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THE VARCHENKO DETERMINANT FOR ORIENTED MATROIDS

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Abstract. We generalize the Varchenko matrix of a hyperplane arrangement to oriented matroids. We show that the celebrated determinant formula for the Varchenko matrix, first proved by Varchenko, generalizes to oriented matroids. It follows that the determinant only depends on the matroid underlying the oriented matroid and analogous formulas hold for closed supertopes in oriented matroids. We follow a proof strategy for the original Varchenko formula first suggested by Denham and Hanlon. Besides several technical lemmas this strategy also requires a topological result on supertopes which is of independent interest. We show that a supertope considered as a subposet of the tope poset has a contractible order complex.

1. Introduction

Let $\mathcal{L}$ be an oriented matroid on a finite ground set E given as a set of covectors $X = (X_e)_{e\in E} \in \{+,-,0\}^E$. We denote by $\mathcal{T} = \mathcal{T}(\mathcal{L})$ the set of topes in $\mathcal{L}$ and call for two topes $P = (P_e)_{e\in E}$ and $Q = (Q_e)_{e\in E}$ the set $\mathrm{Sep}(P,Q) = \{e \in E \mid P_e = -Q_e \neq 0\}$ the *separator* of P and Q. For the oriented matroid $\mathcal{L}$ and a field $\mathbb{K}$ we consider the polynomial ring $\mathbb{K}[U_e \mid e \in E]$ in the set of variables U_e, $e \in E$. We call the following matrix $\mathfrak{V} = \mathfrak{V}(\mathcal{L})$ the *Varchenko matrix* of $\mathcal{L}$. The matrix $\mathfrak{V}$ is the $(\#\mathcal{T} \times \#\mathcal{T})$-matrix over $\mathbb{K}[U_e \mid e \in E]$ with rows and columns indexed by the topes $\mathcal{T}$ in a fixed linear order. For $P, Q \in \mathcal{T}$ the entry $\mathfrak{V}_{P,Q}$ in row P and column Q is given by $\prod\limits_{e\in \mathrm{Sep}(P,Q)} U_e$. In particular, all entries $\mathfrak{V}_{P,P}$ on the diagonal are equal to 1. For $F \in \mathcal{L}$ we set $a(F) := \prod\limits_{\substack{e\in E\\ F_e=0}} U_e$. In this paper we prove:

Theorem 1.1. *Let $\mathfrak{V}$ be the Varchenko matrix of the oriented matroid with covector set $\mathcal{L}$. Then*

$$\det(\mathfrak{V}) \;=\; \prod_{F\in\mathcal{L}} (1 - a(F)^2)^{b_F}.$$

for nonnegative integers b_F.

Note, that a factor $(1 - a(F)^2)$ is zero if and only if F is a tope. In this case it turns out that $b_F = 0$. By the convention $0^0 = 1$ it follows that $\det(\mathfrak{V}) \neq 0$. In Corollary 5.7 we give an alternative formulation of the product formula which will shed more light on the exponents b_F. In particular, it will follow that $\det(\mathfrak{V})$ only depends on the matroid underlying the oriented matroid defined by $\mathcal{L}$.

If $\mathcal{L}$ is given as the set of covectors of a hyperplane arrangement in some $\mathbb{R}^n$ then $\mathfrak{V}$ is the Varchenko matrix of the hyperplane arrangement and Theorem 1.1 is Varchenko's result from [13]. Initially, Varchenko was motivated by the case of the reflection arrangements of the symmetric group. In that case the matrix relates to Drinfeld–Jimbo quantized KacMoody Lie algebras in type A. This relation had been unraveled by Schechtman and Varchenko in [11]. There it is shown that the kernel of specializations of the matrix describe the Serre relations for the algebra. Motivated by these facts Zagier [16] gave a proof of the determinant formula for the reflection arrangement of the symmetric group based on calculations in its group algebra. Work of Hanlon and Stanley [8] ties in the matrix and its kernel with combinatorial aspects of the representation theory of the symmetric group when all variables are substituted by a fixed complex number. For general arrangements of hyperplanes, Denham and Hanlon [6] show that the matrix and its determinant can be used in an approach to determine the Betti numbers of the Milnor fiber of the complexified arrangements; that is the fiber in complex space of the product of linear forms defining the hyperplanes at complex numbers different from 0.

After the original proof in [13] there were attempts in [6] and [7] to provide a cleaner proof of Varchenko's original result. Our approach generalizes ideas from [6] and [7] to oriented matroids and replaces the problematic parts from both works by alternative arguments. Recently, a new proof using a different strategy was published in [1]. We have not studied this proof thoroughly and cannot judge if it generalizes to oriented matroids as well. This paper is not the first to study oriented matroid generalizations of the Varchenko determinant formula. In the works [14, 15] an approach is sketched for proving Theorem 1.1 originally for general oriented matroids in [14] and restricted to oriented matroids that allow a representation as a pseudo point configuration, only, in the subsequent [15]. Despite several attempts we were not able to follow the argumentation of either thesis. Philosophically, our work parallels the article of Byławski and Varchenko [5] who give a matroid generalization of a determinant formula by Schechtman and Varchenko [10] for yet another important class of matrices arising in representation theory. That paper probably also motivated [14] and [15].

Besides amendments and the generalization to oriented matroids the key new ingredient in our proof of Theorem 1.1 is the following result which we consider of independent interest. For its formulation, let $R \in \mathcal{T}$ be a fixed base tope and consider $\mathcal{T}$ as a partially ordered set with order relation $P \preceq_R Q$ if $\mathrm{Sep}(R,P) \subseteq \mathrm{Sep}(R,Q)$. We write $\mathcal{T}_R$ if we consider $\mathcal{T}$ with this partial order. For disjoint subsets $S^+, S^- \subseteq E$ such that $S^+ \cup S^- \neq \emptyset$ the set of topes

$$\mathcal{T}(S^+,S^-) := \left\{T \in \mathcal{T} \mid T_f = + \text{ for all } f \in S^+ \text{ and } T_f = - \text{ for all } f \in S^-\right\}$$

is called a *supertope*. By [4, Proposition 4.2.6] supertopes are exactly the *T-convex sets*, i.e. the sets of topes that contain any shortest path between any of two of its members. We call a supertope $\mathcal{T}(S^+,S^-)$ a *closed supertope*, if for all supertopes $\mathcal{T}(\tilde{S}^+,\tilde{S}^-)$ such that $S^+ \subseteq \tilde{S}^+, S^- \subseteq \tilde{S}^-$ but $(S^+,S^-) \neq (\tilde{S}^+,\tilde{S}^-)$ necessarily $\mathcal{T}(\tilde{S}^+,\tilde{S}^-) \subsetneq \mathcal{T}(S^+,S^-)$. In case the oriented matroid is given by an arrangement of hyperplanes then a closed supertope corresponds to a closed cone cut out by the hyperplanes from the arrangement. Note that our notion of closed supertope is more general than the notion of a cone from [4, Definition 10.1.1 (iii)].

One would expect that T-convex sets as subsets of the tope poset $\mathcal{T}_R$ are contractible. We will show that this is indeed the case.

Theorem 1.2. *Let $R \in \mathcal{T}$ be the base tope of the poset $\mathcal{T}_R$ and $\mathcal{T}(S^+, S^-) \neq \emptyset$ be a supertope. Then $\mathcal{T}(S^+, S^-)$ considered as subposet of $\mathcal{T}_R$ is contractible.*

The paper is organized as follows. In Section 2 we recall some basic notations and results from oriented matroid theory and poset topology. We then use tools from poset topology to derive results on the topology of complexes associated to oriented matroids in Section 3. In Section 4 we provide the proof of Theorem 1.2 and exhibit why we cannot follow the argumentation from [6] and [7]. In Section 5 we prove Theorem 1.1. The key step in the proof is a factorization of the Varchenko matrix, one factor for each element of the ground set E (Proposition 5.3). The key ingredient of the factorization is a result on Möbius numbers which is a direct consequence of Theorem 1.2 (Corollary 4.5). Then the determinant of each factor is analyzed. Möbius number implications of topological results from Section 3 then show that each is block upper triangular with controllable block structure (Lemma 5.6). Now Theorem 1.1 follows via basic linear algebra. As a corollary we give a description of the numbers b_F which implies that the determinant only depends on the matroid underlying the oriented matroid. As a second corollary we show that the result extends to closed supertopes and hence in particular to affine oriented matroids.

2. Background on Oriented Matroids and Poset Topology

2.1. **Poset Topology.** In this paper we will associate various partially ordered sets, posets for short, to oriented matroids. For our purposes it turns out to be useful to consider a poset $\mathcal{P}$ as a topological space. We do this by identifying $\mathcal{P}$ with its order complex, respectively the geometric realization of the order complex. Recall that the *order complex* of a poset P is the simplicial complex whose chains are the linearly ordered subsets of $\mathcal{P}$. Using this identification we can speak about contractible and homotopy equivalent posets. We will employ the following standard tools from poset topology (see [3] for details). For their formulation we denote for a poset $\mathcal{P}$ and $p \in \mathcal{P}$ by $\mathcal{P}_{\leq p}$ the subposet $\{q \in \mathcal{P} \mid q \leq p\}$. Analogously defined are $\mathcal{P}_{<p}$,$\mathcal{P}_{>p}$ and $\mathcal{P}_{\geq p}$. For $p \leq q$ in $\mathcal{P}$ we write $(p,q)_{\mathcal{P}}$ for the *open interval* $\mathcal{P}_{>p} \cap \mathcal{P}_{<q}$ and $[p,q]_{\mathcal{P}}$ for the *closed* interval $\mathcal{P}_{\geq p} \cap \mathcal{P}_{\leq q}$.

Proposition 2.1 (Quillen Fiber Lemma)**.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be posets and $f : \mathcal{P} \to \mathcal{Q}$ a poset map. If for all $q \in \mathcal{Q}$ we have that $f^{-1}(\mathcal{Q}_{\leq q})$ is contractible, then $\mathcal{P}$ and $\mathcal{Q}$ are homotopy equivalent.*

By simple induction on $\#\mathcal{S}$ one derives the following corollary.

Corollary 2.2. *Let $\mathcal{P}$ be a poset and $\mathcal{S}$ a subset such that $\mathcal{P}_{<s}$ is contractible for all $s \in \mathcal{S}$. Then $\mathcal{P} \setminus \mathcal{S}$ and $\mathcal{P}$ are homotopy equivalent.*

We will use poset topology also to prove results on the Möbius number of a poset $\mathcal{P}$. For that we take advantage of the following well known numerical consequence of the fact that two posets are homotopy equivalent. For a poset $\mathcal{P}$ we denote by $\mu(\mathcal{P})$ the *Möbius number* of $\mathcal{P}$ (see [12, Chapter 3]).

Proposition 2.3. *For two homotopy equivalent posets $\mathcal{P}$ and $\mathcal{Q}$ we have $\mu(\mathcal{P}) = \mu(\mathcal{Q})$. In particular, if $\mathcal{P}$ is contractible then $\mu(\mathcal{P}) = 0$.*

2.2. **Oriented Matroids.** As mentioned in Section 1 we consider an oriented matroid $\mathcal{L}$ on ground set E as a set of covectors $X = (X_e)_{e\in E} \in \{+,-,0\}^E$. In our notation we follow [4] which also contains all required background information on oriented matroids. Frequently, we will use the following definitions and notations.

We order the covectors by the product order induced by the order $0 < +,-$ and write $\mathbf{0} = (0)_{e\in E}$ for the unique minimal covector in this order. Following our conventions, for a covector X we write $(\mathbf{0}, X)_{\mathcal{L}}$ for the open interval from $\mathbf{0}$ to X in $\mathcal{L}$. It is well known that the poset of covectors is graded and hence one can assign each covector $X \in \mathcal{L}$ a rank $\mathrm{rank}_{\mathcal{L}}(X)$. The rank $\mathrm{rank}(\mathcal{L})$ of $\mathcal{L}$ is defined as the maximal rank of one of its covectors.

As usual for a covector $X \in \mathcal{L}$ we write X^+ for $\{e \in E \mid X_e = +\}$ and X^- for $\{e \in E \mid X_e = -\}$. In addition, we write $z(X) = \{e \in E \mid X_e = 0\}$ for its zero-set.

Let $A \subseteq E$ be a nonempty set. For $F \in \mathcal{L}$ we denote by $F|_A$ the covector $(F_e)_{e\in A}$. For a set $\mathcal{K}$ of covectors over E we then write $\mathcal{K}|_A$ for the set $\{F|_A \mid F \in \mathcal{K}\}$ of covectors over A. For an oriented matroid $\mathcal{L}$ over E and a nonempty subset $A \subseteq E$ the set of covectors $\mathcal{L}|_A$ defines an oriented matroid called the restriction of $\mathcal{L}$ to A. The contraction of A in $\mathcal{L}$ is the oriented matroid $\mathcal{L}/A$ with covector set $\{F|_{E\setminus A} \mid F \in \mathcal{L}, z(F) \subseteq A\}$. In case $A = \{f\}$ is a singleton we also write $\mathcal{L}/f$ for $\mathcal{L}/A$.

For two covectors $X, Y \in \mathcal{L}$ their *composition* $X \circ Y$ is defined by $(X \circ Y)^+ = X^+ \cup (Y^+ \setminus X^-)$ and $(X \circ Y)^- = X^- \cup (Y^- \setminus X^+)$.

Next we repeat and extend some notation already stated in Section 1. We write $\mathcal{T}(\mathcal{L})$ for the set of topes of $\mathcal{L}$ and simply $\mathcal{T}$ in case there is no danger of ambiguity. For $P, Q \in \mathcal{T}$ we denote by $\mathrm{Sep}(P, Q)$ the *separator* of P and Q. Then for fixed $R \in \mathcal{T}$ the set of topes $\mathcal{T}$ carries a poset structure defined by $P \preceq_R Q$ if and only if $\mathrm{Sep}(R, P) \subseteq \mathrm{Sep}(R, Q)$ (see [4, Definition 4.2.9]). We write $\mathcal{T}_R$ to denote $\mathcal{T}$ with this poset structure. In order to reduce the number of double subscripts we write write $(P, Q)_R$ for $(P, Q)_{\mathcal{T}_R}$ and $[P, Q]_R$ for $[P, Q]_{\mathcal{T}_R}$. For $e \in E$ and $P \in \mathcal{T}$ we say that e *does not define a proper face* of P if the only covector $F \in \mathcal{L}$ with $F \leq P$ and $F_e = 0$ is $F = \mathbf{0}$.

We will frequently encounter the situation where $R, P \in \mathcal{T}$ and $e \in E$ are such that $+ = R_e$ and $- = P_e$. Then after reordering and reorientation we can assume the following.

Situation 2.4. *$R = + \cdots +$ and $P = - \cdots - + \cdots +$ and e is the first coordinate of our sign vectors.*

In the rest of the paper, we will work in the general situation unless there is a technical simplification when assuming Situation 2.4. In that case we will explicitly mention the assumption.

Next we state well known facts about the topology of $\mathcal{L}$ and $\mathcal{T}_R$.

Lemma 2.5 (Lemma 4.3.11 [4])**.** *Let $P, R \in \mathcal{T}$ set*

$$F_R(P) = \{X \in (\mathbf{0}, P)_{\mathcal{L}} \mid z(X) \subseteq \mathrm{Sep}(P, R)\} \subseteq \mathcal{L}.$$

Then $F_R(P)$ is a filter in $(\mathbf{0}, P)_{\mathcal{L}}$. If $P \neq \pm R$ then $F_R(P)$ is contractible.

For a covector X we set $\mathrm{star}(X) := \{T \in \mathcal{T} \mid X \leq T\}$.

Theorem 2.6 (Theorem 4.4.2 [4])**.** *Let $\mathcal{L}$ be an oriented matroid of rank r and $R \in \mathcal{T}$. For $T_1, T_2 \in \mathcal{T}_R$ such that $T_1 \preceq_R T_2$ the order complex of $(T_1, T_2)_R$ is homotopy equivalent to*

(i) a sphere of dimension $r - \mathrm{rank}_{\mathcal{L}}(X) - 2$ if $[T_1, T_2]_R$ equals $\mathrm{star}(X)$ for some covector X,
(ii) a point, i.e. it is contractible, otherwise.

For $e \in E$ and $R \in \mathcal{T}$ we write $\mathcal{T}_{R,e}$ for the poset $\{T \in \mathcal{T} \mid T_e = -R_e\} \cup \{\hat{0}\}$ with $\hat{0}$ as its least element and the remaining poset structure induced from $\mathcal{T}_R$. For $P \in \mathcal{T}_{R,e}$ we write $(\hat{0}, P)_{R,e}$ for the interval from $\hat{0}$ to P in $\mathcal{T}_{R,e}$. We set

$$(\hat{0}, P)^{\triangle}_{R,e} := \{Q \in (\hat{0}, P)_{R,e} \mid \exists X \in \mathcal{L} \;:\; [Q, P]_R = \mathrm{star}(X)\}.$$

The following is an immediate consequence of Theorem 2.6.

Corollary 2.7. *Let $R \in \mathcal{T}$, $e \in E$ and $P \in \mathcal{T}_{R,e}$. Then $(\hat{0}, P)_{R,e}$ and $(\hat{0}, P)^{\triangle}_{R,e}$ are homotopy equivalent.*

Proof. Let

$$S = \{Q \in (\hat{0}, P)_{R,e} \mid \nexists X \in \mathcal{L} \;:\; [Q, P]_R = \mathrm{star}(X)\}.$$

Then $(\hat{0}, P)^{\triangle}_{R,e} = (\hat{0}, P)_{R,e} \setminus S$. For $Q \in S$ Theorem 2.6 implies that $((\hat{0}, P)_{R,e})_{>Q} = (Q, P)_R$ is contractible. Now the assertion follows from Corollary 2.2. ☐

3. Some Oriented Matroid Topology

At the end of the last section we already recalled some known facts about the topology of posets associated to oriented matroids. This section now contains oriented matroid generalizations of topological results stated in [6] and [7] for hyperplane arrangements.

For $P \in \mathcal{T}_{R,e}$ we set $S = E \setminus \mathrm{Sep}(P, R)$ and $S' = \mathrm{Sep}(P, R) \setminus \{e\}$. Let $B \in \mathcal{T}(\mathcal{L}|_{S'})$ be the unique tope from $\mathcal{T}(\mathcal{L}|_{S'})$ such that $B_f = P_f = -R_f$ for all $f \in S'$. Let $G \in \mathcal{T}(\mathcal{L}|_S)$ be the unique tope from $\mathcal{T}(\mathcal{L}|_S)$ such that $G_f = P_f$ for all $f \in S$.

We define

$$W_{R,e}(P) = \{F \in (\mathbf{0}, P)_{\mathcal{L}} \mid F_e = -R_e,\, F|_S = G,\, F|_{S'} \leq B\} \subseteq \mathcal{L}.$$

We consider $W_{R,e}(P)$ as a poset with order relation inherited from $\mathcal{L}$. Assuming Situation 2.4 we have:

$$W_{R,e}(P) = \left\{F \in (\mathbf{0}, P)_{\mathcal{L}} \mid F_e = -,\, F|_S = \{+\}^S,\, F|_{S'} \leq \{-\}^{S'}\right\} \subseteq \mathcal{L}.$$

We consider the following map:

$$\alpha_P : \begin{cases} (\hat{0}, P)^{\triangle}_{R,e} & \to & \mathcal{L} \\ C & \mapsto & \alpha_P(C) := X, \ \substack{\text{for the } X \in \mathcal{L} \\ \text{such that } (C,P)_R = \mathrm{star}(X)} \end{cases}$$

Lemma 3.1. *Let $R \in \mathcal{T}$ and $e \in E$. For $P \in \mathcal{T}_{R,e}$ and $C \in (\hat{0}, P)^{\triangle}_{R,e}$ the following holds:*

(i) $z(\alpha_P(C)) = \mathrm{Sep}(C, P)$ and $\alpha_P(C) \in W_{R,e}(P)$.

(ii) α_P is a poset map from $(\hat{0}, P)^{\triangle}_{R,e}$ to $W_{R,e}(P)$.

(iii) For $F \in W_{R,e}(P)$ the tope $F \circ R$ is the unique maximal element in $\alpha_P^{-1}(W_{R,e}(P)_{\leq F})$.

Proof. We assume Situation 2.4.

(i) By definition $[C, P]_R = \mathrm{star}(X) = [X \circ R, X \circ (-R)]_R$. Hence, $X \leq P$ and $X_e = C_e = P_e = -$, implying $z(X) = \mathrm{Sep}(C, P)$ and $\alpha_P(C) = X \in W_{R,e}(P)$.

(ii) Since $C \preceq_R C'$ in $(\hat{0}, P)^{\triangle}_{R,e}$ implies $\mathrm{Sep}(C', P) \subseteq \mathrm{Sep}(C, P)$ it follows from (i) that $\alpha_P(C) \leq \alpha_P(C')$. Hence α_P is a map of posets.

(iii) Let $F, F' \in W_{R,e}(P)$, $F' \leq F$ and $C = F \circ R$. We have $C' \in \alpha_P^{-1}(F')$ if and only if $[C', P]_R = [F' \circ R, F' \circ (-R)]_R$. As $z(F) \subseteq z(F')$ this implies $\mathrm{Sep}(C', R) \subseteq \mathrm{Sep}(C, R)$ and hence the assertion.

□

The following proposition allows us to determine the topology of the posets $(\hat{0}, P)_{R,e}$ through known results on $W_{R,e}(P)$.

Proposition 3.2. *Let $R \in \mathcal{T}$ and $e \in E$ such that $R_e = +$ and $P \in \mathcal{T}_{R,e}$. Then:*

(i) The order complex of the interval $(\hat{0}, P)_{R,e}$ and the order complex of $W_{R,e}(P)$ are homotopy equivalent.

(ii) For e that do not define a proper face of P the order complex of $W_{R,e}(P)$ is contractible if $\pm R \neq P$ and homotopy equivalent to a $(\mathrm{rank}(\mathcal{L}) - 2)$-sphere if $-R = P$.

Proof. (i) From Corollary 2.7 it follows that $(\hat{0}, P)_{R,e}$ and $(\hat{0}, P)^{\triangle}_{R,e}$ are homotopy equivalent.

Using Lemma 3.1 (iii) it follows that the order complex of each fiber $\alpha_P^{-1}(W_{R.e}(P)_{\leq F})$ for $F \in W_{R,e}(P)$ is a cone and hence contractible. Now the Quillen Fiber Lemma, Proposition 2.1, shows that the order complexes of $(\hat{0}, P)^{\triangle}_{R,e}$ and $W_{R,e}(P)$ are homotopy equivalent.

(ii) Since e does not define a proper face of P, we have $W_{R,e}(P) = F_R(P)$. If $P = -R$ then $W_{R,e}(P) = (\mathbf{0}, P)_{\mathcal{L}}$ and hence is homotopy equivalent to a $(\mathrm{rank}(\mathcal{L}) - 2)$-sphere. If $P \neq \pm -R$ then $W_{R,e}(P) = F_R(P)$ and Lemma 2.5 shows that $W_{R,e}(P)$ is contractible.

□

We summarize the results in the following theorem.

Theorem 3.3. *Let $P \in \mathcal{T}_{R,e}$ such that e does not define a proper face of P. Then the interval $(\hat{0}, P)_{R,e}$ is contractible if $-R \neq P$ and homotopy equivalent to a $(\mathrm{rank}(\mathcal{L}) - 2)$-sphere if $-R = P$.*

Proof. The result is an immediate consequence of Proposition 3.2(i) and (ii). □

The well known connection of homotopy type and Möbius-number from Proposition 2.3 yields.

Corollary 3.4. *Let $P \in \mathcal{T}_{R,e}$ such that e does not define a proper face of P. Then the Möbius number $\mu((\hat{0}, P)_{R,e})$ is 0 if $-R \neq P$ and $(-1)^{\mathrm{rank}(\mathcal{L})}$ if $-R = P$.*

The next result overlaps with Theorem 3.3 but also covers some of the cases where e defines a face of P. Note that e defines a proper face of R if and only if e defines a proper face of $-R$. Hence the interval $(\hat{0}, -R)_{R,e}$ is covered by Theorem 3.3 if e does not define a proper face of R and it is covered by Theorem 3.5 otherwise.

Theorem 3.5. *Let $R \in \mathcal{T}$ and let $e \in E$ define a proper face of R. Let $F \in \mathcal{L}$ be the maximal covector such that $F \leq R$ and $F_e = 0$ and choose $P_{top} \in \mathcal{T}_{R,e} \setminus \mathrm{star}(F)$. Then $(\hat{0}, P_{top})_{R,e}$ is contractible. In particular, $\mu((\hat{0}, P_{top})_{R,e}) = 0$.*

Proof. Let $P \in (\hat{0}, P_{top})_{R,e}$. Then by the gate property [4, Exercise 4.10] the tope $Q = F \circ P \in \mathrm{star}(F)$ is the unique tope in $\mathrm{star}(F)$ such that for all $O \in \mathrm{star}(F)$ we have

$$\begin{aligned} \mathrm{Sep}(P, O) &= \mathrm{Sep}(P, Q) \cup \mathrm{Sep}(Q, O) \\ \emptyset &= \mathrm{Sep}(P, Q) \cap \mathrm{Sep}(Q, O). \end{aligned}$$

Since $F_e = 0$ it also follows that $Q_e = -$. Since $F \leq R$, clearly $\mathrm{Sep}(R, Q) = \mathrm{Sep}(R, F \circ P) \subseteq \mathrm{Sep}(R, P)$ and hence $Q \preceq_R P$. This shows $Q \in (\hat{0}, P_{top})_{R,e}$. Now let $Q \leq_R Q'$. Then $F \circ Q \preceq_R F \circ Q'$. Since $F \leq R$ it follows that $F \circ Q \preceq_R Q$. Obviously $F \circ (F \circ Q) = F \circ Q$.

This shows that the map : $\circ_F : (\hat{0}, P_{top})_{R,e} \to (\hat{0}, P_{top})_{R,e}$ is a closure operator. And hence $(\hat{0}, P_{top})_{R,e}$ is homotopy equivalent to its image (see e.g, [3, Corollary 10.12]).

Since $P_{top} \notin \mathrm{star}(F)$ and $F \circ P_{top} \in \mathrm{star}(F) \cap (\hat{0}, P_{top})_{R,e}$, it also follows that $F \circ Q \preceq_R F \circ P_{top}$ for all $Q \in (\hat{0}, P_{top})_{R,e}$. Hence the image of $\circ_F$ has a maximal element and hence is contractible. □

4. Supertopes

In this section we identify supertopes that are relevant for our purposes and provide the proof of Theorem 1.2. We also deduce Corollary 4.3, which is crucial for the derivation of Theorem 1.1 from Theorem 1.2. Throughout this section we assume Situation 2.4.

In order to apply the Quillen fiber in the proof of Theorem 1.1 we need the following lemma.

Lemma 4.1. *Let $S^+, S^- \subseteq E$ such that $S^+ \cap S^- = \emptyset$, $S^+ \cup S^- \neq E$ and $f \in E \setminus (S^+ \cup S^-)$. Let $\mathcal{T}^f$ denote the set of topes of $\mathcal{L} \setminus f$ and $\mathcal{T}^f_{R \setminus f}$ the corresponding tope poset with base polytope $R \setminus \{f\}$. Consider the poset map $\pi^f : \mathcal{T}(S^+, S^-) \to \mathcal{T}^f(S^+, S^-)$ given by restriction. Let $Q \in \mathcal{T}^f(S^+, S^-)$. Then*

$$(\pi^f)^{-1}(\mathcal{T}^f_{\preceq Q}) = \mathcal{T}(Q^+, S^-).$$

Proof. Let $\tilde{Q} \in \mathcal{T}^f_{\preceq Q} \cap \mathcal{T}^f(S^+, S^-)$. As $R \backslash f$ is all positive, we must have $S^- \subseteq \tilde{Q}^- \subseteq Q^-$ and hence $Q^+ \subseteq \tilde{Q}^+$ implying $(\pi^f)^{-1}(\tilde{Q}) \subseteq \mathcal{T}(Q^+, S^-)$. On the other hand, if $\hat{Q} \in \mathcal{T}(Q^+, S^-)$, then $S^- \subseteq \hat{Q}^- \subseteq Q^- \cup \{f\}$, $S^+ \subseteq Q^+ \subseteq \hat{Q}^+ \cup \{f\}$. As $f \notin S^+ \cup S^-$ we have that $\pi^f(\hat{Q})$ is well defined and $\pi^f(\hat{Q}) \preceq Q$. $\square$

We need another preparatory lemma for the proof of Theorem 1.2.

Lemma 4.2. *Let $\mathcal{L}$ be an oriented matroid on E and $\mathcal{T}$ be the set of its topes. Let $E = S^+ \dot\cup S^- \dot\cup S^*$ be a partition of the ground set into nonempty sets S^+, S^- and S^*. If for all $f \in S^*$ there exists $T^f \in \mathcal{T}$ such that*

$$T^f_g = \begin{cases} + & \textit{if} \quad g \in S^+ \\ - & \textit{if} \quad g \in S^- \\ - & \textit{if} \quad g \in S^* \setminus \{f\} \\ + & \textit{if} \quad g = f, \end{cases}$$

then either there exists a tope $T^{\max} \in \mathcal{T}$ satisfying

$$T^{\max}_g = \begin{cases} + & \textit{if} \quad g \in S^+ \\ - & \textit{if} \quad g \in S^- \\ - & \textit{if} \quad g \in S^*, \end{cases}$$

or there exists a tope $T^{\min} \in \mathcal{T}$ satisfying

$$T^{\min}_g = \begin{cases} + & \textit{if} \quad g \in S^+ \\ - & \textit{if} \quad g \in S^- \\ + & \textit{if} \quad g \in S^* \end{cases}$$

or a covector $Y \in \mathcal{L}$ satisfying

$$Y_g = \begin{cases} + & \textit{if} \quad g \in S^+ \\ - & \textit{if} \quad g \in S^- \\ 0 & \textit{if} \quad g \in S^0 \\ - & \textit{if} \quad g \in S^* \setminus S^0 \end{cases}$$

for some set $\emptyset \neq S^0 \subseteq S^*$.

Hence, for a fixed $R \in \mathcal{T}$ the subposet $\mathcal{T}(S^+, S^-)$ of $\mathcal{T}_R$ either has a unique maximal element or it has a unique minimal element. In particular, it is contractible.

Proof. We proceed by induction on $|S^*|$. If $|S^*| = 1$ the assertion is trivial. If $S^* = \{f, g\}$, then, either f and g are antiparallel and we find a Y as required, or on a shortest path from T^f to T we must pass through $T^{\max}$ or $T^{\min}$. Assume $|S^*| \geq 3$ and let $g \in S^*$. If there exists some $f \in S^* \setminus \{g\}$ such that eliminating g between T^g and T^f yields a covector X^f such that $X^f_f \in \{0, -\}$, then $X \circ T^h$ for $h \in S^* \setminus \{f, g\}$ is an element $T^{\max}$. Hence we may assume that for all $f \in S^* \setminus \{g\}$ we find X satisfying

$$X_h^f = \begin{cases} + & \text{if} \quad h \in S^+ \\ - & \text{if} \quad h \in S^- \\ - & \text{if} \quad h \in S^* \setminus \{f,g\} \\ + & \text{if} \quad h = f \\ 0 & \text{if} \quad h = g. \end{cases}$$

Then the image of X^f in $\mathcal{L}/g$ satisfies the assumptions of the lemma in the oriented matroid $\mathcal{L}/g$. By induction we find either an appropriate $\tilde{Y} \in \mathcal{L}/g$ which clearly yields a Y as required in $\mathcal{L}$, or we find an element $X^{\max} \in \mathcal{L}$ such that

$$X_h^{\max} = \begin{cases} + & \text{if} \quad h \in S^+ \\ - & \text{if} \quad h \in S^- \\ - & \text{if} \quad h \in S^* \setminus \{g\} \\ 0 & \text{if} \quad h = g \end{cases}$$

and $X^{\max} \circ T^f$ is as required for $f \in S^* \setminus g$ and similarly $X^{\min} \circ T^g$ in the remaining case.

The assertion about the topology of $\mathcal{T}(S^+, S^-)$ now follows from the fact that either $T^{\max}$, $T^{\min}$ of $Y \circ (-R)$ has a unique minimal or a unique maximal element.

□

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. We proceed by induction on $|E \setminus (S^+ \cup S^-)|$. If $S^+ \cup S^- = E$, then $\mathcal{T}(S^+, S^-)$ is a singleton and thus contractible. If $S^+ \cup S^- \neq E$, then $S^* := E \setminus (S^+ \cup S^-) \neq \emptyset$. If for all $f \in S^*$ there exists T^f as in Lemma 4.2, $\mathcal{T}(S^+, S^-)$ is contractible by Lemma 4.2. Hence we may assume that there exists $f \in S^*$ such that $T^f \notin \mathcal{T}$. Let $\mathcal{T}^f_{R\setminus\{f\}}$ denote the tope poset in the oriented matroid $\mathcal{L} \setminus f$ with base tope $R \setminus \{f\}$. By inductive assumption its subposet $\mathcal{T}^f(S^+, S^-)$ is contractible. Consider the poset map $\pi^f : \mathcal{T}(S^+, S^-) \to \mathcal{T}^f(S^+, S^-)$ given by restriction. Let $Q \in \mathcal{T}^f(S^+, S^-)$. By Lemma 4.1

$$(\pi^f)^{-1}(\mathcal{T}^f_{\preceq Q}) = \mathcal{T}(Q^+, S^-).$$

Clearly $S^+ \subseteq Q^+$. If $S^+ \subsetneq Q^+$, then $(\pi^f)^{-1}(Q_{\preceq})$ is contractible by inductive assumption. Consider the case that $S^+ = Q^+$. By the choice of f the preimage $(\pi^f)^{-1}(Q)$ is a singleton $\{Z\}$ with $Z = -$. Hence, this is the unique maximal element in $(\pi^f)^{-1}(Q_{\preceq})$ and that fiber is also contractible. Hence by Proposition 2.1 $\mathcal{T}(S^+, S^-)$ and $\mathcal{T}^f(S^+, S^-)$ are homotopy equivalent and the claim follows. □

Corollary 4.3. *Let $R \in \mathcal{T}$ be the base tope of the poset $\mathcal{T}_R$. Let $e \notin S \subseteq E$. Then*

$$\sum_{\substack{Q \in \mathcal{T}(\emptyset, \{e\}) \\ S = \mathrm{Sep}(P,Q) \cap \mathrm{Sep}(Q,R)}} \mu((\hat{0}, Q)_{R,e}) = \begin{cases} -1 & \text{if} \quad S = \emptyset \\ 0 & \text{if} \quad S \neq \emptyset \end{cases}.$$

Proof. We prove the assertion by induction on $\#S$.

If $S = \emptyset$ then

$$\begin{aligned}\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S=\mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)}} \mu((\hat{0},Q)_{R,e}) &= \sum_{\hat{0}<_{R,e}Q\leq_{R,e}P} \mu((\hat{0},Q)_{R,e}) \\ &= -\mu((\hat{0},\hat{0})_{R,e}) \\ &= -1.\end{aligned}$$

Assume $\#S > 0$. Set

$$T^+ = \{f \in E \setminus (S \cup \{e\}) \mid R_f = +\} \text{ and } T^- = \{f \in E \setminus S \mid R_f = -\} \cup \{e\}.$$

Then

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ \mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)\subseteq S}} \mu((\hat{0},Q)_{R,e}) = \sum_{Q\in\mathcal{T}(T^+,T^-)} \mu((\hat{0},Q)_{R,e}) \tag{1}$$

The right hand side of (1) is the sum of Möbius function values from $\hat{0}$ to P where $P \neq \hat{0}$ ranges by Theorem 1.2 over the elements of a contractible poset. By classical Möbius function theory (see e.g. [3, (9.14)]) this sum then is $-\mu(\hat{0},\hat{0}) = -1$ plus the Möbius number of the poset. Since the poset is contractible its Möbius number is 0 and we have shown that:

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ \mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)\subseteq S}} \mu((\hat{0},Q)_{R,e}) = -1 \tag{2}$$

Now rewrite the right hand side of (2) as:

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ \mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)\subseteq S}} \mu((\hat{0},Q)_{R,e}) = \sum_{T\subseteq S} \sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ \mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)=T}} \mu((\hat{0},Q)_{R,e}) \tag{3}$$

By induction the summand $\displaystyle\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ \mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)=T}} \mu((\hat{0},Q)_{R,e})$ is 0 for $T \neq S, \emptyset$ and -1 for $T = \emptyset$. Thus combining (2) and (3) we obtain:

$$\begin{aligned}-1 &= \sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ \mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)\subseteq S}} \mu((\hat{0},Q)_{R,e}) \\ &= -1 + \sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ \mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)=S}} \mu((\hat{0},Q)_{R,e})\end{aligned}$$

From this we conclude

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ \mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)=S}} \mu((\hat{0},Q)_{R,e}) = 0.$$

□

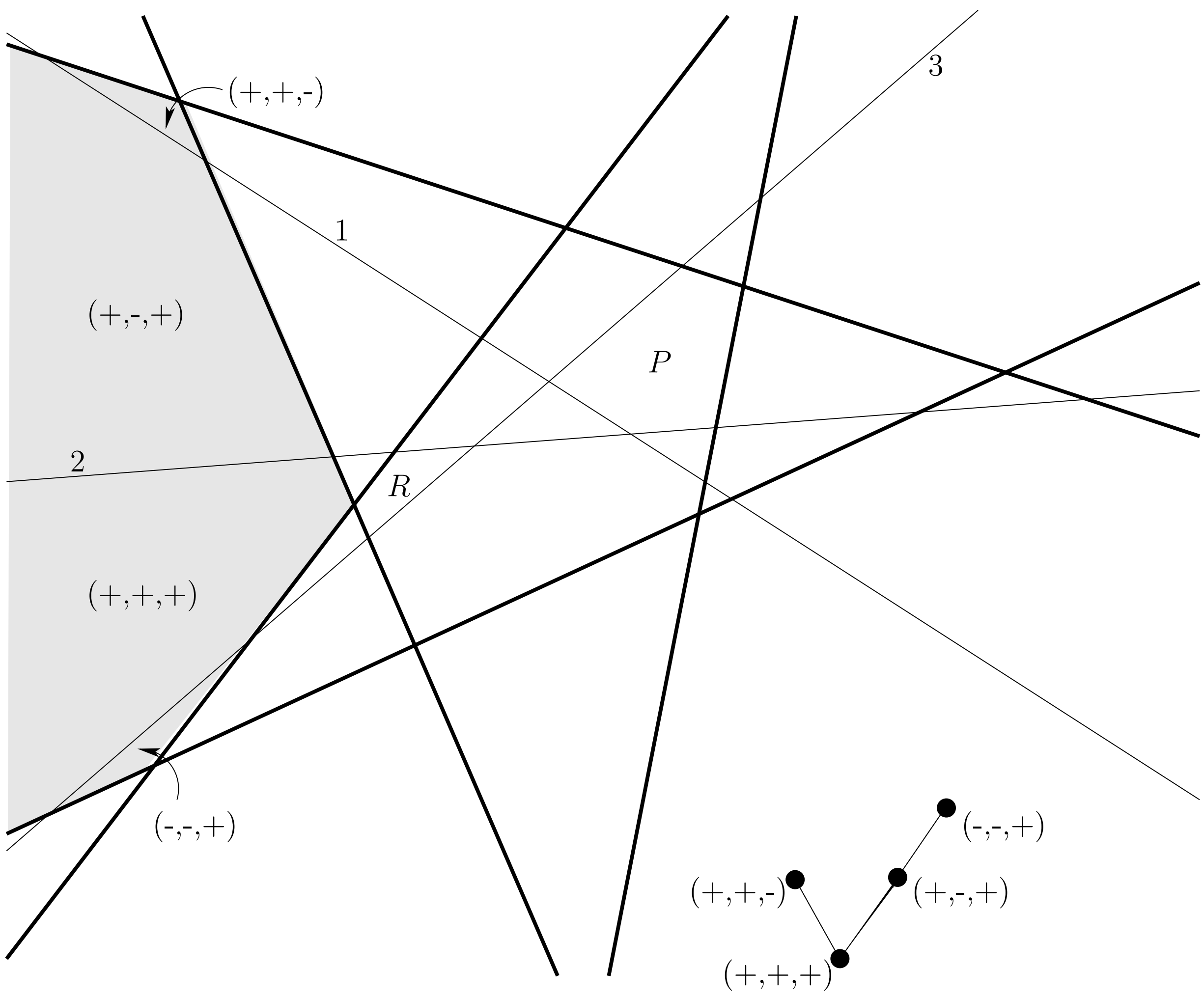

Figure 1. The shaded region has two maximal elements

Remark 4.4. Denham and Hanlon mention in [6] that a "routine argument shows" that the poset on $\{Q \in \mathcal{T}(\emptyset, \{e\}) \mid \mathrm{Sep}(P,Q) \cap \mathrm{Sep}(Q,R) = S\}$ induced by $\mathcal{T}_R$ always contains a unique maximal element. While this can be shown to hold true for line arrangements, it fails already in 3-dimensional hyperplane arrangements. In Figure 1, we provide a counterexample. The element e is supposed to be the drawing plane. The tope R is below and P and the shaded region above e. The separator $\mathrm{Sep}(P,R)$ without e is given by the thin lines, while the intersection of the remaining hyperplanes with e are the bold lines. S is given by the two bold lines that intersect in a vertex at R. The poset induced on the shaded regions is sketched on the bottom of the figure. While it is contractible it has two maximal elements.

5. The Varchenko matrix

In this section we prove Theorem 1.1 and its corollaries. The proof consists of a factorization of the matrix $\mathfrak{V}$ into matrices with controllable determinant.

Recall, that we assume $\mathcal{T}$ to be linearly ordered. For any sign pattern $\epsilon = (\epsilon_1, \epsilon_2) \in \{+,-\}^2$ let $\mathfrak{V}^{e,\epsilon}$ be a $(\ell \times \ell)$-matrix with rows indexed by $\mathcal{T}(\{e\},\emptyset)$ for $\epsilon_1 = +$, $\mathcal{T}(\emptyset,\{e\})$ for $\epsilon_1 = -$ and columns indexed by $\mathcal{T}(\{e\},\emptyset)$ for $\epsilon_2 = +$, $\mathcal{T}(\emptyset,\{e\})$ for $\epsilon_2 = -$. For a tope P indexing a row and a tope Q indexing a column we set $\mathfrak{V}^{e,\epsilon}_{P,Q} = \mathfrak{V}_{P,Q}$. We set $\ell = \#\mathcal{T}(\{e\},\emptyset) = \#\mathcal{T}(\emptyset,\{e\})$. Note that $\ell = \frac{1}{2}\#\mathcal{T}$ is independent of e. We fix a linear ordering on E and set M^e to be the $(\ell\times\ell)$-matrix with rows indexed by $\mathcal{T}(\emptyset,\{e\})$, columns indexed by $\mathcal{T}(\{e\},\emptyset)$ and entries

$$M^e_{Q,R} = \left\{ \begin{array}{cl} -\mu((\hat{0},Q)_{R,e}) \cdot \mathfrak{V}_{Q,R} & \text{if } e \text{ is the maximal element of } \mathrm{Sep}(Q,R) \text{ and} \\ 0 & \text{otherwise,} \end{array} \right.$$

where $Q \in \mathcal{T}(\emptyset,\{e\})$ and $R \in \mathcal{T}(\{e\},\emptyset)$. We write $\mathcal{I}_\ell$ for the $(\ell\times\ell)$-identity matrix and define

$$\mathcal{M}^e = \left(\begin{array}{cc} \mathcal{I}_\ell & M^e \\ M^e & \mathcal{I}_\ell \end{array} \right).$$

Lemma 5.1. *Let e be the maximal element of E. Then $\mathfrak{V}^{e,(-,+)}$ factors as*

$$\mathfrak{V}^{e,(-,+)} \quad = \quad \mathfrak{V}^{e,(-,-)} \cdot M^e. \tag{4}$$

Proof. For $P \in \mathcal{T}(\emptyset,\{e\})$ and $R \in \mathcal{T}(\{e\},\emptyset)$ the entry in row P and column R on the left hand side of (4) is $\mathfrak{V}_{P,R}$. On the right hand side the corresponding entry is:

$$\sum_{Q\in\mathcal{T}(\emptyset,\{e\})} \mathfrak{V}_{P,Q} \cdot M^e_{Q,R} \quad = \quad - \sum_{Q\in\mathcal{T}(\emptyset,\{e\})} \mu((\hat{0},Q)_{R,e}) \cdot \mathfrak{V}_{P,Q} \cdot \mathfrak{V}_{Q,R}$$

By definition for $Q \in \mathcal{T}(\emptyset,\{e\})$ we have

$$\mathfrak{V}_{P,Q} \cdot \mathfrak{V}_{Q,R} \quad = \quad \mathfrak{V}_{P,R} \cdot \prod_{f\in\mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)} U_f^2$$

Thus the claim of the lemma is proved once we have shown that for a fixed subset $S \subseteq E$ and fixed Q,R we have:

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\}) \\ S=\mathrm{Sep}(P,Q)\cap\mathrm{Sep}(Q,R)}} \mu((\hat{0},Q)_{R,e}) = \left\{ \begin{array}{cl} 0 & \text{if } S \neq \emptyset \\ -1 & \text{otherwise.} \end{array} \right. . \tag{5}$$

But this is the content of Corollary 4.3 and we are done. ∎

Next we use the matrices $\mathcal{M}^e$ to factorize $\mathfrak{V}$. The following lemma yields the inductive step in the factorization.

Lemma 5.2. *Let e be the maximal element of E and let $\mathfrak{V}_{U_e=0}$ be the matrix $\mathfrak{V}$ after evaluating U_e to 0. Then*

$$\mathfrak{V} = \mathfrak{V}_{U_e=0} \cdot \mathcal{M}^e$$

Proof. Let $\mathcal{T}(\emptyset, \{e\}) = \{P_1, \ldots, P_\ell\}$ and $\mathcal{T}(\{e\}, \emptyset) = \{P_{\ell+1}, \ldots, P_{2\ell}\}$ be numbered such that $-P_i = P_{\ell+i}$ for $1 \leq i \leq \ell$. Assume the rows and columns of $\mathfrak{V}$ are ordered according to this numbering of $\mathcal{T}$. This yields a block decomposition of $\mathfrak{V}$ as

$$\mathfrak{V} = \begin{pmatrix} \mathfrak{V}^{e,(-,-)} & \mathfrak{V}^{e,(-,+)} \\ \mathfrak{V}^{e,(+,-)} & \mathfrak{V}^{e,(+,+)} \end{pmatrix}.$$

Since $\mathfrak{V}_{P,Q} = \mathfrak{V}_{-P,-Q}$ it follows that $\mathfrak{V}^{e,(-,-)} = \mathfrak{V}^{e,(+,+)}$ and $\mathfrak{V}^{e,(-,+)} = \mathfrak{V}^{e,(+,-)}$. Then by Lemma 5.1 we know $\mathfrak{V}^{e,(-,+)} = \mathfrak{V}^{e,(-,-)} \cdot M^e$ and hence

$$\mathfrak{V}^{e,(+,-)} = \mathfrak{V}^{e,(-,+)} = \mathfrak{V}^{e,(-,-)} \cdot M^e = \mathfrak{V}^{e,(+,+)} \cdot M^e.$$

Thus

$$\begin{aligned} \mathfrak{V} &= \begin{pmatrix} \mathfrak{V}^{e,(-,-)} & 0 \\ 0 & \mathfrak{V}^{e,(+,+)} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{I}_\ell & M^e \\ M^e & \mathcal{I}_\ell \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{V}^{e,(-,-)} & 0 \\ 0 & \mathfrak{V}^{e,(+,+)} \end{pmatrix} \cdot \mathcal{M}^e. \end{aligned} \tag{6}$$

Now the monomial $\mathfrak{V}_{P,Q}$ has a factor U_e if and only if $P \in \mathcal{T}(\emptyset, \{e\})$ and $Q \in \mathcal{T}(\{e\}, \emptyset)$ or $P \in \mathcal{T}(\{e\}, \emptyset)$ and $Q \in \mathcal{T}(\emptyset, \{e\})$. Hence

$$\mathfrak{V}_{U_e=0} = \begin{pmatrix} \mathfrak{V}^{e,(-,-)} & 0 \\ 0 & \mathfrak{V}^{e,(+,+)} \end{pmatrix}. \tag{7}$$

Combining (6) und (7) yields the claim. $\square$

Now we are in position to state and prove the crucial factorization.

Proposition 5.3. *Let $E = \{e_1 \prec \cdots \prec e_r\}$ be a fixed ordering. Then*

$$\mathfrak{V} = \mathcal{M}^{e_1} \cdots \mathcal{M}^{e_r}.$$

Proof. We will prove by downward induction on i that

$$\mathfrak{V} = \mathfrak{V}_{U_i=\cdots=U_r=0} \cdot \mathcal{M}^{e_i} \cdots \mathcal{M}^{e_r}. \tag{8}$$

For $i = r$ the assertion follows directly from Proposition 5.3. For the inductive step assume $i > 1$ and (8) holds for i. We know from Lemma 5.2 that if we choose a linear ordering on E for which e_{i-1} is the largest element then

$$\mathfrak{V} = \mathfrak{V}_{U_{i-1}=0} \cdot \mathcal{N}, \tag{9}$$

where $\mathcal{N} = (N_{Q,R})_{Q,R\in\mathcal{T}}$ is defined as

$$N_{Q,R} = \begin{cases} 1 & \text{if} & Q = R \\ -\mu((\hat{0},Q)_{R,e_{i-1}})\,\mathfrak{V}_{Q,R} & \text{if} & R \in \mathcal{T}(\{e_{i-1}\},\emptyset), Q \in \mathcal{T}(\emptyset,\{e_{i-1}\}) \\ -\mu((\hat{0},-Q)_{-R,e_{i-1}})\,\mathfrak{V}_{Q,R} & \text{if} & -R \in \mathcal{T}(\{e_{i-1}\},\emptyset), -Q \in \mathcal{T}(\emptyset,\{e_{i-1}\}) \\ 0 & \text{otherwise} & \end{cases}.$$

This shows:

$$(N_{Q,R})_{U_i=\cdots=U_r=0} \begin{cases} 1 & \text{if } Q = R \\ -\mu((\hat{0},Q)_{R,e_{i-1}})\,\mathfrak{V}_{Q,R} & \text{if } e_{i-1} \text{ is the largest element in } \operatorname{Sep}(Q,R), \\ & R \in \mathcal{T}(\{e_{i-1}\},\emptyset), Q \in \mathcal{T}(\emptyset,\{e_{i-1}\}) \\ -\mu((\hat{0},-Q)_{-R,e_{i-1}})\,\mathfrak{V}_{Q,R} & \text{if } e_{i-1} \text{ is the largest element in } \operatorname{Sep}(Q,R), \\ & -R \in \mathcal{T}(\{e_{i-1}\},\emptyset), -Q \in \mathcal{T}(\emptyset,\{e_{i-1}\}) \\ 0 & \text{otherwise} \end{cases}.$$

But then $\mathcal{N}_{U_i=\cdots=U_r=0} = \mathcal{M}^{e_{i-1}}$.

Now (9) implies

$$\begin{aligned} \mathfrak{V}_{U_i=\cdots=U_r=0} &= \mathfrak{V}_{U_{i-1}=\cdots=U_r=0} \cdot \mathcal{N}_{U_i=\cdots=U_r=0} \\ &= \mathfrak{V}_{U_{i-1}=\cdots=U_r=0} \cdot \mathcal{M}^{e_{i-1}} \end{aligned}$$

With the induction hypothesis this completes the induction step by

$$\begin{aligned} \mathfrak{V} &= \mathfrak{V}_{U_i=\cdots=U_r=0} \cdot \mathcal{M}^{e_i} \cdots \mathcal{M}^{e_r} \\ &= \mathfrak{V}_{U_{i-1}=U_i=\cdots=U_r=0} \cdot \mathcal{M}^{e_{i-1}} \cdots \mathcal{M}^{e_r}. \end{aligned}$$

For $i = 1$ the matrix $\mathfrak{V}_{U_1=\cdots=U_r=0}$ is the identity matrix. Thus (8) yields:

$$\mathfrak{V} = \mathcal{M}^{e_1} \cdots \mathcal{M}^{e_r}.$$

□

Let $F \in \mathcal{L}$ and $e \in z(F)$ be the maximal element of $z(F)$. Define $\mathcal{T}^{F,e}$ as the set of topes $P \in \mathcal{T}$ such that F is the maximal element of $\mathcal{L}$ for which $F_e = 0$ and $F \le P$.

Proposition 5.4. *For any pair of topes $Q, R \in \mathcal{T}^{F,e}$ we have*

$$\mu((\hat{0},\pm Q)_{\pm R,e}) = \begin{cases} -(-1)^{\operatorname{rank}(\mathcal{L}|_{z(F)})} & \textit{if} & Q_{z(F)} = -R_{z(F)} \\ 0 & \textit{otherwise} & \end{cases}.$$

Proof. By the definition of $\mathcal{T}^{F,e}$ we have $F \le Q, R$. Thus, if we consider the poset $\mathcal{T}_{R_{z(F)},e}$ in the contraction $\mathcal{L}/z(F)$ we find that the interval $(\hat{0},\pm Q)_{\pm R,e}$ is isomorphic to $(\hat{0},\pm Q_{z(F)})_{\pm R_{z(F)},e}$. Furthermore, since F is the maximal element satisfying $F_e = 0$ and $F \le Q$, e does not define a proper face of $Q_{z(F)}$. Hence the claim follows from Corollary 3.4. □

We define $b_{F,e} = 0$ if e is not the maximal element of $z(F)$ and $\frac{1}{2}\#\mathcal{T}^{F,e}$ otherwise. Since $P \mapsto F \circ (-P)$ is a perfect pairing on $\mathcal{T}^{F,e}$ it follows that $\mathcal{T}^{F,e}$ contains an even number of topes. In particular, $b_{F,e}$ is an integer. We denote by $\mathcal{M}^{F,e}$ the submatrix of $\mathcal{M}^e$ obtained by selecting rows and columns indexed by $\mathcal{T}^{F,e}$.

Lemma 5.5. *Let $F \in \mathcal{L}$ and $e \in z(F)$. If $\mathcal{T}^{F,e} \neq \emptyset$. then*

$$\det(\mathcal{M}^{F,e}) = (1 - a(F)^2)^{b_{F,e}}.$$

Proof. By definition of $\mathcal{M}^e$ we obtain that for $Q, R \in \mathcal{T}^{F,e}$ we have

$$\mathcal{M}^e_{Q,R} = \begin{cases} 1 & \text{if} \quad Q = R \\ -\mu((\hat{0}, Q)_{R,e}) \cdot \mathfrak{V}_{Q,R} & \text{if} \quad \begin{array}{c} e \text{ is the largest element of } \mathrm{Sep}(Q,R), \\ R \in \mathcal{T}(\{e\}, \emptyset), Q \in \mathcal{T}(\emptyset, \{e\}) \end{array} \\ -\mu((\hat{0}, -Q)_{-R,e}) \cdot \mathfrak{V}_{Q,R} & \text{if} \quad \begin{array}{c} e \text{ is the largest element of } \mathrm{Sep}(Q,R), \\ -R \in \mathcal{T}(\{e\}, \emptyset), -Q \in \mathcal{T}(\emptyset, \{e\}) \end{array} \\ 0 & \text{otherwise} \end{cases}.$$

If $Q_{z(F)} = -R_{z(F)}$ then $\mathfrak{V}_{Q,R} = a(F)$. Using Proposition 5.4 we find

$$\mathcal{M}^e_{Q,R} = \begin{cases} 1 & \text{if} \quad Q = R \\ -(-1)^{\mathrm{rank}(\mathcal{L}|_S)} a(F) & \text{if} \quad \begin{array}{c} Q = F \circ (-R) \\ e \text{ largest element of } \mathrm{Sep}(Q,R) \end{array} \\ 0 & \text{otherwise} \end{cases}.$$

We order rows and columns of $\mathcal{M}^{F,e}$ so that the elements R and $F \circ (-R)$ are paired in consecutive rows and columns. With this ordering $\mathcal{M}^{F,e}$ is a block diagonal matrix having along its diagonal $b_{F,e}$ two by two matrices

$$\begin{pmatrix} 1 & -(-1)^{\mathrm{rank}(\mathcal{L}|_{z(F)})} a(F) \\ -(-1)^{\mathrm{rank}(\mathcal{L}|_{z(F)})} a(F) & 1 \end{pmatrix}$$

if e is the maximal element of $z(F)$ and identity matrices otherwise. In any case we find $\det(\mathcal{M}^{F,e}) = (1 - a(F)^2)^{b_{F,e}}$ as desired. □

Lemma 5.6. *After suitably ordering $\mathcal{T}$ the matrix $\mathcal{M}^e$ is the block lower triangular matrix with the matrices $\mathcal{M}^{F,e}$ for $F \in \mathcal{L}$ with $F_e = 0$ and $\mathcal{T}^{F,e} \neq \emptyset$ on the main diagonal.*

Proof. We fix a linear ordering of $\mathcal{T}$ such that for fixed $e \in E$ and $F \in \mathcal{L}$ the topes from $\mathcal{T}^{F,e}$ form an interval and such that the topes from $\mathcal{T}^{F,e}$ precede those of $\mathcal{T}^{F',e}$ if $F < F'$.

For this order the claim follows if we show that the entry $(\mathcal{M}^e)_{Q,R}$ is zero whenever $R \in \mathcal{T}^{F,e}$, $Q \in \mathcal{T}^{F',e}$ and $F' < F$.

If $Q_e = R_e$ then by $Q \neq R$ we have $(\mathcal{M}^e)_{Q,R} = 0$. Hence it suffices to consider the case $Q_e \neq R_e$.

If $Q \notin \mathrm{star}(F)$, $Q \in \mathcal{T}(\emptyset, \{e\})$ and $R \in \mathcal{T}(\{e\}, \emptyset)$ then it follows from Theorem 3.5 that $\mu((\hat{0}, Q)_{R,e}) = 0$ and therefore $(\mathcal{M}^e)_{Q,R} = 0$. Analogously if $Q \notin \mathrm{star}(F)$, $-Q \in \mathcal{T}(\emptyset, \{e\})$ and $-R \in \mathcal{T}(\{e\}, \emptyset)$ then $\mu((\hat{0}, -Q)_{-R,e}) = 0$ and therefore $(\mathcal{M}^e)_{Q,R} = 0$.

On the other hand, if $Q \in \mathrm{star}(F)$, then in particular $F \leq Q$. Since by definition of $\mathcal{T}^{F',e}$ we have that F' is the maximal covector such that $F' \leq Q$ and $F'_e = 0$ it follows that $F \leq F'$. Since $F \neq F'$ we must have that $F < F'$, i.e. $(\mathcal{M}^e)_{Q,R}$ is an entry above the diagonal and we are done.

□

Proof of Theorem 1.1. After fixing a linear order on E it follows from Proposition 5.3 that $\det \mathfrak{V}$ is the product of the determinants of $\mathcal{M}^e$ for $e \in E$. By Lemma 5.6 the determinant of each $\mathcal{M}^e$ is a product of determinants of $\mathcal{M}^{F,e}$ for $e \in E$ and $F \in \mathcal{L}$ for which $\mathcal{T}^{F,e} \neq \emptyset$. Then Lemma 5.5 completes the proof. $\square$

As an immediate consequence of the proof we can give a refined version of Theorem 1.1 which also implies that $\det(\mathfrak{V})$ only depends on the matroid $\mathrm{Mat}(\mathcal{L})$ underlying the oriented matroid $\mathcal{L}$.

Let us first state an additional fact about an oriented matroid $\mathcal{L}$ over ground set E and its underlying matroid $\mathrm{Mat}(\mathcal{L})$. Recall, that for a fixed $e \in E$ the bounded topes of the affine oriented matroid defined by e in $\mathcal{L}$ are the topes in $\mathcal{L}$ for which e does not define a proper face. By [4, Theorem 4.6.5] the cardinality of this set of topes is given by twice the β-invariant $\beta(\mathrm{Mat}(\mathcal{L}))$ of $\mathrm{Mat}(\mathcal{L})$ and hence is independent of e.

Corollary 5.7. *Let $\mathfrak{V}$ be the Varchenko matrix of the oriented matroid with covector set $\mathcal{L}$ and $M = \mathrm{Mat}(\mathcal{L})$ the matroid underlying $\mathcal{L}$. Then*

$$\det(\mathfrak{V}) = \prod_{\substack{A \subseteq E \\ A \text{ is closed in } M}} (1 - \prod_{e \in A} U_e^2)^{m_A},$$

where m_A is the product of number of topes in the contraction $\mathcal{L}/A$ and of $\beta(\mathrm{Mat}(\mathcal{L}|_A))$. In particular, it follows that $\det(\mathfrak{V}))$ only depends on the matroid $M = \mathrm{Mat}(\mathcal{L})$.

Proof. Fix $A \subseteq E$. Using the notation of Theorem 1.1 it follows that $m_A = \sum_{\substack{F \in \mathcal{L} \\ z(F) = A}} b_F$. The number of summands equals the number of topes in the contraction $\mathcal{L}/A$ of A. The latter only depends on $\mathrm{Mat}(\mathcal{L}/A)$ which only depends on $\mathrm{Mat}(\mathcal{L})$. By Lemma 5.5 we have $b_F = b_{F,e}$ where e is the maximal element of $z(F)$. Now $b_{F,e}$ is half the number of elements of $\mathcal{T}^{F,e}$, which is the set of topes P for which F is the unique maximal element of $\mathcal{L}$ such that $F \leq P$ and $F_e = 0$. The map sending P to $P_{z(F)}$ is then a bijection between the topes in $\mathcal{T}^{F,e}$ and the topes of $\mathcal{L}|_{z(F)}$ for which e does not define a proper face. As mentioned above, by [4, Theorem 4.6.5] the number of topes in $\mathcal{L}|_{z(F)}$ for which $e \in z(F)$ does not define a proper face in $\mathcal{L}|_{z(F)}$ is independent of e and coincides with twice the beta invariant $\beta(\mathrm{Mat}(\mathcal{L}|_{z(F)}))$. Hence we find that $b_{F,e} = \beta(\mathrm{Mat}(\mathcal{L}|_{z(F)}))$. Since for a nonempty subset the matroid $\mathrm{Mat}(\mathcal{L}|_A)$ depends on A and $\mathrm{Mat}(\mathcal{L})$ only it follows that m_A is an invariant of $\mathrm{Mat}(\mathcal{L})$. $\square$

Finally, as a second corollary we extend Theorem 1.1 to row and column selected submatrices of $\mathfrak{V}$ corresponding to topes in a closed supertope. For oriented matroids coming from hyperplane arrangements this formula can also be found in [1] and [7].

Corollary 5.8. *Let $\mathfrak{V}$ be the Varchenko matrix of the oriented matroid with covector set $\mathcal{L}$. For a subset $E' \subseteq E$ and signs $\epsilon = (\epsilon_e)_{e \in E'} \in \{+,-\}^{E'}$ such that $\mathcal{T}(\epsilon^+, \epsilon^-)$ is a closed supertope let $\mathfrak{V}_\epsilon$ be the matrix constructed from $\mathfrak{V}$ by selecting all rows and columns corresponding to topes $P \in \mathcal{T}$ for which $P_e = \epsilon_e$ for $e \in E'$.*

Then

$$\det(\mathfrak{V}_\epsilon) = \prod_{\substack{F\in\mathcal{L}\\ F_e\neq 0, e\in E'}} (1-a(F)^2)^{b_{F,\epsilon}},$$

for some numbers $b_{F,\epsilon}$.

Proof. Consider the case $E' = \{e\}$. Order $\mathcal{T} = \{P_1 \prec \cdots \prec P_{2s}\}$ such that $(P_1)_e = \cdots = (P_s)_e = +$ and $P_{i+s} = -P_i$ for $1 \leq i \leq s$. Then $\mathfrak{V}_\epsilon$ is a block diagonal matrix with two blocks identical to $\mathfrak{V}_\epsilon$ on the main diagonal. By Theorem 1.1 we have

$$\begin{aligned}\det \mathfrak{V}_{U_e=0} &= \prod_{\substack{F\in\mathcal{L}\\ F_e\neq 0}} (1-a(F))^{b_F} \\ &= \det(\mathfrak{V}_\epsilon)^2.\end{aligned}$$

Since the main diagonal in $\mathfrak{V}_\epsilon$ is constant 1 and since this are the only constant entries it follows that $\det(\mathfrak{V}_\epsilon)$ has constant term $+1$. It follows that

$$\det(\mathfrak{V}_\epsilon) = \prod_{\substack{F\in\mathcal{L}\\ F_e\neq 0}} (1-a(F))^{\frac{b_F}{2}}.$$

Now induction on the cardinality of E' proves the assertion. □

Remark 5.9. If $E' = \{e\}$ in Corollary 5.8, i.e. in the case of an affine oriented matroid, then [4, Theorem 4.6.5] implies, that $\det(\mathfrak{V}_\epsilon)$ is still a matroid invariant.

Remark 5.10. The formulas in Theorem 1.1 and Corollary 5.7 are very explicit in terms of combinatorial invariants of the matroid and are useful for further analysis of the matrix. Nevertheless, it seems computationally hard to write down the formulas in concrete cases. We refer to [9] for an analysis of reflection arrangements of the symmetric groups. The case that originally motivated Varchenko's work.

Remark 5.11. Recently, Bandelt et. al. introduced complexes of oriented matroids (COMs) [2] as families of signed vectors which satisfy the covector elimination axiom and a symmetrized version of the axiom of conformal composition of oriented matroid theory. A COM is an oriented matroid if and only if it contains 0. Examples of COMs are affine oriented matroids and "closed supertopes without boundary". The authors even conjecture that the latter case characterizes COMs. A generalization of Corollary 5.8 to COMs would support that conjecture. Our proof of supertope contractability uses only covector elimination, thus Theorem 1.2, Corollary 4.3 and Proposition 5.3 should generalize to COMs. In order to zero out the Möbius function values below the diagonal of the $\mathcal{M}$ matrices though, we frequently use results about the global topology of an oriented matroid. An exception is Proposition 5.4 which should go for COMs. Note that its proof is based on properties of a proper contraction of a covector. As such a proper contraction of a COM contains 0 it is an oriented matroid. It is not immediately clear, though, that this suffices to generalize Corollary 5.8 without extending all topological results to COMs which should be a non-trivial project on its own.

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