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Application of linear algebra methods to the problem of a polynom decomposition into factors

A well-known problem of the decomposition of a polynom

$$f(x) = f_0 + f_1 x + \ldots + f_n x^n \tag{1}$$

into two factors

$$f(x) = Cg(x)h(x)$$

is discussed. The methods that were elaborated by Hermit, Jakobi, Lobachevski, Schur, Cohn are generally sensitive to calculations errors. At the same time the modern tasks require the application of the stable algorithms, whose work is accompanied with the errors estimation and computation of a effective condition criterion.

A new algorithm for the solution of *the problem of the polynom roots dichotomy by unit circle* is suggested. This problem can be formulated in the following way. Define is any root of polynom (1) lying on the unit circumference. Then if there are not such roots, find a positive number $\rho < 1$ and coefficients of the factors $g(x)$ and $h(x)$ such that:

$$\begin{array}{l} g(x) = (x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_l), \\ h(x) = (\mu_1 x - 1)(\mu_2 x - 1)\ldots(\mu_m x - 1), \end{array} \tag{2}$$

where $l + m = n$ and the next inequalities are thrue:

$$|\lambda_j| \le \rho < 1, \quad |\mu_j| \le \rho < 1\,.$$

Principal features of this algorithm are:
1. calculation of the coefficients of the polynoms $g(x)$ and $h(x)$ with the guaranteed accuracy;
2. diagnostics of the situation, where the roots of $f(x)$ lie too close to unit circle and the calculation with the guaranteed accuracy is impossible.

The algorithm is based on the connection of the investigated polynoms (1) roots with eigenvalues of the matrix pencil (A, B)

$$A = \begin{pmatrix} f_0 & f_1 & \cdots & f_{n-1} \\ & f_0 & \cdots & f_{n-2} \\ & & \ddots & \vdots \\ & & & f_0 \end{pmatrix}, \qquad -B = \begin{pmatrix} f_n & & & \\ f_{n-1} & f_n & & \\ \vdots & \ddots & \ddots & \\ f_1 & \cdots & f_{n-1} & f_n \end{pmatrix}.$$

The norm of the matrix

$$H = \frac{1}{2\pi}\int_0^{2\pi} \left(A - e^{i\varphi}B\right)^{-*}\left(A^*A + B^*B\right)\left(A - e^{i\varphi}B\right)^{-1} d\varphi,$$

is used as a dichotomy quality criterion according to the next estimations

$$\sqrt{\|H\|} \le \left(\frac{1+\rho}{1-\rho}\right)^n, \qquad \rho^n \le \sqrt{\frac{\|H\|-1}{\|H\|+1}}.$$

Denote by $f^{(k)} = (f_0^{(k)}, f_1^{(k)}, \ldots, f_n^{(k)})$, $g^{(k)} = (g_0^{(k)}, g_1^{(k)}, \ldots, g_l^{(k)})$, $h^{(k)} = (h_0^{(k)}, h_1^{(k)}, \ldots, h_m^{(k)})$ the vectors, which are calculated on the k-th algorithm step, $f = (f_0, f_1, \ldots, f_n)$, $g = (g_0, g_1, \ldots, g_l)$, $h = (h_0, h_1, \ldots, h_m)$ – the vectors, whose coordinates are equal to the coefficients of the polynoms (1) and (2).

On the k-th algorithm step the approximations $g_j^{(k)}$, $h_j^{(k)}$ for the coefficients of the polynoms $g(x)$ è $h(x)$ and the approximation $H^{(k)}$ for the matrix H are calculated.

If $\|H^{(k)}\|$ exceed a fixed great value $\omega_{\max}$ (this is evidence of the practical absence of the dichotomy), the calculations process will be interrupt with a corresponding message text.

Algorithm convergence is proved by following estimations

$$\max\left\{\left\|\frac{\|g^{(k)}\|}{\|g\|}g - g^{(k)}\right\|, \left\|\frac{\|h^{(k)}\|}{\|h\|}h - h^{(k)}\right\|\right\} \le \|f\|\, K_p(\omega)\, [\kappa(\omega)]^{2^k},$$

$$\|H - H^{(k)}\| \le \omega K_H(\omega)[\kappa(\omega)]^{2^k+1}$$

where

$$\omega = \|H\|, \qquad \kappa(\omega) = \sqrt{\frac{\omega-1}{\omega-1}},$$

$$K_p(\omega) = 4\sqrt{n}\,\omega\, K_\kappa(\omega)\,, \qquad K_\kappa(\omega) = \frac{1}{(1-\kappa^{\frac{1}{2n}})^{m+1}} \frac{(1+\kappa^{\frac{1}{n}})^{m-1}}{(1-\kappa^{\frac{1}{n}})^{m-1}},$$

$$K_H(\omega) = \frac{4}{1-\kappa}\left(\frac{1+\kappa}{1-\kappa^{2^{k+1}+1}}\left(1+2\sqrt{\omega}\frac{\kappa^{2^k+1}}{1-\kappa^{2^{k+1}+1}}\right)+\frac{1}{2}\right).$$