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The NL-flow polynomial

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Abstract

We generalize the Equivalence Theorem for Nowhere-Zero-Flows to Neumann-Lara-flows (NL-flows). In particular we show that the number of group valued NL-flows of a given digraph is counted by a polynomial in the order of the group.

1 Introduction

Víctor Neumann-Lara [8] introduced the dichromatic number $\vec{\chi}(D)$ of a digraph D, also called acyclic chromatic number, as the smallest integer k such that the vertices V of D can be colored with k colors and each color class induces a directed acyclic graph. In a recent paper [6] we developed a flow theory for the dichromatic number that mimics Tutte's theory of nowhere-zero-flows for the case of the classic chromatic number. The purpose of this paper is to pursue this analogy introducing algebraic NL-flows and a polynomial, which counts these flows.

Our notation is fairly standard and, if not explicitely defined, should follow the books of Diestel [5] for graphs and Björner et. al. [3] for oriented matroids. Note that all our digraphs may have parallel and antiparallel arcs.

2 The Equivalence Theorem

Definition 1. *Let $D = (V, A)$ be a (multi-)digraph. A* Neumann-Lara-k-flow*, a NL-k-flow for short is a map*

$$f : A \to \{0, \pm 1, \dots, \pm(k-1)\},$$

which is a flow

$$\forall v \in V : \sum_{a \in \partial^+(v)} f(a) = \sum_{a \in \partial^-(v)} f(a)$$

such that $D[A/supp(f)]$ is totally cyclic, i.e. every component is strongly connected. If G is a finite Abelian group in additive notation, we define a NL-G-flow *as a map*

$$f : A \to G,$$

which is a flow such that contracting its support yields a totally cyclic digraph.

With this definition we get an equivalence theorem in full analogy to the case of nowhere-zero flows ([9, 7]).

Theorem 2. *Let $D = (V, A)$ be a digraph. Let $k \geq 2$ and G be an Abelian group of order k. Then the following conditions are equivalent:*

(i) There exists a NL-$\mathbb{Z}_k$-flow in D.

(ii) There exists a NL-G-flow in D.

(iii) There exists a NL-k-flow in D.

Proof. $(iii) \Rightarrow (i)$ This is trivial. Just take the integer flow $\mod k$.

$(i) \Rightarrow (iii)$ Let $f : A \to \mathbb{Z}_k$ denote a $\mathbb{Z}_k$ flow. Choose $\tilde{f} \in \{0, \pm 1, \ldots, \pm(k-1)\}^A$ satisfying $\tilde{f}(a) \equiv f(a) \mod k$ for all arcs $a \in A$ and minimizing

$$\sum_{v \in V} \left| \sum_{a \in \partial^+(v)} \tilde{f}(a) - \sum_{a \in \partial^-(v)} \tilde{f}(a) \right|.$$

We claim that $\tilde{f}$ is already a flow. Assume not and denote the set of vertices with excess by $V^+ := \{v \in V \mid \sum_{a \in \partial^+(v)} \tilde{f}(a) > \sum_{a \in \partial^-(v)} \tilde{f}(a)\}$ and by $V^- := \{v \in V \mid \sum_{a \in \partial^+(v)} \tilde{f}(a) < \sum_{a \in \partial^-(v)} \tilde{f}(a)\}$ the set of vertices with demand. By assumption both sets are non-empty. Consider an auxiliary network consisting of the arcs with positive flow in forward direction and backward arcs for each arc with negative flow. Assume there were no directed path from a demand vertex to an excess vertex. Then we would find a cut separating the demand vertices from the excess vertices with the additional property that the forward arcs have a negative flow and the backward arcs have a positive flow. This contradicts the law of flow conservation. Hence we find such a path. Replacing the positive flow values on arcs a by $\tilde{f}(a) - k$ and the negative ones by $\tilde{f}(a) + k$ we find a flow contradicting the minimality of $\tilde{f}$. Hence $\tilde{f}$ must have been a flow, already.

$(i) \iff (ii)$ In the following section we will show that the number of group valued flows is a polynomial in the order of the group, independent of its structure.

□

3 The NL-flow polynomial

The basic observation leading to the definition of the NL-flow polynomial is that a flow is an NL-flow if and only if its support is a dijoin of the digraph.

Definition 3. *Let $D = (V, A)$ denote a directed graph. A set of arcs $S \subseteq A$ is a* dijoin*, if S intersects every non-empty directed cut.*

Proposition 4. *S is a dijoin if and only D/S is totally cyclic.*

Proof. If D/S is totally cyclic, then it does not contain a directed cut. Hence S must have intersected every directed cut. If D/S is not totally cyclic it contains a directed cut, which is a directed cut in D as well. Hence S is not a dijoin. $\square$

As a consequence we find:

Proposition 5. *$f : A \to G$ is an NL-G-flow if and only if* $\mathrm{supp}(f)$ *is a dijoin.*

In the following we will show that given a fixed set S of edges the number of flows that are non-zero on S is a polynomial in the order of the group.

Definition 6. *Let $C \subseteq A$, then $f : C \to G \setminus \{e_G\}$ is a* partial NZ-G-flow *of D, if for every cut $\partial(X) \subseteq C$ of D*

$$\sum_{a \in \partial^+(X)} f(a) = \sum_{a \in \partial^-(X)} f(a).$$

The following proposition is obvious.

Proposition 7. *$f : C \to G \setminus \{e_G\}$ is a partial NZ-G-flow of D if and only if f is a NZ-G-flow of $D/(A \setminus C)$.*

Since the number of NZ-G-flows of $D/(A \setminus C)$ is a given by polynomial it now suffices to count the number of ways in which a partial NZ-G-flow can be extended to a flow of G.

Lemma 8. *Let G be an Abelian group, $M \in \{0, \pm 1\}^{m \times n}$ a totally unimodular matrix of full row rank and $b \in G^m$. Then the number of solutions of $Mx = b$ is $|G|^{(n-m)}$.*

Proof. Choose a basis B of M. Then

$$\begin{array}{rcl} Mx & = & b \\ \iff M_{.B}^{-1} Mx & = & M_{.B}^{-1} b \\ \iff (I_m, \tilde{M}) \begin{pmatrix} x_B \\ x_{\{1,\ldots,n\} \setminus B} \end{pmatrix} & = & M_{.B}^{-1} b, \end{array}$$

where $\tilde{M}$ is a totally unimodular $(m \times (n - m))$-matrix. Thus, for every choice of values for the columns of $\tilde{M}$ we get exactly one solution of the equation. $\square$

As an immediate consequence we have the following theorem:

Theorem 9. *Let $f : C \to G$ define a partial NZ-flow for $C \subseteq A$. Then the number of extensions of f to a G-flow in D is $k^{|A\setminus C|-\mathrm{rk}(A\setminus C)}$, where $\mathrm{rk}(A\setminus C) = |V| - \#$components of $D[A \setminus C]$.*

Summarizing we get

Theorem 10. *Let $C \subseteq A$ and $\phi_C(x)$ denote the flow polynomial of $D/(A \setminus C)$. The number of G-flows that are non-zero on C is*

$$\phi_C(|G|)|G|^{|A\setminus C|-\mathrm{rk}(A\setminus C)}.$$

Note, that the order of ϕ_C is given by the corank of $D/(A \setminus C)$ which computes to

$$\begin{aligned} |C| - \mathrm{rk}(D/(A \setminus C)) &= |C| - |V(D/(A \setminus C))| - \#\text{comp. of } D/(A \setminus C) \\ &= |C| - |V| + \mathrm{rk}(A \setminus C) - \#\text{comp. of } D. \end{aligned}$$

Hence the polynomial $\phi_C(|G|)|G|^{|A\setminus C|-\mathrm{rk}(A\setminus C)}$ has order $|A| - \mathrm{rk}(A)$.

Using Proposition 5 we can now derive the desired result using the principle of inclusion and exclusion.

Definition 11. *Let $D = (V, A)$ be a digraph and $\{S_1, \ldots, S_r\}$ denote its set of inclusionwise minimal dijoins. For $I \subseteq \{1, \ldots, r\}$ let $S_I := \cup_{i\in I} S_i$. As above denote by ϕ_{S_I} the flow polynomial of $D/(A \setminus S_I)$. Then the NL-flow polynomial of D is defined as*

$$\phi^D_{NL}(x) := \sum_{\emptyset \neq I \subseteq \{1,\ldots,r\}} (-1)^{|I|-1} \phi_{S_i}(x) x^{|A\setminus S_I|-\mathrm{rk}(A\setminus S_I)}.$$

Theorem 12. *The number of NL-G-flows of a digraph D depends only on the order k of G and is given by $\phi^D_{NL}(k)$.*

Proof. By Proposition 5 a flow is a NL-flow if and only if its support is a dijoin. If we denote by Σ_i the set of all G-flows with S_i in its support, then the principle of inclusion and exclusion immediately yields that the number of NL-G-flows of D is given by

$$\sum_{\emptyset \neq I \subset \{1,\ldots,r\}} (-1)^{|I|-1} \left| \bigcap_{i\in I} \Sigma_i \right|.$$

Clearly $\bigcap_{i\in I} \Sigma_i$ is the set of flows with $S_I = \bigcup_{i\in I} S_i$ in its support. By Theorem 10 thus $|\bigcap_{i\in I} \Sigma_i| = \phi_{S_i}(|G|)|G|^{|A\setminus S_I|-\mathrm{rk}(A\setminus S_I)}$ and the claim follows. $\square$

4 Oriented Regular Matroids

The equivalence theorem for nowhere-zero-flows has been generalized to regular oriented matroids by Crapo [4] and Arrowsmith and Jaeger [2]. Like them, we can generalize our results of the previous sections to oriented regular matroids. Our main tool will be the following variant of Farkas' Lemma:

Theorem 13 (see 3.4.4 (4P) in [3])**.** *Let $\mathcal{O}$ denote an oriented matroid on a finite set E given by its set of covectors and $\mathcal{O}^*$ its dual. Let $E = P \dot\cup N \dot\cup * \dot\cup O$ be a partition of E and $i_0 \in P$. Either there exists $X \in \mathcal{O}^*$ such that $i_0 \in \operatorname{supp}(X) \subseteq P \cup N \cup *$, $\operatorname{supp}(X) \cap P \subseteq X^+$ and $\operatorname{supp}(X) \cap N \subseteq X^-$ or there exists $Y \in \mathcal{O}$ such that $i_0 \in \operatorname{supp}(Y) \subseteq P \cup N \cup O$, $\operatorname{supp}(Y) \cap P \subseteq Y^+$, $\operatorname{supp}(Y) \cap N \subseteq Y^-$, but not both.*

Definition 14. *Let $\mathcal{O}$ denote the set of covectors of an oriented matroid on a finite set E. We say that $\mathcal{O}$ is* totally cyclic*, if the all $+$-vector is in $\mathcal{O}^*$, i.e. it is a vector. $S \subseteq E$ is a* dijoin*, if $Y \in \mathcal{O} \setminus \{0\}$ and $Y \succeq 0$ implies* $\operatorname{supp}(Y) \cap S \neq \emptyset$*, i.e. S meets every positive cocircuit.*

Proposition 15. *$S \subseteq E$ is a dijoin if and only if $\mathcal{O}/S$ is totally cyclic.*

Proof. Set $P = E \setminus S, * = S$ and $N = O = \emptyset$. Since S is a dijoin, there is no non-zero vector $Y \in \mathcal{O}$ such that $\operatorname{supp}(Y) \subseteq P$ and $\operatorname{supp}(Y) \cap P = Y^+$. Thus, by Theorem 13 for every $e \in P$ there exists $X_e \in \mathcal{O}^*$ such that $e \in \operatorname{supp}(X_e)$ and $\operatorname{supp}(X_e) \cap P = X_e^+$. The composition of these vectors proves that $\mathcal{O}/S$ is strongly connected. Reading the proof backwards yields the other implication. □

It is clear that with this definition we can define the NL-flow polynomial as before. The crucial Lemma 8 for the equivalence of the first two statements in Theorem 2 dealt with totally unimodular matrices anyway. The only implication we are left to verify for an equivalence theorem for NL-flows in regular oriented matroids is (i) implies (iii). The following lemma suffices for that purpose. It could be deduced from Proposition 5 in [2]. We give a short proof for completeness.

Lemma 16. *Let $M \in \{0, \pm 1\}^{m \times n}$ be a totally unimodular matrix and let x denote a $\mathbb{Z}_k$-flow in the corresponding regular matroid $\mathcal{O}$, i.e. $Mx \equiv 0 \mod k$. Then there exists a k-flow $y \in \{0, \pm 1, \dots, \pm(k-1)\}^n$ in $\mathcal{O}$ such that $y \equiv x \mod k$.*

Proof. Choose $y \in \{0, \pm 1, \dots, \pm(k-1)\}^n$ satisfying $y \equiv x \mod k$ that minimizes $\|My\|_1 = \sum_{i=1}^m |(My)_i|$. We claim that y must be as required. Assume not and set $Y^+ := \{i \mid y > 0\}$, $Y^0 := \{i \mid y = 0\}$, $Y^- := \{i \mid y < 0\}$, $P := \{i \mid (My)_i > 0\}$ and $N := \{i \mid (My)_i > 0\}$. By assumption $P \cup N \neq \emptyset$. We consider only the case $P \neq \emptyset$, the other case is similar. Hence let $i_0 \in P$. There cannot exist $u \in \mathbb{R}^m$ such that $u_P \geq 0$, $u_{i_0} > 0$, $u_N \leq 0$, $u^\top A_{Y^+} \leq 0$ and $u^\top A_{Y^-} \geq 0$ for the first three inequalities imply $u^\top Ay > 0$ and the last two $u^\top Ay \leq 0$. Hence by Theorem 13, applied to the pair of oriented matroids defined by the kernel and the row space of the totally unimodular matrix $M := (A, I_P, I_N)$, there exists $(\tilde{y}^\top, z_P^\top, z_N^\top)^\top$ such that $\tilde{y}_{Y^+} \leq 0$, $\tilde{y}_{Y^-} \geq 0$, $\tilde{y}_{Y^0} = 0$, $z_P \geq 0$, $z_{i_0} > 0$, $z_N \leq 0$ such that $M\tilde{y} = -I_P z_P - I_N z_N$. Since M is totally unimodular we may assume that $\tilde{y}, z_P$ and z_N have entries in $\{0, 1, -1\}$ only. Thus, $y + k\tilde{y} \in \{0, \pm 1, \ldots, \pm(k-1)\}^n$ and $y + k\tilde{y} \equiv x \mod k$. But since My is divisible by k we have

$$\|M(y + k\tilde{y})\|_1 = \|My - kI_P z_P - kI_N z_N\|_1 < \|My\|_1,$$

contradicting the choice of y. □

Corollary 17. *Let $\mathcal{O}$ be an oriented regular matroid given by a totally unimodular matrix M. Let $k \geq 2$ and G be an Abelian group of order k. Then the following conditions are equivalent:*

(i) There exists a NL-$\mathbb{Z}_k$-flow in $\mathcal{O}$.

(ii) There exists a NL-G-flow in $\mathcal{O}$.

(iii) There exists a NL-k-flow in $\mathcal{O}$.

5 Open Problems

Our definition of the NL-flow polynomial uses in each term of the inclusion-exclusion formula a flow polynomial which itself uses that formula in its expansion. It might be possible to exploit this and find a simpler closed formula for our polynomial.

Considering the cographic oriented matroid our flow polynomial becomes the NL-coflow polynomial of D which equals the chromatic polynomial [1] for the dichromatic number divided by x. The natural question arises whether, as in the classical case, there exists a meaningful two variable polynomial combining both? Moreover, does such a polynomial or the two single variable polynomials have any meaning in the case of general oriented matroids?

Finally, it might be interesting to study the structure of the NL-flow polynomial for some special classes of orientations of graphs.

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