FernUniversität in Hagen

Bachelorarbeit

The Black–Scholes Equation
A Partial Differential Equation Approach to Option Pricing

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1 Abstract

The pricing of financial derivatives provides a striking example of how abstract mathematical theory can be applied to concrete real-world problems. Among the many models developed in mathematical finance, the Black–Scholes framework [FB73] occupies a central position. Originally derived through probabilistic arguments and hedging considerations, the Black–Scholes equation can also be understood as a linear parabolic partial differential equation (cf. section 1.4 of [DM21]) governing the time evolution of option prices.

The goal of this thesis is to study option pricing from this partial differential equation perspective, with a particular focus on analytical techniques from semigroup theory and spectral analysis, as well as numerical methods for problems where closed-form solutions are no longer available.

We begin by deriving the Black–Scholes equation using the classical delta-hedging argument from [FB73]. While the underlying model assumptions stem from stochastic calculus, the resulting equation is deterministic and amenable to tools from analysis. This observation motivates the mathematical approach taken throughout the thesis: rather than emphasizing probabilistic pricing formulas, we interpret option prices as solutions to evolution equations generated by suitable differential operators.

After introducing the necessary financial and mathematical background, we transform the Black–Scholes equation into the heat equation. This transformation allows us to exploit the well-understood structure of parabolic PDEs and especially the fundamental solution to the heat equation introduced in chapter 2 of [DM21]. Within this framework, we derive explicit pricing formulas for several types of European-style options, including standard European call options as well as binary options such as cash-or-nothing and asset-or-nothing contracts. These examples illustrate how different payoff structures are

encoded as initial conditions and how analytic solutions emerge naturally from the heat equation representation.

The second part of the thesis adopts a functional analytical viewpoint. The Black–Scholes operator is interpreted as the generator of a strongly continuous semigroup on an appropriate Banach space. This perspective embeds option pricing into the general theory of linear evolution equations and provides a unified description of the time evolution of prices. We derive an explicit formula for the associated semigroup as done in [Ein08] and show that it is holomorphic, highlighting its strong regularizing properties.

Building on this framework, we investigate the spectral properties of the Black–Scholes operator on interpolation spaces between L^1 and L^∞. Using results from semigroup theory and the geometry of Banach function spaces, and making great use of [AdP02], we analyze how the spectrum depends on the underlying space and how this, in turn, influences the long-time behavior and stability of solutions. This analysis reveals that qualitative features of option prices are closely tied to the functional setting in which the problem is formulated.

Finally, we turn to American options, where the possibility of early exercise leads to free-boundary problems that cannot be solved in closed form. We follow the arguments given in [Sey15] and reformulate the pricing problem as a linear complementarity problem and discuss numerical methods suitable for its approximation. An implementation is presented to illustrate how the theoretical framework developed earlier can be translated into concrete computational algorithms.

Zusammenfassung

Die Bewertung von Finanzderivaten stellt ein anschaulisches Beispiel dafür dar, wie abstrakte mathematische Theorien zur Modellierung realer wirtschaftlicher Fragestellungen eingesetzt werden können. Eine zentrale Rolle spielt hierbei das Black–Scholes-Modell, das ursprünglich Anfang der 1970er Jahre von Black und Scholes entwickelt wurde [FB73]. Obwohl ihre Herleitung auf stochastischen Modellen und Hedging-Argumenten basierte, führt das Modell auf eine deterministische lineare parabolische partielle Differentialgleichung (cf. Abschnitt 1.4 von [DM21]), die sogenannte Black–Scholes-Gleichung.

Ziel dieser Arbeit ist es, Optionsbewertungen aus der Perspektive partieller Differentialgleichungen zu untersuchen. Dabei liegt der Schwerpunkt weniger auf probabilistischen Methoden, sondern auf analytischen Techniken aus der Theorie der Evolutionsgleichungen, insbesondere der Halbgruppentheorie und der Spektralanalyse, sowie auf numerischen Verfahren für Probleme, bei denen keine geschlossenen Lösungen existieren.

Zu Beginn wird die Black–Scholes-Gleichung mithilfe des klassischen Delta-Hedging-Arguments von [FB73] hergeleitet. Anschließend werden die notwendigen finanzmathematischen und analytischen Grundlagen eingeführt. Ein zentrales Werkzeug ist dabei die Transformation der Black–Scholes-Gleichung in die Wärmeleitungsgleichung. Diese erlaubt es, bekannte Resultate aus der Theorie parabolischer Differentialgleichungen, insbesondere die Fundamentallösung der Wärmeleitungsgleichung vorgestellt in Kapitel 2 von [DM21], direkt anzuwenden.

Auf dieser Grundlage werden explizite Bewertungsformeln für verschiedenen europäische Optionsarten hergeleitet. Neben der klassischen europäischen Call-Option werden auch binäre Optionen betrachtet, darunter Cash-or-Nothing-und Asset-or-Nothing Optionen. Diese Beispiele verdeutlichen, wie unterschiedliche Auszahlungsprofile als Anfangsbedingungen in der transformierten Gleichung erscheinen und wie sich daraus geschlossene Preisformeln ergeben.

Der zweite Schwerpunkt der Arbeit liegt in einer funktionalanalytischen Betrachtung der Black–Scholes-Gleichung. Der zugehörige Differentialoperator wird als Generator einer stark stetigen Halbgruppe auf geeigneten Banachräumen interpretiert. Dieser Zugang ermöglicht eine einheitliche Beschreibung der zeitlichen Entwicklung von Optionspreisen

als Lösung eines abstrakten Cauchy-Problems. Es wird eine explizite Darstellung der erzeugten Halbgruppe angegeben [Ein08] und gezeigt, dass es sich um eine holomorphe Halbgruppe handelt, was auf ausgeprägte Regularisierungseigenschaften der Lösung hinweist.

Darauf aufbauend wird eine Spektralanalyse des Black–Scholes-Operators auf Interpolationsräumen zwischen L^1 und L^∞ durchgeführt. Mithilfe von Resultaten aus der Halbgruppentheorie, und unter großer Hilfe von [AdP02], wird untersucht, wie die spektralen Eigenschaften des Operators von der Geometrie des zugrunde liegenden Funktionenraums abhängen und welche Konsequenzen sich daraus für das Langzeitverhalten und die Stabilität der Lösung ergeben.

Im letzten Teil der Arbeit werden amerikanische Optionen betrachtet, bei denen die Möglichkeit der vorzeitigen Ausübung zu grundlegenden mathematischen Schwierigkeiten führt. Das Bewertungsproblem lässt sich in diesem Fall nicht mehr als reines Anfangs-Randwertproblem formulieren, sondern führt auf ein Freirandproblem. Basierend auf der Argumentation in [Sey15], wird dieses Problem durch eine geeignete Umformulierung als lineares Komplementaritätsproblem dargestellt, was den Einsatz numerischer Verfahren ermöglicht. Abschließend werden Finite-Differenzen-Methoden diskutiert und implementiert, um zu veranschaulichen, wie frühere Resultate in konkrete Algorithmen überführt werden können.

2 Preliminaries

2.1 Financial Background

Financial derivatives such as options are contracts whose value depends on the evolution of an underlying asset, most often a stock. The idea of trading risk through such contracts is centuries old (rice options were already traded in Osaka, Japan, in the 17th century) but it was only in the 20th century that mathematics began to play a systematic role in pricing them. In this chapter we will follow the original hedging method that was used by Black and Scholes (1972) and Merton (1973) to derive the now famous equation named after them.

A central problem in pricing options is the need to model the price of the underlying security over time. Any realistic model must reflect two essential features: the evolution is uncertain, and the price remains strictly positive. Any deterministic model (e.g exponential growth at a fixed rate) ignores the inherent randomness of financial markets, while a simple random walk model allows for negative values, which are not possible in real-world market settings. It is therefore reasonable to assume that the underlying security S follows a so-called Itô process at any time $t \in \mathbb{R}_+$, meaning that

$$dS(t) = \mu\,(t, S(t))\,dt + \sigma\,(t, S(t))\,dW(t), \tag{2.1.1}$$

where $\mu\,(t, S(t))$ is called the drift function and $\sigma\,(t, S(t))$ is the volatility of an increment in $S(t)$. The term $dW(t)$ is called a Wiener-increment. Although the focus of this thesis lies on the partial differential equations aspect of the resulting equation, it is nevertheless worthwhile to note the interesting underlying theory of Wiener processes. We refer to section 7.1 of [Dur19] for a thorough introduction into the associated mathematics.

If we choose $\mu\,(t, S) = \mu S$ and $\sigma\,(t, S) = \sigma S$, where μ and σ are constants, the stock price S is said to follow a geometric Brownian motion (GBM). Intuitively that means that we assume the stock price to be normally distributed. Rearranging (2.1.1) yields

$$\frac{dS}{S} = \mu dt + \sigma dW, \tag{2.1.2}$$

where μdt is the deterministic drift and σdW is the random term. More generally $\mu \in \mathbb{R}$ represents the average rate of return and $\sigma > 0$ measures the intensity of fluctuations.

Another important ingredient in the derivation of the Black–Scholes equation is found in the works of the Japanese mathematician Kiyosi Itô[1], who published multiple papers on stochastic calculus. After introducing the stochastic integral in 1944, he later formulated a theorem that is now known as the famous Itô's Lemma. We just formulate the result that matters most to our discussion here and restrict ourselves to dimension 1 and refer to theorem 6 of [Ito51] and section 6.3 (especially Theorem 6.10) of [MC13] for a more in-depth treatment of the multi-dimensional theory.

Lemma 2.1. *Let $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be of class C^1 in the first variable and of class C^2 in the other and $X(t)$ an Itô process satisfying the differential equation (2.1.1). Then $f(t, X(t))$ is an Itô process with*

$$df(t, X) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X}\mu(t, X) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\sigma^2(t, X)\right] dt + \frac{\partial f}{\partial X}\sigma(t, X)\, dW. \tag{2.1.3}$$

For a geometric Brownian motion, it is therefore

$$df(t, X) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\sigma^2 X^2\right] dt + \frac{\partial f}{\partial X}\sigma X dW. \tag{2.1.4}$$

With this Lemma at hand, the last ingredient we need in order to derive the Black–Scholes PDE is the concept of delta hedging. At its core, delta hedging is a strategy designed to manage the risk that arises from holding derivatives such as options, whose value depends on the underlying asset. An option's delta measures the sensitivity of the options price to small changes in the price of the underlying asset. By dynamically adjusting a portfolio consisting of the option and the underlying asset, an investor can offset this sensitivity, thereby constructing a position that is (at least locally) immune to fluctuations in the underlying market.

To this end, we consider a portfolio consisting of one option and a short position of Δ units of the underlying stock, i.e. we sell Δ shares of the stock at time t. Choosing $\Delta = \frac{\partial V}{\partial S}$ ensures that the dW-terms cancel in the differential of the portfolio. The

[1] *07.09.1915,†10.11.2008

resulting portfolio is therefore locally risk-free.

Mathematically we can denote this hedged portfolio by

$$\Pi = V - \Delta S, \tag{2.1.5}$$

where V is the shorthand-form of the variable $V(t, S(t))$, denoting the price of a derivative security at time t and ΔS stands for the corresponding movement of the underlying stock. With all this, we can now calculate the return on the portfolio as

$$d\Pi = dV - \frac{\partial V}{\partial S} dS. \tag{2.1.6}$$

Applying Lemma 2.1 to this and using the expression for dS we got in (2.1.2), we get

$$d\Pi = \left(-\frac{\partial V}{\partial S}\mu S - \frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \sigma S \left(\frac{\partial V}{\partial S} - \frac{\partial V}{\partial S} \right) dW \tag{2.1.7}$$

which simplifies to

$$d\Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \tag{2.1.8}$$

If we now assume the return r on a risk free portfolio to be constant, meaning that $d\Pi = r\Pi dt$, we finally get

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{2.1.9}$$

This is the well-known **Black–Scholes partial differential equation**.

2.2 Mathematical Background

We quote directly from chapter 5 of [DM21]:

Definition 2.2. *Let X be a Banach space. A family* $(T(t))_{t>0}$ *of bounded linear operators on X is called a* ***strongly continuous semigroup*** *- shortly: a* C_0*-**semigroup** - if*

$$T(t)\,T(s) = T(t+s) \quad \textit{for all } t,s \geq 0 \quad \textit{and} \quad T(0) = Id \tag{2.2.1}$$

and moreover if it is ***strongly continuous****, i.e.,if the* ***orbits***

$$\mathbb{R}_+ \ni t \to T(t)\,x \in X \tag{2.2.2}$$

are continuous for all $x \in X$.

If, furthermore, the condition $t \in \mathbb{R}_+$ is replaced by $t \in \mathbb{R}$ and $T(t)\,T(s) = T(t+s)$ holds for all $t, s \in \mathbb{R}$, we say that the family $(T(t))_{t\in\mathbb{R}}$ is a C_0-**group**.

Next we come to the notion of a **analytic** (or sometimes: **holomorphic**) **semigroups**(**of angle** θ):

Definition 2.3. *Let $\Delta_\theta := \{0\} \cup \{t \in \mathbb{C} : |\arg(t)| < \theta\}$ be the sector of the complex plane of angle θ centered at 0 and symmetric about the real axis. A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is said to be analytic if the following two conditions are satisfied:*

- *For some $\theta \in \left(0, \frac{\pi}{2}\right)$, there is a family $(T(z))_{z\in\Delta_\theta}$ of bounded linear operators on X that extends $(T(t))_{t\geq 0}$ to Δ_θ and satisfies*

$$T(t)\,T(s) = T(t+s) \quad \textit{for all } t,s \in \Delta_\theta \quad \textit{and} \quad T(0) = Id. \tag{2.2.3}$$

- *$\Delta_\theta \setminus \{0\} \ni t \to T(t) \in \mathcal{L}(X)$ is an analytic function.*

To establish existence of integrals of scalar-valued functions in later chapters we introduce the notion of Bochner integrals. Again from [DM21] we quote some easily verifiable properties.

Proposition 2.4. *A measurable function $f : I \to X$ is integrable if and only if*

$$\int_I \|f\| dx < \infty. \tag{2.2.4}$$

Furthermore, if $T : X \to X$ is a bounded linear operator and $f : I \to X$ an integrable function, then $Tf : I \to X$ is also integrable and

$$T\left(\int_I f dx\right) := \int_I T f dx. \tag{2.2.5}$$

Let (Ω, A, μ) *be a measure space. The space* $L^1(\Omega, \mu)$ *is defined as*

$$L^1(\Omega,\mu) := \left\{ f : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |f(x)| d\mu(x) < \infty \right\}. \tag{2.2.6}$$

We say a function is integrable if it belongs to L^1*. Furthermore, for an integrable function* $f : I \to X$ *one has the inequality*

$$\| \int_I f dx \| \leq \int_I \|f\| dx. \tag{2.2.7}$$

We also make use of the following Theorem that goes under the name of **Dominated Convergence Theorem**, which we can also slightly adjust to the case of Bochner Integrals, which we will do in chapter 4.

Theorem 2.5. *Let* $A \supset \mathbb{R}^d$ *be measurable and let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of measurable functions* $f_n : A \to \mathbb{K}$ *that converge pointwise. If there exists an integrable function* $h : A \to [0, \infty]$ *such that* $|f_n(x)| \leq |h(x)|$ *for all* $x \in A$*, then*

$$\int_A \lim_{n\to\infty} f_n(x)\, dx = \lim_{n\to\infty} \int_A f_n(x)\, dx. \tag{2.2.8}$$

This implies, in particular, that $\lim_{n\to\infty} f_n(x)$ *is integrable.*

Following the arguments presented in [JB76], we define two basic functors Δ (intersection) and Σ (sum) from $\mathcal{H}_1$ to $\mathcal{H}$, where we denote by $\mathcal{H}$ any sub-category $\mathcal{N}$ of all normed vector spaces, that is, a class of normed vector spaces together with (some of) the continuous linear maps between them, and $\mathcal{H}_1$ be a compatible couple $\bar{A} = (A_0, A_1)$ (cf. [JB76], section 2.3) as

$$\Delta(\bar{A}) = A_0 \cap A_1, \tag{2.2.9}$$

$$\Sigma(\bar{A}) = A_0 + A_1. \tag{2.2.10}$$

Definition 2.6. *Let* $\bar{A} = (A_0, A_1)$ *be a given couple in* $\mathcal{H}_1$*. Then a space* A *in* $\mathcal{H}$ *will be called an* ***intermediate space*** *between* A_0 *and* A_1 *(or with respect to* $\bar{A}$*) if*

$$\Delta(\bar{A}) \subset A \subset \Sigma(\bar{A}) \tag{2.2.11}$$

with continuous inclusions. The space A is called an ***interpolation space*** *between A_0 and A_1 (or with respect to $\bar{A}$) if in addition*

$$T : \bar{A} \to \bar{A} \text{ implies } T : A \to A, \tag{2.2.12}$$

where the mappings $T : A \to B$ are assumed to all be bounded linear operators from A to B.

Definition 2.7. *Let $\|T\|_{A,B}$ be the norm of the mapping $T : A \to B$ where A and B are interpolation spaces. A generic property of interpolation spaces is*

$$\|T\|_{A,B} \leq C \max\left(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\right) \tag{2.2.13}$$

(cf. section 2.4 of [JB76]). If additionally we have that $C = 1$, we shall say that A and B are ***exact interpolation spaces.***

For use in section 4.2 we note the

Definition 2.8. *A* ***vector lattice*** *(also called* ***Riesz space****) is a real vector space X equipped with a partial order $\leq$ such that:*

1. *$(X, \leq)$ is a lattice, meaning:*

$$\forall x, y \in X, \quad \sup\{x, y\} \text{ and } \inf\{x, y\} \text{ exists.} \tag{2.2.14}$$

2. *The order is compatible with the vector space operations:*

$$x \leq y \implies x + z \leq y + z, \quad z \in X, \tag{2.2.15}$$

$$x \leq y \implies \alpha x \leq \alpha y \quad \forall \alpha \geq 0. \tag{2.2.16}$$

A normed vector lattice $(X, \|\cdot\|)$ is said to have ***order continuous norm*** *if for a sequence $(f_n)_{n\in\mathbb{N}} \in X$*

$$f_n \downarrow 0 \implies \|f_n\| \to 0. \tag{2.2.17}$$

Here $f_n \downarrow 0$ means that

$$f_{n+1} < f_n, \forall n \in \mathbb{N} \text{ and } \inf_{n\in\mathbb{N}} f_n = 0. \tag{2.2.18}$$

Definition 2.9 (Landau-Symbols)**.** *Let* $f, g : \mathbb{N} \to \mathbb{R}$*. We write*

$$f = \mathcal{O}(g) \tag{2.2.19}$$

or

$$f(n) = \mathcal{O}(g(n)) \tag{2.2.20}$$

if there exists a constant C *and a starting point* $n_1 \in \mathbb{N}$*, such that for all* $n \in \mathbb{N}, n \geq n_1$*, it holds that* $|f(n)| \leq Cg(n)$

3 Pricing formulas

In this chapter, we focus on the mathematical derivation of option pricing formulas starting from the Black–Scholes partial differential equation. We begin by transforming the Black–Scholes equation into a heat equation, a classical approach that allows us to leverage well-understood techniques from parabolic partial differential equations. Following this transformation, we investigate how different payoff structures, encoded as initial conditions in the heat equation, determine the resulting option prices. In particular, we consider the standard European call options, as well as binary options, including cash-or-nothing and asset-or-nothing contracts. For each case, we derive a pricing formula in closed form, illustrating how the choice of initial condition directly reflects the financial structure of the option.

3.1 Transformation to the Heat Equation

We start from the Black–Scholes equation (2.1.9) for an options price function $V(t, S(t))$:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{3.1.1}$$

defined for $S > 0$ and $0 \leq t \leq T$. We define by $V(T, S(T)) := \psi(S(T))$ the terminal condition of the option, which specifies the value of the option at maturity. The Black–Scholes PDE evolves backward in time (from the terminal condition $\psi(S(T))$ toward earlier times $t < T$). In the following we will assume σ and r to be constant. We introduce the following transformations of variables:

$$t = T - \frac{2\tau}{\sigma^2} \tag{3.1.2}$$

with $\tau \in \left[0, \frac{1}{2}\sigma^2 T := \tau_{\max}\right]$,

$$S = Ke^{x} \implies x = ln\left(\frac{S}{K}\right). \tag{3.1.3}$$

Here the variable K represents the so-called **strike price**. The strike price is the fixed price at which the holder of an option can buy (for a **call**) or sell (for a **put**) the underlying asset when the option is exercised. Forming the necessary partial differentials:

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial V}{\partial \tau}\left(\frac{-\sigma^2}{2}\right) \tag{3.1.4}$$

and

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\frac{1}{S} \implies S\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}, \tag{3.1.5}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 V}{\partial x^2}\left(\frac{1}{S}\right)^2 - \frac{\partial V}{\partial x}\frac{1}{S^2} \implies S^2\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} \tag{3.1.6}$$

we get the new equation:

$$\frac{-\sigma^2}{2}\frac{\partial V}{\partial \tau} + r\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right) - rV = 0. \tag{3.1.7}$$

To further simplify this equation we introduce the parameter $\lambda := \frac{2r}{\sigma^2}$, thus allowing us to turn the above equation into

$$-\frac{\partial V}{\partial \tau} + (\lambda - 1)\frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial x^2} - \lambda V = 0. \tag{3.1.8}$$

Finally we let

$$V = Ku(\tau, x) \tag{3.1.9}$$

for some unknown function u. Now, by the homogeneity of (3.1.8), we can multiply by K to obtain

$$\frac{\partial u}{\partial \tau} = (\lambda - 1)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} - \lambda u. \tag{3.1.10}$$

This is a typical diffusion equation with drift coefficient $(\lambda - 1)$ and additional decay term $-\lambda u$. One particular choice for u allows us to recover the sought after heat equation.

Namely, we let

$$u\left(\tau,x\right)=e^{ax+b\tau}\cdot v\left(\tau,x\right). \tag{3.1.11}$$

After some straightforward differentiation we get the expression:

$$\frac{\partial u}{\partial\tau}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\left[\lambda-1+2a\right]+\left[a\left(\lambda-1\right)-\lambda+1-b\right]u. \tag{3.1.12}$$

We now set $a=-\frac{(\lambda-1)}{2}$ and $b=-\frac{1}{2}\left(\lambda-1\right)^2-\left(\lambda-1\right)$ to obtain

$$\frac{\partial u}{\partial\tau}=\frac{\partial^2 u}{\partial x^2}\quad\text{for } x\in\mathbb{R},\tau\in\left[0,\tau_{\max}\right]. \tag{3.1.13}$$

This is the heat equation.

3.2 European Options

We now focus our attention to the case where $\psi\left(S\left(T\right)\right)=\max\left(S-K,0\right)$. This is the payoff for a **European call option**. The buyer of the option either earns the difference of the stock price and the strike price if the stock ends above the strike at maturity or nothing. At $t=T$ we have $\tau=0$. Plugging this into (3.1.11) we get

$$u\left(0,x\right)=e^{-\frac{(\lambda-1)x}{2}}v\left(0,x\right)=\frac{\psi\left(S\left(T\right)\right)}{K} \tag{3.2.1}$$

$$\implies v\left(0,x\right)=e^{\frac{(\lambda-1)x}{2}}\frac{\psi\left(S\left(T\right)\right)}{K}. \tag{3.2.2}$$

Now, remembering our transformation of S in (3.1.3) we have

$$\psi\left(S\left(T\right)\right)=\max\left(S-K,0\right)=K\max\left(e^x-1,0\right) \tag{3.2.3}$$

thus allowing us to formulate the initial-value problem for the transformed Black–Scholes equation in the case of an European option as

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \tau \in [0, \tau_{\max}], \\ u(0,x) = \max\left(e^{\frac{(\lambda+1)x}{2}} - e^{\frac{(\lambda-1)x}{2}}, 0\right). \end{cases} \tag{3.2.4}$$

Now we are ready to reap the benefits of our earlier work and simply use the fundamental solution to the heat equation, as introduced for example in chapter 2 of [DM21], to derive a pricing formula. The fundamental solution being

$$u(\tau, x) = \frac{1}{2\sqrt{\pi\tau}} \int_{\mathbb{R}} u(0,s)\, e^{\frac{-(x-s)^2}{4\tau}} ds. \tag{3.2.5}$$

We now let

$$y := \frac{s-x}{\sqrt{2\tau}} \implies dy = \frac{ds}{\sqrt{s\tau}}, \tag{3.2.6}$$

allowing us to write the right-hand side of (3.2.5) as

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u\left(0, x + \sqrt{2\tau} y\right) e^{\frac{-y^2}{2}} dy. \tag{3.2.7}$$

If we now assume $y > \frac{-x}{\sqrt{2\tau}}$, meaning that the payoff is non-zero at maturity, we get the following expression

$$\frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{\frac{(\lambda+1)(x+\sqrt{2\tau}y)}{2}} e^{\frac{-y^2}{2}} dy - \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{\frac{(\lambda-1)(x+\sqrt{2\tau}y)}{2}} e^{\frac{-y^2}{2}} dy. \tag{3.2.8}$$

After some straightforward but space-consuming manipulations of the two integrals we can solve them and get the following expression

$$e^{\frac{(\lambda+1)x}{2}+\frac{(\lambda+1)^2\tau}{4}} \mathcal{N}\left(\frac{x}{\sqrt{2\tau}} + \frac{(\lambda+1)\sqrt{2\tau}}{2}\right) - e^{\frac{(\lambda-1)x}{2}+\frac{(\lambda-1)^2\tau}{4}} \mathcal{N}\left(\frac{x}{\sqrt{2\tau}} + \frac{(\lambda-1)\sqrt{2\tau}}{2}\right), \tag{3.2.9}$$

where $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-z^2}{2}} dz$ is the normal distribution as introduced for example in [Eic22]. Now, all that is left to do, is unwrap this equation in terms of our original variables. We define

$$d_+ := \frac{ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{3.2.10}$$

$$d_- := \frac{ln\left(\frac{S(t)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3.2.11}$$

to obtain the following

Theorem 3.1 (The Black–Scholes Formula). *Let S follow a geometric Brownian motion with volatility σ and interest rate r. The Black-Scholes price at time 0 of a European call with strike K and maturity T can be represented as*

$$V(t, S(t)) = S(t)\mathcal{N}(d_+) - Ke^{-r(T-t)}\mathcal{N}(d_-). \tag{3.2.12}$$

3.3 Binary Options

Binary options, also known as digital options, are a class of financial derivatives that provide a fixed, predetermined payoff depending on whether an underlying asset meets a specified condition at maturity. In contrast to standard European call and put options, which yield a continuous payoff proportional to the difference between the asset price and the strike price, binary options pay out either a fixed amount or nothing at all. In the following we will look at two such options, starting with

3.3.1 Cash-or-nothing options

This type of option pays off a fixed amount A if the underlying asset price S at maturity exceeds the strike price K and zero otherwise. We can denote this particular payoff structure by

$$\psi(S(T)) = A\mathcal{H}(S(T) - K), \tag{3.3.1}$$

where $\mathcal{H}(.)$ is known as the **Heaviside function**. The Heaviside function is defined as follows

$$\mathcal{H}(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x \geq 0. \end{cases} \tag{3.3.2}$$

We also state the following properties of the Heaviside function for later reference.

Lemma 3.2. *Let $\alpha \in \mathbb{R}$. Then one of the three alternatives hold:*

$$\begin{cases} \mathcal{H}(\alpha x) = \mathcal{H}(x) & \textit{if} \quad \alpha > 0, \\ \mathcal{H}(\alpha x) = -\mathcal{H}(x) & \textit{if} \quad \alpha < 0, \\ \mathcal{H}(\alpha x) = 1 & \textit{if} \quad \alpha = 0. \end{cases} \tag{3.3.3}$$

Proof. In all three cases we just have to look at how the variable x is affected by the multiplication with the parameter α. The equality then follows from how the function is defined. □

To obtain the initial condition for this type of option we first note that with Lemma 3.2:

$$\psi(S(T)) = A\mathcal{H}(S(T) - K) = A\mathcal{H}(Ke^x - K) = A\mathcal{H}(e^x - 1) \tag{3.3.4}$$

and thus replicating the calculations in (3.2.2) to get the representation for $u(0, x)$, we get the initial-value problem for the transformed Black–Scholes equation in the case of a cash-or-nothing option as

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \tau \in [0, \tau_{\max}], \\ u(0, x) = \frac{A}{K} e^{\frac{(\lambda-1)x}{2}} \mathcal{H}(e^x - 1). \end{cases} \tag{3.3.5}$$

Theorem 3.3. *The solution to (3.3.5) describes the price of a* ***cash-or-nothing option*** *with interest rate r, strike K and maturity T. We get the closed formula*

$$\hat{V}(t, S(t)) = Ae^{-r\tau}\mathcal{N}(d_-). \tag{3.3.6}$$

3.3.2 Asset-or-nothing options

This type of option is a type of binary option whose payoff depends on whether the underlying asset price at maturity exceeds a predetermined strike price. Formally, the payoff of an European asset-or-nothing call option with strike K and maturity T is given by

$$\psi(S(T)) = S(T)\,\mathcal{H}(S(T) - K)\,. \tag{3.3.7}$$

One immediately sees the resemblance to the cash-or-nothing payoff structure in (3.3.1) and, incidentally, formally repeating the same arguments as in the previous section, under the assumption that (3.1.3) holds, we arrive at the initial condition for the transformed Black–Scholes PDE:

$$u(0, x) = e^{\frac{(\lambda+1)x}{2}}\mathcal{H}(S(T) - K)\,. \tag{3.3.8}$$

Solving the heat equation with this initial condition leads us to a formula for the price of an asset-or-nothing option at any time $t \geq 0$:

Theorem 3.4. *The solution to* (3.3.5) *with initial condition* (3.3.8) *describes the price of an* ***asset-or-nothing option*** *with interest rate* r*, strike* K *and maturity* T*. We get the closed formula*

$$\bar{V}(t, S(t)) = S(t)\,\mathcal{N}(d_+)\,. \tag{3.3.9}$$

We note an interesting connection between the types of options we so far investigated. Citing from chapter 26.10, page 623 of [Hul15]:

> „A regular European call option is equivalent to a long position in an asset-or-nothing call and a short position in a cash-or-nothing call where the cash payoff in the cash-or-nothing call equals the strike price.“

We thus get the relation:

European option = (asset-or-nothing) - K × (cash-or-nothing)

which we can confirm using our formulas in (3.2.12), (3.3.6) and (3.3.9) and assuming A to be one unit of cash.

4 Semigroup Formulation and Spectral Analysis

The aim of this chapter is to investigate the Black–Scholes differential operator as the generator of a strongly continuous semigroup on a suitable Banach space of functions. This viewpoint embeds the pricing dynamics into the general theory of linear evolution equations of the form

$$\frac{du}{dt}(t) = Au(t), \qquad u(0) = u_0, \tag{4.0.1}$$

where A denotes a (possibly unbounded) operator on a Banach space X.

Viewing the equation through this semigroup framework provides a unifying and conceptually elegant description: the solution e^{tA} represents the propagation of prices over time, and the properties of the generator A directly determine the qualitative behavior of the pricing dynamics.

Having established an explicit semigroup representation, we proceed to study the spectrum of the Black–Scholes operator.

4.1 An explicit formula for the C_0-semigroup

Let us first define the Black-Scholes differential operator as

$$Au := \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru, \tag{4.1.1}$$

which allows us to write (2.1.9) as an abstract Cauchy Problem:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + Au(t,x) = 0, & (x,t) \in (0,\infty) \times [0,\tau], \\ u(0,x) = \psi(x). \end{cases} \tag{4.1.2}$$

If the operator A now generates a strongly continuous semigroup on a suitable Banach space X, and $\psi \in D(A)$, then the solution to (4.1.2) can formally be expressed as $u(t) = e^{(T-t)A}\psi$, which would then naturally lead us to a formula for the price of an option, namely:

$$V(t,x) = \left(e^{(T-t)A}\psi\right)(x). \tag{4.1.3}$$

We will follow the arguments presented in [Ein08], chapter 7.3, and look at the Banach space (equipped with the supremum norm $\|.\|_\infty$)

$$X := C_0([0,\infty)) = \left\{ f \in C([0,\infty)); \mathbb{R} | \lim_{x\to\infty} f(x) = 0 \right\}. \tag{4.1.4}$$

The strategy to establish existence of a holomorphic semigroup T in $C_0([0,\infty)])$ generated by A, is to write A as $\alpha\left((B+\gamma)^2 - (1+\gamma)^2\right)$, where B is the generator of a C_0-group and $\gamma \in \mathbb{R}, \alpha > 0$. In order to prove this fact, one first introduces the C_0-group

$$(G(t)f)(x) := f\left(e^t x\right) \tag{4.1.5}$$

for $t \in \mathbb{R}, x > 0$ and $f \in X$.

Proposition 4.1. *The generator of* (4.1.5) *is*

$$Bf := \phi Df, \quad D(B) := \left\{ f \in X \cap C^1(0,\infty) | \phi Df \in X \right\}, \tag{4.1.6}$$

where $\phi(x) := x, x > 0$ is the identity function.

Proof. Let us verify that G in (4.1.5) fulfills the group laws from 2.2 and that it is strongly continuous. We refer to [Ein08], proposition 7.3.1 for a proof that B really is the generator of G.

$$(G(t)G(s)f)(x) = G(t)(y \to f(e^s y))(x) = f\left(e^s e^t x\right) = f\left(e^{t+s}x\right) = (G(t+s)f)(x) \tag{4.1.7}$$

and

$$(G(0)f)(x) = f\left(e^0 x\right) = f(x), \tag{4.1.8}$$

so $G(0) = \mathrm{Id}$. We also observe that

$$\|G(t) f\|_\infty = \sup_{x\geq 0} \left|f\left(e^t x\right)\right| = \sup_{y\geq 0} |f(y)| = \|f\|_\infty \tag{4.1.9}$$

because as x ranges over $[0,\infty)$, so does $y := e^t x$. Therefore $G(t)$ is an isometry and $\|G(t)\| = 1$. To show that $t \to G(t) f$ is strongly continuous, Einemann introduces a dense subspace[1] of $C_0([0,\infty))$, $C_c([0,\infty))$, which denotes all continuous functions with compact support in $[0,\infty)$. $C_c([0,\infty))$ is dense in $C_0([0,\infty))$ equipped with the supremum norm as follows from the standard functional analytic construction of $C_0([0,\infty))$ as the completion of $C_c([0,\infty))$ with respect to the supremum norm, i.e.

$$C_0([0,\infty)) = \overline{C_c([0,\infty))}^{\|\cdot\|_\infty}. \tag{4.1.10}$$

Equivalently, for every $f \in C_0([0,\infty))$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c([0,\infty))$ such that $\|f_n - f\|_\infty \to 0$.

Now, with $f \in C_c$ we get

$$\|G(t) f - f\|_\infty = \sup_{x\geq 0} \left|f\left(e^t x\right) - f(x)\right| = \left|f\left(e^t x_0\right) - f(x_0)\right| \to 0 \;\text{ for } t \to 0, \tag{4.1.11}$$

where $x_0 \in$ supp(f) and since $f \in C_c$, f is uniformly continuous and the above approximation holds. We have thus shown that $G(t) f$ is strongly continuous. $\square$

Once the group properties and strong continuity are established, the next step is to identify its generator, which we, without going into the details of the proof, assume to really be B from (4.1.6).

As earlier mentioned, the Black–Scholes operator turns out to be a shifted and scaled version of B^2. To make sense of the square of a C_0-group generator, we have the following:

Theorem 4.2. *Let C be the generator of a bounded C_0-group $G = (G(t))_{t\in\mathbb{R}}$ on X. Then the operator C^2 with domain $D(C^2) := \{x \in D(C) \,|Cx \in D(C)\}$ generates a bounded holomorphic C_0-semigroup T of angle $\frac{\pi}{2}$ on X given by*

$$T(t)x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} G\left(s\sqrt{2t}\right) x ds, \quad t \geq 0, x \in X. \tag{4.1.12}$$

[1]for the definition of dense we refer to page 290 of [DM21]

Proof. We start by showing that $T(t)x$ is well defined. For that, we observe that the mapping $t \to G(t)\,x$ is strongly continuous (since G is a C_0-group) and the mapping $s \to s\sqrt{2t}$ is continuous on $\mathbb{R}$. As a composition of continuous maps, $s \to G\left(s\sqrt{2t}\right)x$ is continuous on X. Let us now define $F(s) := e^{-\frac{1}{2}s^2} G\left(s\sqrt{2t}\right)x, \;\; t \geq 0, x \in X$. This function, being continuous, is also measurable. Next we claim that

$$\int_{-\infty}^{\infty} \|F(s)\| ds < \infty. \tag{4.1.13}$$

Because G is an isometry, as established earlier, we have that

$$\|F(s)\| = \|e^{-\frac{1}{2}s^2} G\left(s\sqrt{2t}\right)x\| = e^{-\frac{1}{2}s^2}\|G\left(s\sqrt{2t}\right)x\| = e^{-\frac{1}{2}s^2}\|x\|, \tag{4.1.14}$$

leading us directly to the finiteness of (4.1.13). With proposition 2.4 we conclude that $T(t)\,x \in L^1(X)$. In fact, again with proposition 2.4 we can also get the norm estimate

$$\|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} G\left(s\sqrt{2t}\right)x ds\| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\|F(s)\|ds = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi}\|x\| = \|x\|, \tag{4.1.15}$$

leading to $\|T(t)\| \leq 1$. Let us move on to the verification of the semigroup properties in definition 2.2. We quickly verify that

$$T(0)\,x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} G(0)\,x ds = G(0)\,x\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} ds = x. \tag{4.1.16}$$

To show strong continuity at $t = 0$, we fix x and s and observe that $s\sqrt{2t} \to 0$. Now, because G is strongly continuous, we have that $G(t)\,x \to x$ as $t \to 0$, implying $G\left(s\sqrt{2t}\right)x \to x$. Multiplying by a scalar $e^{-\frac{1}{2}s^2}$ gives us pointwise convergence

$$F_t(s) \to e^{-\frac{1}{2}s^2}x \quad \text{as} \quad t \to 0. \tag{4.1.17}$$

Using our bound from (4.1.14), we can use Theorem 2.5 for Bochner Integrals to establish strong continuity at $t = 0$. Moving on to showing that $T(t)T(u) = T(t+u)$. We get

$$T(t)T(u)x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{s^2+r^2}{2}} G\left(s\sqrt{2t}\right) G\left(r\sqrt{2u}\right) xdrds \tag{4.1.18}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2} e^{-\frac{s^2+r^2}{2}} G\left(s\sqrt{2t} + r\sqrt{2u}\right) xdrds, \tag{4.1.19}$$

where we have used the fact that $G(t)G(u) = G(t+u)$. Performing the change of variables $z := s + r$ and using properties of the gaussian kernel, we consequently get

$$T(t)T(u)x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} G\left(z\sqrt{2\,(t+u)}\right) x dz = T(t+u)x. \tag{4.1.20}$$

Thus, the semigroup properties hold and T defines a C_0-semigroup. Next we want to verify that its generator is C^2, namely that

$$\lim_{t \downarrow 0} \frac{T(t)x - x}{t} = C^2 x. \tag{4.1.21}$$

The key point in the proof is that the semigroup $T(t)$ is obtained by averaging the group $G(t)$ against a symmetric Gaussian kernel. Since C generates a C_0-group, vectors in $D\left(C^2\right)$ have group orbits that are twice differentiable at time zero, where we denote the mapping $t \to G(t)x$ as a group orbit. For such vectors, one can expand the group orbit near zero up to second order. When this expansion is inserted into (4.1.12), the first-order term disappears because the Gaussian is symmetric, while the second-order term survives and produces exactly C^2. The remaining higher-order terms vanish in the limit as $t \to 0$, thanks to dominated convergence and the boundedness of the group. □

Having established the integral expression for T, we go on to proof the main theorem of the paper [Ein08] (Theorem 7.3.2), namely:

Theorem 4.3. *The operator* $(\alpha A, D\,(A))$ *generates a bounded holomorphic* C_0*-semigroup* $T = (T(t))_{t \geq 0}$ *of angle* $\frac{\pi}{2}$ *on* X *where* T *is given by*

$$\begin{aligned}(T(t)f)\,(x) &:= \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} \left(G\left(s\sqrt{2\alpha t} + (\beta - \alpha)\,t\right) f\right)(x) ds \\ &= \frac{e^{-\beta t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} f\left(e^{s\sqrt{2\alpha t} + (\beta-\alpha)t} x\right) ds\end{aligned} \tag{4.1.22}$$

for $t \geq 0, f \in X$ *and* $x \geq 0$.

Proof. Thanks to our earlier work, we can show that the operator $(B + \gamma, D\,(B))$ generates a C_0-semigroup $(e^{\gamma t} G(t))_{t \in \mathbb{R}}$. By Theorem 4.2 the operator $\left((B+\gamma)^2, D\,(B^2)\right)$ generates the C_0-semigroup $T_1 = (T_1(t))_{t \geq 0}$ on X with

$$T_1(t)f = \int_{-\infty}^{\infty} \phi(s) e^{\gamma s \sqrt{2t}} G\left(s\sqrt{2t}\right) f dx, \quad t \geq 0, f \in X, \tag{4.1.23}$$

where $\phi(s) := \frac{1}{2\sqrt{2}} e^{-\frac{1}{2}s^2}, s \in \mathbb{R}$. We also observe that

$$D\left(B^2\right) = D\left(A\right). \tag{4.1.24}$$

Remember that in the beginning of this chapter we ventured to write A in the following way

$$A = (B+\gamma)^2 - (1+\gamma)^2. \tag{4.1.25}$$

Thus, with Theorem 4.2 and (4.1.23) we get that A, and thus the Black–Scholes differential operator, generates a bounded holomorphic C_0-semigroup $T_2 = (T_2(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$ on X represented by

$$T_2(t)f := e^{-t(1+2\gamma)} \int_{-\infty}^{\infty} \phi G\left(s\sqrt{2t} + 2\gamma t\right) f ds. \tag{4.1.26}$$

Now, letting $\frac{\beta}{\alpha} = 2\gamma + 1$, we arrive at the fact that αA generates a bounded holomoprhic C_0-semigroup represented by $T(t) := T_2(\alpha t)$ and thus for $f \in X$ we have:

$$T(t)f = e^{-\beta t} \int_{-\infty}^{\infty} \phi(s) G\left(s\sqrt{2\alpha t} + (\beta - \alpha)t\right) f ds. \tag{4.1.27}$$

$\square$

Remark 4.4. *It is interesting to note here that by defining*
$\alpha := \frac{1}{2}\sigma^2 > 0$, $\beta := r$ *and* $\gamma := \frac{1}{2}\left(\frac{\beta}{\alpha} - 1\right) = \frac{r}{\sigma^2} - \frac{1}{2}$, *and solving* (4.1.22), *we can again derive the Black–Scholes Formula* (3.2.12) *through the semigroup approach, showing how versatile they can be applied in the real world.*

4.2 Spectral Analysis

This section investigates the long-time behavior and stability of the Black–Scholes solution $u(t)$ by performing a detailed spectral analysis of the operator B_E, as defined on an (L^1, L^∞)-**interpolation space** E (cf. Definition 2.6)

Loosely speaking, we can imagine this space to be ,,something between L^1 and L^∞", though this might not always be the case. Many spaces are included in (L^1, L^∞)-interpolation spaces such as L^p-spaces, Orlicz spaces, Lorentz spaces and Marcinkiewiecz spaces to name a few.

Following the methodology established by Arendt and de Pagter in [AdP02], the entire asymptotic stability of the system is shown to be dictated by the geometry of the function space E. The reason this is powerful is that we can establish a rigorous, comprehensive mathematical framework for the Black–Scholes partial differential equation that is valid for a wide range of derivative payoffs and initial conditions, which themselves are encoded in the underlying function space!

From here on out we will look at a slightly different operator as we previously did in section 4.1, namely:

$$Bf = x^2 f'' + xf'. \tag{4.2.1}$$

Given an (L^1, L^∞)-interpolation space E, we look only at the part of an operator in E, i.e. for B we get:

$$B_E f = Bf = x^2 f'' + xf', \tag{4.2.2}$$

$$D(B_E) = \{f \in E : Bf \in E\}. \tag{4.2.3}$$

Arendt and de Pagter make great use of the notion of **Boyd-Indices**, as defined by D.W.Boyd in [Boy69], which will later help us determine the spectrum. Let us briefly go over what they are.

Definition 4.5. *For a measurable function f on $[0,\infty)$, while for an f on $[0,1]$, and $t > 0$*

$$D_t f(x) = \begin{cases} f\left(\frac{x}{t}\right) & \text{if } x \le \min(1,t), \\ 0 & \text{if } t < x \le 1, \end{cases} \tag{4.2.4}$$

*is called the **dilation operator**.*

*The **upper and lower Boyd indices** of a **rearrangement-invariant function space**[2] X on $[0,\infty)$ or $[0,1]$ are defined by the respective formulas*

$$\overline{\alpha}_X = \lim_{t\to\infty} \frac{\log\|D_t\|}{\log t}, \quad \underline{\alpha}_X = \lim_{t\downarrow 0} \frac{\log\|D_t\|}{\log t}. \tag{4.2.5}$$

[2]cf. p3 of [AdP02]

The indices effectively give information about how the norm of the space E scales when a function is dilated.

The idea now is to identify the operator B_E with the generator of a dilation group whose spectrum we know and using that spectrum to deduce information about the long-time behavior of the Black–Scholes operator (4.2.2). Let us define a **one-paramter group of dilations** $(T_E(t))_{t\in\mathbb{R}}$ on E by

$$(T_E(t)f)(x) = f\left(e^{-t}x\right) \tag{4.2.6}$$

for all $f \in E, t \in \mathbb{R}, x > 0$ (One notes the similiarity with the C_0-group $G(t)$ in (4.1.5)). Let the generator of (4.2.6) be A_E, more concretely:

$$A_E f = -xf', \tag{4.2.7}$$

$$D(A_E) = \{f \in E : xf' \in E\}, \tag{4.2.8}$$

then the spectrum of A_E can be shown to be completely determined by the Boyd indices, forming a vertical strip in the complex plane:

$$\sigma(A_E) = \{\lambda \in \mathbb{C} : \underline{\alpha}_E \leq \mathrm{Re}\lambda \leq \overline{\alpha}_E\}. \tag{4.2.9}$$

For the proof we refer to [AdP02], Theorem 4.2. The main insight of the paper then is the

Proposition 4.6.

$$B_E = (A_E)^2, \tag{4.2.10}$$

where A_E is the genererator of (4.2.6).

Proof. We verify that

$$A_E^2 f(x) = -x\frac{d}{dx}\left(-xf'(x)\right) \tag{4.2.11}$$
$$= -x\left[\frac{d}{dx}(-x)\cdot f'(x) + (-x)\cdot\frac{d}{dx}\left(f'(x)\right)\right] \tag{4.2.12}$$
$$= -x\left[-1\cdot f'(x) - x\cdot f''(x)\right] \tag{4.2.13}$$
$$= xf'(x) + x^2 f''(x). \tag{4.2.14}$$

Comparing with (4.2.1) concludes the part of the proof showing that both differential operators are identical. To show that their domains match as well is more technically challenging and is done by demonstrating that the functions that satisfy the domain conditions for A_E^2 are exactly the same functions for which the expression $x^2 f''(x)+xf'(x)$ exists in a suitable sense and the resulting function is an element of the space E. □

Having established this link between the operator and the generator, we derive the spectrum of B_E from $\sigma(A_E)$.

Corollary 4.7. *The spectrum of B_E is*

$$\sigma(B_E) = \left\{r + is : \underline{\alpha}_E^2 - \frac{s^2}{4\underline{\alpha}_E^2} \le r \le \overline{\alpha}_E^2 - \frac{s^2}{\overline{\alpha}_E^2}\right\}; \tag{4.2.15}$$

i.e., $\sigma(B_E)$ is the region between two parabolas.

Proof. Since $\sigma(B_E) = \sigma(A_E)^2$ we get with (4.2.9):

$$\sigma(B_E) = \left\{\alpha^2 + 2\alpha\beta i - \beta^2 : \beta\in\mathbb{R}, \underline{\alpha}_E \le \alpha \le \overline{\alpha}_E\right\} \tag{4.2.16}$$
$$= \left\{\alpha^2 - \frac{s^2}{4\alpha^2} + is : s\in\mathbb{R}, \underline{\alpha}_E \le \alpha \le \overline{\alpha}_E\right\}. \tag{4.2.17}$$

□

Quoting directly from [AdP02]:

> „Thus the spectrum of B_E varies very much as a function of the (L^1, L^∞)-interpolation space.“

As we have already established in section 4.1, Theorem 4.3, the Black–Scholes differential operator generates a bounded holomorphic C_0-semigroup of angle $\frac{\pi}{2}$, which also holds true if we restrict ourselves to interpolation spaces. To study the long-time behavior of such a semigroup we make use of the following

Definition 4.8. *The exponential type* $\omega(T)$ *of a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* X *is defined as the infimum of all real numbers* ω *such that the norm of the semigroup grows at most exponentially with rate* ω*, ie.:*

$$\omega(T) = \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0}\|e^{-\omega t}T(t)\| < \infty\right\}. \tag{4.2.18}$$

Proposition 4.9. *If we assume* E *to be an* (L^1, L^∞)*-interpolation space with* ***order continuous norm*** *(cf. Definition 2.8), then the operator* B_E *generates a holomorphic* C_0*-semigroup* V_E *on* E *of angle* $\frac{\pi}{2}$*. The exponential type* $\omega(V_E)$ *of* V_E *is given by*

$$\omega(V_E) = (\overline{\alpha}_E)^2 \tag{4.2.19}$$

and the semigroup V_E *is strongly continuous.*

For the proof we refer to Theorem 5.3 of [AdP02].

This relationship is a direct consequence of the connection $B_E = (A_E)^2$. If, on the other hand, E is **not** order continuous, the semigroup V_E is no longer strongly continuous. Still it is possible to associate a generator A_E to T_E without any further condition on the space and we can still leverage our knowledge about A_E's spectrum.

It is worth to mention here the exceptional smoothing properties of holomorphic (analytical) semigroups. Let $D(A^k)$ be a Banach space (here $k \in \mathbb{N}_0$) for the norm $\|x\|_{D(A^k)} = \|x\| + \|Ax\| + \cdots + \|A^k x\|$, and A be the generator of a semigroup T, then:

$$\frac{d}{dt}T(t)\,x = AT(t)x, \quad (t > 0) \tag{4.2.20}$$

and

$$T(\cdot)\,x \in C^\infty\left((0,\infty), D\left(A^k\right)\right), \tag{4.2.21}$$

where (4.2.21) is not necessarily true if T is not holomorphic (analytic).

Putting all we have established so far together allows us to show well-posedness of the Black-Scholes partial differential equation and deduce long-time behavior of the associated operators spectrum. Furthermore we are enabled to study asymptotics in (and caused by the choice of) the space E. As a sidenote we introduce here the

Definition 4.10. *Let E be an interpolation space and E_n'*[3] *its Köthe dual. Then $\sigma(E, E_n')$ is called the **weak topology** on an interpolation space E induced by its Köthe dual E_n'. Each $g \in E_n'$ is identified with a linear functional ψ_g on E via the dual pairing $\langle f, \psi_g \rangle = \int_0^\infty fg dx$. Then $\sigma(E, E_n')$ is the coarsest topology on E such that all functionals ψ_g are continuous.*

Theorem 4.11. *Let E be an exact (L^1, L^∞)-interpolation space (cf. Definition 2.7) with Köthe dual E_n'. Let $f \in E$, $u(t) = V_E(t)f$. Then u is the **unique solution** of the Cauchy problem*

$$\begin{cases} u \in C^1\left((0,\infty); E\right), \quad u(t) \in D\left(B_E\right) \quad \text{for } t > 0, \\ \frac{\partial}{\partial t} u(t) = B_E u(t) \qquad \text{for } t > 0, \\ \lim_{t \downarrow 0} u(t) = f \quad \text{for } \sigma\left(E, E_n'\right). \end{cases} \tag{4.2.22}$$

Moreover, if we put $u(t,x) = \left(V_E(t)f\right)(x) = u(t)(x)$, then $u \in C^\infty(0,\infty) \times (0,\infty)$ and

$$u_t = x^2 u_{xx} + x u_x \qquad (t > 0, x > 0). \tag{4.2.23}$$

This is Theorem 5.8 of [AdP02].

Proof. We know that u is a solution to (4.2.22). To show it is uniqueness we make us of the fact that the operator B_E generates a holomorphic semigroup V_E on an exact (L^1, L^∞)-interpolation space E. This semigroup then explicitly provides the solution to the equation as $u(t,x) = \left(V_E(t)f\right)(x)$. The key challenge in the proof is that V_E is not always strongly continuous in the norm of E unless E has order continuous norm. To ensure uniqueness in the general case, we can show that V_E is always continuous with respect to $\sigma(E, E_n')$. This formulation ensures that for any initial data $f \in E$, there is exactly one function u that satisfies the differential equation.

Coming to the regularity results. We know that if $f \in D(A_E)$ we have $f \in L^1_{loc}(0,\infty)$ and $xf' \in E \subset L^1_{loc}(0,\infty)$. Hence $f \in C(0,\infty)$. We then use induction to show that

[3] the space of all functionals given by a measurable function, cf. p2 of [AdP02]. Formally: $E_n' = \left\{g \in L^0(0,\infty) : \int_0^\infty |fg| dx < \infty \forall f \in E\right\}$. Quoting directly from [JB76], p63: Let supp$f$ denote the support of f, i.e. any measurable set E, such that $f = 0$ outside E and $f \neq 0$ almost everywhere on E. We thus define L^0 to be the space of all measurable functions f, s.t $\|f\|_{L_0} = \mu(\text{supp} f) < \infty$, with μ being a measure.

the solution $u(t,\cdot)$ remains in the domain of higher powers of the generator. Namely

$$D\left(A_E^{k+1}\right) \subset C^k(0,\infty) \text{ for all } k \in \mathbb{N}. \tag{4.2.24}$$

If the initial data f is in the initital domain of B_E then by (4.2.21) and (4.2.10) we get that

$$V_E(\cdot) f \in C^\infty\left((0,\infty), D\left(B_E^k\right)\right) = C^\infty\left((0,\infty); D\left(A_E^{2k}\right)\right) \text{ for all } f \in E. \tag{4.2.25}$$

From that we can deduce that $u \in C^\infty(0,\infty)\times(0,\infty)$. This then concludes the proof. □

Let us summarize our findings.

- **Well-posedness:** The Black–Scholes operator B_E generates a holomorphic semigroup V_E on E. This ensures that for any initial condition $f \in E$, there exists a unique solution to the equation $u_t = x^2 u_{xx} + x u_x$ where $t > 0, x > 0$. This formualtion is to be understood in the terms of $\sigma(E, E_n')$-continuity, allowing for valid solutions even in spaces where traditional strong continuity might fail.
- **Regularity:** Because V_E is a holomorphic semigroup, the solution $u(t,\cdot)$ enjoys significant smoothing for $t > 0$. However, the continuity of the solution as $t \downarrow 0$ depends strictly on the space E: the solution is strongly continuous if and only if E has an order continuous norm. In all other cases, it remains continuous only in the weaker topology induced by the Köthe dual E_n'.
- **Asymptotics and spectrum:** The asymptotic behavior and exponential growth bounds of the solution are entirely determined by the Boyd indices $\underline{\alpha}_E$ and $\overline{\alpha}_E$. The spectrum of the underlying generator A_E is the vertical strip $\{\lambda \in \mathbb{C} : \underline{\alpha}_E \leq \text{Re}\lambda \leq \overline{\alpha}_E\}$, wich in turn determines the spectral properties of the Black–Scholes operator B_E.

To conclude this section, let us look at three particular choices for the space E and the consequences for the solution to (4.2.23):

Example 4.12. ***1.*** *$E = L^p$ with $(1 < p < \infty)$: In these standart spaces, the Boyd indices are given by*

$$\underline{\alpha}_E = \overline{\alpha}_E = \frac{1}{p}. \tag{4.2.26}$$

Because L^p has order continuous norm, the solution is strongly continuous at $t = 0$.

2. *$E = L^p \cap L^q(0,\infty), 1 \leq p \leq q \leq \infty$, (equipped with the norm $\|f\|_E = max(\|f\|_p, \|f\|_q)$):*

then

$$\underline{\alpha}_E = \frac{1}{q}, \quad \overline{\alpha}_E = \frac{1}{p}, \tag{4.2.27}$$

which looking back at (4.2.15) *creates a broader strip for the spectrum of* B_E.

3. $E = L^\infty$*: In this instance, the solution* $u(t, \cdot)$ *does not converge to the initial data* f *in the norm of the space as* $t \to 0$*. Instead, convergence occurs pointwise almost everywhere and in the weak sense of the Köthe dual, illustrating the necessity of the generalized framework introduced in [AdP02].*

5 American options and numerical methods

Unlike European options, American options may be exercised at any time before maturity. This additional flexibility leads to a fundamentally different mathematical structure: The option value is no longer determined by a terminal-value problem for the Black–Scholes equation alone, but gives rise to a free-boundary problem, where the region in which the equation is satisfied is itself unknown.

In this chapter, we study the pricing of American options within the Black–Scholes framework with an emphasis on numerical methods. We first describe how early exercise divides the domain into a continuation region and a stopping region and explain the role of the associated early-exercise boundary. Rather than computing this boundary explicitly, the pricing problem is reformulated in terms of differential inequalities.

This reformulation leads to a linear complementarity problem, which incorporates both the Black–Scholes dynamics and the early-exercise constraint in a single framework. After introducing this formulation, the problem is discretized using finite-difference methods.

5.1 Free-Boundary Problems

In the following we will additionally assume a continuous flow of dividends in the Black–Scholes model, meaning that the stock price is assumed to continuously lose a fixed percentage of its value over time to dividend payments. The constant dividend rate is denoted by $\delta \geq 0$. With this in mind, one can re-derive the Black–Scholes partial differential equation in (2.1.9) as

$$\frac{\partial V}{\partial t} + (r - \delta)\, S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{5.1.1}$$

We denote by $V^{\cdot}(t,S)$ the solution to the initial value problem

$$\begin{cases} (5.1.1) \quad S>0, t\in[0,T], \\ V^{\cdot}(0,S)=\psi(S(T)), \end{cases} \tag{5.1.2}$$

where $V^{Am}(t,S)$ represents the price of an American option and $V^{Eu}(t,S)$ the price of an European option respectively. The assumption $\delta>0$ is important.

Proposition 5.1. *If $\delta=0$ and $r>0, t<T$, we get the equality*

$$V^{Am}(t,S)=V^{Eu}(t,S). \tag{5.1.3}$$

Proof. In the case of a call-option we get the proof, as shown in [Sey15]:

$$V_C^{Am}\geq V_C^{Eu}\geq S-Ke^{T-t}>S-K=\psi\left(V^{Am}(T)\right). \tag{5.1.4}$$

This then implies $V_C^{Am}>$ payoff, resulting in a loss if exercised before the time T. □

Intuitively this means that early exercise of the American option would be unwise, collapsing the theory to the European type investigated in chapter 3.

Following the arguments presented in chapter 4.5 of [Sey15] we have the following

Theorem 5.2. *For each $t>0$ there exists a S^* such that*

(1) $V^{Am}(t,S^*)=K-S^*$,

(2) $\frac{\partial V^{Am}(t,S^*)}{\partial S}=-1$,

(3) $V^{Am}(t,S)>K-S$ *for* $S>S^*$,

(4) $V^{Am}(t,S)=K-S$ *for* $S\leq S^*$.

*This S^* is unique and defines a function, which we will denote $S_f(t)$. The "f" stands for **free boundary**.*

Proof. In the case of an European put option we have the inequality

$$V_P^{Eu}(0,t)=Ke^{-r(T-t)}<K, \tag{5.1.5}$$

which implies that there exists a S^* such that

$$V_P^{Eu}(t, S^*) = K - S^*. \tag{5.1.6}$$

To prove (1) we first assume that

$$V_P^{Am} > K - S \tag{5.1.7}$$

for all $S > 0$. Then

$$-V_P^{Am} + K - S < 0, \tag{5.1.8}$$

meaning that exercising the put would lead to a loss for all $S > 0$. Thus, as before mentioned in the case of $\delta = 0$, we get $V_P^{Am} = V_P^{Eu} > K - S$ contradicting 5.1.6. For the proof of (2) we refer to page 427 of [Sey15]. The points (3) and (4) follow immediately from the previous two points.

By [Sey15], section 4.5, we have that V_P^{Am} is monotonic decreasing and convex. Together with points (3) and (4) this shows that S^* must be unique. □

Property (2) of the above Theorem is often called smooth pasting.

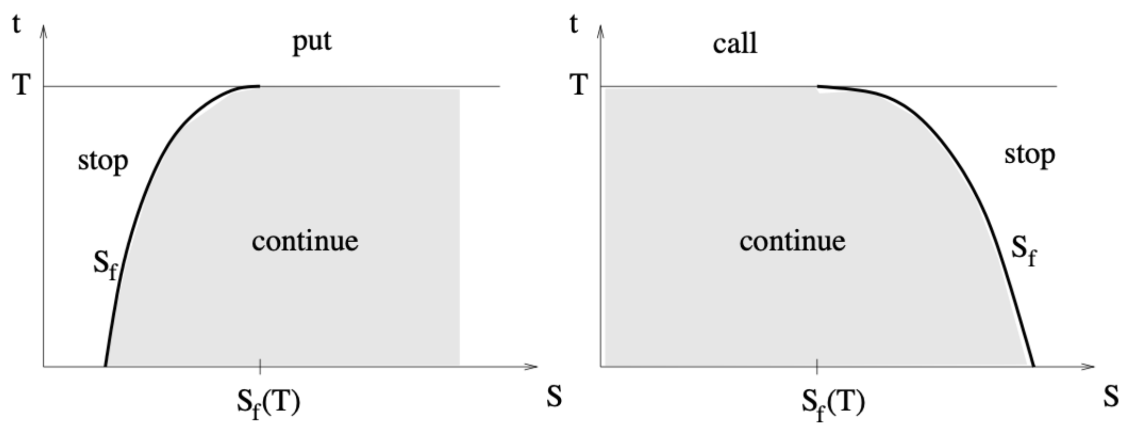

Figure 5.1: credit: [Sey15]

The curve $S_f(t)$ is called the **early-exercise curve** and the region where $S > S_f$ is called the **continuation region** and the region where $S \leq S_f$ is called the **stopping region** for the two following reasons:

(1.) When $V_P > (K - S)^+ := \max(K - S, 0)$, exercising the option results in $-V + K - S < 0$, which is a loss. The holder is advised to hold ("continue") the option when $S \geq S^*$.

(2.) When $S < S^*$, i.e. the stock S surpasses the curve, then exercising results in a profit because the amount K can be invested and we get

$$Ke^{r(T-t)} - K = K\left(e^{r(T-t)} - 1\right) > Se^{\delta(T-t)}, \tag{5.1.9}$$

if additionally $r(T-t) < 1$. Thus the holder is advised to sell ("stop holding")

For a call-option the oppositie is true, i.e hold when $S < S^*$ and buy when $S \geq S^*$. Quoting directly from [Sey15], p.431:

> „**Free-Boundary Problem** means: The Black–Scholes equation is valid only in the continuation region, not in the stopping region (cf. figure 5.1). Hence the domain for the BS equation for an American-style put is
>
> $$S_f(t) < S < \infty. \tag{5.1.10}$$
>
> The left-hand boundary $S_f(t)$ is "free" in the sense that it is unknown initially. It is calculated numerically, based on the additional boundary condition provided by the condition $\frac{\partial V}{\partial S} = -1$. This condition fixes the location of $S_f(t)$"

We also note the following important properties of $S_f(t)$ for a **put** under the Black–Scholes model:

Proposition 5.3. *1.) $S_f(t)$ is contiuously differentiable for $t < T$,*

2.) $S_f(t)$ is monotonic increasing,

3.)

$$S_f(T) := \lim_{\substack{t \to T \\ t < T}} S_f(t) = \begin{cases} K & \text{for } 0 \leq \delta \leq r, \\ \frac{r}{\delta}K & \text{for } r < \delta. \end{cases} \tag{5.1.11}$$

*For a **call**, $S_f(t)$ in point (2.) is decreasing instead of increasing and we get*

$$S_f(T) := \lim_{\substack{t \to T \\ t < T}} S_f(t) = \max\left\{K, \frac{r}{\delta}K\right\} \tag{5.1.12}$$

For a proof we refer again to section 4.5 of [Sey15].

5.2 Linear Complementarity Formulation of American Options

After introducing American options as free-boundary problems, we now reformulate the pricing problem in a way that avoids explicit tracking of the unknown early-exercise boundary. The key idea is to replace the free-boundary fromulation by a **global inequality problem**.

We start by introducing the notation:

$$\mathcal{L}_{BS}(V) := \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)\, S \frac{\partial V}{\partial S} - rV. \tag{5.2.1}$$

With that, equation (5.1.1) can be written as:

$$\frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) = 0. \tag{5.2.2}$$

In the stopping region, however, the option value is fixed at the payoff and no longer evolves freely. For a put and $S < S_f$, the identity $V \equiv \psi$ (remember the payoff function ψ defined in chapter 3) holds.

Proposition 5.4.

$$\begin{cases} V^{Am} = K - S, \\ \frac{\partial V^{\cdot}}{\partial S} = -1, \\ \frac{\partial^2 V^{\cdot}}{\partial S^2} = 0, \\ \frac{\partial V^{\cdot}}{\partial t} = 0. \end{cases} \tag{5.2.3}$$

Proof. For an American put we have

$$\psi\,(t, S(t)) = \max\,(K - S, 0)\,. \tag{5.2.4}$$

In the stopping region, i.e. for $S < S_f$ (cf. figure 5.1), exercising immediately is optimal. Immediate exercise corresponds to the stopping time $\tau = t$, hence

$$V_P^{Am}\,(t, S) = K - S \tag{5.2.5}$$

for all $S < S_f$. From that fact, computing the respective partial derivatives yield the other equations. $\square$

Pluggin the equalties from (5.2.3) into (5.2.1), we get:

$$\frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) = \delta S - rK. \tag{5.2.6}$$

For $S < S_f(T)$ one can conclude, by looking at the two possibilities $r < \delta$ or $r \geq \delta$, that

$$\frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) \leq 0 \tag{5.2.7}$$

holds for all $S > 0, 0 \leq t \leq T$.

The Equation (5.2.7) is known as a **differential inequality** for obvious reasons. Transforming the problem in the above manner, ensures that we are no longer dependent on the free-boundary. Its real usefulness, though, only reveals itself later on, when we introduce what is known as linear complementarity problems.

Remark 5.5. *Remembering the transformations in section 3.1, we can also express* (5.2.2) *as*

$$\frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) = -\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial x^2} = 0. \tag{5.2.8}$$

Before coming to the main definition of this section, let us take a moment to consider some aspects which will become important for our numerical implementation. It is necessary, for numerical purposes, to truncate the infinite stip, on which the heat equation is defined, to a rectangle of the form

$$(\tau, x) \in [0, \tau_{\max}] \times [x_{\min}, x_{\max}]\,. \tag{5.2.9}$$

To make the problem well-posed, we need additionally impose boundary-conditions for the sides $x_{\min}$ and $x_{\max}$.

For reference we also note that, comparing with page 404 of [Sey15], the interval should obey a certain range, namely, it should fulfill the inequalities

$$x_{\min} < \min\left\{0, \log\frac{S}{K}\right\}, \qquad \max\left\{0, \log\frac{S}{K}\right\} < x_{\max}. \tag{5.2.10}$$

The inequalities in (5.2.10) stem from our logarithmic transformation $x = \ln\left(\frac{S}{K}\right)$ in

(3.1.3) and are justified by the asymptotic behavior of the option value, since for sufficiently small (respectively large) stock prices the value of an American put coincides with its payoff (respectively tends to zero), cf. section 4.4 of [Sey15]. These boundary conditions will later be encoded in the functions r_1 and r_2 in section 5.4.

Remark 5.6. *From (5.2.10) one can see that in general*

$$x_{\min} < 0 < x_{\max}. \tag{5.2.11}$$

This is by design and makes sure that the left boundary for a put and the right boundary for a call lies in the stopping region where the price is known.

Coming back to the problem at hand, the main insight of [Sey15] now is the following: The system

$$\left.\begin{aligned} \left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u-g) = 0, \\ \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0, \\ u - g \geq 0, \end{aligned}\right\} \tag{5.2.12}$$

is called the **linear complementarity problem** for an American put. The function g is defined as

$$g(\tau, x) := \exp\left\{\frac{1}{4}(q+1)^2\tau\right\} \max\left\{e^{\frac{1}{2}(q-1)x} - e^{\frac{1}{2}(q+1)x}, 0\right\}, \tag{5.2.13}$$

where $q := \frac{2r}{\sigma^2}$, and comes from expressing the side condition

$$V_P^{Am}(t, S) \geq \max(K - S, 0) = K \max(1 - e^x, 0) \tag{5.2.14}$$

in the transformed variables τ, x. The boundary-and initial conditions of (5.2.12) are defined as

$$\begin{aligned} u(0, x) &= g(0, x), \\ u(\tau, x_{\min}) &= g(\tau, x_{\min}), \\ u(\tau, x_{\max}) &= g(\tau, x_{\max}). \end{aligned} \tag{5.2.15}$$

The problem (5.2.12) is an example of the more general

Definition 5.7. *Let $M \in \mathbb{R}^{n\times n}$ be a square matrix and $q \in \mathbb{R}^n$. The problem: find $w = (w_1, \cdots, w_n)^T$* [1]*, $z = (z_1, \cdots, z_n)^T$ satisfying*

$$\begin{cases} w - Mz = q, \\ w \geq 0, \quad z \geq 0 \quad \text{and } w_i z_i = 0 \forall i, \end{cases} \tag{5.2.16}$$

is called a ***linear complementarity problem (LCP)****.*

We refer to [Mur88] for an in-depth discussion of the interesting theory behind LCPs.

The formulation in (5.2.12) may be interpreted as an obstacle problem on a fixed spatial domain, a classic problem in analysis, which is more amenable to analytical considerations and, in particular, to numeric approximation. Intuitively, one tries to compute a function that wraps around a fixed obstacle. Later on that obstacle will turn out to be the payoff function representing the value obtained by immediate exercise. After discretization in time and space, the continuous complementarity system gives rise to a finite-dimensional LCP, forming the basis for numerical methods, which we shortly discuss in the following chapter.

5.3 Intermission - Numerical methods

Let us begin by recalling some basic numerical methods as, for example, throroughly discussed in [Lin24].

We introduce a grid of equidistantly spaced grid points x_i

$$\cdots < x_{i-1} < x_i < x_{i+1} < \cdots, \tag{5.3.1}$$

where we denote by $h := x_{i+1} - x_i$ the distance between two points. Using a Taylor expansion, we can obtain the representation

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(\xi) \quad \text{for a } \xi \in (x, x+h) \text{ and } f \in C^2. \tag{5.3.2}$$

Introducing the notation $f_i = f(x_i)$, we can now write:

[1] For a vector $\mathbf{x}$, $\mathbf{x}^T$ is defined as the transpose of $\mathbf{x}$.

$$f'(x_i) = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2) \quad \text{for } f \in C^3,$$
$$f''(x_i) = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + O(h^2) \quad \text{for } f \in C^4, \tag{5.3.3}$$
$$f'(x_i) = \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h} + O(h^2) \quad \text{for } f \in C^3.$$

Recall that after our transformations in section 3.1 we are no longer in $(t, S(t))$-space but in (τ, x)-space. The transformed Black–Scholes PDE in 3.1.13 is now posed on the rectancle $(\tau, x) \in [0, \tau_{max}] \times [x_{min}, x_{max}]$. For numerical purposes, we now discretizes this strip as follows:

$$\Delta x := \frac{x_{\max} - x_{\min}}{m}, \qquad x_i := x_{\min} + i \cdot \Delta x, i = 0, 1, \cdots m,$$
$$\Delta \tau := \frac{\tau_{\max}}{\nu_{\max}}, \qquad \tau_\nu := \nu \cdot \Delta \tau, \nu = 0, \cdots, \nu_{\max}. \tag{5.3.4}$$

$u_{\nu,i} := u(\tau_\nu, x_i)$ is the value of u at the node (τ_ν, x_i). Approximations of $u_{\nu,i}$ are denoted by $w_{\nu,i}$. The values for $w_{0,i} = u(0, x_i)$ are determined by the specific initial conditions.

If in 3.1.13, we replace

$$\frac{\partial u_{\nu,i}}{\partial \tau} := \frac{\partial u(\tau_\nu, x_i)}{\partial \tau} = \frac{u_{\nu+1,i} - u_{\nu,i}}{\Delta \tau} + O(\Delta \tau) \text{ and}$$
$$\frac{\partial^2 u_{\nu,i}}{\partial x^2} = \frac{u_{\nu,i+1} - 2u_{\nu,i} + u_{\nu,i-1}}{\Delta x^2} + O\left(\Delta x^2\right), \tag{5.3.5}$$

drop the O-error terms, replace $u \to w$, we can obtain the **difference equation**:

$$\frac{w_{\nu+1,i} - w_{\nu,i}}{\Delta \tau} = \frac{w_{\nu,i+1} - 2w_{\nu,i} + w_{\nu,i-1}}{\Delta x^2}. \tag{5.3.6}$$

We see that all right-hand terms are at time level ν. We obtain the values for the time $\nu + 1$ as

$$w_{\nu+1,i} = w_{\nu,i} + \frac{\Delta \tau}{\Delta x^2}\left(w_{\nu,i+1} - 2w_{\nu,i} + w_{\nu,i-1}\right). \tag{5.3.7}$$

Letting $\lambda := \frac{\Delta \tau}{\Delta x^2}$, we get the **explicit method**

$$w_{\nu+1,i} = \lambda w_{\nu,i-1} + (1 - 2\lambda) w_{\nu,i} + \lambda w_{\nu,i+1}. \tag{5.3.8}$$

However, [Sey15] spends quite some effort showing:

$$\lambda \leq \frac{1}{2} \iff \Delta\tau \leq \frac{1}{2}\Delta x^2, \tag{5.3.9}$$

meaning that refining space forces us to refine time quadratically. For American options, however, we want to be able to take small time steps and have fine spatial resolution near the free boundary, making the explicit method unpractical. For that reason we turn our attention to whats known as **implicit methods.**

The backward difference quotient

$$\frac{\partial u_{\nu,i}}{\partial \tau} = \frac{u_{\nu,i} - u_{\nu-1,i}}{\Delta\tau} + O\left(\Delta\tau\right) \tag{5.3.10}$$

leads to

$$-\lambda w_{\nu,i+1} + (1 + 2\lambda)\, w_{\nu,i} - \lambda w_{\nu,i-1} = w_{\nu-1,i}. \tag{5.3.11}$$

Letting

$$w^{(\nu)} := \left(w_{\nu,1}, \cdots, w_{\nu,m-1}\right)^T, \tag{5.3.12}$$

we can solve for $w^{(\nu)}$ with the help of the $(m-1) \times (m-1)$-matrix

$$A := \begin{pmatrix} 1+2\lambda & -\lambda & & 0 \\ -\lambda & \ddots & \ddots & \\ & \ddots & \ddots & -\lambda \\ 0 & & -\lambda & 1+2\lambda \end{pmatrix}. \tag{5.3.13}$$

The vector $w^{(\nu)}$ is defined implicitly as the solution of the system $Aw^{(\nu)} = w^{(\nu-1)}$, or

$$Aw^{(\nu+1)} = w^{(\nu)}, \nu = 0, \cdots, \nu_{\max} - 1. \tag{5.3.14}$$

Before going into the implementation of a numerical method for American options, let us investigate the stability of the aforementioned method. [Sey15] introduces two Lemmata for that purpose.

Lemma 5.8. *Let $\rho(A) := \max_k |\mu_k^A|$ where μ^A is an eigenvalue of A. Then*

$$\rho(A) < 1 \iff A^\nu z \to 0 \quad \textit{for all } z \textit{ and } \nu \to \infty. \tag{5.3.15}$$

Proof. Suppose $\rho(A) < 1$. We know that there exists a matrix norm $\|.\|_\epsilon$ such that $\|A\|_\epsilon < \rho(A) + \epsilon$. Choose $\epsilon > 0$ such that $\|A\|_\epsilon$. Then

$$\begin{aligned} \|\lim_{\nu\to\infty} A^\nu\|_\epsilon &\le \lim_{\nu\to\infty} \|A^\nu\|_\epsilon \\ &\le \lim_{\nu\to\infty} \|A^\nu\|_\epsilon^\nu \\ &= 0. \end{aligned} \tag{5.3.16}$$

□

Lemma 5.9. *For*

$$G = \begin{pmatrix} \alpha & \beta & & & 0 \\ \gamma & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta \\ 0 & & & \gamma & \alpha \end{pmatrix} \in \mathbb{R}^{n\times n}, \tag{5.3.17}$$

the eigenvalues are

$$\mu_k^G = \alpha + 2\beta\sqrt{\frac{\gamma}{\beta}}\cos\frac{k\pi}{N+1}, \qquad k = 1, \cdots, n. \tag{5.3.18}$$

Proof. With the eigenvectors

$$v^{(k)} = \left(\sqrt{\frac{\gamma}{\beta}}\sin\frac{k\pi}{n+1}, \left(\sqrt{\frac{\gamma}{\beta}}\right)^2 \sin\frac{2k\pi}{n+1}, \cdots, \left(\sqrt{\frac{\gamma}{\beta}}\right)^n \sin\frac{nk\pi}{n+1}\right)^T, \tag{5.3.19}$$

check $Gv^{(k)} = \mu^G v^{(k)}$. □

Let $e^{(\nu)} := \bar{w}^{(\nu)} - w^{(\nu)}$ be the error that occurs when calculating the system (5.3.14) ($w^{(\nu)}$ is the exact vectors of $w^{(\nu+1)} = Aw^{(\nu)} + d^{(\nu)}$, where above $d^{(\nu)} = 0$, and $\bar{w}^{(\nu)}$ is the version that has been calculated by a computer and is subjected to rounding erros (cf. [Lin24])).

Applying Lemma 5.9 directly to the matrix (5.3.13), with $n = m - 1, \alpha = 2$ and $\beta = \gamma = -1$, we get that

$$\mu_k^A = 1 - 4\lambda \sin^2 \frac{k\pi}{2m}. \tag{5.3.20}$$

By Lemma 5.8:

$$\begin{aligned} \text{stability} &\iff \left|1 - 4\lambda \sin^2 \frac{k\pi}{2m}\right| < 1, \quad k = 1, \cdots, m-1 \\ &\iff \lambda > 0 \text{ and } -1 < 1 - 4\lambda \sin^2 \frac{k\pi}{2m} \text{ or } \frac{1}{2} > \lambda \sin^2 \frac{k\pi}{2m}. \end{aligned} \tag{5.3.21}$$

That is why in (5.3.8) we chose $0 < \lambda < \frac{1}{2}$. A has the structure

$$A = I - \lambda \cdot \underbrace{\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{pmatrix}}_{:=G} \tag{5.3.22}$$

$\implies \mu^A = 1 - \lambda\mu^G$ and since $Aw^{(\nu+1)} = w^{(\nu)} \implies w^{(\nu+1)} = (I - \lambda \cdot G)^{-1} w^{(\nu)}$ we have that $\mu^{A^{-1}} = \frac{1}{1+\lambda|\mu^G|}$.

Proposition 5.10. *The implicit scheme is stable because*

$$e^{(\nu+1)} = Ae^{(\nu)} \implies e^{(\nu)} = A^{(\nu)} e^{(0)} \tag{5.3.23}$$

and by the above results we have $e^{(\nu)} \to 0$ for every initial error $e^{(0)}$ and every $\lambda > 0$. We can choose $\Delta\tau$ and Δx independent of each other.

[Sey15] goes on to show one particular weakness of the implicit (and the explicit) scheme, namely, that the accuracy of the first order in $\Delta\tau$ is of the order

$$O\left(\Delta x^2\right) + O\left(\Delta\tau\right). \tag{5.3.24}$$

To combat this weakness we introduce one last piece in the puzzle that will reveal a numerical algorithm that can deal with American options. For that, we assemble the forward quotient for ν

$$\frac{w_{\nu+1,i} - w_{\nu,i}}{\Delta\tau} = \frac{w_{\nu,i+1} - 2w_{\nu,i} + w_{\nu,i+1}}{\Delta x^2} \tag{5.3.25}$$

and the backward quotient for $\nu + 1$

$$\frac{w_{\nu+1,i} - w_{\nu,i}}{\Delta\tau} = \frac{w_{\nu+1,i+1} - 2w_{\nu+1,i} + w_{\nu+1,i-1}}{\Delta x^2}. \tag{5.3.26}$$

If we add both equations, we get the following scheme that is known under the name **Crank–Nicolson** [CN47], named after the original inventors Crank.J and Nicolson.P who deviced the method in 1947:

$$\frac{w_{\nu+1,i} - w_{\nu,i}}{\Delta\tau} = \frac{1}{2\Delta x^2}\left(w_{\nu,i+1} - 2w_{\nu,i} + w_{\nu,i-1} + w_{\nu+1,i+1} - 2w_{\nu+1,i} + w_{\nu+1,i-1}\right). \tag{5.3.27}$$

Theorem 5.11. *For the scheme in* (5.3.27) *the following assertions hold:*

1.) For $u \in C^4$ the method is of the order $O(\Delta x^2) + O(\Delta\tau^2)$,

2.) For each ν a system of linear equations in tridiagonal[2] form must be solved,

3.) The method is stable for all $\Delta\tau > 0$.

For the lenghty proof we refer to section 4.3 of [Sey15].

For the following algorithm we further define the matrix B as:

$$B := \begin{pmatrix} 1-\lambda & \frac{\lambda}{2} & & 0 \\ \frac{\lambda}{2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & & \frac{\lambda}{2} & 1-\lambda \end{pmatrix}. \tag{5.3.28}$$

Algorithm 5.12. *(Crank–Nicolson)*

start: *Choose $m, \nu_{\max}$; calculate $\Delta x, \Delta\tau$. Let*

$w_i^{(0)} = u(0, x_i) \quad (0 \le i \le m)$

LR-decomposition[3] (or RL-decomposition) of A

loop: *for $\nu = 0, 1, \cdots, \nu_{\max} - 1$:*

$c := Bw^{(\nu)} + 0$ *(preliminary boundary conditions 0)*

[2] cf. p.171 Definition 6.3.4 of [Lin24]
[3] cf. [Lin24]

Solve $Ax = c$ (using LR/RL-decomposition)
$w^{(\nu+1)} := x$

5.4 Implementation

To obtain a well-posed problem on a bounded interval, one must prescribe two spatial boundary conditions in addition to the initial condition. We differentiate between a call and a put and define the two functions r_1 and r_2:

$$\text{call:}\quad r_1(\tau, x) := 0, \quad r_2(\tau, x) := \exp\left(\tfrac{1}{2}\left(q_\delta + 1\right) x + \tfrac{1}{4}\left(q_{\delta+1}\right)^2 \tau\right).$$

$$\text{put:}\quad r_1(\tau, x) := \exp\left(\tfrac{1}{2}\left(q_\delta - 1\right) x + \tfrac{1}{4}\left(q_\delta - 1\right)^2 \tau\right), \quad r_2(\tau, x) := 0.$$

For the finite interval $a := x_{\min} \le x \le x_{\max} =: b$ we let

$$\begin{aligned} w_{\nu,0} &= r_1\left(\tau_\nu, a\right), \\ w_{\nu,m} &= r_2\left(\tau_\nu, b\right). \end{aligned} \tag{5.4.1}$$

In the Crank–Nicolson scheme this leads to additional terms which we can include into our system of equations (5.3.14) by adding the term

$$d^{(\nu)} := \frac{\lambda}{2} \cdot \begin{pmatrix} w_{\nu,0} + w_{\nu+1,0} \\ 0 \\ \vdots \\ 0 \\ w_{\nu,m} + w_{\nu+1,m} \end{pmatrix}. \tag{5.4.2}$$

We thus get the new system

$$Aw^{(\nu+1)} = Bw^{(\nu)} + d^{(\nu)}. \tag{5.4.3}$$

Gathering everything we achieved in the previous three sections, we now discretize the linear complementarity problem (5.2.12) with the grid introduced at the start of section 5.3:

$$\frac{w_{\nu+1,i}-w_{\nu,i}}{\Delta\tau}=\theta\frac{w_{\nu+1,i+1}-2w_{\nu+1,i}+w_{\nu+1,i-1}}{\Delta x^2}+(1-\theta)\,\frac{w_{\nu,i+1}-2w_{\nu,i}+w_{\nu,i-1}}{\Delta x^2}. \quad (5.4.4)$$

Remark 5.13. *For $\theta = 0$ we get the explicit method and for $\theta = \frac{1}{2}$ the Crank-Nicolson scheme.*

With $\lambda = \frac{\Delta\tau}{(\Delta x)^2}$ the inequality in (5.2.12) becomes

$$w_{\nu+1,i}-\lambda\theta\left(w_{\nu+1,i+1}-2w_{\nu+1,i}+w_{\nu+1,i-1}\right)-w_{\nu,i}-\lambda\left(1-\theta\right)\left(w_{\nu,i+1}-2w_{\nu,i}+w_{\nu,i-1}\right)\geq 0. \quad (5.4.5)$$

We further define the terms

$$\begin{aligned} b_{\nu,i} &:= w_{\nu,i}+\lambda\left(1-\theta\right)\left(w_{\nu,i+1}-2w_{\nu,i}+w_{\nu,i-1}\right), \quad \text{for } i=2,\cdots,m-2,\\ b_{\nu,1} &:= w_{\nu,1}+\lambda\left(1-\theta\right)\left(w_{\nu,2}-2w_{\nu,1}+g_{\nu,0}\right)+\lambda\theta g_{\theta+1,0},\\ b_{\nu,m-1} &:= w_{\nu,m-1}+\lambda\left(1-\theta\right)\left(g_{\nu,m}-2w_{\nu,m-1}+w_{\nu,m-2}\right)+\lambda\theta g_{\nu+1,m}, \end{aligned} \quad (5.4.6)$$

where $g_{\nu,i} := g\left(\tau_\nu, x_i\right)$ and $b_{\nu,1}, b_{\nu,m-1}$ encode the boundary conditions. Defining the vectors $b^{(\nu)} := \left(b_{\nu,1},\cdots,b_{\nu,m-1}\right)^T$ and $w^{(\nu)}, g^{(\nu)}$ likewise and letting the matrix A be

$$A := \begin{pmatrix} 1+2\lambda\theta & -\lambda\theta & & 0\\ -\lambda\theta & \ddots & \ddots & \\ & \ddots & \ddots & -\lambda\theta\\ 0 & & -\lambda\theta & 1+2\lambda\theta \end{pmatrix} \in \mathbb{R}^{(m-1)\times(m-1)}, \quad (5.4.7)$$

we get the new problem:

$$\left.\begin{aligned} \left(Aw^{(\nu+1)}-b^{(\nu)}\right)^T\left(w^{(\nu+1)}-g^{(\nu+1)}\right) &= 0,\\ Aw^{(\nu+1)} &\geq b^{(\nu)} \quad \text{for all } \nu,\\ w^{(\nu)} &\geq g^{(\nu)}. \end{aligned}\right\} \quad (5.4.8)$$

This constitutes the following algorithm

Algorithm 5.14. *For $\nu = 0, 1, \cdots, \nu_{\max -1}$:*

$$\textit{compute } g := g^{(\nu+1)}, b := b^{(\nu)}, \textit{ as above;}$$
$$\textit{compute } w \textit{ as the solution of } (5.4.8)$$
$$\textit{set } w^{(\nu+1)} := w$$

This then means that at each time-step $\nu \in [0, \nu_{\max} - 1]$ a LCP must be solved. Let us quickly outline the proof of the solvability of (5.4.8) as first shown by [PJ89] in 1989.

Proposition 5.15. *The problem (5.4.8) has a unique solution at each time $\nu \in [0, \nu_{\max} - 1]$.*

Proof. The authors first introduce the transformations

$$x := w^{(\nu+1)} - g^{(\nu+1)}, \quad y := Aw^{(\nu+1)} - b^{(\nu)} \tag{5.4.9}$$

for a fixed $\nu \in [0, \cdots, \nu_{\max} - 1]$. Applying these transformations, we can represent (5.4.8) as the equivalent problem

$$\begin{cases} x_i y_i = 0 \forall i, \\ x \geq 0, \\ y \geq 0. \end{cases} \tag{5.4.10}$$

We also define $\hat{b} := Ax - y = b^{(\nu)} - Ag^{(\nu+1)}$, which implcitly implies $Ax - \hat{b} \geq 0$. Next we define a quadratic functional

$$G(x) := \frac{1}{2} x^T A x - \hat{b}^T x, \tag{5.4.11}$$

where A is the Matrix from (5.4.7). After using Lemma 5.9 we get that $\mu_k^A \geq 1$, which implies that A is positive definite. Differentiating G with respect to x yields

$$\frac{\partial G}{\partial x} = Ax - \hat{b}, \quad \frac{\partial^2 G}{\partial x^2} = A. \tag{5.4.12}$$

The right equation of (5.4.12) then implies that A is the Hessian matrix[4] of G. It is well known, cf. [SB04], that a functional is **strictly convex** if its Hessian is positive definite. Therefore G is strictly convex. We also note that the constraint $x \geq 0$ defines a closed convex set. By the Karush-Kuhn-Tucker (KKT) theorem (cf. [Kuh51]), a

[4]The Hessian $\nabla^2 f(x)$ of a function f is defined as $\nabla^2 f(x)_{ij} := \frac{\partial^2 f(x)}{\partial x_i \partial_j}, \quad i, j = 1, \cdots, n.$

strictly convex functional on a convex set has a unique minimizer and satisfies exactly the complementarity conditions in (5.4.10) plus the property

$$\frac{\partial G}{\partial x_i} + y_i \frac{\partial G}{\partial x_i} + 0 = \frac{\partial G}{\partial x_i} - y_i = 0. \tag{5.4.13}$$

Hence, the LCP in (5.4.10) has a unique solution and by substituting back, so does the LCP in (5.4.8). □

As a consequence, the associated linear system $Aw^{(\nu+1)} = b^{(\nu)}$ admits a unique solution at every time-step $\nu \in [0, \nu_{\max} - 1]$ and can be solved efficiently by Gaussian elimination.

Following the approach suggested by Brennan and Schwartz in [MJB77], the complementarity conditions are enforced directly during the solution process. First, the system $Aw^{(\nu+1)} = b^{(\nu)}$ is transformed via a suitable RL-decomposition into an equivalent triangular system denoted by $\tilde{A}w^{(\nu+1)} = \tilde{b}^{(\nu)}$. Then, the solution is obtained componentwise by substituting sequentially and imposing the constraint pointwise by

$$w_i^{(\nu+1)} := \max\left\{\tilde{w}_i^{(\nu+1)}, g_i\right\}, \tag{5.4.14}$$

where $\tilde{w}_i^{(\nu+1)}$ denotes the value obtained from the triangular system that we get after applying Gaussian elimination.

For American put options, the early exercise region is known to occur at low asset prices. Consequently, the algorithm proceeds in forward order, starting from the lower boundary, which requires a decomposition leading to a lower triangular system. For American call options with dividends, the procedure is applied analogously in reverse order.

The actual algorithm is split into two phases. For a put option it can be summarized as:

Algorithm 5.16. *(put)*

1st phase: *Calculate the RL-decomposition of A. Then set $\tilde{A} = L$ and obtain $\tilde{b}$ from $R\tilde{b} = b$ (backward loop)*

2nd phase: *(forward loop) Start with $i = 1$. Calculate the next component of $\tilde{A}w = \tilde{b}$ which we will call $\tilde{w}_i$. Set $w_i := \max(\tilde{w}_i, g_i)$.*

Regarding the accuracy of this method, we want to quote directly from the original paper by Brennan and Schwartz, i.e. [MJB77]:

> „As a crude test of the accuracy of the above procedure several puts were valued assuming a zero rate of interest, and the result compared to the corresponding Black–Scholes solution. The numerical method was accurate to withing about 0.1% at option prices corresponding to the exercise price. "

At the early exercise boundary $S_f(t)$ the American option value is continuous and satisfies the smooth-pasting property from Theorem 5.2, but generally fails to be twice continuously differentiable with respect to the underlying asset price. Since the Black–Scholes operator involves second-order derivatives, this loss of regularity deteriorates the local truncation error of finite difference schemes near the free boundary. As a consequence, the global convergence order of Crank–Nicolson-based methods for American options may be reduced, despite the fact that the Brennan–Schwartz algorithm solves the resulting discrete problem exactly.

5.5 Conclusive remarks

In this chapter, we discretized the initial-boundary value problem (5.2.12)+(5.2.15) and arrived at the fully discrete formulation (5.4.8), which takes the form of a linear complementarity problem at each time step. Proposition 5.15 ensures that this discrete problem admits a unique solution, thereby establishing well-posedness of the numerical scheme on the discrete level. It should be emphasized, however, that this result does not by itself imply well-posedness of the original continuous problem (5.2.12)+(5.2.15). To establish existence and uniqueness of solutions to the continuous variational inequality, one would additionally need to prove convergence of the form

$$w_{\Delta t,\Delta x} \to V^{Am} \tag{5.5.1}$$

in a suitable topology and for $(\Delta t, \Delta x) \to 0$ and that the limit is uniquely determined. Such analysis, which typically relies on stability and compactness arguments is beyond the scope of the present thesis.

The Brennan–Schwartz algorithm provides an elegant and problem-specific solution to valuing American options, that unlike European options, have no closed formula for pricing them. By exploiting the tridiagonal structure and monotonicity properties of the matrix arising from the Crank–Nicolson discretization, the algorithm computes the unique solution of the discrete complementarity problem in a single sweep, meaning from left to right along the grid, without the need for iterative methods. As a result, the algorithm introduces no additional approximation error beyond that of the underlying finite difference scheme. What we ultimately obtain from the algorithm is a full numerical approximation of the American option value on the chosen grid, together with an implicit approximation of the early exercise boundary.

From a financial perspective, the Brennan–Schwartz algorithm represented a significant advance. It was among the first methods to make the numerical valuation of American options computationally feasible in practice, combining efficiency, stability and conceptual clarity. The ideas introduced in this algorithm influenced a wide range of later methods for free boundary problems and complementarity formulations in quantitative finance. Some of them can be found in [SI07].

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