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Computing the NL-flow polynomial

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Computing the NL-flow polynomial

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Abstract

In 1982 Víctor Neumann-Lara [6] introduced the dichromatic number of a digraph D as the smallest integer k such that the vertices V of D can be colored with k colors and each color class induces a directed acyclic graph. In [4] a flow theory for the dichromatic number transferring Tutte's theory of nowhere-zero-flows from classic graph colorings has been developed and in [2] and [5] this analogy has been pursued by introducing algebraic NL-flows and a polynomial counting these flows.
In [5] we asked for a simpler closed formula for that polynomial. We answer this question to the positive and present a different approach for computing this NL-flow polynomial. Furthermore we discuss computational aspects of its computation for orientations of complete graphs and obtain a closed formula in the acyclic case.

1 Introduction, definitions and previous results

Large parts of graph theory have been driven by the Four Color Problem. In particular it inspired William T. Tutte to develop his theory of Nowhere-Zero-Flows [7].
In 1982 Víctor Neumann-Lara [6] introduced the dichromatic number of a digraph D as the smallest integer k such that the vertices V of D can be colored with k colors and each color class induces a directed acyclic graph. Moreover, in 1985 he conjecture, that every orientation of a simple planar graph can be acyclically colored with two colors. This intrigueing problem led us to trying to for an analogy follow Tutte's road map and vdevelop a corresponding flow theory, which we named Neumann-Lara-flows.

Definition 1. *Let $D = (V, A)$ be a digraph. A* NL-k-flow *is a map*

$$f : A \to \{0, \pm 1, ..., \pm (k-1)\},$$

satisfying Kirchhoff's law of flow conservation

$$\forall v \in V : \sum_{a \in \partial^+(v)} f(a) = \sum_{a \in \partial^-(v)} f(a),$$

such that $D[A/supp(f)]$ is totally cyclic, i.e. every component is strongly connected. If G is an Abelian group, then an NL-G-flow *is a map*

$$f : A \to G \setminus \{0_G\},$$

satisfying Kirchhoff's law of flow conservation.

As it is proven in [5], [2] a flow is a NL-flow if and only if its support is a dijoin, i.e. a set of arcs $S \subseteq A$, intersecting every directed cut in the given digraph $D = (V, A)$. This observation leads to the following definition.

Definition 2. *Let $D = (V, A)$ be a digraph and $\{S_1, .., S_r\}$ denote its set of inclusionwise minimal dijoins. For $I \subseteq \{1, ..., r\}$ let $S_I := \cup_{i \in I} S_i$. Denote by ϕ_{S_I} the flow polynomial of $D/(A \setminus S_I)$. Then the NL-flow polynomial of D is defined as*

$$\phi^D_{NL}(x) := \sum_{\emptyset \neq I \subset \{1,..,r\}} (-1)^{|I|-1} \phi_{S_I}(x) x^{|A \setminus S_I| - rk(A \setminus S_I)},$$

where $rk(G) := n - c$ is the rank of a graph G with n vertices and c connected components.

In [5], [2] it is shown that the number of NL-G-flows of a digraph D and a group G of order k is given by $\phi^D_{NL}(k)$. Clearly, this definition seems quite cumbrous and its computation takes some time. Moreover, in [5] we asked for a simpler closed formula for that polynomial. In order to develop such a formula we use a kind of generalization of the well-known inclusion-exclusion formula, the Möbius inversion (see for instance [1]).

Definition 3. *Let $(P, \leq)$ be a finite poset, then the* Möbius function *is defined as follows*

$$\mu : P \times P \to \mathbb{Z},\ \mu(x, y) := \begin{cases} 0 & , \text{ if } x \not\leq y \\ 1 & , \text{ if } x = y \\ -\sum_{x \leq z < y} \mu(x, z) & , \text{ otherwise .} \end{cases}$$

Proposition 1. *Let $(P, \leq)$ be a finite poset, $f, g : P \to \mathbb{K}$ functions and μ the Möbius function. Then the following equivalence holds*

$$f(x) = \sum_{y \leq x} g(y), \text{ for all } x \in P \iff g(x) = \sum_{y \leq x} \mu(y, x) f(y), \text{ for all } x \in P.$$

2 Our results

In order to derive the new formula for the NL-flow-polynomial of a given digraph $D = (V, A)$ we use Proposition 1 with $f_k, g_k : 2^A \to \mathbb{Z}$, such that $f_k(B)$ indicates all G-flows and $g_k(B)$ all NL-G-flows in the subgraph of D induced by $B \in 2^A$ for some fixed Abelian group G of order k.
The basic observation that a flow is an NL-flow iff its support is a dijoin (see [5]) encourages to consider the following poset $(\mathcal{C}, \supseteq)$, where for $B \subseteq A$

$$\mathcal{C}_B := \{B \setminus C \mid \exists\, C_1, ..., C_r \text{ directed cuts of } D[B], \text{ such that } C = \bigcup_{i=1}^{r} C_i\}.$$

Using this we find

Theorem 1.

$$\begin{aligned} \phi^D_{NL}(k) = g_k(A) &= \sum_{B \in \mathcal{C_A}} \mu(B, A) f_k(B) \\ &= \sum_{B \in \mathcal{C_A}} \mu(B, A) k^{|B| - rk(B)}. \end{aligned} \quad (1)$$

Proof. By Proposition 1 for the first equality it suffices to show that $f_k(B) = \sum_{\tilde{B}\in\mathcal{C}_B} g_k(\tilde{B})$ holds for all subsets B of A. Given a flow on B we set

$$\tilde{B} = B \setminus \bigcup\{C_i | C_i \text{ is a directed cut in } D[B] \text{ and } f_k(C_i) = 0\}.$$

Then clearly $\tilde{B} \in \mathcal{C}_B$ and $f_{k|\tilde{B}}$ is a NL-G-flow on $\tilde{B}$. On the other hand $f_{k|\hat{B}}$ is clearly not an NL-G flow for any other set $\tilde{B} \neq \hat{B} \in \mathcal{C}_B$. Hence the first equality follows. The second is clear since $f_k(B) = k^{|B|-rk(B)}$. □

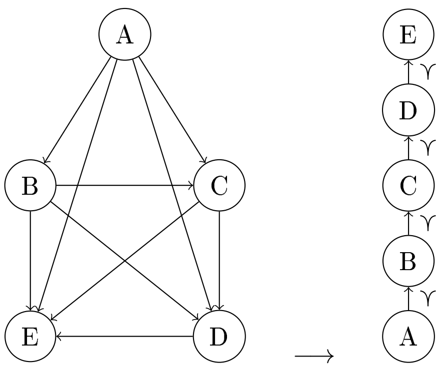
3 Orientations of complete digraphs

3.1 Complete acyclic digraphs

As an application we examine complete acyclic digraphs $D = (V, A)$. Recall that all acyclic digraphs with $n \geq 1$ vertices are isomorphic, thus the NL-flow polynomial does not depend on the orientation of the given digraph.
Moreover acyclic digraphs allow a topological ordering (see [3]), which is an ordering of the vertices $v_1, ..., v_n$ of D such that for every arc $(v_i, v_j) \in A$ we have $i < j$.
In the complete case this ordering is even unique since complete acyclic digraphs contain a hamiltonian path:

Proposition 2. *Every complete acyclic digraph allows a unique topological ordering.*

Proof. Define a poset $(V, \prec)$ by letting $x \prec y$ to be true, for any two vertices $x, y \in V$, whenever there exists a directed path from x to y. Obviously, since D is complete and acyclic, this poset is even totally ordered. With these definitions, a topological ordering of the given digraph correlates to this total order (see [3]), hence, it is unique. □



As one can see in the above picture, every arc in the right graph corresponds to exactly one directed cut in the left graph. Particularly, $C \subseteq A$ is a dicut if and only if the following properties

(1) if $(x, y) \in C,\ x \prec y$, then $(x, z) \in C$, for all $z \succ y$,

(2) if $(y,z) \in C,\ y \prec z$, then $(x,z) \in C$, for all $x \prec y$ and

(3.1) if $(x,z) \in C,\ x \prec z$, then $(x,y) \in C$, for all $x \prec y \prec z$ or

(3.2) if $(x,z) \in C,\ x \prec z$, then $(y,z) \in C$, for all $x \prec y \prec z$

are satisfied.
Now, recall that a complete acyclic digraph with $n \geq 1$ vertices has exactly $n-1$ dicuts, in the following denoted by $C_1, ..., C_{n-1}$. As a result the above defined poset $(\mathcal{C}, \supseteq)$ admits a simple structure.

Lemma 1. *Let $D = (V, A)$ be a complete acyclic digraph with $|V| = n \geq 2$ and $(\mathcal{C}, \supseteq)$ as above. Then $\mathcal{C}$ is isomorphic to $2^{[n-1]}$.*

Proof. Denote for some set J of indices $C_J := \cup_{j \in J} C_j$. Thus the elements of $\mathcal{C}$ are $A \setminus C_J$, for $J \subseteq [n-1]$ and the following map

$$\varphi : \mathcal{C} \to 2^{[n-1]},\ \varphi(A \setminus C_J) := J$$

is well-defined since there are exactly $n-1$ dicuts. Moreover each set of indices $J \in 2^{[n-1]}$ induces exactly one element in $\mathcal{C}$, hence φ is bijective.
Now, let $A \setminus C_J \supseteq A \setminus C_I$ for some $I, J \subseteq [n-1]$, thus $C_J \subseteq C_I$ and let $j \in J$. Then, for all $(x,y) \in C_j$ there is some $i \in I$ such that $(x,y) \in C_i$. Since C_j and C_i are dicuts, they satisfy the above properties $(1),(2),(3.1)$ or $(1),(2),(3.2)$. So, assume $j \neq i$ and, without loss of generality, let $(x,z) \in C_j$ with $x \prec z \prec y$. Then $i \neq i' \in I$ exists with $(x,z) \in C_{i'}$, otherwise $j = i$ would hold.
All in all there are at least $n-1$ arcs in C_j, so $|I| \leq n-1$, hence $j \in I$ anyway and φ is an order isomorphism. □

As a result we can write (1) as

$$\phi_{NL}^D(k) = \sum_{J \in 2^{[n-1]}} (-1)^{|J|}\ k^{|A \setminus \cup_{i \in J} C_i| - rk(A \setminus \cup_{i \in J} C_i)}, \tag{2}$$

since $\mu(J, 2^{[n-1]}) = (-1)^{|2^{[n-1]} \setminus J|} = (-1)^{|J|}$, for all $J \in 2^{[n-1]}$. This immediately leads to the following theorem.

Theorem 2. *Let $D = (V, A)$ be a complete acyclic digraph with $|V| = n$. For $1 \leq p \leq n$ denote by $(k_1, ..., k_p)$ the composition of n into p parts, i.e. $\sum_{i=1}^{p} k_i = n$, with $k_i \geq 1$, $i = 1, ..., p$. Then the NL-flow polynomial is given by*

$$\phi_{NL}^D(x) = \sum_{p=1}^{n} (-1)^{p-1} \sum_{(k_1,...,k_p)} \prod_{i=1}^{p} x^{\binom{k_i - 1}{2}}.$$

Proof. Let $n \geq 2$, otherwise we have $\Phi_{NL}^D(x) = 1$, the empty flow. For $J \in 2^{[n-1]}$ let $D[C_J]$ denote the subgraph of D induced by $A \setminus \cup_{i \in J} C_i$ and $p = |J| + 1$ the number of connected components in $D[C_J]$. We only have to count the number of arcs in $D[C_J]$, since the rank is given by $n - p$.
Deleting $|J|$ dicuts of the given complete digraph yields a subgraph with p strongly connected components, each containing $k_i \geq 1$, $i = 1, .., p$, vertices and thus $\binom{k_i}{2}$ arcs, satisfying $\sum_{i=1}^{p} k_i = n$.

Since the digraph is complete and acyclic, every combination is presumed, hence, with (2), the number of NL-k-flows is given by

$$\sum_{p=1}^{n}(-1)^{p-1}\sum_{\substack{(k_1,\ldots,k_p)\\ \sum_{i=1}^{p}k_i=n}} k^{\sum_{i=1}^{p}\binom{k_i}{2}-(n-p)}.$$

The claim follows, using $\binom{m}{2}-(m-1)=\binom{m-1}{2}$, for all $m\in\mathbb{N}$. □

Now we can compute several NL-flow polynomials of complete acyclic digraphs with n vertices in comparably short time:

$n=1:$
$$1$$
$n=2:$
$$0$$
$n=3:$
$$x-1$$
$n=4:$
$$x^3-2\,x+1$$
$n=5:$
$$x^6-2\,x^3+x$$
$n=6:$
$$x^{10}-2\,x^6+x^3-x^2+2\,x-1$$
$n=7:$
$$x^{15}-2\,x^{10}+x^6-2\,x^4+2\,x^3+3\,x^2-4\,x+1$$
$n=8:$
$$x^{21}-2\,x^{15}+x^{10}-2\,x^7+x^6+6\,x^4-4\,x^3-3\,x^2+2\,x$$

Obviously there are a lot of regularities and we can explicitely give the exponent of the two leading terms and their coefficients.

Proposition 3. *Let $D=(V,A)$ be a complete acyclic digraph with $n\geq 1$ vertices.*

(i) The leading term of $\Phi^D_{NL}(x)$ equals $x^{\binom{n-1}{2}}$.

(ii) Assume $n\geq 4$. Then the second term with highest exponent equals $-2x^{\binom{n-2}{2}}$.

Proof. We only need to consider the case where $p=1$, since the exponent of $\Phi^D_{NL}(x)$ is maximum for $k_1=n$. The next lower exponent occurs when $p=2$, having $k_1=n-1$, $k_2=1$ and vice versa. □

Let us now look at the constant term of the polynomial.

Lemma 2. *Let $D=(V,A)$ be a complete acyclic digraph with $n\geq 3$ vertices and $c(n)$ denote the constant term of $\Phi^D_{NL}(x)$. Then the following recursion holds*

$$c(n)=-(c(n-1)+c(n-2)).$$

Proof. Since we are interested in the constant term of $\Phi_{NL}^{D}(x)$ we only need to consider the cases where $k_i \in \{1,2\}$ for all $1 \le i \le n$ and get the following distinction.

$$\begin{aligned} c(n) &= \sum_{\substack{k_2+\ldots+k_p=n-1\\ k_1=1\\ k_i\in\{1,2\}}} (-1)^{p-1} + \sum_{\substack{k_2+\ldots+k_p=n-2\\ k_1=2\\ k_i\in\{1,2\}}} (-1)^{p-1} \\ &\overset{r:=p-1}{=} - \sum_{\substack{k_1+\ldots+k_r=n-1\\ k_i\in\{1,2\}}} (-1)^{r-1} - \sum_{\substack{k_1+\ldots+k_r=n-2\\ k_i\in\{1,2\}}} (-1)^{r-1} \\ &= -(c(n-1)+c(n-2)). \end{aligned}$$

□

This observation yields the following proposition.

Proposition 4. *Let $D=(V,A)$ be a complete acyclic digraph with $n \ge 1$ vertices, then the constant term of $\Phi_{NL}^{D}(x)$ is given by*

$$c(n) = \Phi_{NL}^{D}(0) = \begin{cases} -1 & \text{, if } n \text{ mod } 3 = 0, \\ 1 & \text{, if } n \text{ mod } 3 = 1, \\ 0 & \text{, if } n \text{ mod } 3 = 2. \end{cases}$$

Proof. Lemma 2 immediately yields

$$c(n+3) = -\big(c(n+2)+c(n+1)\big) = -\Big(-\big(c(n+1)+c(n)\big)+c(n+1)\Big) = c(n)$$

and the base cases from above prove the claim. □

Observing the linear term we get:

Proposition 5. *Let $D=(V,A)$ be a complete acyclic digraph with $n \ge 4$ vertices, then the linear term of $\Phi_{NL}^{D}(x)$ is given by*

$$l(n) = \frac{1}{3} \begin{cases} n & \text{, if } n \text{ mod } 3 = 0, \\ -2(n-1) & \text{, if } n \text{ mod } 3 = 1, \\ n-2 & \text{, if } n \text{ mod } 3 = 2. \end{cases}$$

Proof. In this case exactly one part of the composition, call it k_j, equals 3, while the other parts have to be either 1 or 2. Let $c(n)$ be the constant term of $\Phi_{NL}^{D}(x)$, then we have

$$\begin{aligned} l(n) = &\sum_{\substack{k_1+\ldots+k_{p-1}=n-1\\ j\neq p\\ k_i\in\{1,2\}, i\neq j}} (-1)^{p-1} + \sum_{\substack{k_1+\ldots+k_{p-1}=n-2\\ j\neq p\\ k_i\in\{1,2\}, i\neq j}} (-1)^{p-1} + \sum_{\substack{k_1+\ldots+k_{p-1}=n-3\\ j=p\\ k_i\in\{1,2\}}} (-1)^{p-1} \\ &= -l(n-1) - l(n-2) - c(n-3) \end{aligned}$$

Now we can proceed per induction, using Proposition 4.

$$
\begin{aligned}
l(n+1) &= -l(n) - l(n-1) - c(n-2)\\
&\overset{IV}{=} -\frac{1}{3}\begin{cases} n \\ -2(n-1) \\ n-2 \end{cases} - \frac{1}{3}\begin{cases} (n-1)-2 \\ n-1 \\ -2((n-1)-1) \end{cases} - \begin{cases} 1 & \text{, if n mod 3 = 0} \\ 0 & \text{, if n mod 3 = 1} \\ -1 & \text{, if n mod 3 = 2} \end{cases}\\
&= \frac{1}{3}\begin{cases} -2((n+1)-1) & \text{, if n+1 mod 3 = 1} \\ (n+1)-1 & \text{, if n+1 mod 3 = 2} \\ n+1 & \text{, if n+1 mod 3 = 0} \end{cases}
\end{aligned}
$$

□

3.2 Complete digraphs

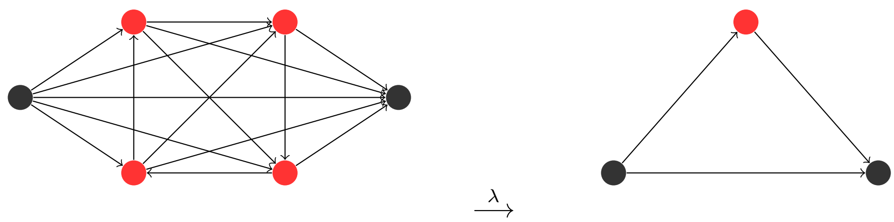
Considering an arbitrary complete digraph $D = (V, A)$ the NL-flow polynomial depends on its orientation. Let $d \in \mathbb{N}$ denote the number of maximal strongly connected components and denote their vertex sets with $S_1, ..., S_d$. Since we cannot cut through cycles there are exactly $d - 1$ dicuts and the poset $\mathcal{C}$ is isomorphic to $2^{[d-1]}$. Similarly as in (2) we conclude

$$\phi_{NL}^{D}(k) = \sum_{J \in 2^{[d-1]}} (-1)^{|J|}\, k^{|A \setminus \cup_{i \in J} C_i| - rk(A \setminus \cup_{i \in J} C_i)}, \tag{3}$$

where $C_i, i = 1, ..., d-1$ denote the dicuts in D.
Recall that the maximal strongly connected components form a partition of the given digraph. Consequently we consider the following map

$$
\begin{aligned}
\lambda : V &\to \{1, ..., d\}\\
v &\mapsto i, \text{ with } v \in S_i,
\end{aligned}
$$

which induces the complete acyclic digraph on d vertices.



As a result of Proposition 2 the vertices of $D[\lambda(V)]$ can be ordered topologically, thus the strongly connected components of D allow a similar ordering.

Theorem 3. *Let $D = (V, A)$ be a complete digraph with $d \geq 1$ strongly connected components, each containing $k_1, ..., k_d$ vertices, such that the subgraph of D induced by $\lambda(V)$ is topologically ordered. For $1 \leq p \leq d$ consider the composition $(d_1, ..., d_p)$ of d into p parts, i.e. $\sum_{i=1}^{p} d_i = d$, with $d_i \geq 1$, for all*

$1 \leq i \leq p$. *Then the NL-flow polynomial is given by*

$$\phi_{NL}^{D}(x) = \sum_{p=1}^{d} (-1)^{p-1} \sum_{(d_1,...,d_p)} \prod_{j=1}^{p} x^{\binom{n_j - 1}{2}}, \text{ with}$$

$$n_j := \sum_{s=\delta(j-1)+1}^{\delta(j)} k_s \text{ and } \delta(j) := \sum_{r=1}^{j} d_r.$$

Proof. Denote the strongly connected components of D with $K_1, ..., K_d$, such that the topologically ordering of $\lambda(V)$ is preserved. Analoguesly to the proof of Theorem 2 we only have to count the number of vertices in each partition of $D[\lambda(V)]$ induced by some composition $(d_1, ..., d_p)$, where each vertex $1 \leq v \leq d$ corresponds to a strongly connected component K_v, each containing k_v vertices in D.

So, let $(d_1, ..., d_p)$ be an arbitrary composition of d with p parts, hence there are d_j, $1 \leq j \leq p$, vertices in each part of $D[\lambda(V)]$. Let D_j denote the set of vertices in the corresponding strongly connected components in D. Then

$$D_1 = \bigcup_{i=1}^{d_1} K_i, \ D_2 = \bigcup_{i=d_1+1}^{d_1+d_2} K_i \ , ..., \ D_p = \bigcup_{i=d_1+...+d_{p-1}+1}^{d_1+...+d_p} K_i.$$

Thus there are

$$|D_j| = \sum_{i=\sum_{r=1}^{j-1} d_r + 1}^{\sum_{r=1}^{j} d_r} k_i$$

vertices in the $j-$th corresponding part of D. □

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