Principal frequencies and inradius

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References

This is part of an ongoing research project on

"**Geometry of principal frequencies**"

in collaboration with Francesca Bianchi (Ferrara/Parma),
Francesca Prinari (Pisa) and Anna Chiara Zagati (Ferrara/Parma)

In particular, in the last part I will present some results from

- Bianchi - B., accepted on **Ann. Mat. Pura Appl.** (2022)

For more contributions `https://cvgmt.sns.it/person/198/`

Plan of the talk

What is the principal frequency?

Principal frequency VS. volume

Principal frequency VS. inradius

The case of fractional Sobolev spaces

What is the principal frequency?

Principal frequency VS. volume

Principal frequency VS. inradius

The case of fractional Sobolev spaces

Drums

Take a vibrating membrane fixed at the boundary of a set $\Omega \subset \mathbb{R}^2$

This is a superposition of a discrete set of **stationary vibrations**

$$U(x,t) = \sum_{k=1}^{\infty} u_k(x) \left(\alpha_k \cos\left(\sqrt{\lambda_k(\Omega)}\, t\right) + \beta_k \sin\left(\sqrt{\lambda_k(\Omega)}\, t\right)\right)$$

The **eigenpair** $(u_k, \lambda_k(\Omega))$ solves

$$-\Delta u_k = \lambda_k(\Omega)\, u_k \text{ in } \Omega, \qquad u_k = 0 \text{ on } \partial\Omega$$

- $\lambda_k(\Omega)$ is the $k-$**th eigenvalue of the Dirichlet-Laplacian**
- u_k is a $k - th$ **eigenfunction**
- $k \mapsto \sqrt{\lambda_k(\Omega)}$ **increasing** (it is the frequency of vibration)
- $\sqrt{\lambda_1(\Omega)}$ is the **principal frequency**, corresponds to the gravest tone of the drum

Heat conductors

Take a bounded heat conductor $\Omega \subset \mathbb{R}^3$ with uniform initial temperature

Put it in a cold basin. The time evolution of the temperature obeys

$$\begin{cases} \Delta U = \partial_t U, & \text{in } \Omega \times (0, +\infty) \\ U = 0, & \text{in } \partial\Omega \times (0, +\infty) \\ U = 1, & \text{at } t = 0 \end{cases}$$

and it is a superposition of a discrete set of **stationary heat-waves**

$$U(x,t) = \sum_{k=1}^{\infty} \alpha_k \, u_k(x) \, e^{-\lambda_k(\Omega)\, t}$$

Long-time behavior

$$U(t,x) \sim e^{-\lambda_1(\Omega)\, t} \qquad \text{for } t \to +\infty$$

$\lambda_1(\Omega)$ dictates the rate of heat dissipation

The principal frequency $\lambda_1(\Omega)$

Variational definition

$$\lambda_1(\Omega) = \inf_{u \in W_0^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla u|^2 \, dx \, : \, \int_\Omega |u|^2 \, dx = 1 \right\}$$

i.e. this is the sharp constant in the **Poincaré inequality**

$$\lambda_1(\Omega) \int_\Omega |u|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx$$

for functions u "zero on the boundary $\partial\Omega$"

Remarks

1. Definition makes (mathematical) sense in every dimension N
2. Definition makes sense for **a general open set** Ω, it is not necessary that the spectrum of the Dirichlet-Laplacian is discrete

Goal

In general, it is difficult to compute exactly $\lambda_1(\Omega)$

- Is it possible to give estimates on $\lambda_1(\Omega)$?

- Possibly in terms of simple **geometric** features of Ω?

A classic

Faber-Krahn inequality

For every $\Omega \subset \mathbb{R}^N$ open set with finite volume

$$\lambda_1(\Omega) \geq \frac{\lambda_1(B_1)\,|B_1|^{2/N}}{|\Omega|^{2/N}}$$

Equality holds if and only if Ω is a ball.

Remarks

- **Isoperimetry**: among sets with fixed volume, balls minimize λ_1
- **Geometric estimate**: λ_1 can be estimated <u>from below</u> by the volume

Proof of the Faber-Krahn inequality

1. The proof is based on the **isoperimetric inequality**
2. take $u \in W_0^{1,2}(\Omega)$ positive with $\displaystyle\int_\Omega |u|^2\,dx = 1$
3. define u^* its <u>symmetric decreasing rearrangement</u>, i.e. u^* is a radially symmetric decreasing function, defined on the ball Ω^* centered at the origin with $|\Omega^*| = |\Omega|$
4. u and u^* are **equi-measurable**, i.e.
$$\left|\left\{x \,:\, u(x) > t\right\}\right| = \left|\left\{x \,:\, u^*(x) > t\right\}\right|$$
5. in particular $\displaystyle\int_\Omega u^2\,dx = \int_{\Omega^*} (u^*)^2\,dx = 1$
6. $u^* \in W_0^{1,2}(\Omega^*)$ and
$$\boxed{\int_\Omega |\nabla u|^2\,dx \geq \int_{\Omega^*} |\nabla u^*|^2\,dx}$$
(Pólya-Szegő principle)

The Pólya-Szegő principle

We set $\mu(t) := \left|\left\{x \,:\, u(x) > t\right\}\right|$ **distribution function**

$$\int_\Omega |\nabla u|^2\,dx \overset{Coarea}{=} \int_0^{+\infty} \left(\int_{\{u=t\}} |\nabla u|^2 \frac{d\sigma}{|\nabla u|} \right) dt$$

$$\overset{Jensen}{\geq} \int_0^{+\infty} \left(\int_{\{u=t\}} |\nabla u| \frac{d\sigma}{|\nabla u|} \right)^2 \frac{dt}{\int_{\{u=t\}} |\nabla u|^{-1}\,d\sigma}$$

$$= \int_0^{+\infty} \frac{(\mathrm{Perimeter}(\{u > t\}))^2}{-\mu'(t)}\,dt$$

$$\overset{Isoperimetry}{\geq} \int_0^{+\infty} \frac{(\mathrm{Perimeter}(\{u^* > t\}))^2}{-\mu'(t)}\,dt$$

$$= \int_{\Omega^*} |\nabla u^*|^2\,dx$$

Reverse Faber-Krahn?

Faber-Krahn gives a geometric lower bound in terms of $|\Omega|^{-2/N}$

Is it possible to revert this estimate?

Is $\lambda_1(\Omega)$ equivalent to $|\Omega|^{-2/N}$? **NO**

Counter-examples

- Take a *slab–type* sequence $\Omega_n = (-n, n)^{N-1} \times (-1, 1)$

$$\lambda_1(\Omega_n) \to \left(\frac{\pi}{2}\right)^2 \qquad \text{and} \qquad |\Omega_n|^{-2/N} \to 0 \qquad \text{as } n \to \infty$$

Important: $(\pi/2)^2$ coincides with λ_1 for the interval $(-1, 1)$

- any set such that $\lambda_1(\Omega) > 0$ and $|\Omega| = +\infty$

An important class of sets having

$$\lambda_1(\Omega) > 0 \qquad \text{and} \qquad |\Omega| = +\infty$$

is that of *infinite curved wave-guides in* $\mathbb{R}^2$ *or in* $\mathbb{R}^3$

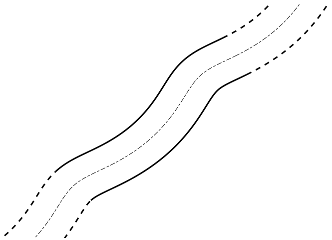


Figure: The tubular neighborhood of an unbounded planar curve

A humble criticism to Faber-Krahn

It is <u>useless</u> for sets like these

Inradius of a set

For an open set $\Omega \subset \mathbb{R}^N$, this is

$$r_\Omega = \sup \left\{ r > 0 \, : \, \exists \text{ a ball of radius } r \text{ contained in } \Omega \right\}$$

Essentially, this is the radius of a largest ball inscribed in Ω

The inradius r_Ω is a measure of "fatness", in a sense

By recalling the initial physical models, one could guess

$$r_\Omega \text{ is large} \qquad \Longleftrightarrow \qquad \text{the drum has a very low gravest tone}$$

or also

$$r_\Omega \text{ is large} \qquad \Longleftrightarrow \qquad \text{the heat dissipation is slow}$$

Is it true?

Is it possible to relate r_Ω and $\lambda_1(\Omega)$?

A very simple sharp estimate

For every ball $B_r(x_0) \subset \Omega$ we have

$$\lambda_1(\Omega) \leq \lambda_1(B_r(x_0)) = \frac{\lambda_1(B_1)}{r^2}$$

By arbitrariness of the ball, we get

$$\lambda_1(\Omega) \leq \frac{\lambda_1(B_1)}{r_\Omega^2}$$

Remark

This is a quantitative version of the statement "*if Ω contains large balls, then the first eigenvalue must be small*"

In particular, if $r_\Omega = +\infty$ then $\lambda_1(\Omega) = 0$ and the set **does not** support the Poincaré inequality (ex. $\mathbb{R}^N$, cones etc.)

A reverse estimate?

Is it possible to have

$$\lambda_1(\Omega) \geq \frac{c}{r_\Omega^2}$$

for some uniform $c > 0$?

False in general!

Take $\Omega = \mathbb{R}^2 \setminus \mathbb{Z}^2$, in this case

$$r_\Omega = \sqrt{2} \qquad \text{but} \qquad \lambda_1(\Omega) = 0$$

The second fact is due to **points have zero capacity** in dimension $N \geq 2$, thus $W_0^{1,2}(\mathbb{R}^2 \setminus \mathbb{Z}^2) = W_0^{1,2}(\mathbb{R}^2)$ and

$$\lambda_1(\mathbb{R}^2 \setminus \mathbb{Z}^2) = \lambda_1(\mathbb{R}^2) = 0$$

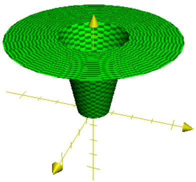
A glimpse of capacity

In which sense "points have zero capacity"?

- Take an open bounded set Ω, let us try to compare

$$W_0^{1,2}(\Omega) \qquad \text{and} \qquad W_0^{1,2}(\Omega \setminus \{x_0\})$$

- Take a function $u \in W_0^{1,2}(\Omega)$, to make it admissible for $W_0^{1,2}(\Omega \setminus \{x_0\})$ we need to "force it" to vanish at x_0
- we multiply u by a "funnel–type" function η_ε ($\varepsilon \ll 1$ is the radius of the hole)



- How much does it cost to "brutally" jump at 0 in a point?
- ...i.e. how much $\int_\Omega |\nabla(u\,\eta_\varepsilon)|^2$ increases w.r.t $\int_\Omega |\nabla u|^2$?
- the crucial question of course is

$$\text{how large} \qquad \int_\Omega |\nabla \eta_\varepsilon|^2 \qquad \text{is?}$$

- $|\nabla \eta_\varepsilon|$ is large....but it is integrated on a very small region!
- if we choose η_ε accurately (not as in the picture!) we have

$$\int_\Omega |\nabla \eta_\varepsilon|^2 \to 0$$

- η_ε is chosen so at "to pay as less as possible", i.e. through a minimization problem

$$\inf \left\{ \int_\Omega |\nabla \eta|^2 \ : \ \begin{array}{c} \eta \equiv 1 \text{ for } |x - x_0| > \varepsilon \\ \eta \equiv 0 \text{ for } |x - x_0| < \varepsilon^2 \end{array} \right\}$$

- thus $W_0^{1,2}(\Omega \setminus \{x_0\}) = W_0^{1,2}(\Omega)$

The role of topology and geometry

What if we can not cheat by drilling "*holes*"?

For example, we could ask what happens for

1. **simply connected sets**
2. **convex sets**

These are two classes of open sets for which very often things "work better"

Is it possible to have

$$\lambda_1(\Omega) \geq \frac{c}{r_\Omega^2}$$

with some uniform $c > 0$?

Simply connected sets for $N \geq 3$

- Take a ball B and remove n radial segments, with an endpoint on ∂B and the other at distance $1/n$ from the center
- We call $\{\Omega_n\}_{n\in\mathbb{R}}$ this sequence of open simply connected sets
- Segments in dimension $N \geq 3$ have **zero capacity**, thus again

$$\lambda_1(\Omega_n) = \lambda_1(B)$$

  while by a clever choice of the segments, we have $r_{\Omega_n} \to 0$
- **Conclusion**: for simply connected sets in dimension $N \geq 3$, we can not have

$$\lambda_1(\Omega) \geq \frac{c}{r_\Omega^2}$$

  with some uniform $c > 0$ (observe that the example above is even *contractible*, not just simply connected)

Convex sets

Theorem [Hersch-Protter]

For every $\Omega \subset \mathbb{R}^N$ open convex set

$$\lambda_1(\Omega) \geq \left(\frac{\pi}{2}\right)^2 \frac{1}{r_\Omega^2}$$

Inequality is sharp and the equality sign is never attained for bounded sets

Trivia

- Hersch proved the result in 1960 for $N = 2$
- Protter extended it to $N \geq 3$ in 1981 (incomplete argument)
- To the best of my knowledge, the first complete proof is by Kajikiya in 2015 (!), by an elegant and completely different strategy. This is based on the fact that "*the distance function is superharmonic on a convex set*"

A brief summary

- for every open set

$$\lambda_1(\Omega) \leq \frac{\lambda_1(B_1)}{r_\Omega^2}$$

- for **convex sets**, we have the (sharp) reverse estimate

$$\lambda_1(\Omega) \geq \left(\frac{\pi}{2}\right)^2 \frac{1}{r_\Omega^2}$$

- for general open sets, the reverse estimate

$$\lambda_1(\Omega) \geq \frac{c}{r_\Omega^2}$$

  with a uniform constant is **false**, even among **simply connected sets** in $N \geq 3$

Question

What happens for simply connected sets in dimension $N = 2$?

Simply connected sets in $\mathbb{R}^2$

Theorem [Makai (1965), Hayman (1977)]

Let $\Omega \subset \mathbb{R}^2$ be an open simply connected set, then

$$\lambda_1(\Omega) \geq \frac{c}{r_\Omega^2}$$

for a uniform constant $c > 0$

Trivia

- Makai found $c = 1/4$, which nowadays is known to be not optimal
- Hayman found $c = 1/900$, much worse than Makai's one...
- the sharp constant is **still unknown** (the best known result is due to Bañuelos & Carroll (1994))

A glimpse of proofs

Makai's proof

- it starts as Faber-Krahn's proof
- in Pólya-Szegő principle $\int_{\Omega} |\nabla u|^2 \, dx \geq \int_{\Omega^*} |\nabla u^*|^2 \, dx$...
- ...use **Bonnesen–type inequality** in place of **Isoperimetric inequality** (i.e. an "improved" isoperimetric inequality, with remainder term depending on r_Ω)

$$\ell(\partial\Omega) \geq \frac{|\Omega|}{r_\Omega} + \pi \, r_\Omega$$

Hayman's proof

- even if the constant is much worse, the proof is **elementary** (NO Coarea, NO isoperimetric inequality, NO rearrangements)
- it is just based on a quite simple **covering lemma** in terms of "boundary disks" (i.e. disks centered at $\partial\Omega$)...
- ...and a Poincaré inequality for "boundary disks"

$$\frac{c}{r^2}\int_{B_r}|u|^2\,dx \le \int_{B_r}|\nabla u|^2\,dx$$

- the radius r of the covering can be chosen in such a way that
  1. $r \sim r_\Omega$
  2. the disks do not overlap "too much"
- the result is then obtained by "patching" together all the Poincaré inequalities above

What is the principal frequency?

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The case of fractional Sobolev spaces

The fractional Laplacian

We consider the same kind of issues for the **fractional Laplacian**

$$(-\Delta)^s u(x) = \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(x+h)}{|h|^{2\,s}} \, \frac{dh}{|h|^N} \qquad \boxed{0 < s < 1}$$

The Fourier side

It is the pseudo-differential operator whose symbol is given by $|\xi|^{2\,s}$

The probabilistic side

The infinitesimal generator of an isotropic stable stochastic process with stationary and independent increments

The variational side

As $-\Delta$ is the first variation of the Dirichlet integral, $(-\Delta)^s$ is the first variation of the **fractional** $s-$**Dirichlet Integral**

$$\big[u\big]^2_{W^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \frac{\delta_h u}{|h|^s} \right|^2 \, dx \right) \, \frac{dh}{|h|^N}$$

A glimpse of interpolation

The quantity $\left[u\right]^2_{W^{s,2}(\mathbb{R}^N)}$ is "intermediate" between L^2 norm and Dirichlet integral

More precisely, it can be obtained by the $K-$**method** in **real interpolation** (Lions, Petree,...), with interpolation parameter s

Asymptotics I – Maz'ya-Shaposhnikova

$$s\left[u\right]^2_{W^{s,2}(\mathbb{R}^N)} \sim \int_{\mathbb{R}^N} |u|^2\,dx \qquad \text{for } s \searrow 0$$

Asymptotics II – Bourgain-Brezis-Mironescu

$$(1-s)\left[u\right]^2_{W^{s,2}(\mathbb{R}^N)} \sim \int_{\mathbb{R}^N} |\nabla u|^2\,dx \qquad \text{for } s \nearrow 1$$

First eigenvalue of the fractional Dirichlet-Laplacian

Notation

$$W^{s,2}(\mathbb{R}^N) = \Big\{ u \in L^2(\mathbb{R}^N) \,:\, [u]_{W^{s,2}(\mathbb{R}^N)} < +\infty \Big\}$$

$$\widetilde{W_0^{s,2}}(\Omega) = \text{`` closure of } C_0^\infty(\Omega) \text{ in } W^{s,2}(\mathbb{R}^N)\text{''}$$

Remark

Functions $u \in \widetilde{W_0^{s,2}}(\Omega)$ are considered as defined on the whole $\mathbb{R}^N$, with the **nonlocal boundary condition** $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$

Variational characterization

$$\lambda_1^s(\Omega) = \inf_{u \in \widetilde{W_0^{s,2}}(\Omega)} \left\{ [u]^2_{W^{s,2}(\mathbb{R}^N)} \,:\, \int_\Omega |u|^2 \, dx = 1 \right\}$$

sharp constant in the **fractional Poincaré inequality**

$$\lambda_1^s(\Omega) \int_\Omega |u|^2 \, dx \le [u]^2_{W^{s,2}(\mathbb{R}^N)}$$

for functions u "zero on the boundary $\mathbb{R}^N \setminus \Omega$"

The fractional Makai-Hayman inequality

Theorem [Bianchi - B.]

Let $\boxed{1/2 < s < 1}$, *for an open simply connected set* $\Omega \subset \mathbb{R}^2$

$$\lambda_1^s(\Omega) \geq \frac{c_s}{r_\Omega^{2\,s}}$$

for a uniform constant $c_s > 0$. Moreover, we have

$$c_s \sim \frac{1}{1-s} \quad \text{as } s \nearrow 1 \qquad \text{and} \qquad c_s \sim s - \frac{1}{2} \quad \text{as } s \searrow 1/2$$

Remarks

- NO reasonable "*Coarea–type trick*" for the fractional $s-$Dirichlet integral...we use a Hayman–type elementary proof
- By using *Bourgain-Brezis-Mironescu* we recover as $s \nearrow 1$ the classical Makai-Hayman inequality
- the result does not cover the case $s \leq 1/2$ and the constant deteriorates as $s \searrow 1/2$...why?

The fractional Makai-Hayman non-inequality

Theorem [Bianchi - B.]

There exists a sequence of open simply connected sets $\{\Omega\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ *such that for every* $\boxed{0 < s \leq 1/2}$

$$\lambda_1^s(\Omega_n) \to 0 \qquad \textit{and} \qquad 0 < r_{\Omega_n} \leq C$$

The fractional Makai-Hayman <u>*does not hold*</u> *for* $\boxed{0 < s \leq 1/2}$

Construction

- The sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ is constructed by taking the squares $(-n, n) \times (-n, n)$ and removing a periodic array of horizontal segments
- **Crucial point**: segments have **zero** $s-$**capacity** for $s \leq 1/2$. The <u>borderline case</u> $s = 1/2$ is delicate

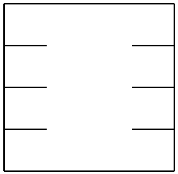

Figure: The set Ω_n with $n = 2$...

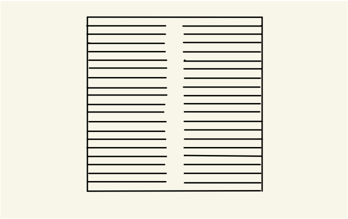

Figure: ...and the set Ω_n with $n = 10$ (the scales are different)

Thanks for your kind attention