \cup l_2)$. Then $l_p := (l_1 \vee p) \wedge (l_2 \vee p)$ is a line for every $p \in E \setminus e$ and coplanar to l_1 and l_2. By Proposition 2 it must be disjoint from l_1 and l_2 and from e. This together with Proposition 2 implies that for $q \in E \setminus e$ with $p \neq q$ we must have either $l_p \wedge l_q = 0$ or $l_p = l_q = p \vee q$. We denote the set of lines constructed this way by Δ. Now we choose a line $l_{p^*} \in \Delta$ and for each $r \in e \setminus (l_1 \cup l_2)$ we get a line $l_r = e \wedge (l_{p^*} \vee r)$. Let Σ be the set of lines obtained in that way. It is clear that Σ is a line partition of $e \setminus (l_1 \cup l_2)$. Again, Proposition 2 implies that these lines must be pairwise disjoint and disjoint from l_1, l_2, l_{p^*} and all lines $l_q \in \Delta$. Now, the set $\Gamma = l_1 \cup l_2 \cup \Sigma \cup \Delta$ is the desired set of lines partitioning E. Obviously, it holds $\Gamma \subseteq \mathcal{M}$. □

A non-trivial and non-principal modular cut in a matroid always contains a non-modular pair of flats. Proposition 5 implies, that in a hypermodular rank-4 matroid it even must contain two disjoint coplanar lines. By Lemma 3 we, thus, get a set of pairwise disjoint lines that partition the ground set. Moreover we have:

Theorem 10. *(i) Under the assumptions of Lemma 3 the following two statements are equivalent:*
(a) There exists a single-element extension M' where $\mathrm{cl}_{M'}(l_1)$ and $\mathrm{cl}_{M'}(l_2)$ intersect.
(b) The modular cut generated by l_1 and l_2 in M contains a set of pairwise coplanar lines, l_1 and l_2 among them, partitioning the groundset $E(M)$.
(ii) If a single-element extension M' as in (i) exists, then the restriction to M of any line in M' is a line.
(iii) If there is no single-element extension as in (i), the matroid M contains two non-coplanar lines l_3, l_4 such that l_i and l_j are coplanar for all $i \in \{1,2\}$ and $j \in \{3,4\}$ and no three of them are coplanar, i.e., it has the Vámos matroid containing l_1 and l_2 as a restriction.

Proof. (i) By Lemma 3 the modular cut $\mathcal{M}$ generated by l_1 and l_2 contains a set of lines partitioning the groundset E. Since any two of these lines intersect in the extension M' in the new point, they must be coplanar.

On the other hand, if we have a set Γ of pairwise coplanar lines partitioning the groundset E, l_1 and l_2 among them, these lines must form the minimal elements of a modular cut. This is seen as follows. Consider the set $\mathcal{M}$ of flats in M which are elements or supersets of elements of Γ. Any two lines of $\mathcal{M}$ are disjoint and coplanar, hence they do not form a modular pair. For $p \in E$ let l_p denote the line in Γ containing p and let $h \in \mathcal{M}$ be a hyperplane containing p. Then h contains l_p or some other line l that is coplanar with l_p. Since in the second case $l_p \le l \vee p \le h$ we always have $l_p \le h$. Let $h_1 \neq h_2$ be two hyperplanes in $\mathcal{M}$, let $l = h_1 \wedge h_2$ and $p \neq q$ be two points on l. Then $l_p \le h_i$ and $l_q \le h_i$ for $i \in \{1,2\}$ implying $l_p = l_q = l$. Finally, consider a hyperplane h and a line l. If they are a modular pair then they must intersect in a point r, hence $l = l_r$ and $l \le h$. Thus $\mathcal{M}$ is a modular cut defining a single-element extension where l_1 and l_2 intersect.

(ii) Let p denote the new point and l a line containing p. Let q be another point on l. Then q is contained in a line l_q in M of the partition of $E(M)$ in lines. In M' we obtain $\{p,q\} \subseteq \mathrm{cl}_{M'}(l_q)$. Since $\{p,q\} \subseteq l$ we obtain $\mathrm{cl}_{M'}(l_q) = l$ hence the restriction of l to M is the line l_q.

(iii) Let $\Gamma = l_1 \cup l_2 \cup \Sigma \cup \Delta$ be the line-partition of the groundset E from the proof of Lemma 3. By (i) there exist l_3 and l_4 in $\Gamma \setminus \{l_1, l_2\}$ that are not coplanar and hence $l_3 \cup l_4 \not\subseteq \mathrm{cl}(l_1 \cup l_2) = e$. If $l_3, l_4 \in \Delta$ we are done hence we may assume that $l_3 = l_r \in \Sigma$ and $l_4 = l_q \in \Delta$ where $l_q = (l_1 \vee q) \wedge (l_2 \vee q)$ and $l_r = e \wedge (l_{p^*} \vee r)$, as in the proof of Lemma 3. Since l_{p^*} and l_3 are coplanar we conclude $l_{p^*} \neq l_4$. If l_{p^*} and l_4 are not coplanar, we replace l_3 by l_{p^*} and are done. Hence we may assume that they are coplanar. The hyperplanes $l_4 \vee r$ and $l_{p^*} \vee r$ intersect in the line $l_3' = (l_4 \vee r) \wedge (l_{p^*} \vee r)$. Assuming $l_3' \le e$ yields $l_3' = (l_4 \vee r) \wedge (l_{p^*} \vee r) \wedge (l_1 \vee l_2) = l_r = l_3$, contradicting l_3 and l_4 being not coplanar. Hence l_3' intersects e only in r. Furthermore, by Proposition 2, l_3' must be disjoint from l_{p^*} and l_4. Choose p' on l_3' but not on e and define $l_3'' := l_{p'} \in \Delta$. We claim that $l_{p'}$ must be noncoplanar with at least one of l_{p^*} or l_4. Otherwise, we would have

$$l_3'' = (l_{p^*} \vee l_{p'}) \wedge (l_4 \vee l_{p'}) = (l_{p^*} \vee p') \wedge (l_4 \vee p') = (l_{p^*} \vee l_3') \wedge (l_4 \vee l_3') = l_3'$$

which is impossible since $l_3'' \in \Delta$ is disjoint from e. □

The absence of a configuration in Theorem 10 (iii) is called bundle condition in the literature.

Definition 2. *A matroid M of rank at least 4 satisfies the* bundle condition *if for any four disjoint lines l_1, l_2, l_3, l_4 of M, no three of them coplanar, the following holds: If five of the six pairs (l_i, l_j) are coplanar, then all pairs are coplanar.*

Since a non-modular pair of hyperplanes together with the entire groundset always forms a modular cut that is not principal, OTE matroids must be hypermodular. Hence, Theorem 10 has the following corollary:

Corollary 2. *Let M be an OTE matroid of rank 4. If the bundle-condition in M holds, then M is modular.*

Proof. Let M be a rank-4 OTE-matroid that is not modular. Then, because M is hypermodular and because of Proposition 5 it contains two disjoint coplanar lines. From Theorem 10 (iii) follows that the bundle-condition does not hold in M. □

5. Embedding Theorems

With these results, we can prove a first embedding theorem. Assertion (iii) is a result of Kahn [4].

Theorem 11. *Let M be a hypermodular rank-4 matroid with a finite or countably infinite groundset. Then M is embeddable in an OTE matroid $\overline{M}$ of rank 4 where the restriction of any line of $\overline{M}$ is a line in M. Furthermore:*

- *(i) $\overline{M}$ is finite if and only if M is finite.*
- *(ii) The simplification of $\overline{M}/p$ is isomorphic to the simplification of M/p for every $p \in E(M)$.*
- *(iii) If M fulfills the bundle-condition then $\overline{M}$ is modular.*

Proof. Let P be a list of all disjoint coplanar pairs of lines of M. Clearly, P is finite or countably infinite. We inductively define a chain of matroids $M = M_0, M_1, M_2 \ldots$ as follows: Let $M_0 = M$, suppose M_{i-1} has already been defined for an $i \in \mathbb{N}$. Let l_{i1} and l_{i2} denote the pair of disjoint lines in the list at index i. If l_{i1} and l_{i2} are not intersectable in the matroid M_{i-1}, set $M_i = M_{i-1}$. Otherwise, let M_i be the single-element extension of M_{i-1} corresponding to the modular cut $\mathcal{M}_{i-1}$ generated by l_{i1} and l_{i2} in M_{i-1}.

By Theorem 10 (ii), the restriction of a line in M_{i+1} is a line in M_i and hence is also a line in M. As a consequence also the restriction of a plane in M_{i+1} is a plane in M hence two planes in M_{i+1} intersect in a line. Thus all matroids M_i are hypermodular of rank 4. Now let $\overline{M}$ be the set system $(\overline{E}, \overline{\mathcal{I}})$ where $\overline{\mathcal{I}} \subseteq \mathcal{P}(\overline{E})$, $\overline{E} = \bigcup_{i=0}^{\infty}(E(M_i))$ and $I \in \overline{\mathcal{I}}$ if and only if I is independent in some M_i. Clearly, $\overline{\mathcal{I}}$ satisfies the independence axioms of matroid theory. We call $\overline{M}$ the *union of the chain of matroids*. The matroid $\overline{M}$ is hypermodular of rank 4 and has no new lines as well.

Assume there were a modular cut $\overline{\mathcal{M}}$ in $\overline{M}$ that is not principal. By Proposition 5 it contains a pair of disjoint coplanar lines. The restriction of this pair in M is on the list, say with index i. The modular cut $\mathcal{M}_{i-1}$ generated by these two lines in M_{i-1} must contain $\mathrm{cl}_{M_{i-1}}(\emptyset)$, otherwise the lines would intersect in M_i, hence also in $\overline{M}$. Since $\{\mathrm{cl}_{\overline{M}}(X) | X \in \mathcal{M}_i\} \subseteq \overline{\mathcal{M}}$ we also must have $\mathrm{cl}_{\overline{M}}(\emptyset) \in \overline{\mathcal{M}}$, a contradiction to $\overline{\mathcal{M}}$ not being principal. Thus, $\overline{M}$ is OTE. If M is finite, so is the list P and hence $\overline{M}$ proving (i).

It suffices to show that for every point $p \in E(M)$ every point $q \in E(\overline{M}) - (E(M) \cup p)$ is parallel to a point in M/p. As the restriction of the line spanned by p and q in $\overline{M}$ is a line in M it contains a point different from p and (ii) follows. Finally, (iii) is Corollary 2. □

This embedding theorem has the following corollary:

Corollary 3. *Kantor's conjecture is reducable to the rank-4 case.*

Proof. Assume Kantor's conjecture holds for rank-4 matroids. Let M be a finite hypermodular matroid of rank $n > 4$. All contractions of M by a flat of rank $n-4$ are finite hypermodular matroids of rank 4, hence are embeddable into a modular matroid. Using Theorem 11, it is easy to see that these contractions are also *strongly embeddable* (as defined in [5], Definition 2) into a modular matroid. Hence the matroid M satisfies the assumptions of Theorem 2 in [5], and thus is embedabble into a modular matroid, implying the general case of Kantor's Conjecture. □

Similarly, it is easy to show that our Conjecture 1 is reducible to the rank-4 case. We have a second embedding theorem:

Theorem 12. *Let M be a matroid of finite rank on a set E where E is finite or countably infinite. Then M is embeddable in an OTE matroid of the same rank.*

Proof. We proceed similar to the proof of Theorem 11. Let P be the list of all intersectable non-modular pairs of M. We build a chain of matroids $M = M_0, M_1 \ldots$, where each matroid M_{i+1} is the extension of M_i, where the modular defect of the i-th pair on the list can no longer be decreased. Let $\overline{M}$ be the union of the extension chain as in the proof before. Then $\overline{M}$ is a matroid of finite rank with a finite or countably infinite ground set. If there still are intersectable non-modular pairs in $\overline{M}$ we repeat the process and obtain $\overline{M}_1$. This yields a chain of matroids $\overline{M}, \overline{M}_1, \overline{M}_2, \ldots$. Let $\overline{\overline{M}}$ be the union of that extension chain. Clearly, $\overline{\overline{M}}$ is a matroid. We claim it is OTE. For assume it had a non-trivial modular cut generated by a non-modular pair of intersectable flats f_1, f_2. Since their rank is finite, there exists an index k such that the matroid $\overline{M}_k$ contains a basis of f_1 as well as of f_2. But then in the matroid $\overline{M}_{k+1}$ the pair would not be intersectable anymore and we get a contradiction. Thus, $\overline{\overline{M}}$ is an OTE matroid. □

We have a similar result for hypermodular matroids:

Theorem 13. *Every matroid M of finite rank r with finite or countably infinite groundset is embeddable in a infinite hypermodular matroid $\overline{\overline{M}}$ of rank r.*

Proof. The proof mimics the one of Theorem 12, except that we have only the non-modular pairs of hyperplanes in the list. This generalizes the technique of *free closure* of rank-3 matroids and it is not difficult to show (see e.g. Kantor [5], Example 5) that if M is non-modular (hence $\mathrm{r} \geq 3$), every contraction of $\overline{\overline{M}}$ by a flat of rank $\mathrm{r}-3$ in $\overline{\overline{M}}$ is an infinite projective non-Desarguesian plane and hence $\overline{\overline{M}}$ must be infinite, too. □

6. On the Non-Existence of Certain Modular Pairs in Extensions of OTE Matroids

In order to prove that the proper amalgam exists for any two extensions of a finite rank-4 OTE matroid we need some technical lemmas. We will show that certain modular pairs cannot exist in extensions of rank-4 OTE matroids. We need some preparations for that.

Proposition 6. *Let M be matroid with groundset T, let (X,Y) be a modular pair of subsets of T and let $Z \subseteq X \setminus Y$. Then $(X \setminus Z, Y)$ is a modular pair, too.*

Proof. Submodularity implies $\mathrm{r}(X \cup Y) - \mathrm{r}(X) \leq \mathrm{r}((X \setminus Z) \cup Y) - \mathrm{r}(X \setminus Z)$. Using modularity of (X,Y) we find

$$\begin{aligned} \mathrm{r}(X \setminus Z) + \mathrm{r}(Y) &= \mathrm{r}(X \cup Y) + \mathrm{r}((X \setminus Z) \cap Y) - \mathrm{r}(X) + \mathrm{r}(X \setminus Z) \\ &\leq \mathrm{r}((X \setminus Z) \cup Y) + \mathrm{r}((X \setminus Z) \cap Y) \end{aligned}$$

and another application of submodularity implies the assertion. □

By (D) we abbreviate the following list of assumptions:

- M is a matroid with groundset T and rank function r.
- M' is an extension of M with rank function r' and groundset E'.
- X and Y are subsets of E' such that $X \cap T = l_X$ and $Y \cap T = l_Y$ are two disjoint coplanar lines in M.
- $X \cap Y$ is a flat in M'.

Proposition 7. *Assume (D) and, furthermore, that $X \setminus T \subseteq Y$ and that (X,Y) is a modular pair of sets in M'. Then $x \notin \mathrm{cl}_{M'}(Y)$ for all $x \in l_X$.*

Proof. Assume to the contrary that there exists $x \in l_X$ with $x \in \mathrm{cl}_{M'}(Y)$. Then coplanarity of l_X and l_Y implies

$$X \cap T = l_X \subseteq l_X \vee l_Y = x \vee l_Y \subseteq \mathrm{cl}_{M'}(Y).$$

Hence $X \subseteq \mathrm{cl}_{M'}(Y)$, implying $\mathrm{r}'(Y) = \mathrm{r}'(X \cup Y)$ and modularity of (X,Y) yields $\mathrm{r}'(X) = \mathrm{r}'(X \cap Y)$, a contradiction, because $X \cap Y$ is a flat in M' and a proper subset of X. □

Lemma 4. *Assume (D) and that M is of rank 4 (the rank of M' may be larger) and, furthermore,*

- *(X,Y) is a modular pair of sets in M' with $X \setminus T \subseteq Y$ and $T \nsubseteq \mathrm{cl}_{M'}(X \cup Y)$ and*
- *$l' \subseteq T$ is a line disjoint coplanar to l_X and l_Y, not lying in $l_X \vee l_Y$.*

Then $X' = (X \setminus T) \cup l'$ implies $\mathrm{r}'(X') = \mathrm{r}'(X)$.

Proof. Choose $x \in l_X$ and $x' \in l' = X' \cap T$. Because l_X and l_Y are coplanar and $X \setminus T \subseteq Y$ we conclude $\mathrm{cl}_{M'}(x \cup Y) = \mathrm{cl}_{M'}(X \cup Y)$. Similarly, we get $\mathrm{cl}_{M'}(x' \cup Y) = \mathrm{cl}_{M'}(X' \cup Y)$.

By assumption M, being of rank 4, is spanned by l', l_X and l_Y and hence $T \subseteq \mathrm{cl}_{M'}(\{x, x'\} \cup Y)$. If we had $x' \in \mathrm{cl}_{M'}(x \cup Y)$, then this would imply that $T \subseteq \mathrm{cl}_{M'}(x \cup Y) = \mathrm{cl}_{M'}(X \cup Y)$, contradicting the assumptions, thus $x' \notin \mathrm{cl}_{M'}(x \cup Y)$. In particular $x' \notin \mathrm{cl}_{M'}(X)$.

Proposition 7 yields $x \notin \mathrm{cl}_{M'}(Y)$. If we had $x \in \mathrm{cl}_{M'}(x' \cup Y)$ using the exchange-axiom of the closure-operator we would find $x' \in \mathrm{cl}_{M'}(x \cup Y)$ which is impossible. Hence we obtain $x \notin \mathrm{cl}_{M'}(x' \cup Y) = \mathrm{cl}_{M'}(X' \cup Y)$. In particular $x \notin \mathrm{cl}_{M'}(X')$.

The choice of x and x' implies $\mathrm{cl}_M(l_X \cup x') = \mathrm{cl}_M(l' \cup x)$ and using $X \setminus T = X' \setminus T$ we obtain $\mathrm{cl}_{M'}(X \cup x') = \mathrm{cl}_{M'}(X' \cup x)$. We conclude

$$\mathrm{r}'(X') + 1 = \mathrm{r}'(X' \cup x) = \mathrm{r}'(X \cup x') = \mathrm{r}'(X) + 1,$$

hence $\mathrm{r}'(X') = \mathrm{r}'(X)$. □

Lemma 5. *Assume (D), M is a rank-4 OTE matroid and $X \setminus T \subseteq Y$, $Y \setminus T \subseteq X$ and $T \nsubseteq \mathrm{cl}_{M'}(X \cup Y)$. Then (X, Y) is not a modular pair in M'.*

Proof. OTE matroids are hypermodular, hence M is hypermodular, OTE and of rank 4. By Theorem 10 (iii), it has two lines l_1 und l_2 that span M but are both disjoint coplanar to l_X and l_Y and disjoint to $l_X \vee l_Y$.

Assume that (X, Y) were a modular pair in M'. Let $X' = (X \setminus T) \cup l_1$ and $Y' = (Y \setminus T) \cup l_2$. Then by Lemma 4

$$\mathrm{r}'(X') = \mathrm{r}'(X) \text{ and } \mathrm{r}'(Y') = \mathrm{r}'(Y). \tag{4}$$

Since $T \subseteq \mathrm{cl}_M(l_1, l_2) \subseteq \mathrm{cl}_{M'}(X' \cup Y')$ and $T \nsubseteq \mathrm{cl}_{M'}(X \cup Y)$ we get

$$\mathrm{r}'(X \cup Y) < \mathrm{r}'(X \cup Y \cup T) = \mathrm{r}'(X' \cup Y' \cup T) = \mathrm{r}'(X' \cup Y'). \tag{5}$$

By definition $X' \cap Y' = (X \setminus T) \cap (Y \setminus T) = X \cap Y$ and hence by sumodularity

$$\begin{aligned} \mathrm{r}'(X \cup Y) + \mathrm{r}'(X \cap Y) &< \mathrm{r}'(X' \cup Y') + \mathrm{r}'(X' \cap Y') && \text{by (5)} \\ &\le \mathrm{r}'(X') + \mathrm{r}'(Y') \\ &= \mathrm{r}'(X) + \mathrm{r}'(Y) && \text{by (4)} \end{aligned}$$

contradicting (X, Y) being a modular pair. □

We come to the main result of this section.

Theorem 14. *Let M be a rank-4 OTE matroid with groundset T and M' an extension of M with ground set E'. Let $X, Y \subseteq E'$ be sets such that $X \cap Y$ is a flat in M' and the restrictions $l_X = X \cap T$ and $l_Y = Y \cap T$ are disjoint coplanar lines in M. If $T \nsubseteq \mathrm{cl}_{M'}(X \cup Y)$ then (X, Y) is not a modular pair in M'.*

Proof. Assume to the contrary that (X, Y) were a modular pair in M'. Let $X' = (X \cap T) \cup (X \cap Y)$ and $Y' = (Y \cap T) \cup (X \cap Y)$. Applying Proposition 6 twice, we find that the pair (X', Y') is modular in M', too, and satisfies the assumptions of Lemma 5 yielding the required contradiction. □

By contraposition we get

Corollary 4. *Let M be a rank-4 OTE matroid with groundset T and M' an extension of M. Let (X, Y) be a modular pair of flats in M' such that $(X \cap T, Y \cap T)$ is a non-modular pair in M. Then $T \subseteq \mathrm{cl}_{M'}(X \cup Y)$.*

Regarding the case that $(X \cap T, Y \cap T)$ is a disjoint line-plane pair, we show the following.

Lemma 6. *Let M be a rank-4 OTE matroid with groundset T and rank function r and let M' be an extension of M with groundset E' and rank function r'. Assume that $X, Y \subseteq E'$ are sets such that $X \cap T = e_X$ is a plane, $Y \cap T = l_Y$ a line disjoint from e_X in M, and that $X \cap Y$ is a flat in M'. Assume that there exists a line $l_X \subseteq e_X$ coplanar with l_Y such that $\mathrm{r}'((X \cap Y) \cup e_X) = \mathrm{r}'((X \cap Y) \cup l_X) + 1$. Then (X, Y) is not a modular pair in M'.*

Proof. Assume, for a contradiction, (X, Y) were a modular pair in M' and let $X' = (X \cap Y) \cup e_X$. Since $X' = X \setminus Z$ with $Z = X \setminus (Y \cup e_X) \subseteq X \setminus Y$, we find that by Proposition 6, (X', Y) is a modular pair in M', too. Let $X'' = (X \cap Y) \cup l_X$. By assumption $\mathrm{r}'(X') = \mathrm{r}'(X'') + 1$ and $X'' \cap T$ is a line disjoint from and coplanar to l_Y. Moreover $X'' \cap Y = X \cap Y$, thus $X'' \cap Y$ is a flat in M'. Furthermore submodularity implies $\mathrm{r}'(X' \cup Y) \leq \mathrm{r}'(X'' \cup Y) + 1$. Because (X', Y) is a modular pair we obtain:

$$\begin{aligned} \mathrm{r}'(X'' \cup Y) + 1 + \mathrm{r}'(X'' \cap Y) &\geq \mathrm{r}'(X' \cup Y) + \mathrm{r}'(X' \cap Y) \\ &= \mathrm{r}'(X') + \mathrm{r}'(Y) = \mathrm{r}'(X'') + 1 + \mathrm{r}'(Y) \end{aligned}$$

and again submodularity of r' implies that equality must hold throughout. Hence (X'', Y) is a modular pair and

$$\mathrm{r}'(X'' \cup Y) + 1 = \mathrm{r}'(X' \cup Y) = \mathrm{r}'(X' \cup Y \cup T) = \mathrm{r}'(X'' \cup Y \cup T)$$

implying $T \nsubseteq \mathrm{cl}_{M'}(X'' \cup Y)$. The pair (X'', Y) now contradicts Theorem 14. □

7. The Proper Amalgam

We prove Theorem 5 by constructing the *proper amalgam* of two given extensions of a rank-4 OTE matroid. In this section we define this amalgam and we analyse some of its properties. Throughout, if not mentioned otherwise, we assume the following situation.

Let M be a matroid with groundset T and rank function r and M_1 and M_2 be extensions of M with groundsets E_1 resp. E_2 and rank functions r_1 resp. r_2, where $E_1 \cap E_2 = T$ and $E_1 \cup E_2 = E$. All matroids are of finite rank with finite or countably infinite ground set. We define two functions $\eta : \mathcal{P}(E) \to \mathbb{Z}$ und $\xi : \mathcal{P}(E) \to \mathbb{Z}$ by

$$\eta(X) = \mathrm{r}_1(X \cap E_1) + \mathrm{r}_2(X \cap E_2) - \mathrm{r}(X \cap T)$$
$$\text{and } \xi(X) = \min\{\eta(Y) \colon Y \supseteq X\}.$$

The following is immediate:

Proposition 8. *The function ξ is subcardinal, finite and monotone. That is,*

R1 : $0 \leq \xi(X) \leq |X|$, *for all* $X \subseteq E$.

R1a : *For all* $X \subseteq E$ *there exist an* $X' \subseteq X, |X'| < \infty$, *such that* $\xi(X) = \xi(X')$.

R2 : *For all* $X_1 \subseteq X_2 \subseteq E$ *we have* $\xi(X_1) \leq \xi(X_2)$.

Moreover $\xi(X) \leq \eta(X)$ *for all* $X \subseteq E$.

If ξ is submodular on $\mathcal{P}(E)$, then ξ is the rank function of an amalgam of M_1 and M_2 along M (see eg. [7], Proposition 11.4.2). This amalgam, if it exists, is called the *proper amalgam of M_1 and M_2 along M*.

Now let $\mathcal{L}(M_1, M_2)$ be the set of all subsets X of E, so that $X \cap E_1$ and $X \cap E_2$ are flats in M_1 resp. M_2. Then it is easy to see that $\mathcal{L}(M_1, M_2)$ with the inclusion-ordering is a complete lattice of subsets of E. Let $\wedge_{\mathcal{L}}$ and $\vee_{\mathcal{L}}$ be the meet resp. the join of this lattice. Clearly, for two sets $X, Y \in \mathcal{L}(M_1, M_2)$ we have $X \wedge_{\mathcal{L}} Y = X \cap Y$ and $X \vee_{\mathcal{L}} Y \supseteq X \cup Y$. We need two results from [7].

Lemma 7 (see [7] Prop. 11.4.5.)**.** *For all* $X \subseteq E$

$$\xi(X) = \min\{\eta(Y) \colon Y \in \mathcal{L}(M_1, M_2) \text{ and } Y \supseteq X\}.$$

Lemma 8 (see [7] Lemma 11.4.6.)**.** *Let $Y \subseteq E$ and Z be the smallest element of $\mathcal{L}(M_1, M_2)$ containing Y, then $\eta(Z) \leq \eta(Y)$ holds.*

The proof of Lemma 11.4.6 in [7] must be slightly modified in the end in order to make it work for matroids of finite rank but infinite groundset as well.

Proof. As in [7] for all $X \subseteq E$ we define $\phi_1(X) = \mathrm{cl}_1(X \cap E_1) \cup (X \cap E_2)$ and $\phi_2(X) = (X \cap E_1) \cup \mathrm{cl}_2(X \cap E_2)$. Following [7] we derive

$$\eta(\phi_i(X)) \leq \eta(X) \text{ for all } X \subseteq E \text{ and } i = 1, 2.$$

Now let Z be the minimal element in $\mathcal{L}(M_1, M_2)$ such that $Y \subseteq Z$ and choose $Y \subseteq W \subseteq Z$ maximal with

$$\eta(W) \leq \eta(Y).$$

From $Y \subseteq W \subseteq \phi_i(W) \subseteq Z$ and $\eta(\phi_i(W)) \leq \eta(W)$ follows $\phi_i(W) = W$ for $i = 1, 2$ and hence $W = Z \in \mathcal{L}(M_1, M_2)$ and Lemma 8 follows, also implying Lemma 7. □

Note that the proof of this lemma and part (R1a) of Proposition 8 imply that Theorem 1 holds for infinite matroids of finite rank as well. Now we generalize a result of Ingleton (cf. [7], Theorem 11.4.7):

Theorem 15. *Assume that for any pair (X, Y) of sets of $\mathcal{L}(M_1, M_2)$ the inequality defining submodularity is satisfied for at least one of η or ξ. Then ξ is submodular on $\mathcal{P}(E)$ and the proper amalgam of M_1 and M_2 along M exists.*

Proof. Let $X_1, X_2 \subseteq E$. By Lemma 7 we find $Y_i \in \mathcal{L}(M_1, M_2)$ such that $X_i \subseteq Y_i$ and $\xi(X_i) = \eta(Y_i)$ for $i = 1, 2$. From $\eta(Y_i) = \xi(X_i) \leq \xi(Y_i) \leq \eta(Y_i)$ we conclude that $\xi(X_i) = \xi(Y_i) = \eta(Y_i)$. By assumption either η or ξ or both are submodular on the pair of flats (Y_1, Y_2). Furthermore, $X_1 \cap X_2 \subseteq Y_1 \cap Y_2 = Y_1 \wedge_{\mathcal{L}} Y_2$ and $X_1 \cup X_2 \subseteq Y_1 \cup Y_2 \subseteq Y_1 \vee_{\mathcal{L}} Y_2$. Hence, by Proposition 8

$$\xi(X_1 \cap X_2) + \xi(X_1 \cup X_2) \leq \xi(Y_1 \wedge_{\mathcal{L}} Y_2) + \xi(Y_1 \vee_{\mathcal{L}} Y_2).$$

Thus, if η is submodular on (Y_1, Y_2)

$$\begin{aligned} \xi(X_1 \cap X_2) + \xi(X_1 \cup X_2) &\leq \eta(Y_1 \wedge_{\mathcal{L}} Y_2) + \eta(Y_1 \vee_{\mathcal{L}} Y_2) \\ &\leq \eta(Y_1) + \eta(Y_2) = \xi(X_1) + \xi(X_2) \end{aligned}$$

and otherwise

$$\xi(X_1 \cap X_2) + \xi(X_1 \cup X_2) \leq \xi(Y_1) + \xi(Y_2) = \xi(X_1) + \xi(X_2).$$

Hence ξ is submodular on $\mathcal{P}(E)$ and the proper amalgam exists. □

Lemma 8 immediately yields

Lemma 9. *If X, Y are in $\mathcal{L}(M_1, M_2)$, then $\eta(X \cup Y) \geq \eta(X \vee_{\mathcal{L}} Y)$. Moreover we have $\xi(X \cup Y) = \xi(X \vee_{\mathcal{L}} Y)$.*

We finish this section with a small lemma.

Lemma 10. *Additionally to the assumptions from the second paragraph of this section let M be of rank 4. Let $X \in \mathcal{L}(M_1, M_2)$ with $\mathrm{r}(X \cap T) \geq 2$. Then $\xi(X) = \eta(X)$.*

Proof. Assume there exists $Y \supseteq X$ such that $\xi(X) = \eta(Y) < \eta(X)$. Then $\mathrm{r}(Y \cap T) > \mathrm{r}(X \cap T)$. Hence there exists an element $t \in (Y \cap T) \setminus X$, and because $X \cap E_1, X \cap E_2$ and $X \cap T$ are flats we get

$$\begin{aligned}\eta(X \cup t) &= \mathrm{r}_1((X \cup t) \cap E_1) + \mathrm{r}_2((X \cup t) \cap E_2) - \mathrm{r}((X \cup t) \cap T)\\ &= \mathrm{r}_1(X \cap E_1) + 1 + \mathrm{r}_2(X \cap E_2) + 1 - \mathrm{r}(X \cap T) - 1 = \eta(X) + 1.\end{aligned}$$

But since M is of rank 4 and $\mathrm{r}((X \cup t) \cap T) \geq 3$, the decrease of η for supersets of $X \cup t$ is bounded by 1 and thus $\eta(Y) \geq \eta(X \cup t) - 1 = \eta(X)$, a contradiction. □

8. Proof of Theorem 5

Our proof of Theorem 5 may be considered as a generalization of the proof of Proposition 11.4.9. in [7]. Oxley refers to unpublished results of A.W. Ingleton. We start with a lemma.

Lemma 11. *Let M be a rank-4 OTE matroid with ground set T. Let M_1 and M_2 be two extensions of M with the ground sets E_1, E_2 and rank functions r_1, r_2. Let $E_1 \cap E_2 = T$ and $E_1 \cup E_2 = E$ and let η, ξ and $\mathcal{L}(M_1, M_2)$ be defined as in Section 7.*

Let (X, Y) be a pair of elements of $\mathcal{L}(M_1, M_2)$ that violates the submodularity of η. Then

- *(i)* $\eta(X) + \eta(Y) - \eta(X \cap Y) - \eta(X \cup Y)$ $= \delta(X \cap E_1, Y \cap E_1) + \delta(X \cap E_2, Y \cap E_2) - \delta(X \cap T, Y \cap T) = -1.$
- *(ii)* *$(X \cap E_i, Y \cap E_i)$ is a modular pair in M_i for $i = 1, 2$.*
- *(iii)* *$(X \cap T, Y \cap T)$ are two disjoint coplanar lines or a disjoint line-plane pair in M.*
- *(iv)* *$\eta(X) = \xi(X)$ and $\eta(Y) = \xi(Y)$.*

Proof. For part (i) a straightforward computation yields the first equality. The second one follows from the fact that OTE-matroids are hypermodular and that the modular defect in a hypermodular rank-4 matroid is bounded by 1. Parts (ii) and (iii) are immediate from (i) and part (iv) follows from Lemma 10. □

Lemma 12. *Under the assumptions of Lemma 11, let (X, Y) be a pair of elements of $\mathcal{L}(M_1, M_2)$ such that the submodularity of η in $\mathcal{L}(M_1, M_2)$ is violated, and either $\xi(X \cup Y) < \eta(X \cup Y)$ or $\xi(X \cap Y) < \eta(X \cap Y)$. Then ξ is submodular for (X, Y) in $\mathcal{L}(M_1, M_2)$.*

Proof. Recall that $\xi(X \cup Y) \leq \eta(X \cup Y)$ and $\xi(X \cap Y) \leq \eta(X \cap Y)$ and $\xi(X \cap Y) = \xi(X \wedge_{\mathcal{L}} Y)$ as well as $\xi(X \cup Y) = \xi(X \vee_{\mathcal{L}} Y)$ by Lemma 9. Moreover by Lemma 11 (iv), $\eta(X) = \xi(X)$ and $\eta(Y) = \xi(Y)$. Altogether this implies

$$\begin{aligned}&\xi(X) + \xi(Y) - \xi(X \wedge_{\mathcal{L}} Y) - \xi(X \vee_{\mathcal{L}} Y)\\ &= \xi(X) + \xi(Y) - \xi(X \cap Y) - \xi(X \cup Y)\\ &> \eta(X) + \eta(Y) - \eta(X \cap Y) - \eta(X \cup Y) = -1\end{aligned}$$

proving the assertion. □

We are now ready to tackle the proof of Theorem 5 which is an immediate consequence of the following:

Theorem 16. *Let M be a rank-4 OTE matroid. Then for any pair of extensions of M the proper amalgam exists.*

Proof. Let T denote the ground set of M and M_1, M_2 be two extensions of M with ground sets E_1, E_2 and rank functions r_1, r_2, such that $E_1 \cap E_2 = T$ and $E_1 \cup E_2 = E$. We show that for these two extensions the proper amalgam exists. Let η and ξ be defined as in the previous section. By Lemma 15 it suffices to show that for each pair (X, Y) of elements of $\mathcal{L}(M_1, M_2)$ either η or ξ is submodular.

By cases, we check all possible pairs (X, Y) of sets of $\mathcal{L}(M_1, M_2)$ where the submodularity of η could be violated, and show that $\xi(X \cup Y) < \eta(X \cup Y)$ or $\xi(X \cap Y) < \eta(X \cap Y)$ and hence (by Lemma 12) ξ is submodular on (X, Y).

By Lemma 11, $(X \cap E_i, Y \cap E_i)$ are modular pairs of flats in M_i for $i = 1, 2$ and $(X \cap T, Y \cap T)$ is a pair of disjoint coplanar lines or a disjoint line-plane-pair.

Disjoint coplanar lines: Assume $X \cap T = l_X$ and $Y \cap T = l_Y$ are two disjoint coplanar lines. By Corollary 4 the fact that $(X \cap E_i, Y \cap E_i)$ are modular pairs for $i = 1, 2$ implies that $T \subseteq \mathrm{cl}_{M_i}((X \cup Y) \cap E_i)$ for $i = 1, 2$. Let $t \in T \setminus \mathrm{cl}_M(l_X \cup l_Y)$. Then

$$\begin{aligned} & \eta(X \cup Y \cup t) \\ = \;& r_1((X \cup Y \cup t) \cap E_1) + r_2((X \cup Y \cup t) \cap E_2) - \mathrm{r}((X \cup Y \cup t) \cap T) \\ = \;& r_1((X \cup Y) \cap E_1) + r_2((X \cup Y) \cap E_2) - \mathrm{r}((X \cup Y) \cap T) - 1 \\ = \;& \eta(X \cup Y) - 1. \end{aligned}$$

Hence $\xi(X \cup Y) < \eta(X \cup Y)$.

Disjoint point-line pair: Assume $X \cap T = e_X$ is a plane and $Y \cap T = l_Y$ is a line disjoint from e_X. By Lemma 6 for every line $l \subseteq e_X$ such that $\mathrm{r}(l \vee l_Y) = 3$ we must have

$$r_i((X \cap Y \cap E_i) \cup e_X) = r_i((X \cap Y \cap E_i) \cup l) \text{ for } i = 1, 2. \tag{6}$$

Choose a point $p_1 \in e_X$. Since M must be hypermodular $l_X = (e_X \wedge (l_Y \vee p_1))$ is a line in M and $p_1 \in l_X$. Since $Y \cap E_1$ is a flat in M_1 not containing p_1 and $X \cap Y \cap E_1$ is a flat in M_1 disjoint from T we have

$$r_1((Y \cup p_1) \cap E_1) = r_1(Y \cap E_1) + 1 \tag{7}$$
$$r_1((X \cap Y \cap E_1) \cup p_1) = r_1(X \cap Y \cap E_1) + 1. \tag{8}$$

Choose a second point $p_2 \in l_X$ such that $p_2 \neq p_1$. Since l_X and l_Y are coplanar, we obtain

$$p_2 \in l_X \subseteq \mathrm{cl}_M(p_1 \cup l_Y) = \mathrm{cl}_M(p_1 \cup (Y \cap T)) \subseteq \mathrm{cl}_{M_1}(p_1 \cup (Y \cap E_1))$$

and thus

$$r_1((Y \cup l_X) \cap E_1) = r_1((Y \cup \{p_1, p_2\}) \cap E_1) = r_1((Y \cup p_1) \cap E_1). \tag{9}$$

Furthermore, since $\{p_1, p_2\} \subseteq l_X \subseteq X$:

$$r_1((X \cup Y \cup \{p_1, p_2\}) \cap E_1) = r_1((X \cup Y) \cap E_1) \tag{10}$$

Using these equations and the modularity of $(X \cap E_1, Y \cap E_1)$ in M_1 we compute

$$\begin{aligned}
& r_1(X \cap E_1) + r_1((Y \cup \{p_1, p_2\}) \cap E_1) \\
\overset{(9)}{=} \;& r_1(X \cap E_1) + r_1((Y \cup p_1) \cap E_1) \\
\overset{(7)}{=} \;& r_1(X \cap E_1) + r_1(Y \cap E_1) + 1 \\
\overset{(Mod.)}{=} \;& r_1((X \cup Y) \cap E_1) + r_1(X \cap Y \cap E_1) + 1 \\
\overset{(10)}{=} \;& r_1((X \cup Y \cup \{p_1, p_2\}) \cap E_1) + r_1(X \cap Y \cap E_1) + 1 \\
\overset{(8)}{=} \;& r_1((X \cup Y \cup \{p_1, p_2\}) \cap E_1) + r_1((X \cap Y \cap E_1) \cup p_1) \\
\leq \;& r_1((X \cup Y \cup \{p_1, p_2\}) \cap E_1) + r_1((X \cap Y \cap E_1) \cup \{p_1, p_2\})
\end{aligned}$$

By submodularity of r_1 the last inequality must hold with equality and hence

$$r_1((X \cap Y \cap E_1) \cup l_X) = r_1((X \cap Y \cap E_1) \cup p_1). \tag{11}$$

By symmetry (8) and (11) are also valid for r_2 and E_2. Recalling that $X \cap Y \cap T = \emptyset$, we compute

$$\begin{aligned}
\eta((X \cap Y) \cup e_X) &= \left[\sum_{i=1}^{2} r_i((X \cap Y \cap E_i) \cup e_X)\right] - \mathrm{r}(e_X) \\
&\overset{(6)}{=} \left[\sum_{i=1}^{2} r_i((X \cap Y \cap E_i) \cup l_X)\right] - 3 \\
&\overset{(11)}{=} \left[\sum_{i=1}^{2} r_i((X \cap Y \cap E_i) \cup p_1)\right] - 3 \\
&\overset{(8)}{=} \left[\sum_{i=1}^{2} (r_i(X \cap Y \cap E_i) + 1)\right] - \mathrm{r}(X \cap Y \cap T) - 3 \\
&= \eta(X \cap Y) - 1.
\end{aligned}$$

Hence $\xi(X \cap Y) < \eta(X \cap Y)$. □

9. Conclusion

Now if we put the embedding theorems together with Theorem 5, we get the equivalence of three conjectures:

Theorem 17. *The following statements are equivalent:*

(i) *All finite sticky matroids are modular. (SMC)*

(ii) *Every finite hypermodular matroid is embeddable in a modular matroid. (Kantor's Conjecture)*

(iii) *Every finite OTE matroid is modular.*

Proof. $(i) \Rightarrow (ii)$ These two statements can be reduced to the rank-4 case (see Theorem 2 and Corollary 3). Now consider a finite hypermodular rank-4 matroid M. Because of Theorem 11, it can be embedded into a finite rank-4 OTE matroid M' that is sticky due to Theorem 5. If (i) holds then M' is modular and M can be embedded into a modular matroid and (ii) holds.

$(ii) \Rightarrow (iii)$ Let M be a finite OTE matroid. It is also hypermodular. If (ii) holds, it is embeddable into a modular matroid. Since M is OTE, it must itself already be modular.

$(iii) \Rightarrow (i)$ Let M be a finite sticky matroid. Because of Theorem 3 it must be an OTE matroid and, if (iii) holds, must be modular and (i) holds. □

A slightly weaker conjecture than the (SMC) in the finite case, which could also hold in the infinite case, is the generalization of Theorem 5 to arbitrary rank.

Conjecture 4. *A matroid is sticky if and only if it is an OTE matroid.*

Our proof of Theorem 5 frequently uses the fact that we are dealing with rank 4 matroids. We think there is a way to avoid Lemma 10, but the case checking in the proof of of Theorem 16 seems to become tedious even for ranks only slightly larger than 4. Moreover, we need a generalization of Theorem 10 (iii) in order to generalize Lemma 4.

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