Immanuel Albrecht:

Duality Respecting Representations and Compatible Complexity Measures for Gammoids

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Duality Respecting Representations and
Compatible Complexity Measures for
Gammoids

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Abstract

We show that every gammoid has special digraph representations, such
that a representation of the dual of the gammoid may be easily obtained
by reversing all arcs. In an informal sense, the duality notion of a poset
applied to the digraph of a special representation of a gammoid commutes
with the operation of forming the dual of that gammoid. We use these special
representations in order to define a complexity measure for gammoids, such
that the classes of gammoids with bounded complexity are closed under
duality, minors, and direct sums.

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A well-known result due to J.H. Mason is that the class of gammoids is closed
under duality, minors, and direct sums [5]. Furthermore, it has been shown by
D. Mayhew that every gammoid is also a minor of an excluded minor for the class
of gammoids [6], which indicates that handling the class of all gammoids may get
very involved. In this work, we introduce a notion of complexity for gammoids
which may be used to define subclasses of gammoids with bounded complexity,
that still have the desirable property of being closed under duality, minors, and direct sums; yet their representations have a more limited number of arcs than the general class of gammoids.

1 Preliminaries

In this work, we consider matroids to be pairs $M = (E, \mathcal{I})$ where $E$ is a finite set and $\mathcal{I}$ is a system of independent subsets of $E$ subject to the usual axioms ([7], Sec. 1.1). If $M = (E, \mathcal{I})$ is a matroid and $X \subseteq E$, then the restriction of $M$ to $X$ shall be denoted by $M|X$ ([7], Sec. 1.3), and the contraction of $M$ to $X$ shall be denoted by $M.X$ ([7], Sec. 3.1). Furthermore, the notion of a digraph shall be synonymous with what is described more precisely as finite simple directed graph that may have some loops, i.e. a digraph is a pair $D = (V, A)$ where $V$ is a finite set and $A \subseteq V \times V$. Every digraph $D = (V, A)$ has a unique opposite digraph $D^{opp} = (V, A^{opp})$ where $(u, v) \in A^{opp}$ if and only if $(v, u) \in A$. All standard notions related to digraphs in this work are in accordance with the definitions found in [2]. A path in $D = (V, A)$ is a non-empty and non-repeating sequence $p = p_1p_2 \ldots p_n$ of vertices $p_i \in V$ such that for each $1 \leq i < n, (p_i, p_{i+1}) \in A$. By convention, we shall denote $p_n$ by $p_{-1}$. Furthermore, the set of vertices traversed by a path $p$ shall be denoted by $|p| = \{p_1, p_2, \ldots, p_n\}$ and the set of all paths in $D$ shall be denoted by $P(D)$.

Definition 1.1. Let $D = (V, A)$ be a digraph, and $X, Y \subseteq V$. A routing from $X$ to $Y$ in $D$ is a family of paths $R \subseteq P(D)$ such that

(i) for each $x \in X$ there is some $p \in R$ with $p_1 = x$,

(ii) for all $p \in R$ the end vertex $p_{-1} \in Y$, and

(iii) for all $p, q \in R$, either $p = q$ or $|p| \cap |q| = \emptyset$.

We shall write $R : X \Rightarrow Y$ in $D$ as a shorthand for “$R$ is a routing from $X$ to $Y$ in $D$”, and if no confusion is possible, we just write $X \Rightarrow Y$ instead of $R$ and $R : X \Rightarrow Y$. A routing $R$ is called linking from $X$ to $Y$, if it is a routing onto $Y$, i.e. whenever $Y = \{p_{-1} \mid p \in R\}$.

Definition 1.2. Let $D = (V, A)$ be a digraph, $E \subseteq V$, and $T \subseteq V$. The gammoid represented by $(D, T, E)$ is defined to be the matroid $\Gamma(D, T, E) = (E, \mathcal{I})$ where

$$\mathcal{I} = \{X \subseteq E \mid \text{there is a routing } X \Rightarrow T \text{ in } D\}.$$
The elements of $T$ are usually called sinks in this context, although they are not required to be actual sinks of the digraph $D$. To avoid confusion, we shall call the elements of $T$ targets in this work. A matroid $M' = (E', I')$ is called gammoid, if there is a digraph $D' = (V', A')$ and a set $T' \subseteq V'$ such that $M' = \Gamma(D', T', E')$.

**Theorem 1.3** ([5], Corollary 4.1.2). Let $M = (E, I)$ be a gammoid and $B \subseteq E$ a base of $M$. Then there is a digraph $D = (V, A)$ such that $M = \Gamma(D, B, E)$.

For a proof, see J.H. Mason’s seminal paper *On a Class of Matroids Arising From Paths in Graphs* [5].

### 2 Special Representations

**Definition 2.1.** Let $(D, T, E)$ be a representation of a gammoid. We say that $(D, T, E)$ is a duality respecting representation, if

$$\Gamma(D^{opp}, E \setminus T, E) = (\Gamma(D, T, E))^*$$

where $(\Gamma(D, T, E))^*$ denotes the dual matroid of $\Gamma(D, T, E)$.

**Lemma 2.2.** Let $(D, T, E)$ be a representation of a gammoid with $T \subseteq E$, such that every $e \in E \setminus T$ is a source of $D$, and every $t \in T$ is a sink of $D$. Then $(D, T, E)$ is a duality respecting representation.

**Proof.** We have to show that the bases of $N = \Gamma(D^{opp}, E \setminus T, E)$ are precisely the complements of the bases of $M = \Gamma(D, T, E)$ ([7], Thm. 2.1.1). Let $B \subseteq E$ be a base of $M$, then there is a linking $L : B \Rightarrow T$ in $D$, and since $T$ consists of sinks, we have that the single vertex paths $\{x \in P(D) \mid x \in T \cap B\} \subseteq L$. Further, let $L^{opp} = \{p_1p_2\ldots p_n \mid p_1, p_2, \ldots, p_n \in L\}$. Then $L^{opp}$ is a linking from $T$ to $B$ in $D^{opp}$ which routes $T \setminus B$ to $B \setminus T$. The special property of $D$, that $E \setminus T$ consists of sources and that $T$ consists of sinks, implies, that for all $p \in L$, we have $|p| \cap E = \{p_1, p_{-1}\}$. Observe that thus

$$R = \{p \in L^{opp} \mid p_1 \in T \setminus B\} \cup \{x \in P(D^{opp}) \mid x \in E \setminus (T \cup B)\}$$

is a linking from $E \setminus B = (T \cup (E \setminus T)) \setminus B$ onto $E \setminus T$ in $D^{opp}$, thus $E \setminus B$ is a base of $N$. An analog argument yields that for every base $B'$ of $N$, $E \setminus B'$ is a base of $M$. Therefore $\Gamma(D^{opp}, E \setminus T, E) = (\Gamma(D, T, E))^*$.\[\square\]
Definition 2.3. Let $M$ be a gammoid and $(D, T, E)$ with $D = (V, A)$ be a representation of $M$. Then $(D, T, E)$ is a standard representation of $M$, if $(D, T, E)$ is a duality respecting representation, $T \subseteq E$, every $t \in T$ is a sink in $D$, and every $e \in E \setminus T$ is a source in $D$.

The name standard representation is justified, since the real matrix $A \in \mathbb{R}^{T \times E}$ obtained from $D$ through the Lindström Lemma [4, 1] is a standard matrix representation of $\Gamma(D, T, E)$ up to possibly rearranging the columns ([8], p.137).

Theorem 2.4. Let $M = (E, I)$ be a gammoid, and $B \subseteq E$ a base of $M$. There is a digraph $D = (V, A)$ such that $(D, B, E)$ is a standard representation of $M$.

Proof. Let $D_0 = (V_0, A_0)$ be a digraph such that $\Gamma(D_0, B, E) = M$ (Theorem 1.3). Furthermore, let $V$ be a set with $E \subseteq V$ such that there is an injective map $': V_0 \rightarrow V \setminus E, v \mapsto v'$. Without loss of generality we may assume that $V = E \cup V'_0$. We define the digraph $D = (V, A)$ such that

$$A = \{(u', v') \mid (u, v) \in A_0\} \cup \{(b', b) \mid b \in B\} \cup \{(e, e') \mid e \in E \setminus B\}.$$

For every $X \subseteq E$, we obtain that by construction, there is a routing $X \Rightarrow B$ in $D_0$ if and only if there is a routing $X \Rightarrow B$ in $D$. Therefore $(D, B, E)$ is a representation of $M$ with the additional property that every $e \in E \setminus B$ is a source in $D$, and every $b \in B$ is a sink in $D$. Thus $(D, B, E)$ is a duality respecting representation of $M$ (Lemma 2.2). \qed

3 Gammoids with Low Arc-Complexity

Definition 3.1. Let $M$ be a gammoid. The arc-complexity of $M$ is defined to be

$$C_A(M) = \min \{|A| \mid ((V, A), T, E) \text{ is a standard representation of } M\}.$$

Lemma 3.2. Let $M = (E, I)$ be a gammoid, $X \subseteq E$. Then the inequalities $C_A(M|X) \leq C_A(M)$, $C_A(M,X) \leq C_A(M)$, and $C_A(M) = C_A(M^*)$ hold.

Proof. Let $M$ be a gammoid and let $(D, T, E)$ be a standard representation of $M$ with $D = (V, A)$ for which $|A|$ is minimal among all standard representations of $M$. Then $(D^\text{opp}, E \setminus T, E)$ is a standard representation of $M^*$ that uses the same number of arcs. Thus $C_A(M) = C_A(M^*)$ holds for all gammoids $M$. Let $X \subseteq E$. If $T \subseteq X$, then $(D, T, X)$ is a standard representation of $M|X$. \qed
and therefore $C_A(M|X) \leq C_A(M)$. Otherwise let $Y = T \setminus X$, and let $B_0 \subseteq X$ be a set of maximal cardinality such that there is a routing $R_0: B_0 \rightarrow Y$ in $D$. Let $D' = (V, A')$ be the digraph that arises from $D$ by a sequence of operations as described in Theorem 4.1.1 [5] and Corollary 4.1.2 [5] with respect to the routing $R_0$. Observe that every $b \in B_0$ is a sink in $D'$ and that $|A'| = |A|$. We argue that $(D', (T \cap X) \cup B_0, X)$ is a standard representation of $M|X$: Let $Y_0 = \{p_{-1} | p \in R_0\}$ be the set of targets that are entered by the routing $R_0$. It follows from Corollary 4.1.2 [5] that the triple $(D', (T \cap X) \cup B_0 \cup (Y \setminus Y_0), E)$ is a representation of $M$. The sequence of operations we carried out on $D$ preserves all those sources and sinks of $D$, which are not visited by a path $p \in R_0$. So we obtain that every $e \in E \setminus (T \cup B_0)$ is a source in $D'$, and that every $t \in T \cap X$ is a sink in $D'$. Thus the set $T' = (T \cap X) \cup B_0$ consists of sinks in $D'$, and the set $X \setminus T' \subseteq E \setminus (T \cup B_0)$ consists of sources in $D'$. Therefore $(D', (T \cap X) \cup B_0, X)$ is a standard representation, and we give an indirect argument that $(D', (T \cap X) \cup B_0, X)$ represents $M|X$. Clearly, $(D', (T \cap X) \cup B_0 \cup (Y \setminus Y_0), X)$ is a representation of $M|X$. Since we assume that $(D', (T \cap X) \cup B_0, X)$ does not represent $M|X$, there must be a set $X_0 \subseteq X$ such that there is a routing $Q_0: X_0 \rightarrow (T \cap X) \cup B_0 \cup (Y \setminus Y_0)$ and such that there is no routing $X_0 \rightarrow (T \cap X) \cup B_0$, both in $D'$. Thus there is a path $q \in Q_0$ with $q_{-1} \in Y \setminus Y_0$ and $q_1 \in X$. Consequently we have a routing $Q'_1 = \{q_1\} \cup \{b \in P(D') | b \in B_0\}$ in $D'$. This implies that there is a routing $B_0 \cup \{q_1\} \rightarrow Y$ in $D$, a contradiction to the maximal cardinality of the choice of $B_0$ above. Thus our assumption must be wrong, and $(D', (T \cap X) \cup B_0, X)$ is a standard representation of $M|X$. Consequently $C_A(M|X) \leq C_A(M)$ holds again. Finally, we have $C_A(M|X) = C_A((M^*|X)^*) = C_A(M^*|X) \leq C_A(M^*) = C_A(M)$. 

**Definition 3.3.** Let $f : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ be a function. We say that $f$ is super-additive, if for all $n, m \in \mathbb{N} \setminus \{0\}$

$$f(n + m) \geq f(n) + f(m)$$

holds.

**Definition 3.4.** Let $f : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ be a super-additive function, and let $M = (E, \mathcal{T})$ be a gammoid. The $f$-width of $M$ shall be

$$W_f(M) = \max \left\{ \frac{C_A((M|Y)|X)}{f(|X|)} \mid X \subseteq Y \subseteq E \right\}.$$

**Theorem 3.5.** Let $f : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ be a super-additive function, and let $0 < q \in \mathbb{Q}$. Let $\mathcal{W}_{f,q}$ denote the class of gammoids $M$ with $W_f(M) \leq q$. The class $\mathcal{W}_{f,q}$ is closed under duality, minors, and direct sums.
Proof. Let $M = (E, T)$ be a gammoid and $X \subseteq Y \subseteq E$. It is obvious from Definition 3.4 that $W_f((M, Y) \mid X) \leq W_f(M)$, and consequently $W_{f,q}$ is closed under minors. Since $C_A(M) = C_A(M^*)$ and since every minor of $M^*$ is the dual of a minor of $M$ ([7], Prop. 3.1.26), we obtain that $W_f(M) = W_f(M^*)$. Thus $W_{f,q}$ is closed under duality.

Now, let $M = (E, T)$ and $N = (E', T')$ with $E \cap E' = \emptyset$ and $M, N \in W_{f,q}$. The cases where either $E = \emptyset$ or $E' = \emptyset$ are trivial, now let $E \neq \emptyset \neq E'$. Furthermore, let $X \subseteq Y \subseteq E \cup E'$. The direct sum commutes with the forming of minors in the sense that

$$(M \oplus N, Y) \mid X = ((M, Y \cap E) \mid X \cap E) \oplus ((N, Y \cap E') \mid X \cap E').$$

Let $(D, T, E)$ and $(D', T', E')$ be representations of $M$ and $N$ where $D = (V, A)$ and $D' = (V', A')$ such that $V \cap V' = \emptyset$. Then $((V \cup V', A \cup A'), T \cup T', E \cup E')$ is a representation of $M \oplus N$, and consequently $C_A(M \oplus N) \leq C_A(M) + C_A(N)$ holds for all gammoids $M$ and $N$, thus we have

$$C_A\left((M \oplus N, Y) \mid X\right) \leq C_A(M_{X,Y}) + C_A(N_{X,Y})$$

where $M_{X,Y} = (M, Y \cap E) \mid X \cap E$ and $N_{X,Y} = (N, Y \cap E') \mid X \cap E'$. The cases where $C_A(M_{X,Y}) = 0$ or $C_A(N_{X,Y}) = 0$ are trivial, so we may assume that $X \cap E \neq \emptyset \neq X \cap E'$. We use the super-additivity of $f$ at $(\ast)$ in order to derive

$$\frac{C_A\left((M \oplus N, Y) \mid X\right)}{f(|X|)} \leq \frac{C_A(M_{X,Y}) + C_A(N_{X,Y})}{f(|X|)} \leq \frac{q \cdot f(|X \cap E|) + q \cdot f(|X \cap E'|)}{f(|X|)} \overset{(\ast)}{\leq} q.$$

As a consequence we obtain $W_f(M \oplus N) \leq q$, and therefore $W_{f,q}$ is closed under direct sums. 

4 Further Remarks and Open Problems

Let $r, n \in \mathbb{N}$ with $n \geq r$, the uniform matroid of rank $r$ on $n$ elements is the matroid $U_{r,n} = \{\{1, 2, \ldots, n\}, \mathcal{I}_{r,n}\}$ where $\mathcal{I}_{r,n} = \{X \subseteq \{1, 2, \ldots, n\} \mid |X| \leq r\}$. Let $T = \{1, 2, \ldots, r\}$, $X = \{r + 1, r + 2, \ldots, n\}$, and $D = (X \cup T, X \times T)$. Then $U_{r,n} = \Gamma(D, T \cup X)$. Thus $C_A(U_{r,n}) \leq r \cdot (n - r)$. Unfortunately, we were not able to find a known result in graph or digraph theory that implies:
Conjecture 4.1.

\[ C_A(U_{r,n}) = r \cdot (n - r). \]

A slightly weaker version is the following:

Conjecture 4.2. For every \( q \in \mathbb{Q} \) there is a gammoid \( M = (E, \mathcal{I}) \) with

\[ C_A(M) \geq q \cdot |E|. \]

For the rest of this work, we set \( f : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}, x \mapsto \max \{1, x\} \), we denote \( W_f \) by \( W \), and we fix an arbitrary choice of \( q \in \mathbb{Q} \) with \( q > 0 \). Clearly, if Conjecture 4.2 holds, then \( (W_{f,q})_{q \in \mathbb{N} \setminus \{0\}} \) is strictly monotonous sequence of subclasses of the class of gammoids, such that every subclass is closed under duality, minors, and direct sums. For which super-additive \( f \) and \( q \in \mathbb{Q} \setminus \{0\} \) may \( W_{f,q} \) be characterized by finitely many excluded minors? For which such classes can we list a sufficient (possibly infinite) set of excluded minors that decide class membership of \( W_{f,q} \)?

A consequence of a result of S. Kratsch and M. Wahlström ([3], Thm. 3) is, that if a matroid \( M = (E, \mathcal{I}) \) is a gammoid, then there is a representation \((D, T, E)\) of \( M \) with \( D = (V, A) \) and \( |V| \leq \text{rk}_M(E)^2 \cdot |E| + \text{rk}_M(E) + |E| \). It is easy to see that if \( M \in W_{f,q} \), then there is a representation \((D, T, E)\) of \( M \) with \( D = (V, A) \) and \( |V| \leq \lfloor 2q \cdot f(|E|) \rfloor \), since every arc is only incident with at most two vertices. Therefore, deciding \( W_{f,q} \)-membership with an exhaustive digraph search appears to be easier than deciding gammoid-membership with an exhaustive digraph search.

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References


