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**On the Geometry of Holmsen's Combinatorial Version of the Colorful
Carathéodory**

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On the Geometry of Holmsen's Combinatorial Version of the Colorful Carathéodory

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Abstract

Carathéodory's Theorem of convex hulls plays an important role in convex geometry. In 1982, Bárány formulated and proved a more general version, called the Colorful Carathéodory. This colorful version was even more generalized by Holmsen in 2016. He formulated a combinatorial extension in [4] and found a topological proof. Taking a dual point of view we gain an equivalent formulation of Holmsen's result that has a more geometric meaning.

1 Carathéodory's Theorem

The following theorem, named after Constantin Carathéodory, plays an important role in convex geometry since it gives an upper bound to the length of the convex combination of a point lying in a convex set.

Theorem 1 (Carathéodory). *For $P \subseteq \mathbb{R}^d$ ($d \in \mathbb{N}$) and a point $x \in \text{conv } P$ there are $d + 1$ points $p_0, \dots, p_d \in P$ such that $x \in \text{conv}\{p_0, \dots, p_d\}$.*

Bárány proved the Colorful Carathéodory [2] in 1982, a more general version of Carathéodory's Theorem. In this version, the set P is divided into different subsets. If these sets are disjoint, we can interpret them as colors, such that each point gets a color dependent on the subset in which the point is included.

Theorem 2 (Colorful Carathéodory, Bárány (1982) [2]). *Consider $d + 1$ sets P_0, \dots, P_d in \mathbb{R}^d ($d \in \mathbb{N}$). If $x \in \mathbb{R}^d$ is in all convex hulls $\text{conv } P_i$ ($0 \leq i \leq d$), there are points $p_i \in P_i$ in every P_i ($0 \leq i \leq d$) such that x is in the convex hull of those $d + 1$ points, i.e. $x \in \text{conv}\{p_0, \dots, p_d\}$.*

This theorem tells us that the point x is in a colorful simplex, which means that each corner is in a different color, i.e. belongs to a different set P_i .

It is even possible to extend the Colorful Carathéodory. In [4] Holmsen considered a combinatorial and a topological extension of Theorem 2 which are equivalent to each other.

2 Introduction to Matroid and Oriented Matroid Theory

To understand the combinatorial version, we will shortly introduce the necessary terms in the theory of matroids and oriented matroids. For detailed information we refer to [5], [3] and [1].

Definition 1. A matroid \mathcal{M} on a ground set E with a collection \mathcal{C} of circuits is an ordered pair $\mathcal{M} = (E, \mathcal{C})$ such that the following conditions hold:

(M1) $\emptyset \notin \mathcal{C}$.

(M2) For $X, Y \in \mathcal{C}$ with $X \subseteq Y$ is $X = Y$.

(M3) If $X, Y \in \mathcal{C}$, $X \neq Y$ and $e \in X \cap Y$, there is a $Z \in \mathcal{C}$ such that $Z \subseteq (X \cup Y) \setminus \{e\}$.

The elements of \mathcal{C} are called circuits. A loop is an element $x \in E$ such that $\{x\}$ is a circuit. An independent set of a matroid $\mathcal{M} = (E, \mathcal{C})$ is a subset $X \subseteq E$ such that no subset of X is a circuit. A maximal independent set is called a basis.

Definition 2. The rank of a set $X \subseteq E$ is the cardinality of the inclusion-maximal independent subset of X . We denote the rank of X as $\rho(X)$. The rank of a matroid \mathcal{M} is $\rho(E)$.

Definition 3. A subset $D \subseteq E$ of a matroid \mathcal{M} is a double circuit if

$$\begin{aligned} \rho(D) &= |D| - 2 \quad \text{and} \\ \forall e \in D : \rho(D \setminus \{e\}) &= |D| - 2. \end{aligned}$$

Definition 4. The dual matroid \mathcal{M}^* of a matroid $\mathcal{M} = (E, \mathcal{C})$ is a matroid on the same ground set E such that a set is independent if and only if the complement contains a basis of \mathcal{M} . It is $\mathcal{M}^{**} = \mathcal{M}$. The circuits of the dual matroids are called cocircuits of \mathcal{M} .

To define the notion of oriented matroids, we need to introduce signed sets. A signed subset of a finite set E is an ordered pair $X = (X^+, X^-)$ with $X^+, X^- \subseteq E$ such that $X^+ \cap X^- = \emptyset$. We write $-X = (X^-, X^+)$ and $\underline{X} = X^+ \cup X^- \subseteq E$.

Definition 5. A pair $\mathcal{O} = (E, \mathcal{C})$ is an oriented matroid if

(O1) $(\emptyset, \emptyset) \notin \mathcal{C}$ and if $X \in \mathcal{C}$, then $-X \in \mathcal{C}$.

(O2) For two sets $X, Y \in \mathcal{C}$ with $\underline{X} \subseteq \underline{Y}$, it is either $X = Y$ or $X = -Y$.

(O3) For $X, Y \in \mathcal{C}$ with $X \neq -Y$ and $e \in X^+ \cap Y^-$ there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$.

The signed sets in \mathcal{C} are called (signed) circuits of \mathcal{O} and a positive circuit is a (signed) circuit (X^+, \emptyset) of \mathcal{O} . If $\underline{X} \subseteq S$ for a set S and a signed set X , we say X is contained in S .

It is easy to see that the set $\underline{\mathcal{C}} = \{\underline{X} : X \in \mathcal{C}\}$ for a circuit system \mathcal{C} of an oriented matroid is the collection of circuits of a matroid. This matroid is called the underlying matroid of \mathcal{O} . The rank of an oriented matroid is defined as the rank of its underlying matroid.

Definition 6. Two signed sets X, Y are orthogonal if $\underline{X} \cap \underline{Y} = \emptyset$ or the restrictions of X and Y to their intersection $\underline{X} \cap \underline{Y}$ is neither equal nor opposite.

For any oriented matroid $\mathcal{O} = (E, \mathcal{C})$, there is a unique maximal family \mathcal{C}^* such that X and Y are orthogonal for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}^*$. This set is a set of circuits of an oriented matroid on the ground set E , called the *dual oriented matroid* of \mathcal{O} . The circuits of the dual matroid are called *cocircuits*. The matroid underlying the dual oriented matroid is the dual of the underlying matroid.

3 Holmsen's Theorem

The following theorem is the theorem of Holmsen stated in [4]. In this section we will only show the connection of this version to the Colorful Carathéodory (Theorem 2).

Theorem 3 (Holmsen). *Let \mathcal{O} be an oriented matroid on the ground set E with rank r . Consider a matroid \mathcal{M} on the same ground set E and with rank function ϱ such that $\varrho(E) > r$. If every subset $S \subseteq E$ with $\varrho(E \setminus S) < r$ contains a positive circuit of \mathcal{O} , there exists a positive circuit of \mathcal{O} which is independent in \mathcal{M} .*

At first glance you may not notice the connection of this theorem to the Colorful Carathéodory Theorem. For this reason, we will recall from [4] that the Colorful Carathéodory is a special case.

Holmsen's Theorem \Rightarrow Colorful Carathéodory. Let P_0, P_1, \dots, P_d be subsets of \mathbb{R}^d and $x \in \mathbb{R}^d$ a point such that $x \in \text{conv } P_i$ for all $i \in \{0, \dots, d\}$. Consider the disjoint union $E = P_0 \dot{\cup} P_1 \dot{\cup} \dots \dot{\cup} P_d$ as a multiset of points in \mathbb{R}^d . We assume $x \notin E$ (if x is already in one of the sets P_i ($i \in \{0, \dots, d\}$), the statement is trivial). For every inclusion-minimal subset $S \subseteq E$ with a linear dependency $\sum_{p \in S} \alpha_p(p - x) = 0$ ($\alpha_p \in \mathbb{R}$), we consider the two subsets of S :

$$\begin{aligned} S^+ &= \{p \in S : \alpha_p > 0\}, \\ S^- &= \{p \in S : \alpha_p < 0\}. \end{aligned}$$

All signed subsets (S^+, S^-) constructed in such a way build an oriented matroid $\mathcal{O} = (E, \mathcal{C})$ where \mathcal{C} is the set of signed subsets (S^+, S^-) .

This so constructed oriented matroid has the rank r equal to dimension of the vector space spanned by $E \cup \{x\}$.

Observe that for any set $S \subseteq E$ and any $y \in \text{conv } S$ there is linear dependency with only positive coefficients. This shows that

$$x \in \text{conv } S \quad \Leftrightarrow \quad S \text{ contains a positive circuit of } \mathcal{O}.$$

By the preliminaries of the Colorful Carathéodory x is contained in all $\text{conv } P_i$. By this observation we know that all P_i contain a positive circuit.

Furthermore, we define the matroid \mathcal{M} by its independent sets. So let $I \subseteq E$ be independent if and only if $|I \cap P_i| \leq 1$ for all $i \in \{0, \dots, d\}$. This is a so called partition matroid and hence it fulfills the matroid axioms. We need to check whether every subset $S \subseteq E$ with $\varrho(E \setminus S) < r$ contains a positive circuit in order to apply Theorem 3. The condition $\varrho(E \setminus S) < r$ means there are at most $r - 1$ elements in $E \setminus S$, so this set has an empty intersection with $d + 1 - (r - 1) = d - r + 2 \geq 2$ of the $d + 1$ sets P_0, \dots, P_d . This shows that S

contains two of the sets P_0, \dots, P_d . Each P_i ($i = 0, \dots, d$) contains a positive circuit and so does S itself.

Now all necessary conditions of Theorem 3 are fulfilled, so there is a positive circuit C of \mathcal{O} such that $C \in \mathcal{I}$. By the observation above this means $x \in \text{conv } C$. In C there is at most one point of every set P_i so we can extend C to a set consisting of exactly one point of every P_i as mentioned in the Colorful Carathéodory Theorem. \square

4 Dual of Holmsen's Theorem

Using the basics introduced previously we are finally able to reformulate Holmsen's Theorem.

Claim 1: *We can replace the condition “for all $S \subseteq E$ with $\varrho(E \setminus S) < r$ ” by “for all $S \subseteq E$ with $\varrho(E \setminus S) = r - 1$ ” in Theorem 3.*

Proof. Every subset S with $\varrho(E \setminus S) = r - 1$ fulfills obviously the condition that the rank of the complement is smaller than r . If we start with a subset $S \subseteq E$ such that $\varrho(E \setminus S) < r - 1$, we extend the set $E \setminus S$ to $E \setminus S'$ such that $\varrho(E \setminus S') = r - 1$ (possible since the rank of \mathcal{M} is greater than r). This implies $S' \subseteq S$. The smaller set S' contains by assumption an positive circuit which is also contained in S . \square

Claim 2: *It is enough to show Theorem 3 for matroids with rank $r + 1$. In particular, matroids of rank $r + 2$ or higher need not to be considered explicitly.*

Proof. To see this we assume that \mathcal{O} is an oriented matroid of rank r and \mathcal{M} a matroid of rank $> r + 1$, both on the same ground set E . We delete elements of E such that, we get a submatroid \mathcal{M}' of \mathcal{M} of rank $r + 1$ on the ground set $E' \subset E$. Let $r - k$ for $k \in \mathbb{N}_0$ be the rank of the oriented matroid \mathcal{O}' , the restriction of \mathcal{O} to E' . We add k new elements e_1, \dots, e_k as a loop to \mathcal{M}' and in such a way that they are coloops in the underlying matroid of \mathcal{O}' . The new matroid is denoted by $\mathcal{M}' + \{e_1, \dots, e_k\}$ and the oriented matroid by $\mathcal{O}' + \{e_1, \dots, e_k\}$. The rank of the new matroid $\mathcal{M}' + \{e_1, \dots, e_k\}$ with elements $E' \cup \{e_1, \dots, e_k\}$ is $r + 1$ since the added elements do not appear in any bases. The rank of the new oriented matroid increases by k . We need to show that the condition “for all $S \subseteq E' \cup \{e_1, \dots, e_k\}$ with $\varrho((E' \cup \{e_1, \dots, e_k\}) \setminus S) = r - 1$, S contains a positive circuit of $\mathcal{O}' + \{e_1, \dots, e_k\}$ ” holds in the constructed matroid and oriented matroid. This is true since for any such subset S with $\{e_1, \dots, e_k\} \subseteq S$, we get a positive circuit C of $\mathcal{O}' + \{e_1, \dots, e_k\}$ contained in S . Since a coloop is in no circuit, we get $C \cap \{e_1, \dots, e_k\} = \emptyset$. For any other subset S with $\varrho((E' \cup \{e_1, \dots, e_k\}) \setminus S) = r - 1$, we can extend the set S by the elements e_1, \dots, e_k and get a positive circuit without $\{e_1, \dots, e_k\}$, so the positive circuit is contained in S itself. So the precondition is true. Hence there is a positive circuit C' of $\mathcal{O}' + \{e_1, \dots, e_k\}$ (with $C' \cap \{e_1, \dots, e_k\} = \emptyset$) which is independent in $\mathcal{M}' + \{e_1, \dots, e_k\}$. This implies that C' is independent in \mathcal{M} as well. This shows that the condition “ $\varrho(E) > r$ ” can be replaced by “ $\varrho(E) = r + 1$ ”. \square

So we showed that Theorem 3 is an equivalent formulation of

Theorem 4. *Let \mathcal{O} be an oriented matroid on the ground set E with rank r . Consider a matroid \mathcal{M} on the same ground set E and with rank function ϱ such that $\varrho(E) = r + 1$. If every subset $S \subseteq E$ with $\varrho(E \setminus S) = r - 1$ contains a positive circuit of \mathcal{O} , there exists a positive circuit of \mathcal{O} which is independent in \mathcal{M} .*

By the Topological Representation Theorem every oriented matroid can be realized as an oriented pseudosphere arrangement. A circuit in such an arrangement is a collection of half-spaces with empty intersection. Since cocircuits correspond to vertices in the arrangement and so they are easier to visualize, we will present a dual version of Holmsen's Theorem.

Theorem 5 (Dual version of Holmsen's Theorem). *Let \mathcal{M} be a matroid with rank function ϱ such that $\varrho(E) = r - 1$. Furthermore we consider an oriented matroid \mathcal{O} with rank r on the same ground set E . If every double circuit of \mathcal{M} contains a positive cocircuit of \mathcal{O} , then there exists a positive cocircuit in \mathcal{O} , whose complement spans \mathcal{M} .*

Proof. Let \mathcal{M}, \mathcal{O} be as in the theorem. From the rank formula

$$\varrho_{\mathcal{M}^*}(X) = |X| + \varrho_{\mathcal{M}}(E \setminus X) - \varrho_{\mathcal{M}}(E),$$

we see that $r^* := |E| - r$ is the rank of the dual oriented matroid \mathcal{O}^* and

$$\begin{aligned} \varrho_{\mathcal{M}^*}(E) &= |E| + \varrho_{\mathcal{M}}(E \setminus E) - \varrho_{\mathcal{M}}(E) \\ &= |E| + \varrho_{\mathcal{M}}(\emptyset) - (r - 1) = r^* + 1. \end{aligned}$$

So $\mathcal{M}^*, \mathcal{O}^*$ fulfill the assumptions of Theorem 4. Furthermore every inclusion-minimal subset $S \subseteq E$ with $\varrho_{\mathcal{M}^*}(E \setminus S) = r^* - 1$ is a double circuit in its dual, which follows from the rank formula and of $\mathcal{M}^{**} = \mathcal{M}$:

$$\varrho_{\mathcal{M}}(S) = |S| + \varrho_{\mathcal{M}^*}(E \setminus S) - \varrho_{\mathcal{M}^*}(E) = |S| + r^* - 1 - (r^* + 1) = |S| - 2.$$

Since S is inclusion-minimal with this property, the complement of $S \setminus \{e\}$ for every $e \in S$ has rank $\varrho_{\mathcal{M}^*}(E \setminus (S \setminus \{e\})) = r^*$. This implies that

$$\varrho_{\mathcal{M}}(S \setminus \{e\}) = |S| - 1 + r^* - (r^* + 1) = |S| - 2$$

for every $e \in E$. So S is a double circuit in \mathcal{M} . Every set $S' \subseteq E$ which is not inclusion-minimal contains an inclusion-minimal set S and hence it contains a double circuit. Every double circuit contains a positive cocircuit of \mathcal{O} by assumption and so every S contains a positive circuit of \mathcal{O}^* . So by Theorem 4, we know that there is a positive circuit of \mathcal{O}^* which is independent in \mathcal{M}^* . This is a positive cocircuit of \mathcal{O} whose complement spans \mathcal{M} . \square

The dual version mentioned in Theorem 5 is an equivalent formulation of Holmsen's Theorem. The proof that Holmsen's Theorem follows from the dual version mentioned above works analogously.

5 Complementary Positive Cocircuits

We will now think of an oriented matroid of rank $d + 1$ as an arrangement of signed pseudospheres in S^d . The positive cocircuits are the vertices of the main

polytope. The complement of a cocircuit corresponds exactly to the spheres going through a vertex of the polytope. There are at least d spheres intersecting in a vertex v . So if the complement of a positive cocircuit is considered to span the matroid \mathcal{M} of rank d , this means that the elements corresponding to the spheres, intersecting in the cocircuit, span the matroid. If the complement of a positive cocircuit C does not span the matroid, the rank is less than $\varrho(E) = d$, so

$$\varrho(E \setminus C) \leq d - 1 \leq |E \setminus C| - 1.$$

This shows that $E \setminus C$ contains a circuit of \mathcal{M} .

If the vertex v of the main polytope is degenerate and $E \setminus C$ is non-spanning or $\varrho(E \setminus C) \leq d - 2$ there exists a vertex V' of the main polytope such that the set of spheres intersecting in v' is disjoint from those intersecting in v . We call such a vertex *complementary vertex*.

We will now study in which cases the union of the elements intersecting in two adjacent (in the 1-skeleton of the face lattice of the positive polytope) positive cocircuits contain a double circuit of the matroid, i.e. in which case the edge connecting those vertices has a complementary vertex. For this reason note that:

Proposition 1. *A set $X \subseteq E$ contains a double circuit if and only if the rank of X is $\varrho(X) \leq |X| - 2$.*

Proposition 2. *Every non-spanning set in a matroid of rank d with more than d elements contains a double circuit.*

So by the last Proposition 2, we may assume that in each vertex there are exactly d intersecting elements.

Let C_1 and C_2 be two adjacent positive cocircuit and each complement does not span the matroid. So

$$\varrho(E \setminus C_i) \leq d - 1 \text{ for } i = 1, 2.$$

If $(E \setminus C_1) \cap (E \setminus C_2)$ is independent, the submodularity of the rank function shows

$$\begin{aligned} \varrho((E \setminus C_1) \cup (E \setminus C_2)) &\leq \varrho(E \setminus C_1) + \varrho(E \setminus C_2) - \varrho((E \setminus C_1) \cap (E \setminus C_2)) \\ &\leq (d - 1) + (d - 1) - |(E \setminus C_1) \cap (E \setminus C_2)| \\ &\leq d - 1. \end{aligned}$$

Furthermore $(E \setminus C_1) \cup (E \setminus C_2)$ contains $d + 1$ elements, so $(E \setminus C_1) \cup (E \setminus C_2)$ contains a double circuit if $(E \setminus C_1) \cap (E \setminus C_2)$ is independent.

In a similar way, we get that $(E \setminus C_1) \cup (E \setminus C_2)$ contains a double circuit if either the additional element of $E \setminus C_1$ or the additional element of $E \setminus C_2$ does not increase the rank.

By the assumption of the dual version (Theorem 5) every double circuit contains a positive cocircuit. So the elements intersecting in the two cocircuits C_1 and C_2 contain a positive cocircuit.

The remaining part is that both elements increase the rank. In this case the rank of the union is

$$\begin{aligned}\varrho(E \setminus C_1 \cup E \setminus C_2) &= \varrho(E \setminus C_1 \cap E \setminus C_2) + 2 \\ &< |E \setminus C_1 \cap E \setminus C_2| + 2 \\ &= |E \setminus C_1 \cup E \setminus C_2|.\end{aligned}$$

In this case, the union does not contain a double circuit if the intersection does not contain one. On the other hand $E \setminus C_1 \cup E \setminus C_2$ has rank d and still contains a spanning positive cocircuit if C_1, C_2 do not span the matroid.

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