Control and Observation for Heat Equations

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Controllability

Let $T > 0$. Let Ω be a Lipschitz domain in $\mathbb{R}^N$.
Let ω be a measurable subset of Ω.

Let Δ be the Laplace operator with Dirichlet boundary condition if $\partial\Omega \neq \emptyset$.

Consider the controlled heat equation

$$\begin{cases} \partial_t u - \Delta u = \chi_\omega f, & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0. \end{cases}$$

- Well-posedness, smooth regularity

- Null Controllability: For each $u_0 \in L^2(\Omega)$, there exists a control $f \in L^2(\omega \times (0,T))$ such that at the final time $u(\cdot,T) = 0$.

Observability

Consider the free heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0. \end{cases}$$

- Final-state Observability: $\exists C = C(\Omega, \omega, T)$ such that

$$\|u(\cdot, T)\|_{L^2(\Omega)} \leq C \|u\|_{L^2(\omega \times (0,T))}, \quad \forall u_0 \in L^2(\Omega).$$

Note: The difficulty comes from the partial observation: $\omega \subset \Omega$.

One can also consider the case of boundary control or observation!

Outline

Controllability and Observability for

- Heat equation in bounded domains
- Heat equation in $\mathbb{R}^N$
- Heat equation with time-space dependent potentials in $\mathbb{R}^N$
- Semilinear heat equation in $\mathbb{R}^N$

§1. Necessary and sufficient conditions for bounded domains

Abstract Theorem: The controlled heat equation is null controllable in time $T > 0$ if and only if the following observability inequality holds:

$$\|e^{T\Delta} f\|^2_{L^2(\Omega)} \leq C \int_0^T \|e^{t\Delta} f\|^2_{L^2(\omega)}\, dt \quad \forall f \in L^2(\Omega).$$

Theorem (Spectral method): Let $\mathcal{E}_\lambda$ be the family of spectral projection operators of $-\Delta$ (with Dirichlet B.C.) up to the value λ. Assume

$$\exists c > 0 \quad : \quad \|\mathcal{E}_\lambda f\|_{L^2(\Omega)} \leq e^{c\sqrt{\lambda}} \|\mathcal{E}_\lambda f\|_{L^2(\omega)} \;\; \forall f \in L^2(\Omega) \;\; \forall \lambda > 0.$$

Then

$$\exists C > 0 \quad : \quad \|e^{T\Delta} f\|^2_{L^2(\Omega)} \leq C e^{C/T} \int_0^T \|e^{t\Delta} f\|^2_{L^2(\omega)}\, dt \quad \forall f \in L^2(\Omega).$$

Known results for spectral inequality

Case 1: When Ω is C^2-smooth and ω is a nonempty open subset.

G. Lebeau, L. Robbiano, E. Zuazua, Q. Lü, ...

Case 2: When Ω is locally star-shaped and ω is a Lebesgue measurable set of positive measure.

J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, G. Lebeau, I. Moyano, ...

Recently, such a kind of spectral inequality has been established for the elliptic operator $-\mathrm{div}(A(\cdot)\nabla)$

G. Lebeau, I. Moyano, N. Burq, L. Escauriaza, C. Zhang

Moreover, one can derive the following refined observability from measurable sets in time and space variables:

$$\|e^{T\Delta}u_0\|_{L^2(\Omega)} \leq N(\Omega, T, \mathcal{D}) \int_{\mathcal{D}} |e^{t\Delta}u_0(x)|\, dxdt, \;\forall\, u_0 \in L^2(\Omega),$$

where $\mathcal{D}$ is a Lebesgue measurable subset of $\Omega \times (0, T)$.

We have extended such a result to the case of parabolic equations with analytic coefficients in bounded domains.

The proof therein is mainly based on quantitative estimates from measurable sets for real analytic functions.

Propagation of estimate for real-analytic functions

Theorem

Assume that $f : B_{2R} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ *is real-analytic in* B_{2R} *verifying*

$$|\partial_x^\alpha f(x)| \le \frac{M|\alpha|!}{(\rho R)^{|\alpha|}}\ , \quad \text{when} \quad x \in B_{2R},\ \ \alpha \in \mathbb{N}^n,$$

for some $M > 0$ *and* $0 < \rho \le 1$. *Let* $E \subset B_R$ *be a measurable set with positive measure. Then, there are positive constants* $N = N(\rho, |E|/|B_R|)$ *and* $\theta = \theta(\rho, |E|/|B_R|)$, *with* $\theta \in (0,1)$, *such that*

$$\|f\|_{L^\infty(B_R)} \le N \left(\fint_E |f|\, dx \right)^{\theta} M^{1-\theta}.$$

S. Vessella, A continuous dependence result in the analytic continuation problem, Forum Math., 11 (1999), 695–703.

Application to the time optimal control problem

For each $M > 0$, define

$$\mathcal{U}_M = \{g \text{ measurable on } \omega \times \mathbb{R}^+ : \ |g(\cdot,\cdot)| \leq M\}.$$

Let $u_0 \in L^2(\Omega) \setminus \{0\}$. Consider the minimal time control problem:

$$(TP)_M : \ T_M \equiv \min_{g \in \mathcal{U}_M} \{t > 0 : \ u(x,t;g) = 0 \text{ for a.e. } x \in \Omega\},$$

where $u(\cdot,\cdot\,;g)$ is the solution to

$$\begin{cases} \partial_t u - \Delta u = \chi_\omega g, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0, & \text{in } \Omega. \end{cases} \tag{1}$$

Existence of optimal controls is classical.

Bang-bang Property

Theorem

Problem $(TP)_M$ has the bang-bang property: any minimal time control g satisfies that $|g(x,t)| = M$ for a.e. $(x,t) \in \omega \times (0, T_M)$. Consequently, this problem has a unique minimal time control.

Other applications in

Shape optimization problem (Zuazua, Trélat, etc.)

Rapid stabilization problem (Wang, etc.)

§2. Previous results for unbounded domains

Let $E \subset \mathbb{R}^n$ be an open set.

•(V. R. Cabanillas, S. B. de Menezes, E. Zuazua 01) The null controllability holds when observations are made over a subset $E \subset \Omega$, with $\Omega \backslash E$ bounded.

•(L. Miller 05) A sufficient condition for null controllability is that E satisfies

$$\exists r, \delta > 0 \ \ : \ \ \forall y \in \mathbb{R}^n, \ \exists x \in E \ : \ B_r(x) \subset E, \ \|y - x\| < \delta.$$

• (X. Zhang, E. Zuazua 08) Observability and controllability for Kirchhoff plate systems with potentials in unbounded domain

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Thick set

Let $E \subset \mathbb{R}^n$ be measurable.

We say E is *thick* if there exist $\gamma \in (0, 1]$ and $L > 0$ such that

$$|E \cap Q_L(x)| \geq \gamma L^n \quad \forall x \in \mathbb{R}^n.$$

Here $Q_L(x)$ is a cube with length L and center x.

In each cube with the side length L, $|E|$ is bigger than or equals to γL^n. So E distributes almost equally in $\mathbb{R}^n$, and parameters γ and L characterize the distribution of E.

We also refer to E as (γ, L)-thick. Such kind of sets arose from studies of the uncertainty principle for harmonic functions. Sometimes, people also say it as relative dense sets.

The structure of a set of γ-thickness at scale L may be very complicated.

Simple Examples:

- The set $\bigcup_{n\in\mathbb{Z}}[n - \frac{1}{1000}, n + \frac{1}{1000}]$ is $\frac{1}{500}$-thick at scale 1 in $\mathbb{R}$.
- A periodic arrangement of balls is a think set.

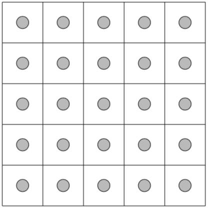

Figure: periodic configuration

On the whole space $\mathbb{R}^n$

Theorem 1 (M.Wang-G.Wang-C.Zhang-Y.Zhang, JMPA, 2019)
Let $\Omega = \mathbb{R}^n$ and $T > 0$. The statements are equivalent:

(i) The set ω is γ-thick at scale L for some $\gamma > 0$ and $L > 0$.

(ii) The set ω satisfies the spectral inequality

$$\int_{\mathbb{R}^n} |f(x)|^2 \, \mathrm{d}x \le e^{C(1+N)} \int_{\omega} |f(x)|^2 \, \mathrm{d}x \quad \forall f \in L^2(\mathbb{R}^n) \ \text{ with } \ \mathit{supp}\, \widehat{f} \subset B_N.$$

(iii) The set ω satisfies the interpolation inequality

$$\int_{\mathbb{R}^n} |e^{T\Delta} f(x)|^2 \, \mathrm{d}x \le e^{C(1+\frac{1}{T})} \Big(\int_{\omega} |e^{T\Delta} f(x)|^2 \, \mathrm{d}x \Big)^{\theta} \Big(\int_{\mathbb{R}^n} |f(x)|^2 \, \mathrm{d}x \Big)^{1-\theta} \quad \forall f \in L^2(\mathbb{R}^n).$$

(iv) The set ω satisfies the observability inequality

$$\int_{\mathbb{R}^n} |e^{T\Delta} f(x)|^2 \, \mathrm{d}x \leq C \int_0^T \int_E |e^{t\Delta} f(x)|^2 \, \mathrm{d}x \, \mathrm{d}t \qquad \forall f \in L^2(\mathbb{R}^n).$$

(v) The heat equation in $\mathbb{R}^n$ is null controllable with control restricted on ω.

Remarks

• The interpolation inequality is a kind of quantitative unique continuation for the heat equation. It provides a Hölder-type propagation of smallness for solutions. In fact, if $\int_E |u(T,x)|^2 \, \mathrm{d}x = \delta$, then $\int_{\mathbb{R}^n} |u(T,x)|^2 \, \mathrm{d}x$ is bounded by $C\delta^\theta$.

• (J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, JEMS,14) It was proved that any subset ω of positive measure satisfies the Hölder-type interpolation inequality for the heat equation in C^1-smooth bounded domains.

Conclusion

In bounded domains, the heat equation is null controllable whenever the control set is of positive measure.

While, in the whole space, the heat equation is null controllable when the control set is thick.
This result is also obtained independently by M. Egidi and I. Veselic.

Further research topics:

How about the answer for general elliptic operators and unbounded domains?

In particular: half-space, cones, infinite strip, exterior domain

Recently, there are many interesting advances on these topics by M. Egidi, I. Nakić, A. Seelmann, I. Veselić, M. Täufer, M. Tautenhahn.

§3. Extend to the heat equation with bounded potentials

Consider the heat equation with a *time and space* dependent potential

$$\begin{cases} \partial_t \varphi - \Delta \varphi + a\varphi = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ \varphi(\cdot, 0) = \varphi_0 & \text{in } \mathbb{R}^N \end{cases}$$

with $\varphi_0 \in L^2(\mathbb{R}^N)$ and $a \in L^\infty(\mathbb{R}^N \times (0,T))$.

• Well-posedness: weak solution (also called J.L.Lions' transposition solution) & mild solution

$$\varphi \in L^2(0,T; H^1(\mathbb{R}^N)) \cap C([0,T]; L^2(\mathbb{R}^N)).$$

Recall: For linear parabolic equations in $\mathbb{R}^N$, the observability inequality holds when the observation is a subset ω, with $\mathbb{R}^N \backslash \omega$ bounded. (E. Zuauza, etc., 01)

Difficulty:

- $\mathbb{R}^N \setminus \omega$ is unbounded
- Spectral inequality cannot be applied

Our Observation

Quantitative unique continuation principles on multiscale structures for Schrödinger and second-order elliptic operators in large domains have been recently well studied by M. Egidi, I. Nakić, A. Seelmann, I. Veselić, M. Täufer, M. Tautenhahn, etc.

An important feature in those works is that the observation subdomain satisfies a so-called equidistributed set.

The methods utilized in these papers are almost based on the spectral inequality.

Unfortunately, they are not valid any more for the case that the coefficients in parabolic equations are time-dependent.

Theorem 2 (Y. Duan, L. Wang, C. Zhang, SICON, 20)
Let $0 < r_1 < r_2 < \infty$. Assume that there is a sequence $\{x_i\}_{i\geq 1} \subset \mathbb{R}^N$ so that

$$\mathbb{R}^N = \bigcup_{i\geq 1} Q_{r_2}(x_i) \quad \text{with} \quad \text{int}(Q_{r_2}(x_i)) \bigcap \text{int}(Q_{r_2}(x_j)) = \emptyset.$$

Let

$$\omega \triangleq \bigcup_{i\geq 1} \omega_i \quad \text{with } \omega_i \text{ being an open set and } \ B_{r_1}(x_i) \subset \omega_i \subset B_{r_2}(x_i).$$

Then the corresponding observability inequality holds, say,

$$\int_{\mathbb{R}^n} |\varphi(x,T)|^2 \, dx \leq C \int_0^T \int_\omega |\varphi(x,t)|^2 \, dxdt \quad \forall \varphi_0 \in L^2(\mathbb{R}^n).$$

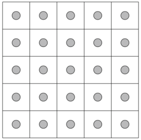

Figure: Observation over a sequence of balls

Key Step: two-ball and one-cylinder inequality

Let $0 < r < R < +\infty$ and $\delta \in (0,1]$. Then there are constants C and $\gamma \in (0,1)$ so that

$$\int_{B_R(x_0)} |\varphi(x,T)|^2 \mathrm{d}x \leq \left[C \int_{T/2}^{T} \int_{Q_{2R_0}(x_0)} \varphi^2(x,t) \mathrm{d}x\mathrm{d}t \right]^{\gamma} \times \left(\int_{B_r(x_0)} |\varphi(x,T)|^2 \mathrm{d}x \right)^{1-\gamma},$$

where $R_0 \triangleq (1+2\delta)R$.

Important fact: $x_0 \in \mathbb{R}^n$ is arbitrary.

We use the "frequency function method" to derive the above quantitative estimate of unique continuation for the heat equation with a bounded potential. This method is usually used by people to study the unique continuation of parabolic operators.

Monotonicity formula

Let Ω be a bounded and convex subset in $\mathbb{R}^N$. Let $r > 0$, $\lambda > 0$, $T > 0$ and $x_0 \in \Omega$. Denote by

$$G_\lambda(x,t) \triangleq \frac{1}{(T-t+\lambda)^{N/2}} e^{-\frac{|x-x_0|^2}{4(T-t+\lambda)}}, \quad t \in [0,T].$$

Set

$$N_{\lambda,r}(t) \triangleq \frac{\int_{B_r(x_0)\bigcap\Omega} |\nabla u(x,t)|^2 G_\lambda(x,t)\mathrm{d}x}{\int_{B_r(x_0)\bigcap\Omega} |u(x,t)|^2 G_\lambda(x,t)\mathrm{d}x}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} N_{\lambda,r}(t) \leq \frac{1}{T-t+\lambda} N_{\lambda,r}(t) + \frac{\int_{B_r(x_0)\bigcap\Omega} |(\partial_t u - \Delta u)(x,t)|^2 G_\lambda(x,t)\mathrm{d}x}{\int_{B_r(x_0)\bigcap\Omega} |u(x,t)|^2 G_\lambda(x,t)\mathrm{d}x}.$$

Null Controllability

Linear Case:

$$\begin{cases} \partial_t y - \Delta y + b(x,t)y = \chi_\omega u & \text{in } \mathbb{R}^N \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $y_0 \in L^2(\mathbb{R}^N)$ is an initial state, $b \in L^\infty(\mathbb{R}^N \times (0,T))$.

By duality between observability and controllability, the answer for the null controllability is positive for this kind of parabolic equation.

§4. Nonlinear Case

Consider the controlled semilinear heat equation

$$\begin{cases} \partial_t y - \Delta y + f(y) = \chi_\omega u & \text{in} \quad \mathbb{R}^N \times (0,T), \\ y(0) = y_0 & \text{in} \quad \mathbb{R}^N, \end{cases}$$

where $y_0 \in L^2(\mathbb{R}^N)$, $u \in L^2(\mathbb{R}^N \times (0,T))$, and the nonlinearity f satisfies a globally Lipschitz condition.

In bounded domains case:

The usual approach:
(i) null controllability of the linearized system
(ii) fixed-point theory.

When Ω is a general unbounded domain, however, the above approach cannot be directly applied because of the lack of compactness of Sobolev's embedding, which is one of the main ingredients used in (ii).

More precisely, the following Aubin-Lions lemma cannot be applied directly:

$$\{y \mid y \in L^2(0,T;H_0^1(\Omega)), y_t \in L^2(0,T;H^{-1}(\Omega))\}$$

compactly embeds into $L^2(0,T;L^2(\Omega))$.

Approximation strategy

Inspired by a recent work
A. Seelmann, I. Veselić. Exhaustion approximation for the control problem of the heat or Schrödinger semigroup on unbounded domains. Arch. Math. (Basel) 115 (2020), 195-213,
we propose the following strategy:

Step 1. Consider the null control problem in bounded and large cubes Q_k.

Step 2. Show that the above corresponding null controls converge weakly in $L^2(\omega \times (0,T))$ as $k \to \infty$ to a null control for our problem in the whole space.

$$\begin{cases} \partial_t y_k - \Delta y_k + f(y_k) = \chi_{\omega \cap Q_k} u_k & \text{in } Q_k \times (0,T), \\ y_k = 0 & \text{on } \partial Q_k \times (0,T), \\ y_k(\cdot,0) = y_k^0 & \text{in } Q_k. \end{cases}$$

$$\Downarrow \quad \text{As } k \text{ goes to infinity} \quad \Downarrow$$

$$\begin{cases} \partial_t y - \Delta y + f(y) = \chi_{\omega} u & \text{in } \mathbb{R}^N \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \mathbb{R}^N. \end{cases}$$

The main point is to find a uniform bound for the control u_k for all $k \in \mathbb{N}$.

Assumptions

(H_1). There is a sequence $\{x_i\}_{i=1}^{\infty} \subset \mathbb{R}^N$ so that

$$\mathbb{R}^N = \bigcup_{i=1}^{\infty} Q_{r_2}(x_i), \ \mathrm{int}(Q_{r_2}(x_i)) \bigcap \mathrm{int}(Q_{r_2}(x_j)) = \emptyset \text{ for each } i \neq j.$$

Moreover, $\omega \triangleq \bigcup_{i=1}^{\infty} \omega_i$, where ω_i is an open set and

$B_{r_1}(x_i) \subset \omega_i \subset B_{r_2}(x_i)$ for each $i \in \mathbb{N}$.

(H_2). For each $n \in \mathbb{N}$, $\Omega_n \triangleq \mathrm{int}\left(\bigcup_{i \in I_n} Q_{r_2}(x_i)\right)$ is convex, with

$\mathrm{Card}\,(I_n) < \infty$. In addition, $I_n \subsetneq I_m$ when $n < m$, and

$$\bigcup_{n=1}^{\infty} \Omega_n = \mathbb{R}^N.$$

(H_3). The function $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous and satisfies that $f(0) = 0$.

Theorem (Joint with L. Wang, preprint)

Assume $(H_1)-(H_3)$ *hold. Then there is a positive constant* κ *so that for any* $n \in \mathbb{N}$ *and any* $y_0 \in L^2(\mathbb{R}^N)$*, there is a control* $u_n \in L^2(\mathbb{R}^N \times (0,T))$*, with the uniform bound*

$$\|u_n\|_{L^2(\mathbb{R}^N\times(0,T))} \leq \kappa \|y_0\|_{L^2(\mathbb{R}^N)},$$

so that the corresponding solution $y_n \in C([0,T]; L^2(\Omega_n))$ *to the following semilinear heat equation*

$$\begin{cases} \partial_t y_n - \Delta y_n + f(y_n) = \chi_{\omega\cap\Omega_n} u_n & \text{in} \quad \Omega_n \times (0,T), \\ y_n = 0 & \text{on} \quad \partial\Omega_n \times (0,T), \\ y_n(\cdot,0) = z_0(\cdot) & \text{in} \quad \Omega_n, \end{cases}$$

satisfies that $y_n(\cdot,T) = 0$ *over* Ω_n.

Hence, one could derive the null controllability for the semilinear heat equation with control acted on an equidistrubuted set in $\mathbb{R}^N$.

Theorem

Assume that (H_1) *and* (H_3) *hold. Let* $T > 0$ *and* E *be a subset of positive measure in* $(0, T)$*. Then, for each initial value* $y_0 \in L^2(\mathbb{R}^N)$*, there is a control* $u \in L^2(0, T; L^2(\mathbb{R}^N))$ *with an upper bound*

$$\|u\|_{L^2(\mathbb{R}^N \times (0,T))} \leq \kappa \|y_0\|_{L^2(\mathbb{R}^N)},$$

so that the corresponding solution

$$\begin{cases} \partial_t y - \Delta y + f(y) = \chi_\omega \chi_E u & \text{in} \quad \mathbb{R}^N \times (0, T), \\ y(0) = y_0 & \text{in} \quad \mathbb{R}^N, \end{cases}$$

satisfies that $y(T) = 0$ *in* $\mathbb{R}^N$.

Questions

- unbounded potentials
- variable coefficients parabolic equations
- Other equations (particularly, Stochastic heat equations)
- super-linear case

Thank you !