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Kirchberger's Theorem for Complexes of Oriented Matroids

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Kirchberger's Theorem for Complexes of Oriented Matroids

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Abstract

The separation theorem of Kirchberger can be proven using a combination of Farkas' Lemma and Carathéodory's Theorem. Since those theorems are at the heart of oriented matroids, we are interested in a generalization of Kirchberger's Theorem to them. This has already been done for rank 3 oriented matroids. Here we prove it for complexes of oriented matroids, which are a generalization of oriented matroids.

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In order to introduce Kirchberger's Theorem we use a picture from [9]. Imagine we have black and white sheep in a meadow and we want to decide whether they can be separated by a straight fence. Kirchberger gives an answer to this question.

Theorem 1 (Kirchberger's Theorem). *Let V and W be finite subsets of \mathbb{R}^n . If every set $C \subseteq V \cup W$ of $n + 2$ or fewer points can be strictly separated into the sets $V \cap C$ and $W \cap C$, then V can be strictly separated from W , i.e. one can find $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $a^T v - \alpha < 0$ for all $v \in V$ and $a^T w - \alpha > 0$ for all $w \in W$*

For our example this means if every set of four sheep can be separated by a straight fence, all sheep can be separated. The original proof of Kirchberger in 1902 is really long and hard to understand [7]. Nowadays easier proofs are known. One possibility is to prove it using Helly's Theorem like in [2] or [8]. There is also a simpler proof which is basically a combination of Carathéodory's Theorem and Farkas' Lemma which can be found in [10]. Because those two theorems are crucial for *oriented matroids (OMs)*, it is natural to generalize

Kirchberger's Theorem to them as well. This has been already done for pseudo-line arrangements, i.e. OMs of rank 3 [3], [5]. We will prove it for *complexes of oriented matroids (COMs)*. COMs have been introduced in [1] under the name conditional oriented matroids as a common generalization of oriented matroids, affine oriented matroids, and lopsided sets. Before we define COMs and some of their properties, we need the following definitions

Definition 2. Let $\mathcal{L} \subseteq \{0, +, -\}^E$ be a set of sign vectors on a finite ground set E . The composition of two sign vectors X and Y is defined as

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{if } X_e = 0, \end{cases} \forall e \in E.$$

The separator of X and Y is defined as

$$S(X, Y) = \{e \in E : X_e = -Y_e \neq 0\}.$$

The support of X is defined as

$$\underline{X} = \{e \in E : X_e \neq 0\}.$$

Let us introduce three axioms for systems of sign vectors.

(FS) Face Symmetry

$$\forall X, Y \in \mathcal{L} : X \circ (-Y) \in \mathcal{L}$$

(SE) Strong Elimination

$$\begin{aligned} \forall X, Y \in \mathcal{L} \forall e \in S(X, Y) \exists Z \in \mathcal{L} : \\ Z_e = 0 \text{ and } \forall f \in E \setminus S(X, Y) : Z_f = (X \circ Y)_f. \end{aligned}$$

(C) Composition

$$\forall X, Y \in \mathcal{L} : X \circ Y \in \mathcal{L}.$$

Now we are in the position to define the term COM.

Definition 3 (Complex of Oriented Matroids (COM)). Let E be a finite set and $\mathcal{L} \subseteq \{0, +, -\}^E$. The pair (E, \mathcal{L}) is called a COM, if \mathcal{L} satisfies (FS) and (SE). The elements of \mathcal{L} are called covectors.

Note that (FS) implies (C). Indeed, by (FS) we first get $X \circ -Y \in \mathcal{L}$ and then $X \circ Y = (X \circ -X) \circ Y = X \circ -(X \circ -Y) \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

Let $\mathcal{M} = (E, \mathcal{L})$ be a COM. In the following we assume that \mathcal{M} is simple, i.e. $\forall e \in E : \{X_e | X \in \mathcal{L}\} = \{+, -, 0\}$ and $\forall e \neq f \in E : \{X_e X_f | X \in \mathcal{L}\} = \{+, -, 0\}$. In this setting the sign-vectors in \mathcal{L} of full support are called *topes* and \mathcal{T} is the set of all *topes* of \mathcal{M} . A COM \mathcal{M} is an *oriented matroid (OM)* [4], if $\mathbf{0} \in \mathcal{L}$.

The *restriction* of a sign-vector $X \in \{0, \pm\}^E$ to $E \setminus F$, $F \subseteq E$, denoted by $X \setminus F \in \{0, +, -\}^{E \setminus F}$, is defined by $(X \setminus F)_e = X_e$ for all $e \in E \setminus F$. Given a system of sign vectors $\mathcal{M} = (E, \mathcal{L})$ and $F \subseteq E$, the *contraction* of F is the system of sign vectors $\mathcal{M}/F = (E \setminus F, \mathcal{L}/F)$, where $\mathcal{L}/F = \{X \setminus F : X \in \mathcal{L} \text{ and } \underline{X} \cap F = \emptyset\}$. It has been shown in [1] and we will implicitly make use of it that the class of COMs is closed under contractions. Let us look at an example of a COM:

Example 4. Let $E = \{v_1, \dots, v_m\} \subset \mathbb{R}^n$. We look at the following functions

$$(a, \alpha) : E \rightarrow \{+, -, 0\}$$

$$v_i \rightarrow \text{sign}(a^T v_i - \alpha),$$

where $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ and $i = 1 \dots n$. We claim that the collection of those functions induce a COM with ground set E and covectors $(\text{sign}(a^T v_1 - \alpha), \dots, \text{sign}(a^T v_n - \alpha))$. Let X be induced by (a, α) and Y be induced by (b, β) . We set

$$\epsilon = \min \left\{ \frac{|a^T v_i - \alpha|}{|b^T v_i - \beta|} : |a^T v_i - \alpha| \cdot |b^T v_i - \beta| \neq 0 \right\}.$$

Now the sign vector $X \circ -Y$ can be induced by

$$(c, \gamma) = (a, \alpha) - \frac{\epsilon}{2}(b, \beta).$$

One can see this by looking at

$$(X \circ -Y)_i = \text{sign}(c^T v_i - \gamma) = \text{sign}((a^T v_i - \alpha) - \frac{\epsilon}{2}(b^T v_i - \beta)).$$

This equals $X_i = \text{sign}(a^T v_i - \alpha)$, if $X_i \neq 0$ and $-Y_i = -\text{sign}(b^T v_i - \beta)$, if $X_i = 0$. Since (c, γ) is in our collection of functions, we see that $X \circ -Y$ is in the COM, so face symmetry is fulfilled. Let us look at strong elimination. Let $e \in S(X, Y)$ and w.l.o.g. $(a^T v_e - \alpha) < 0$ and $(b^T v_e - \alpha) > 0$. If we look at the vector

$$Z_i = \text{sign}((b^T v_e - \alpha)(a^T v_i - \alpha) - (a^T v_e - \alpha)(b^T v_i - \beta))$$

we see that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for $f \in E \setminus S(X, Y)$. Furthermore the function that induces Z is in our collection, so strong elimination is fulfilled as well and we have a COM. Note that if we set $a = (0, \dots, 0)$ and $\alpha = 0$ we get the sign vector $X = (0, \dots, 0)$, so our COM is in particular an OM.

Before we go on to Kirchberger's Theorem for COMs we need to define the rank of a COM.

Definition 5 (Rank of a COM). The rank $r(\mathcal{M})$ of a COM $\mathcal{M} = (E, \mathcal{L})$ is defined as

$$r(\mathcal{M}) = \max_{A \subseteq E} \{ |A| \mid \mathcal{L} \setminus (E \setminus A) = \{0, +, -\}^{|A|} \}.$$

So let $\mathcal{M} = (E, \mathcal{L})$ be a COM of rank r on a ground set E with $|E| = n$. We say two sets $V, W \subseteq E$ are *separable* if there exists a covector $X = (X^+, X^-)$ such that $V \subseteq X^+$ and $W \subseteq X^-$. Our sheep correspond now to the elements of E and as above we want to know if we can separate them. W.l.o.g. assume that we want to know if we can separate the first k elements of E from the last $n - k$ elements, i.e. we want to know if the vector

$$\underbrace{(+, +, \dots, +)}_k, \underbrace{(-, \dots, -, -)}_{n-k} \quad (1)$$

is a *tope* of \mathcal{M} . Our theorem will say that if for all $C \subseteq E$ with $|C| = n - (r + 1)$ the sets $V \cap C$ and $W \cap C$ can be separated in \mathcal{M}/C (i.e. (1) restricted to $E \setminus C$ is a covector of \mathcal{M}/C), then V and W can be separated in \mathcal{M} (i.e. (1) is a covector of \mathcal{M}). Let us demonstrate this in our example.

Example 6 (Example 4 continued). *If now*

$$X = \underbrace{(+, +, \dots, +)}_k, \underbrace{(-, \dots, -, -)}_{n-k}$$

is a *tope* of the COM in Example 4 this means that there is an (a, α) such that $av - \alpha = 0$ separates v_1, \dots, v_k and v_{k+1}, \dots, v_n strictly. So in this case Kirchberger's Theorem for COMs will say that whenever $r + 1$ elements of E can be separated strictly, then all of them can be separated strictly which is Kirchberger's Theorem in its original version. Let us look at the rank of our COM. We may assume (e.g. by induction over the dimension) that v_1, \dots, v_n span \mathbb{R}^n affinely. Therefore we will find vectors $v_{i_1}, \dots, v_{i_{n+1}}$ that span an n -simplex. It is easy to see (e.g. by induction) that one gets every possible sign vector within those simplex spanning elements by using a proper separating hyperplane. This shows by Definition 5 that the rank r of our COM is at least $n + 1$. We will show that the rank is exactly $n + 1$. If we look at $n + 2$ or more vectors, i.e. $V = v_{i_1}, \dots, v_{i_{n+2}}$, one would find by Radon's Theorem [2] a Radon Partition $(P_1, P_2) \subseteq V$, which is a partition where $\text{conv}(P_1) \cap \text{conv}(P_2) \neq \emptyset$. If we now look for the sign vector which has minus entries in P_1 and plus entries in P_2 we will see that this pattern can not be induced. Either some points of P_1 are in $\text{conv}(P_2)$, then it is obvious that they can not be separated from P_2 or there are two points of P_1 where the connecting line intersects $\text{conv}(P_2)$, so we also do not find an hyperplane which separates them from P_2 . So the required sign vector can not be induced which shows that the rank is exactly $n + 1$. By that we see that $r + 1 = n + 2$ which explains why we have $n + 2$ in Kirchberger's original theorem and $r + 1$ in the theorem for COMs.

In order to simplify the proof we will formulate the theorem on a reorientation of \mathcal{M} (i.e. \mathcal{M} with some flipped signs, which does not affect the general structure), where we do not look for the sign vector (1) but for the vector $R = (+, \dots, +)$ of length n .

Theorem 7 (Kirchberger’s Theorem for COMs). *Let $\mathcal{M} = (E, \mathcal{L})$ be a COM of rank r and $|E| = n$. If for all $C \subseteq E$ with $|C| = n - (r + 1)$ the sign vector $R \setminus C$ is a tope of \mathcal{M}/C , then R is a tope of \mathcal{M} .*

We need the following lemma for our proof, which is a generalization of [6, Lemma 4]. The OM of the following example will play a major role in our proof.

Example 8. *Let us look at a special case of Example 4. Take the points $\{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{e_n - e_1\}$, where e_i are the unit vectors. Any $n - 1$ of them are linear independent but all n of them are not. We call such structures a directed cycle and the corresponding COM (OM) \mathcal{C}_n .*

Lemma 9. *Let $\mathcal{M} = (E, \mathcal{L})$ be a COM with tope set \mathcal{T} , such that for all $f \in E$ there exists $T^f \in \mathcal{T}$ such that*

$$T_g^f = \begin{cases} - & \text{if } g \in E \setminus \{f\} \\ + & \text{if } g = f. \end{cases}$$

If $R \notin \mathcal{T}$, then $\mathcal{M} = \mathcal{C}_{|E|}$.

Proof. We will show by induction that all covectors which contain exactly one minus-entry and at least one plus-entry are in \mathcal{L} . Since then in particular all covectors which contain exactly one plus-entry and one minus-entry exist in \mathcal{L} , we get by (SE) that $\mathbf{0} \in \mathcal{L}$. Together, we can conclude that $\mathcal{M} = \mathcal{C}_n$, since we obtain all its covectors by composition of those vectors. Since \mathcal{C}_n is uniform no other oriented matroid can contain these covectors.

So let $T^f \in \mathcal{T}$ for all $f \in E$ and $R \notin \mathcal{T}$. We will use induction over the number of zero-entries in the covectors, i.e. we want to show that for every $n = 0, \dots, |E| - 2$ all sign-vectors with n zero entries, one minus-entry and $|E| - (n + 1)$ plus-entries are covectors of \mathcal{M} .

$n = 0$: By the existence of T^f here is nothing to show. We fix $n > 0$ and assume that all covectors with n or less zero-entries, exactly one minus entry and at least one plus-entry exist in \mathcal{L} .

$n \rightarrow n + 1 \leq |E| - 2$: We now look for a covector with zero-entries in the i -th position, $i \in I \subset E$, $|I| = n + 1$, a minus-entry in the j -th position, $j \notin I$ and $+$ everywhere else. We choose an $\hat{i} \in I$ and take two covectors with 0 in $I \setminus \hat{i}$. One of them should have its $-$ in the \hat{i} -th position and the other one at the j -th position. W.l.o.g. those two covectors look like this:

$$\begin{array}{c} (0, \dots, 0, \overbrace{-}^{\hat{i}}, +, +, \dots, +) \\ (0, \dots, 0, \underbrace{+}_{I \setminus \hat{i}}, \underbrace{-}_j, +, \dots, +). \end{array}$$

They exist because $|I \setminus \hat{i}| = n$, so the induction hypotheses holds. If we now perform strong elimination with those two covectors we get (again w.l.o.g) the

covector

$$X = (\underbrace{0, \dots, 0}_{I \setminus \hat{i}}, \underbrace{0}_{\hat{i}}, \underbrace{*}_{j}, +, \dots, +).$$

If $*$ was $+$, then $X \circ T^j = R$. Since $R \notin \mathcal{T}$ we have $* = -$ and have the covector we were looking for. \square

We will now prove Theorem 7 by contraposition.

Proof. Suppose that R does not exist in \mathcal{L} . Let now $D \subseteq E$ be of maximal cardinality such that $R \setminus D$ does not exist in \mathcal{M}/D . Since we choose D to be maximal, we have that

$$\begin{pmatrix} -, & +, & \dots, & +, \\ +, & -, & \dots, & +, \\ \vdots & & \ddots & \vdots \\ +, & +, & \dots, & - \end{pmatrix} \in \mathcal{L}/D.$$

Indeed, since D is maximal for every $f \in E \setminus D$ there is a tope in \mathcal{L}/D with $T_f^f = +$ being its only positive entry. By Lemma 9 we have $\mathcal{M}/D = \mathcal{C}_{|Y|}$, where $Y = E \setminus D$ and $\mathcal{C}_{|Y|}$ the directed cycle of $|Y|$ elements. Since \mathcal{M} has rank r , the cycle can have at most $r + 1$ elements, i.e. $|Y| \leq r + 1$. So we found an D with $|D| = n - |Y|$ such that $R \setminus D$ does not exist in \mathcal{M}/D . Since $|Y| \leq r + 1$ we can conclude that we will also find an C with $|C| = n - (r + 1) \leq n - |Y|$ where $R \setminus C$ will not exist in \mathcal{M}/C . This finishes our contraposition. \square

Since every OM is a COM, the statement for OMs is a direct corollary of Theorem 7.

Corollary 10 (Kirchberger's Theorem for OMs). *Let $\mathcal{O} = (E, \mathcal{L})$ be a OM of rank r on E , $|E| = n$. If for all $C \subseteq E$ with $|C| = n - (r + 1)$ the sign-vector $R \setminus C$ exists in \mathcal{O}/C , then R exists in \mathcal{O} .*

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