Bounds for asymptotically stable matrix exponentials

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July 13, 2000

A matrix A is *asymptotically (or exponentially) stable*, if

$$\|e^{At}\| \le Ce^{-\beta t}, \quad \text{for some } C, \beta > 0$$

($\|\cdot\|$ is the spectral norm). We present new estimates for C, β above as well as for the difference

$$\|e^{Bt} - e^{At}\|.$$

We will use the solution X of the *Ljapunov equation*

$$A^*X + XA = -I$$

— inspired by the bound

$$\|e^{At}\| \le \sqrt{\|X\|\|X^{-1}\|}e^{-\frac{t}{2\|X\|}}. \tag{1}$$

(Godunov et al. 1990). An advantage of this bound is its independence of the space dimension, and the spectral geometry of A; a shortcoming is the factor $\|X\|\|X^{-1}\|$ which may be large. Our second ingredient is the *logarithmic norm*

$$\delta = \delta(A) = \sup_{\|\psi\|=1} \Re(A\psi, \psi)$$

Our new estimate reads

$$\|e^{At}\| \leq$$

$$\begin{cases} (1+\delta\|X\|)^{\frac{1}{2}+\frac{1}{2\delta\|X\|}} e^{-\frac{t}{2\|X\|}}, & t \geq t_\delta \\ e^{\frac{\delta t}{2}}, & t \leq t_\delta \end{cases} \tag{2}$$

for $t_\delta = \frac{1}{\delta}\ln(1+\delta\|X\|)$ and $\delta \neq 0$ and

$$\|e^{At}\| \leq \begin{cases} e^{-\frac{t-\|X\|}{2\|X\|}}, & t \geq \|X\| \\ 1, & t \leq \|X\| \end{cases} \tag{3}$$

for $\delta = 0$.

The case $\delta = 0$ (dissipative A) has special interest. In fact, damped vibrational system

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{4}$$

with M, C, K real symmetric, M, K positive definite and C positive semidefinite, gives rise to the dissipative matrix

$$A = \begin{bmatrix} 0 & L_1^T L_2^{-T} \\ L_2^{-1} L_1 & -L_2^{-1} C L_2^{-T} \end{bmatrix}. \tag{5}$$

with $K = L_1 L_1^T$, $M = L_2 L_2^T$.

In this case our estimate improves (1), for $\|X\|\|X^{-1}\| > e \approx 2.7$, for $\|X\|\|X^{-1}\| \leq e$ it is weaker. As δ grows from $-1/\|X\|$ to infinity our estimate gradually looses against (1).

Comparison with known bounds. The van Loan bound (1977) reads

$$\|e^{At}\| \le \sum_{k=0}^{n-1} \frac{(t\|M\|)^k}{k!} e^{\alpha(A)t},$$

where M is the strict upper triangle in the Schur form of A and $\alpha(A)$ is the maximum real part of the eigenvalues of A. This has about the same complexity as our estimate (the common way to compute the Ljapunov solution goes via the Schur form). A stronger bound holds with $\|M^k\|$ instead of $\|M\|^k$, but then the complexity is $\mathcal{O}(n^4)$. Both van Loan bounds depend on the dimension as well as on the non-normality of A.

We compare the bounds on the linear damped system on Fig. 1

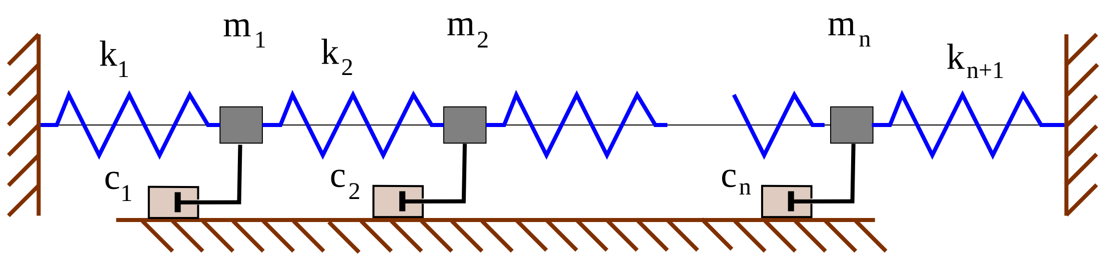

Figure 1: The n-mass oscillator with dampers

Fig. 2 below displays the four bounds for $n = 5$ and

1. Homogeneous parameters, $m_i = c_i = k_i = 1$.
2. As above, except $c_2 = \cdots = c_5 = 0$.
3. As above, except $k_1 = 0, k_2 = 0.01$.

Lines: Solid blue: the two van Loan bounds, dashed green: Godunov bound, solid red: our new bound.

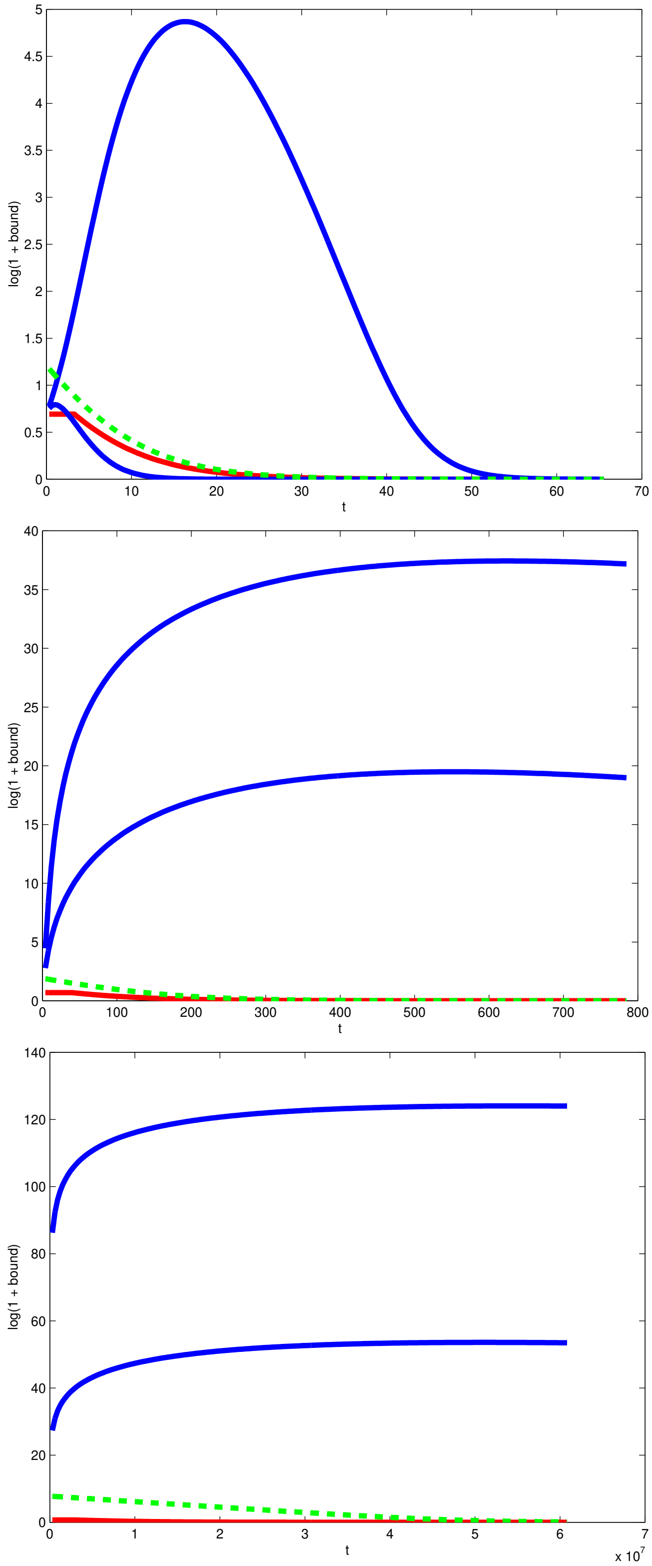

Figure 2:

Perturbation bounds. We factorise the perturbation:

$$B - A = W = GUG^*.$$

If both A and B are asymptotically stable then

$$\|e^{Bt} - e^{At}\| \le \|U\| \times$$

$$\min\{\sqrt{\|Y_B(GG^*)\|\|X_A(GG^*)\|}, \sqrt{\|Y_A(GG^*)\|\|X_B(GG^*)\|}\},$$

where $X_C(D)$ and $Y_C(D)$ solve the 'mutually dual' Ljapunov equations

$$C^*X + XC = -D, \quad YC^* + CY = -D,$$

respectively. Choose $G = G^* = I, U = W$. Then

$$\|e^{Bt} - e^{At}\| \le$$

$$\min\{\sqrt{\|Y_B(I)\|\|X_A(I)\|}, \sqrt{\|Y_A(I)\|\|X_B(I)\|}\}\|W\|. \qquad (6)$$

The novelty is that the bounds on the right hand side are independent of t. This might be considered a shortcoming — one would expect the bound to vanish for $t \to 0$. Such a bound can, however, not be made uniform, if no additional information on A is known. In fact, in infinite dimensional space the exponential function may be only strongly continuous and the bound may not vanish at $t = 0$! If e.g. $\|Y_A(I)^{-1}\|$ is known then the factor $\|Y_A(I)\|$ above can be substituted by

$$\|Y_A(I)\|(1 - \exp(-t\|Y_A(I)^{-1}\|))$$

etc. which vanishes at 0.

All estimates above can be modified to accomodate the case, where only A is known to be asymptotically stable and the perturbation $W = B - A$ is small enough.

As a last example we perturb the damping parameters on Fig. 1 as follows

$$c_i \mapsto c_i + \delta c_i, \quad |\delta c_i| \le \varepsilon c_i, \quad i = 1, \dots, n, \quad 0 < \varepsilon < 1.$$

The choice

$$G = \begin{bmatrix} 0 & 0 \\ 0 & L_2^{-1}\mathrm{diag}(\sqrt{|c_i|}) \end{bmatrix}$$

leads to the estimate

$$\|e^{Bt} - e^{At}\| \le \frac{\varepsilon}{2\sqrt{1-\varepsilon}}$$

This bound is — apart from the relative error ε — completely independent of the properties of either A or B!

These perturbation bounds may be combined with the bounds (2) and (3) to get decay at infinity as well.

As it can be imagined from the absence of the space dimension, our bounds, in fact, hold for general exponentially stable semigroups in Hilbert spaces.