FernUniversität in Hagen

On the Existence of Chaotic C_0-Semigroups on Quantum Graphs

Master's Thesis by Volker Kempf

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CHAPTER 1

Introduction

The foundations of graph theory are usually traced back to the problem *The Seven Bridges of Königsberg*, which was solved as early as 1736 by Leonard Euler with methods that are now considered graph theoretical. In contrast, the mathematical concept of *quantum graphs*, understood as metric graphs equipped with certain operators, is relatively new and has only in recent years become a separate area of research. First survey articles and dedicated books, e.g. [8, 23, 26], were published in the last two decades. Although the field seems to be a recent trend, this impression may be deceiving, because works that deal with structures that fit today's definition of a quantum graph started to appear already in the 1930s, in a wide variety of applications, among others in physics, chemistry and biology.

The range of applications for the quantum graph as a simplification of physical systems is immense, the book [8] gives an extensive overview, including hundreds of references for a multitude of separate research areas. Equipping metric graphs with a variety of differential operators makes it possible to study heat conduction, diffusion or wave propagation in a one dimensional complex, to name only a few fields of application.

Unfortunately, the term quantum graph is not consistently defined. A metric or equivalently weighted graph structure is always neccessary to set up the appropriate function spaces, but while some authors explicitly demand the self-adjointness of the used operator, see e.g. [4], others accept a more general definition, see e.g. [8]. For the purpose of our study, we will see in Theorem 5.1 that self-adjointness of the quantum graph operator makes generation of a chaotic semigroup impossible. That is why we follow the latter idea and introduce a quantum graph $\Gamma = (\mathcal{G}, A, VC)$ in Chapter 4 as the triple of a metric graph $\mathcal{G}$, equipped with a linear operator A and an appropriate set of vertex conditions VC that contrary to most applications do not necessarily result in a self-adjoint operator. We will see that the vertex conditions, which can be thought of as boundary conditions of the operator on each edge, already describe the complete topology of the graph.

The second mathematical concept used in this thesis in addition to quantum graphs are one-parameter semigroups of linear operators, or just semigroups for short, and in Chapter 3 we establish the theory to the extent needed in this thesis. A strongly continuous semigroup $(T(t))_{t\geq 0}$ is a family of bounded linear operators, so that for $t \in [0, \infty)$ and for an initial value f the map $t \mapsto T(t)f$ can be thought of as the solution to the abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t), & \text{for } t \geq 0, \\ u(0) = f, \end{cases}$$

which, if we consider the variable t to represent time, is a Banach space valued initial value problem. In this case the operator A is called the *generator* of the semigroup. Since operator semigroups can also be seen as dynamical systems, the question of stability is an important factor when analyzing their properties. From the particular viewpoint of this thesis, the different kinds of stability, see e.g. [15], are not relevant, and we will focus on chaotic behavior.

Chaoticity is in general a very undesirable characteristic for dynamical systems, which was first systematically examined in [11], and since then in various different publications for a variety of distinct cases, see e.g. [9, 17, 18, 24, 25, 28]. Not only is chaoticity undesirable, it also seems not to be very common, as not many explicit examples of chaotic semigroups have been discovered yet. Some are known in the context of first and second order partial differential equations, see e.g. [24, 25, 28] and [10], respectively, and a larger collection of examples can be found in [17, Chapter 7]. Also unlike e.g. in the case of contractive semigroups, where every semigroup operator is by definition a contraction itself, it was shown in [7] that there are chaotic strongly continuous semigroups that do not contain any chaotic operator.

In Section 3.2 we present the conditions for an operator A to be the generator of a strongly continuous semigroup, and Section 3.3 is dedicated to the study of chaotic semigroups and the spectral characteristics of their generators.

Examination of the semigroup that is generated by the operator of a quantum graph combines the two mentioned areas of research, and was subject of a number of publications, see e.g. [14, 20–22, 26]. In Chapter 5 we will follow this approach and investigate if certain quantum graph operators generate chaotic semigroups, which it seems has not yet been addressed in any existing articles. To this end, we first make some general observations about semigroups on quantum graphs and possible chaoticity in Section 5.1 and 5.2, then we define and study a model problem of diffusion-advection-reaction type and some variations.

CHAPTER 2

Preliminaries

As a start, this chapter introduces aspects concerning graph theory, functional analysis and operator theory that will be utilized later on. It is also used to establish the necessary notational framework for the thesis. Most definitions and theorems are tailored to the needs of this thesis and not presented in the most general form, we refer to [1, 13, 29, 30] and the references in the sections below for further information on the subjects.

2.1. Graph Theory

The following basic notations, definitions and theorems are, where not otherwise specified, based on [8] and [13]. In this section we introduce the notions of graph and digraph as well as further basic concepts concerning these structures. The assumptions on the topology of graphs and digraphs that we make at the end of the section will be in effect for the rest of the thesis.

Before we can give the formal definition of a graph, we need the following convention. Let V be a set and $k \in \mathbb{N}$. Then we denote by $\binom{V}{k}$ the set of subsets of V with exactly k elements, i.e.

$$\binom{V}{k} = \{U \subset V : \ |U| = k\},$$

where $|\cdot|$ denotes the cardinality of a set.

Definition 2.1. A triple $G = (V, E, \partial)$ of sets V, E and the *incidence map* $\partial : E \to \binom{V}{2} \cup \binom{V}{1}$ is called *graph*. Elements of V and E are called *vertices* and *edges*, respectively.

Example 2.2. Figure 1 illustrates the graph $G_b = (V, E, \partial)$, where

$$V = \{v_i : \ i \in \{1, \ldots, 8\}\}, \ E = \{e_i : \ i \in \{1, \ldots, 7\}\},$$
$$\partial(e_1) = \{v_1, v_2\}, \quad \partial(e_2) = \{v_1, v_3\}, \quad \partial(e_3) = \{v_1, v_4\}, \quad \partial(e_4) = \{v_1, v_5\},$$
$$\partial(e_5) = \{v_2, v_4\}, \quad \partial(e_6) = \{v_6, v_7\}, \quad \partial(e_7) = \{v_8\}.$$

The edge e_7 is an example of a loop, see Definition 2.6(iv).

Remark 2.3. It is possible to describe the topology of a graph with two matrices,

- the adjacency matrix, that contains the information about which vertices are connected by an edge, and
- the incidence matrix, that describes which vertices are the endpoints of which edge.

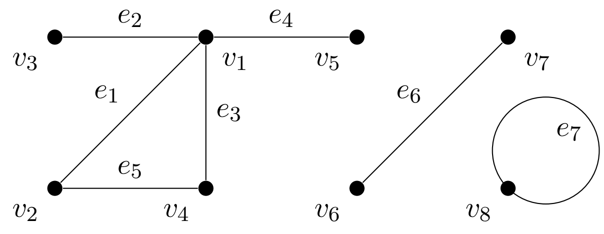


FIGURE 1. The graph G_b.

These matrices are especially important when working with discrete operators on graphs, but since we want to study the continuous structure of metric and quantum graphs, we do not go into more detail on these concepts, see any book on graph theory, e.g. [13], for more information.

The edges of a graph do not have any direction and only represent links between the connected vertices. For our purposes, it is preferable to use edges with an orientation, which leads to the next definition.

Definition 2.4. A triple $D = (V, E, \partial)$ of two sets V, E and the *incidence map* $\partial : E \to V \times (V \cup \{\infty\})$ is called a *directed graph* or *digraph.*

Example 2.5. Figure 2 illustrates the digraph $D_b = (V, E, \partial)$, where

$$V = \{v_i : \ i \in \{1, \dots, 7\}\}, \ E = \{e_i : \ i \in \{1, \dots, 7\}\},$$
$$\partial(e_1) = (v_1, v_2), \quad \partial(e_2) = (v_1, v_3), \quad \partial(e_3) = (v_4, v_7), \quad \partial(e_4) = (v_5, v_1),$$
$$\partial(e_5) = (v_2, v_4), \quad \partial(e_6) = (v_7, v_6), \quad \partial(e_7) = (v_2, \infty).$$

The edge e_7 is an example of a lead, see Definition 2.6(iv).

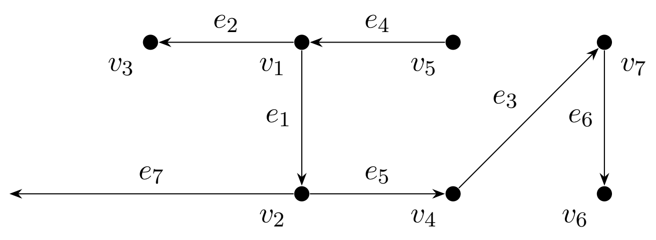


FIGURE 2. The digraph D_b.

We now introduce some important terminology concerning graphs and digraphs.

Definition 2.6. Let the vertices v, w and edges in the following list be either of a graph or digraph, depending on the context.

(i) An edge $e \in E$ with $\partial(e) = \{v, w\}$ is called *incident* to the vertices $v, w \in V$. An edge $e \in E$ with $\partial(e) = (v, w)$ is called *positively incident* to v, *negatively incident* to w and *incident* to both vertices. In the digraph case, the vertices v, w are called *initial* and *terminal* vertex of e, respectively. When the relation to the edge is important, these are alternatively written as e^{init} and e^{term}.

(ii) The sets E_v^+, E_v^- denote the sets of edges that are positively and negatively incident to $v \in V$, respectively. The set E_v, denotes the set of edges incident to $v \in V$.

(iii) The vertex $v \in V$ is called *isolated*, if $E_v = \emptyset$.

(iv) The edge $e \in E$ is called
- *directed edge* or *arc*, if $\partial(e) = (v, w)$,
- *loop*, if $\partial(e) = (v, v)$ for a digraph, or $|\partial(e)| = 1$ for a graph,
- *lead*, if $\partial(e) = (v, \infty)$,
- *reversal of* $f \in E$ with $\partial(f) = (v, w)$, if $e = (w, v)$ and we write $e = \overline{f}$.

A lead can be thought of as incident to only one vertex, with the other end leading to infinity.

(v) The edges $e_1, e_2 \in E$, $e_1 \neq e_2$ with $\partial(e_i) \in V \times V$ for $i = 1, 2$ are called
- *parallel*, if $\partial(e_1) = \partial(e_2)$,
- *anti-parallel*, if $\partial(e_1) = \partial(\overline{e_2})$,

Note that we exclude leads from the definition of parallel edges, i.e. there can be multiple leads incident to one vertex. This is consistent with the assumption that ∞ is not simply another vertex, but rather the concept of the edges going on indefinitely.

(vi) A (di-)graph is called
- *finite*, if $n \in \mathbb{N}$ exists so that $|E| = n$, otherwise it is called *infinite*,
- *locally finite*, if for every $v \in V$ a $n_v \in \mathbb{N}$ exists so that $|E_v| \leq n_v$,
- *simple*, if no loops or parallel edges exist.

Further on we need the concept of connection in graphs and digraphs, which basically means that when choosing two arbitrary vertices it is possible to find a string of edges that connects them. We state this notion mathematically precise in the following definitions.

Definition 2.7. Let $G = (V, E, \partial)$ be a graph. For two vertices $v, w \in V$, $v \neq w$, a graph $P = (\overline{V}, \overline{E}, \overline{\partial})$, where $\overline{V} \subset V$, $\overline{E} \subset E$ and for every $e \in \overline{E}$, $\overline{\partial}(e) = \partial(e)$ holds, is called *v, w-path in G*, if

- $v, w \in \overline{V}$,
- there are no parallel edges or loops in $\overline{E}$,
- there is exactly one edge $e \in \overline{E}$ incident to v and exactly one edge $f \in \overline{E}$ incident to w, the case $e = f$ may occur if $\overline{V} \setminus \{v, w\} = \emptyset$,
- for all $u \in \overline{V} \setminus \{v, w\}$ there are exactly two edges $e, f \in \overline{E}$ with $\overline{\partial}(e) \cap \overline{\partial}(f) = \{u\}$.

The graph G is called *connected*, if a v, w-path exists for all $v, w \in V$.

In order to transfer this concept to digraphs, we define the structures of an *orientation* of a graph and the *underlying graph* of a digraph.

Definition 2.8. Let I be an arbitrary index set, $D = (V_D, E_D, \partial_D)$ a digraph with $E_D = \{e_i\}_{i \in I}$, and $G = (V_G, E_G, \partial_G)$ a graph with $E_G = \{f_i\}_{i \in I}$. The digraph D is called *orientation* of G, if $V_D = V_G$ and for every edge f_i with

$\partial_G(f_i) = \{v_j, v_k\}$, there holds

$$\partial_D(e_i) = (v_j, v_k) \quad \text{or} \quad \partial_D(e_i) = (v_k, v_j)$$

for the arc e_i.

The graph G is called *underlying graph* of D, if $V_G = V_D$ and for every arc e_i with $\partial_D(e_i) = (v_j, v_k)$ there holds

$$\partial_G(f_i) = \{v_j, v_k\}$$

for the edge f_i.

For finding an orientation of a graph, we assume an arbitrary direction for every edge, and for finding the underlying graph of a digraph, we forget the direction of every arc. In general more than one orientation of a graph exist, but the underlying graph of a digraph D is unique and we denote it by $U(D)$. Using these terms, we can carry the concept of connection over to digraphs.

Definition 2.9. Let D be a digraph and $U(D)$ the corresponding underlying graph. Then D is called *connected*, if $U(D)$ is connected according to Definition 2.7.

We have seen from the preceding definitions, that the topological structure of graphs and digraphs can become very complicated even for relatively small numbers of vertices and edges. That is why for the purpose of this thesis, we restrict our analysis to subcategories of graphs which retain a certain level of regularity.

Assumption 2.10. For the rest of this thesis we will only consider graphs and digraphs that are

(i) simple,
(ii) connected,
(iii) locally finite and
(iv) have no isolated vertices.

Remark 2.11. The restriction to simple graphs is not really a limitation for most applications in quantum graphs, as described in [8, Remark 1.3.3] and [26, Remark 2.39]. Parallel edges and loops can be broken up by inserting new vertices somewhere along the edges. In the setting of quantum graphs with second order differential operators, see Chapter 4, we then need to prescribe certain vertex conditions at the inserted vertices, where we essentially claim continuity of the functions and continuity of the first derivative in these vertices.

2.2. Functional Analysis

Now we recall a few definitions from functional analysis that will be used in the next chapters. This section is based on [2] and [1].

First we recall that a normed space X is called separable, if a dense subset $D \subset X$, i.e. $\overline{D} = X$, exists, that is at most countably infinite. Here $\overline{D}$ denotes the topological closure of a set.

We use the standard declaration for the *Lebesgue spaces*

$$L^p(\Omega) = \{f : \Omega \to \mathbb{C} : \ \|f\|_{L^p(\Omega)} < \infty\},$$

in the sense of equivalence classes of functions, where $\Omega \subset \mathbb{R}$ is an open set, and

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$
$$\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |f(x)| \quad \text{for } p = \infty.$$

This one-dimensional setting is sufficient for the purposes of this thesis. A more general definition can be found in [2] or [30]. For $1 < p < \infty$, $q \in \mathbb{R}$ is called *conjugated exponent of* p, if $p^{-1} + q^{-1} = 1$. For the case $p = 1$ we set $q = \infty$ and vice versa.

Since dealing with quantum graphs involves differential operators, we use the spaces $C^k(\Omega)$, $k \in \mathbb{N}$, of k-times continuously differentiable functions and the following Sobolev spaces.

Definition 2.12. Let $k \geq 0$ be an integer, $p \in [1, \infty]$ and Ω an open subset of $\mathbb{R}$. The function space

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : \; f^{(i)} \in L^p(\Omega), 0 \leq i \leq k\},$$

where $f^{(i)}$ denotes the i-th weak derivative of f, is called *Sobolev space*. Equipped with the norms

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{i=0}^{k} \|f^{(i)}\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$
$$\|f\|_{W^{k,\infty}(\Omega)} = \max_{0\leq i\leq k} \|f^{(i)}\|_{L^\infty(\Omega)} \quad \text{for } p = \infty,$$

the Sobolev spaces $W^{k,p}(\Omega)$ are Banach spaces, see [1]. In the case $p = 2$ we write $H^k(\Omega) = W^{k,2}(\Omega)$.

The spaces $L^2(\Omega)$ and $H^k(\Omega)$ equipped with the inner products

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)\overline{g(x)} \,\mathrm{d}x,$$
$$(f, g)_{H^k(\Omega)} = \sum_{i=0}^{k} (f^{(i)}, g^{(i)})_{L^2(\Omega)},$$

become Hilbert spaces. Here $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Later in Section 4.2 we use metric graphs as topological setting, which are collections of one-dimensional intervals with certain endpoints identified with each other. To define the appropriate function spaces on this type of structure, we need to consider the direct sum of a set of Banach spaces.

Definition 2.13. Let $\{X_i\}_{i\in I}$ be a family of Banach spaces, where I is an arbitrary index set. The space

$$X = \bigoplus_{i\in I} X_i = \left\{ (f_i)_{i\in I} \in \prod_{i\in I} X_i : \; f_i \neq 0 \text{ only for finitely many } i \right\},$$

where $\prod_{i\in I} X_i$ denotes the Cartesian product of the spaces $\{X_i\}_{i\in I}$, is called *direct sum* of $\{X_i\}_{i\in I}$. Using one of the norms

$$\|f\|_{X^p} = \left(\sum_{i\in I} \|f_i\|_{X_i}^p\right)^{\frac{1}{p}} \quad \text{for } 1 \le p < \infty,$$
$$\|f\|_{X^\infty} = \sup_{i\in I} \|f_i\|_{X_i} \quad \text{for } p = \infty,$$

where $f = (f_i)_{i\in I} \in X$, to complete the space X, we get the Banach spaces X^p and X^∞. If $|I| < \infty$, we have

$$\bigoplus_{i\in I} X_i = \prod_{i\in I} X_i.$$

2.3. Operator Theory

To conclude this preliminary chapter, we present relevant definitions and results from the study of linear operators on Hilbert and Banach spaces, based on [30].

Definition 2.14. Let X_1, X_2 be normed spaces. A mapping $A : D(A) \to X_2$, where $D(A)$ is a linear subspace of X_1, satisfying

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g),$$

for all $f, g \in D(A)$ and $\alpha, \beta \in \mathbb{C}$, is called *linear operator from* X_1 *in* X_2 and *linear operator in* X, if $X = X_1 = X_2$.

For a linear operator A we will usually write Af instead of $A(f)$. We now collect some useful terms in connection with linear operators.

Definition 2.15. Let X_1, X_2, X_3 be normed spaces and A, B linear operators from X_1 in X_2.

(i) The linear subspace $D(A)$ of X_1 from Definition 2.14 and the set $R(A) = \{Af : f \in D(A)\} \subset X_2$ are called *domain* and *range* of A, respectively.

(ii) The linear operator A is called
- *extension* of B, if $D(B) \subset D(A)$ and $Af = Bf$ for all $f \in D(B)$, we then write $B \subset A$,
- *equal* to B, if A is an extension of B and B is an extension of A, we then write $A = B$,
- *densely defined*, if its domain is dense in X_1, i.e. $\overline{D(A)} = X_1$,
- *bounded*, if there is $C \in \mathbb{R}$ so that for all $f \in X_1$ the estimate $\|Af\|_{X_2} \le C\|f\|_{X_1}$ holds,
- *closed*, if for every convergent sequence $(f_k)_{k\in\mathbb{N}}$ in $D(A)$ with $f_k \to f$ for which the sequence $(Af_k)_{k\in\mathbb{N}}$ is convergent in X_2 with $Af_k \to g$ follows that $f \in D(A)$ and $Af = g$.

(iii) The linear operator A in a Banach space X is called
- *dissipative*, if the estimate $\|(\lambda I - A)f\| \ge \lambda\|f\|$ holds for all $\lambda > 0$ and $f \in D(A)$,
- *m-dissipative*, if it is dissipative and $R(I - A) = X$ holds.

(iv) The linear operator A in a separable Banach space X is called

- *hypercyclic*, if $f \in X$ exists, so that the set $\{f\} \cup \{A^n f : n \in \mathbb{N}\}$ is dense in X,
- *chaotic*, if it is hypercyclic and the set $\{f \in X :$ there is a $n \geq 1$ so that $A^n f = f\}$ is dense in X.

(v) Let C be a linear operator from X_1 in X_2 and D a linear operator from X_2 in X_3. Then the *product* DC is a linear operator from X_1 in X_3 that is defined by

$$(DC)f = D(Cf) \quad \text{for } f \in D(DC),$$
$$D(DC) = \{f \in D(C) : \; Cf \in D(D)\}.$$

As is the case in real and complex analysis, the notion of continuity is of importance in the setting of functional analysis, which leads to the following definition.

Definition 2.16. Let X_1, X_2 be normed spaces. The mapping $A : D(A) \to X_2$, $D(A) \subset X_1$, is called *continuous in* $f \in D(A)$, if $Af_k \to Af$ holds for every sequence $(f_k)_{k \in \mathbb{N}} \subset D(A)$ with $f_k \to f$. A is called *continuous*, if it is continuous in every $f \in D(A)$.

This definition can be hard to verify, but for linear operators continuity can be characterized more conveniently: A linear operator is continuous if and only if it is bounded, see e.g. [30, Theorem 2.1].

We denote by $\mathcal{B}(X_1, X_2)$ the space of bounded linear operators A from X_1 in X_2 for which $D(A) = X_1$. If $X = X_1 = X_2$, we write $\mathcal{B}(X)$. The *operator norm* of a linear operator $A \in \mathcal{B}(X_1, X_2)$ is well-defined by

$$\|A\|_{\mathcal{B}(X_1,X_2)} = \sup_{\|f\|_{X_1}=1} \|Af\|_{X_2},$$

and

$$\|A\| = \inf\{C \in \mathbb{R} : \; \|Af\| \leq C\|f\| \text{ for all } f \in D(A)\}$$

holds. Furthermore the operator norm is sub-multiplicative, i.e.

$$\|AB\| \leq \|A\| \, \|B\| \tag{2.1}$$

holds for $A, B \in \mathcal{B}(X)$. As already done in this section, we will drop the index of any norm and just write $\|\cdot\|$, if no confusion can arise.

Definition 2.17. Let X be a complex normed space. The space

$$X' = \{F : X \to \mathbb{C} : \; F \text{ linear and continuous}\},$$

is called (*topological*) *dual space* of X and the elements of X' are called *linear continuous functionals* on X.

With the operator norm $\|\cdot\|_{\mathcal{B}(X,\mathbb{C})}$ the dual space X' is a normed space and since $\mathbb{C}$ is complete, X' is a Banach space, see [30, Satz 2.12].

Definition 2.18. A Banach space X is called *reflexive*, if the natural linear injection $J : X \to X''$ into its second dual space $X'' = (X')'$, which is defined by

$$Jf(g) = g(f), \quad f \in X, \quad g \in X',$$

is surjective.

For $p \in [1, \infty)$ the dual spaces $L^p(\Omega)'$ of the Lebesgue spaces can be identified with $L^q(\Omega)$, where q is the conjugated exponent of p, as the following proposition shows.

Proposition 2.19. *Let $q \in \mathbb{R}$ be the conjugated exponent of $p \in [1, \infty]$. Then every function $f \in L^q(\Omega)$ generates a linear bounded functional*

$$F_f(g) = \int_\Omega f(x)g(x)\,\mathrm{d}x,$$

in $L^p(\Omega)$. The mapping $f \mapsto F_f$ is linear and isometric. For $p \in [1, \infty)$ the mapping $L^q(\Omega) \to L^p(\Omega)'$, $f \mapsto F_f$ is surjective, i.e. every linear bounded functional in $L^p(\Omega)$ is generated by a function $f \in L^q(\Omega)$.

Proof. The proof can be found in [30, Satz 2.16]. □

By the above proposition, the dual spaces of the Lebesgue spaces with $p \in [1, \infty)$ are isometrically isomorphic to the Lebesgue spaces with conjugated exponents q, i.e. $L^p(\Omega)' \cong L^q(\Omega)$, see also [1, Theorem 2.44 and 2.45]. By [1, Theorem 2.46], the Lebesgue spaces $L^p(\Omega)$ are reflexive for $p \in (1, \infty)$. While $L^1(\Omega)$ is separable, its dual space that is isometrically isomorphic to $L^\infty(\Omega)$ is not, which means that neither of these spaces can be reflexive.

From linear algebra we are familiar with the concept of the spectrum, i.e. the eigenvalues, of a linear operator in finite-dimensional vector spaces. For the general infinite-dimensional case additional concepts are necessary. For a linear operator A in a normed space X we use the standard definitions of the *resolvent set*

$$\rho(A) = \{\lambda \in \mathbb{C} :\ (\lambda I - A) \text{ injective}, \overline{R(\lambda I - A)} = X, (\lambda I - A)^{-1} \text{ bounded}\},$$

the *resolvent*

$$R(\cdot, A) : \rho(A) \to \mathcal{B}(X),\ \lambda \mapsto (\lambda I - A)^{-1},$$

and the *spectrum*

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

If X is a Banach space and A a closed operator, the definition of the resolvent set can be simplified to

$$\rho(A) = \{\lambda \in \mathbb{C} :\ (\lambda I - A) \text{ bijective}\} = \{\lambda \in \mathbb{C} :\ (\lambda I - A)^{-1} \in \mathcal{B}(X)\}.$$

For a comprehensive introduction to spectral theory of linear operators see e.g. [30]. We denote the standard decomposition of the spectrum into the disjoint parts *continuous*, *residual* and *point* spectrum by $\sigma_c(A)$, $\sigma_r(A)$ and $\sigma_p(A)$, respectively. For any linear operator A we call the number

$$s(A) = \sup\{\mathrm{Re}(\lambda) :\ \lambda \in \sigma(A)\}$$

spectral bound of A.

We now get to the fundamental term of the adjoint operator, which will lead to the concept of self-adjoint operators. These linear operators form an important category and appear often in applications.

Definition 2.20. Let X_1, X_2 be Hilbert spaces and A a linear operator from X_1 in X_2 with $\overline{D(A)} = X_1$. Then the linear operator A^* from X_2 in X_1 defined by

$$\begin{aligned} D(A^*) = \{g \in X_2 :\ &\text{there is a } h \in X_1, \\ &\text{where } (Af, g)_{X_2} = (f, h)_{X_1} \text{ for all } f \in D(A)\}, \\ A^* g = h \qquad &\text{for } g \in D(A^*),\ \text{if for all } f \in D(A) : (Af, g)_{X_2} = (f, h)_{X_1}, \end{aligned}$$

is unique and called the *adjoint operator of* A. If $X_1 = X_2$ and $A = A^*$, then A is called *self-adjoint*.

A very fundamental result of operator theory in Hilbert spaces is that the spectrum of every self-adjoint operator A is real, i.e. $\sigma(A) \subset \mathbb{R}$, see e.g. [30, Satz 5.14].

CHAPTER 3

Chaotic C_0-Semigroups

Describing the solutions of abstract Cauchy problems, a form of evolution equation, one-parameter operator semigroups are closely connected to dynamical systems. In Section 3.1 and 3.2 we give a short introduction on the basics of the subject, for a more general examination of the subject and the proofs of the stated results we refer to [15, 16, 27].

In the context of semigroups describing dynamical systems, an important area of research is stability, which has a strong connection to the spectral properties of the operators and the generator of the semigroup, see e.g. [15, Section V.3]. As a kind of antithesis to stability, we are interested in the concept of hypercyclicity and chaoticity of strongly continuous semigroups, and we explore these concepts in Section 3.3.

When we mention a semigroup we always mean a one-parameter semigroup of linear operators, and we will use the terms C_0- and *strongly continuous* semigroup interchangeably.

3.1. Operator Semigroups

In general, a parameterized family of operators $(T(t))_{t\geq 0}$ is an operator semigroup, if it satisfies a certain semigroup property, which we state in the next definition. To give a semigroup more structure there are different concepts, like uniform continuity or strong continuity. While uniform continuity is a very nice property, the class of uniformly continuous operator semigroups is too small to encompass the arising problems from applications, and the weaker concept of strong continuity is needed.

Definition 3.1. Let X be a Banach space and $(T(t))_{t\geq 0}$ a family of bounded linear operators in X. The family $(T(t))_{t\geq 0}$ is called *strongly continuous* (or C_0-)*semigroup* (*of linear operators*), if the semigroup property

$$\text{(SP)} \qquad \begin{cases} T(t+s) = T(t)T(s) & \text{for } t, s \geq 0, \\ T(0) = I, \end{cases}$$

holds and the *orbit maps* $\xi_f : [0, \infty) \to X$, $t \mapsto \xi_f(t) = T(t)f$, are continuous for every $f \in X$. The range of the orbit map $R(\xi_f) = \{T(t)f : \ t \geq 0\}$ is called *orbit of* f.

Strongly continuous semigroups can be characterized in the following way.

Lemma 3.2. *Let X be a Banach space and $(T(t))_{t\geq 0}$ a semigroup on X. Then the following statements are equivalent:*

(i) $(T(t))_{t\geq 0}$ *is strongly continuous.*
(ii) $\lim_{t\searrow 0} T(t)f = f$, *for* $f \in X$.

(iii) *There are constants* $\delta > 0$ *and* $M \geq 1$ *and a dense subset* $D \subset X$*, so that*
- $\|T(t)\| \leq M$*, for* $t \in [0, \delta]$*,*
- $\lim_{t \searrow 0} T(t)f = f$*, for* $f \in D$*.*

PROOF. The proof can be found in [15, Proposition I.1.3]. □

Proposition 3.3. *Let* $(T(t))_{t\geq 0}$ *be a strongly continuous semigroup. Then there are constants* $\omega \in \mathbb{R}$*,* $M \geq 1$*, so that for* $t \geq 0$

$$\|T(t)\| \leq M\mathrm{e}^{\omega t} \tag{3.1}$$

holds.

PROOF. For the proof see [15, Proposition I.1.4]. □

Using the above proposition, we can identify several categories of C_0-semigroups. A strongly continuous semigroup is called *bounded*, if $\omega = 0$ is possible and *contractive*, if $\omega = 0$ and $M = 1$ is possible. It is called *isometric*, if $\|T(t)f\| = \|f\|$ for all $t \geq 0$ and all $f \in X$. We call the number

$$\omega_0 = \inf \left\{ \omega \in \mathbb{R} : \text{ there is } M_\omega \geq 1 \text{ so that } \|T(t)\| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0 \right\}$$

growth bound of the semigroup $(T(t))_{t\geq 0}$.

3.2. The Generator of a C_0-Semigroup

It is possible to show, see e.g. [16, Lemma II.1.1], that given a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X, if for $f \in X$ the orbit map ξ_f is differentiable on $\mathbb{R}_+$, then it is right differentiable at $t = 0$. So for the subspace consisting of those $f \in X$ with differentiable orbit map, the right derivative at $t = 0$ yields an operator which is characteristic of the semigroup.

Definition 3.4. Let X be a Banach space and $(T(t))_{t\geq 0}$ a strongly continuous semigroup on X. The operator A defined by

$$Af = \frac{\mathrm{d}\xi_f}{\mathrm{d}t}(0) = \lim_{h \searrow 0} \frac{(T(h)f - f)}{h},$$
$$D(A) = \{ f \in X \ : \ \xi_f \text{ is differentiable in } [0, \infty)\},$$

is called *generator* of $(T(t))_{t\geq 0}$.

Operator semigroups appear naturally when we examine the Banach space valued abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t) & \text{for } t \geq 0, \\ u(0) = f, \end{cases} \tag{ACP}$$

that can be seen as a type of evolution equation. Here A is a linear operator in a Banach space X and $f \in X$. The variable t represents time and a function $u : [0, \infty) \to X$, $t \mapsto u(t)$, is called a *classical solution* of (ACP), if

- u satisfies (ACP),
- $u \in C^1([0, \infty))$ and
- $u(t) \in D(A)$ for all $t \geq 0$.

It turns out that if the operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$, then for every initial value $f \in D(A)$, the function $t \mapsto u(t) = T(t)f$ is a classical solution of (ACP). In order to analyze problems of this type, it is important to know characterizations of the operator A from (ACP) which make it the generator of a strongly continuous semigroup.

It is relatively easy to prove, see e.g. [16, Theorem I.3.7], that for a Banach space X, the operators $A \in \mathcal{B}(X)$ generate semigroups $(T(t))_{t\geq 0}$ of the form

$$T(t) = \mathrm{e}^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!},$$

which are uniformly continuous in the sense that the mapping $t \mapsto T(t) \in \mathcal{B}(X)$ is continuous for $t > 0$ in the operator norm topology on $\mathcal{B}(X)$. This definition of course implies strong continuity. Uniform continuity is a nice property, but many natural semigroups do not meet this condition, see e.g. the examples in [16, Section I.4.c].

That is why we need to consider the weaker concept of strong continuity instead and must find additional criteria for generators of these semigroups. In preparation for these theorems, we need the following lemmata.

Lemma 3.5. *The generator of a strongly continuous semigroup is a closed and densely defined linear operator. It is unique.*

Proof. For the proof we refer to [15, Theorem II.1.4]. □

Lemma 3.6. *Let X be a Banach space, $(T(t))_{t\geq 0}$ a strongly continuous semigroup on X with generator A and constants $\omega, M \in \mathbb{R}$, $M \geq 1$, so that*

$$\|T(t)\| \leq M\mathrm{e}^{\omega t}$$

holds for $t \geq 0$, see Proposition 3.3.
Then the following properties hold for A:

- (i) *If $\lambda \in \mathbb{C}$ so that $R_\lambda f := \int_0^\infty \mathrm{e}^{-\lambda s} T(s) f \,\mathrm{d}s$ exists for all $f \in X$, then $\lambda \in \rho(A)$ and $R(\lambda, A) = R_\lambda$.*
- (ii) *If $\mathrm{Re}(\lambda) > \omega$, then $\lambda \in \rho(A)$ and the resolvent is given by the integral expression in* (i).
- (iii) $\|R(\lambda, A)\| \leq \frac{M}{\mathrm{Re}(\lambda) - \omega}$ *for all λ with $\mathrm{Re}(\lambda) > \omega$.*

Proof. The proof can be found in [15, Theorem II.1.10]. □

As an immediate implication of the preceding lemma, we can state the following relation between growth bound of a semigroup and spectral bound of its generator.

Corollary 3.7. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup and A its generator, then*

$$-\infty \leq s(A) \leq \omega_0 < \infty$$

holds.

In Definition 3.4 of the generator, the strongly continuous semigroup is assumed, and the generator is constructed. In contrast, the following theorems define the characteristics a given linear operator needs in order to be the generator of a strongly continuous semigroup.

Theorem 3.8 (Hille-Yosida, [15, Theorem II.3.5])**.** *Let X be a Banach space and A a linear operator in X. Then the following statements are equivalent:*

(i) *A is the generator of a strongly continuous contractive semigroup.*

(ii) *A is closed, densely defined and for $\lambda > 0$*

$$\|\lambda R(\lambda, A)\| \leq 1 \quad \textit{and} \quad \lambda \in \rho(A)$$

holds.

(iii) *A is closed, densely defined and for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$*

$$\|R(\lambda, A)\| \leq \frac{1}{\mathrm{Re}(\lambda)} \quad \textit{and} \quad \lambda \in \rho(A)$$

holds.

Proof. We only state the idea of the proof. Reviewing Lemma 3.5 and Lemma 3.6, there is only (ii)⇒(i) left to show. We use the *Yosida approximants*

$$A_n = nAR(n, A) = n^2 R(n, A) - nI, \quad n \in \mathbb{N},$$

which are bounded and thus generate uniformly continuous semigroups $(T_n(t))_{t\geq 0}$. The proof is then completed by proving the existence of operators $T(t)f = \lim_{n\to\infty} T_n(t)f$, showing that they form a contractive semigroup with generator A. □

The above theorem characterizes the generator of contractive strongly continuous semigroups, the next theorem does the same for the general case.

Theorem 3.9 (Feller-Miyadera-Phillips)**.** *Let X be a Banach space, A a linear operator in X and $\omega \in \mathbb{R}$, $M \geq 1$ constants. Then the following statements are equivalent:*

(i) *A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ with*

$$\|T(t)\| \leq M\mathrm{e}^{\omega t} \quad \textit{for } t \geq 0.$$

(ii) *A is closed, densely defined and for $\lambda > \omega$ and $n \in \mathbb{N}$*

$$\|((\lambda - \omega)R(\lambda, A))^n\| \leq M \quad \textit{and} \quad \lambda \in \rho(A)$$

holds.

(iii) *A is closed, densely defined and for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega$ and $n \in \mathbb{N}$*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\mathrm{Re}(\lambda) - \omega)^n} \quad \textit{and} \quad \lambda \in \rho(A)$$

holds.

Proof. For the proof we refer to [15, Theorem II.3.8]. □

Theorem 3.9 shows that for a given linear operator A to be the generator of a strongly continuous semigroup, the following conditions must be satisfied:

(i) The spectrum $\sigma(A)$ must be contained in a left half plane.

(ii) The linear operator A must be closed and densely defined.

(iii) Estimates of the form

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\mathrm{Re}(\lambda) - \omega)^n},$$

must hold for the resolvent on a right half plane.

Every strongly continuous semigroup can be re-scaled to a bounded semigroup, see [15, Section I.1.10], and for every bounded semigroup, an equivalent norm can be constructed to make the semigroup contractive, as [15, Lemma 3.10] shows. This motivates a closer look at contractive semigroups, for which a characterization of the generator alternate to Theorem 3.8 is possible, without having to rely on the estimates for the resolvent.

Theorem 3.10 (Lumer-Phillips)**.** *Let X be a Banach space and A a densely defined, dissipative operator in X. Then the following statements are equivalent:*

(i) *The closure $\overline{A}$ of A is the generator of a contractive semigroup.*
(ii) *The range $R(\lambda I - A)$ of $(\lambda I - A)$ is dense in X for some, and thus all, $\lambda > 0$.*

Proof. The proof can be found in [15, Theorem II.3.15]. □

From the above theorem we get the two following corollaries, which can be applied to easily show the generator property of an operator.

Corollary 3.11. *Let X be a Banach space, A a densely defined, closed and dissipative linear operator in X and let $\lambda > 0$ exist with $\lambda \in \rho(A)$. Then A generates a contractive semigroup.*

Proof. The assumptions satisfy all conditions of Theorem 3.10, and for a closed operator $\overline{A} = A$ holds. □

Corollary 3.12. *Let X be a Banach space and A a densely defined dissipative operator in X. If the adjoint operator A^* is dissipative, then the closure $\overline{A}$ of A generates a contractive strongly continuous semigroup.*

Proof. See [15, Corollary II.3.17] for a proof. □

We conclude this section with some results from spectral and perturbation theory for generators of operator semigroups, which we will use in the examples for chaotic semigroups in Section 3.3 and Section 5.4.

The first proposition states that the generator property of a linear operator is preserved under a bounded perturbation.

Proposition 3.13. *Let X be a Banach space, A a linear operator that generates a strongly continuous semigroup on X and $B \in \mathcal{B}(X)$. Then $C = A + B$, $D(C) = D(A)$, generates a strongly continuous semigroup on X.*

Proof. See [16, Theorem III.1.3] for a proof. □

If we relax the condition $B \in \mathcal{B}(X)$ and allow unbounded perturbations, we need additional assumptions so that the resulting operator still generates a strongly continuous semigroup.

Proposition 3.14. *Let X be a Banach space and A a linear operator that generates a contractive C_0-semigroup on X. Let B be a dissipative linear operator with $D(B) \supset D(A)$ and*

$$\|Bf\| \leq \alpha\|Af\| + \beta\|f\|,$$

for $f \in D(A)$, where $0 \leq \alpha < 1$ and $\beta \geq 0$. Then $A + B$ generates a contractive C_0-semigroup.

Proof. For the proof see [27, Corollary 3.3.3]. □

The next proposition is called *spectral mapping theorem for the point spectrum* and relates the point spectrum of the generator to the spectra of the semigroup operators.

Proposition 3.15. *Let X be a Banach space and A a linear operator that generates the strongly continuous semigroup $(T(t))_{t\geq 0}$. Then the identities*

(i) $\sigma_p(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma_p(A)}$, *for $t \geq 0$,*
(ii) $\ker(\lambda I - A) = \bigcap_{t\geq 0} \ker(\mathrm{e}^{\lambda t} I - T(t))$, *for $\lambda \in \mathbb{C}$,*
(iii) $\ker(\mathrm{e}^{\lambda t} I - T(t)) = \overline{\mathrm{span}} \left\{\bigcup_{n\in\mathbb{Z}} \ker\left(\left(\lambda + \frac{2\pi n i}{t}\right) I - A\right)\right\}$, *for $t > 0$,*

hold.

Proof. For the proof we refer to [16, Theorem IV.3.7 and Corollary IV.3.8]. □

3.3. Hypercyclicity and Chaoticity

We now investigate some aspects of the dynamic behavior of strongly continuous semigroups. In finite-dimensional spaces every linear operator A can be represented by a matrix and is always bounded, thus always generates a uniformly continuous semigroup that takes the form of the exponential matrix

$$T(t) = \mathrm{e}^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \quad t \geq 0.$$

For these matrix semigroups there are three typical types of behavior for $t \to \infty$: The semigroup e^{tA} can be

- *stable*, i.e. $\lim_{t\to\infty} \|T(t)\| = 0$, which is equivalent to all eigenvalues of A having negative real part, see the Liapunov stability theorem, e.g. in [16, Theorem I.2.10],
- *periodic*, i.e. there is $t_0 > 0$ so that $e^{t_0 A} = I$, which is in our finite-dimensional case equivalent to $\sigma(A) \subset \alpha i\mathbb{Z}$ for some $\alpha > 0$ and the corresponding eigenvectors spanning the space X, see e.g. [16, Theorem IV.2.26],
- *divergent*, i.e. $\|T(t)\| \to \infty$ as $t \to \infty$.

Additional concepts of long term behavior can occur in infinite dimensions, which we will examine in this section. We mainly follow [11] and [17] for our observations.

Let X be a Banach space, $(T(t))_{t\geq 0}$ a strongly continuous semigroup on X and A its generator. We will need the following subsets of X:

$$X_\infty = \{f \in X : \forall \varepsilon > 0\ \exists (g \in X, t > 0) : (\|g\| < \varepsilon \wedge \|T(t)g - f\| < \varepsilon)\},$$
$$X_0 = \{f \in X : \lim_{t\to\infty} T(t)f = 0\},$$
$$X_p = \{f \in X : \exists t > 0 : T(t)f = f\}.$$

The elements of X_p are called *periodic points* of $(T(t))_{t\geq 0}$.

Definition 3.16. Let X be a separable Banach space and $(T(t))_{t\geq 0}$ a strongly continuous semigroup on X. The semigroup $(T(t))_{t\geq 0}$ is called *hypercyclic*, if $f \in X$ exists, so that the range $R(\xi_f)$ of its orbit map is dense in X. It is called *chaotic*, if additionally X_p is dense in X.

The book of Devaney [12] includes a definition of chaos, which required an arbitrary continuous mapping in a metric space to

- be transitive,
- have a dense set of periodic points and
- have sensitive dependence on the initial conditions,

in order to be chaotic. The third item turned out to be redundant, as proved in [6], and our definition is consistent with this approach.

While hypercyclicity is rather common, as it was showed in [9] that every separable infinite-dimensional Banach space admits a uniformly continuous hypercyclic semigroup, chaoticity on the other hand seems to be different. Only a few examples of chaotic semigroups are known, many of them are summed up in [17, Chapter 7], and a full characterization of the generator of a chaotic strongly continuous semigroup has not been found yet.

The first sufficient spectral condition for the generator of a chaotic semigroup, which was later generalized in [5], was given in [11]. In order to present the proof of this theorem, we need the following lemmata.

Lemma 3.17 ([11, Theorem 2.2])**.** *Let X be a separable Banach space and $(T(t))_{t\geq 0}$ a strongly continuous semigroup on X. Then the following statements are equivalent:*

(i) *$(T(t))_{t\geq 0}$ is hypercyclic.*
(ii) *For all $g, h \in X$ and $\varepsilon > 0$ exist $f \in X$ and $t > 0$ so that $\|g - f\| < \varepsilon$ and $\|h - T(t)f\| < \varepsilon$.*
(iii) *For all $\varepsilon > 0$ there is a dense subset $D \subset X$ so that for all $h \in D$ there is a dense subset $\widehat{D} \subset X$ so that for all $g \in \widehat{D}$ there are $f \in X$ and $t > 0$ so that $\|g - f\| < \varepsilon$ and $\|h - T(t)f\| < \varepsilon$.*

Proof. (i) ⇒ (ii): Let $e \in X$ have dense orbit $R(\xi_e) = \{T(t)e : t \geq 0\}$. Such an element exists due to the assumed hypercyclicity of $(T(t))_{t\geq 0}$. Then for $s > 0$ the set $\{T(t)e : t > s\}$ is also dense. Now take $g, h \in X$ and $\varepsilon > 0$, then there is $s > 0$ so that $\|g - T(s)e\| < \varepsilon$ and there is $u > s$ so that $\|h - T(u)e\| < \varepsilon$. Finally set $f = T(s)e$ and $t = u - s$.

(ii) ⇒ (i): Let $(h_i)_{i\in\mathbb{N}}$ be a dense sequence in X. We now construct two sequences $(g_i)_{i\in\mathbb{N}} \subset X$ and $(t_i)_{i\in\mathbb{N}} \subset [0, \infty)$ in the following way:

- Set $g_1 = h_1$ and $t_1 = 0$.

- For $n > 1$ find g_n, t_n so that

$$\|g_n - g_{n-1}\| \leq \frac{2^{-n}}{\sup\{\|T(t_j)\| : \ j < n\}}, \tag{3.2}$$

and

$$\|h_n - T(t_n)g_n\| \leq 2^{-n}.$$

Then (3.2) implies that $\|g_n - g_{n-1}\| \leq 2^{-n}$, so that the sequence $(g_i)_{i\in\mathbb{N}}$ has a limit f. Using the above two inequalities, we get the estimate

$$\begin{aligned}\|h_n - T(t_n)f\| &\leq \|h_n - T(t_n)g_n\| + \|T(t_n)\| \, \|g_n - f\| \\ &\leq \|h_n - T(t_n)g_n\| + \sum_{i=n+1}^{\infty} \|T(t_n)\| \, \|g_i - g_{i-1}\| \\ &\leq 2^{-n} + \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n+1}.\end{aligned}$$

Given $h \in X$ and $\varepsilon > 0$, arbitrarily large $n \in \mathbb{N}$ exist, so that $\|h - h_n\| < \frac{\varepsilon}{2}$. With n large enough so that $2^{-n+1} < \frac{\varepsilon}{2}$, we have

$$\|h - T(t_n)f\| \leq \|h - h_n\| + \|h_n - T(t_n)f\| < \varepsilon.$$

This implies that the orbit $R(\xi_f)$ of f is dense.

(ii) $\Rightarrow$ (iii): Put $D = \widehat{D} = X$.

(iii) $\Rightarrow$ (ii): Let $h \in X$, $\varepsilon > 0$ and choose $\tilde{h} \in D$ so that $\|h - \tilde{h}\| < \frac{\varepsilon}{2}$. Then specify $\widehat{D}$ in accordance with (iii) using $\tilde{h}$ instead of h and $\frac{\varepsilon}{2}$ instead of ε. Choose for $g \in X$ a $\tilde{g} \in \widehat{D}$ so that $\|\tilde{g} - g\| < \frac{\varepsilon}{2}$. Now pick t and e according to (iii) with $\frac{\varepsilon}{2}$, $\tilde{g}$, $\tilde{h}$ instead of ε, g, h and get the estimates

$$\begin{aligned}\|T(t)f - h\| &\leq \|T(t)f - \tilde{h}\| + \|\tilde{h} - h\| < \varepsilon, \\ \|f - g\| &\leq \|f - \tilde{g}\| + \|\tilde{g} - g\| < \varepsilon.\end{aligned}$$

□

Lemma 3.18 ([11, Theorem 2.3])**.** *Let X be a separable Banach space and $(T(t))_{t\geq 0}$ a strongly continuous semigroup on X. If X_0 and X_∞ are dense subsets of X, then $(T(t))_{t\geq 0}$ is hypercyclic.*

Proof. We use Lemma 3.17(iii), setting $D = X_\infty$, which is independent of ε, and setting $\widehat{D} = X_0$, which is independent of ε and h. Let $h \in X_\infty$ and $g \in X_0$. Then for every $\varepsilon > 0$ exist arbitrarily large $t > 0$ and $e \in X$ so that

$$\|T(t)e - h\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|e\| < \varepsilon.$$

As $g \in X_0$, we get $\|T(t)g\| < \frac{\varepsilon}{2}$ for t sufficiently large. Setting $f = g + e$, we see that the estimates

$$\|h - T(t)f\| \leq \|h - T(t)e\| + \|T(t)g\| < \varepsilon$$

and

$$\|g - f\| = \|e\| < \varepsilon$$

hold. □

We can now prove the following spectral conditions for the generator of a chaotic semigroup. The theorem is taken from [11] and is sometimes called *Desch-Schappacher-Webb-* or (DSW)-criterion for chaoticity. In the following, $\langle \cdot, \cdot \rangle$ denotes the canonical duality pairing of elements of the considered space with elements of its dual space. Before stating the theorem, we introduce the definition of weak holomorphy of a function.

Definition 3.19. Let $U \subset \mathbb{C}$ be an open set and $f : U \to X$ a function. f is called *weakly holomorphic*, if for every $f_* \in X'$ the complex valued mapping $z \mapsto \langle f(z), f_* \rangle$ is holomorphic on U.

It is clear that every holomorphic function is weakly holomorphic.

Theorem 3.20 ([11, Theorem 3.1]). *Let X be a separable Banach space, $(T(t))_{t\geq 0}$ a strongly continuous semigroup on X with A its generator, $U \subset \sigma_p(A)$ an open subset of the point spectrum of A and $f^\lambda \in \ker(\lambda I - A)$ an eigenvector to the eigenvalue $\lambda \in U$. If*

(i) $U \cap i\mathbb{R} \neq \emptyset$,
(ii) *for each $f_* \in X'$ the function $F_{f_*} : U \to \mathbb{C}$, $F_{f_*}(\lambda) = \langle f_*, f^\lambda \rangle$ is analytic and*
(iii) *F_{f_*} does not vanish identically unless $f_* = 0$,*

then $(T(t))_{t\geq 0}$ is chaotic.

Proof. First we prove the following claim: Let $V \subset U$ be an arbitrary subset that has an accumulation point in U, then $Y_V = \operatorname{span}\{f^\lambda : \lambda \in V\}$ is dense in X. We prove this claim by contradiction: The Hahn-Banach theorem, see e.g. [2, Theorem 4.15], guarantees the existence of $f_* \in X'$ so that $f_* \neq 0$ and $\langle f_*, f \rangle = 0$ for all $f \in Y_V$. This implies that $F_{f_*}(\lambda) = \langle f_*, f^\lambda \rangle = 0$ for all $\lambda \in V$. The set V having an accumulation point in U and F_{f_*} being analytic on U implies by the identity theorem that $F_{f_*} \equiv 0$ on U, which contradicts the assumption.

We will now show that $(T(t))_{t\geq 0}$ is hypercyclic by showing that the subspaces X_0 and X_∞ are dense in X, which implies the assertion according to Lemma 3.18. Then we show that additionally X_p is dense, which is the definition of chaoticity for a C_0-semigroup.

Let V be a subset of $\{\lambda \in U : \operatorname{Re}(\lambda) < 0\}$ that has an accumulation point in U. Then for $\lambda \in V$ the eigenvector f^λ is contained in X_0. X_0 is a subspace, which means that $Y_V \subset X_0$. The density of Y_V implies the density of X_0 in X.

Now let V be a subset of $\{\lambda \in U : \operatorname{Re}(\lambda) > 0\}$ that has an accumulation point in U. The equation

$$\sum_{j=1}^{n} \alpha_j f^{\lambda_j} = T(t) \left(\sum_{j=1}^{n} \alpha_j \mathrm{e}^{-\lambda_j t} f^{\lambda_j} \right)$$

holds, which means that for every $f \in Y_V$ also $f \in X_\infty$ holds. As before, density of Y_V implies the density of X_∞.

Now let $V = \{\lambda_1, \lambda_2, \ldots\}$, where $\operatorname{Re}(\lambda_i) = 0$ and $\operatorname{Im}(\lambda_i) \in \mathbb{Q}$ for all $i \in \mathbb{N}$ and so that $(\lambda_i)_{i\in\mathbb{N}}$ is a convergent sequence. Then each f^{λ_i} is periodic with a period which is a rational multiple of π. This also means that every linear combination

$$\sum_{i=1}^{n} \alpha_i \lambda_i$$

is periodic. Therefore the dense subspace Y_V is contained in X_p which again implies the density of X_p. □

In [5] it was shown that the restriction $U \subset \sigma_p(A)$ in Theorem 3.20 is indeed implicit. While the (DSW)-criterion can be used to check if the semigroup generated by a given operator A is chaotic, the next theorems give conditions by which this can be dismissed. First we need another lemma, the proof can be reviewed in [17, Theorem 7.16].

Lemma 3.21. *Let $(T(t))_{t\geq 0}$ be a hypercyclic C_0-semigroup, and $p \neq 0$ a polynomial. Then for all $t > 0$ the operator $p(T(t))$ has dense range.*

Theorem 3.22 ([17, Proposition 7.18])**.** *Let X be a separable complex Banach space and A a linear operator in X that is the generator of a chaotic C_0-semigroup. Then for all $n \in \mathbb{N}$*

$$|\sigma_p(A) \cap i\mathbb{R}| > n$$

holds.

Proof. By assumption the set X_p is dense in X. By the point spectral mapping theorem for semigroups, see Proposition 3.15,

$$\begin{aligned} X_p = \bigcup_{t>0} \ker(I - T(t)) &= \bigcup_{t>0} \overline{\operatorname{span}} \left\{ \bigcup_{n\in\mathbb{Z}} \ker\left(\frac{2\pi n i}{t} I - A\right) \right\} \\ &\subset \overline{\operatorname{span}} \left\{ \bigcup_{t>0} \bigcup_{n\in\mathbb{Z}} \ker\left(\frac{2\pi n i}{t} I - A\right) \right\} \end{aligned}$$

holds and the density of X_p implies $X = \overline{\operatorname{span}} \left\{ \bigcup_{\lambda\in i\mathbb{R}} \ker(\lambda I - A) \right\}$.

Assume $n \in \mathbb{N}$ exists, so that $|\sigma_p(A) \cap i\mathbb{R}| \leq n$. Then finite collections of $n_k \in \mathbb{Z}$, $t_k > 0$, $k \in \{1, \dots, N\}$ would exist, so that

$$X_p \subset \overline{\operatorname{span}} \left\{ \bigcup_{k=1}^{N} \ker\left(\frac{2\pi n_k i}{t_k} I - A\right) \right\}. \tag{3.3}$$

Again by Proposition 3.15

$$\ker\left(\frac{2\pi n_k i}{t_k} I - A\right) \subset \ker(I - T(t_k)) \tag{3.4}$$

holds for $k \in \{1, \dots, N\}$. From (3.3) and (3.4) follows that the linear operator

$$T = (I - T(t_1)) \cdots (I - T(t_N))$$

vanishes on X_p and since this set is dense this implies $T = 0$. This contradicts Lemma 3.21 by which T should have dense range. □

Combining the results from Corollary 3.7 and Theorem 3.22, we get another criterion.

Corollary 3.23. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with growth bound ω_0. If $\omega_0 < 0$, then $(T(t))_{t\geq 0}$ is not chaotic.*

The assertion of this corollary is of course also immediately clear from the characterization of uniform exponential stability of a strongly continuous semigroup using the growth bound, see e.g. [15, Section V.3.b].

Theorem 3.24 ([11, Theorem 3.3])**.** *Let X be a separable complex Banach space and $(T(t))_{t\geq 0}$ a hypercyclic semigroup on X with generator A. Then the following statements hold:*

(i) *If $f_* \in X'$, $f_* \neq 0$, then the range of its orbit map $R(\xi^*_{f_*}) = \{T^*(t)f_* : \ t \geq 0\}$ is unbounded.*
(ii) *The point spectrum $\sigma_p(A^*)$ of the dual of the generator is empty.*

Proof. (i): By the hypercyclicity assumption, there is $f \in X$ so that the range of its orbit $R(\xi_f) = \{T(t)f : \ t \geq 0\}$ is dense in X. Assume that for $f_* \in X'$, $f_* \neq 0$ the orbit is bounded, that means $\|T^* f_*\| \leq M$ for $M > 0$ and for all $t \geq 0$. Let $g \in X$ be so that $|\langle f_*, g\rangle| > 3M\|f\|$, and choose $t > 0$ so that $\|T(t)f - g\| \leq M\frac{\|f\|}{\|f_*\|}$. Then the estimate

$$\begin{aligned} 3M\|f\| &< |\langle f_*, g\rangle| \leq |\langle f_*, T(t)f\rangle| + |\langle f_*, T(t)f - g\rangle| \\ &= |\langle T^*(t)f_*, f\rangle| + |\langle f_*, T(t)f - g\rangle| \leq M\|f\| + \|f_*\|\frac{M\|f\|}{\|f_*\|} = 2M\|f\| \end{aligned}$$

holds, which is a contradiction.

(ii): Assume $\lambda \in \sigma_p(A^*)$, which means $A^* f_* = \lambda f_*$ for $f_* \neq 0$. Then $T^*(t)f_* = \mathrm{e}^{\lambda t} f_*$, and by (i) we see that $\mathrm{Re}(\lambda) > 0$. The equation

$$\langle f_*, T(t)f\rangle = \mathrm{e}^{\lambda t}\langle f_*, x\rangle$$

means that either $f_*(R(\xi_f)) = 0$ or

$$|\langle f_*, T(t)f\rangle| \geq |\langle f_*, f\rangle|, \quad \text{for } t \geq 0.$$

The first case would imply $f_* = 0$, because the range $R(\xi_f)$ of the orbit of f is dense in X. Now let $g \in X$ be so that $\langle f_*, f\rangle = 0$ and choose t so that the inequality

$$\|T(t)f - g\| \leq \frac{|\langle f_*, f\rangle|}{2\|f_*\|}$$

holds. Then we can estimate

$$|\langle f_*, T(t)f\rangle| \leq |\langle f_*, g\rangle| + \|f_*\|\|T(t)f, g\| < \frac{1}{2}|\langle f_*, x\rangle|,$$

which is a contradiction to the second case. □

Remark 3.25. We briefly summarize the negative indicators for chaoticity. A linear operator A does not generate a chaotic C_0-semigroup, if

(i) it does not have an infinite number of eigenvalues on the imaginary axis, see Theorem 3.22,
(ii) its adjoint operator has eigenvalues, i.e. $\sigma_p(A^*) \neq \emptyset$, see Theorem 3.24.

To illustrate the concept of chaoticity, we will give an example of a chaotic semigroup using the (DSW)-criterion.

Example 3.26 ([11, Example 4.12])**.** Let $X = L^2([0,\infty))$ and consider the problem

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= a\frac{\partial^2}{\partial x^2}u(x,t) + b\frac{\partial}{\partial x}u(x,t) + cu(x,t), \\ u(0,t) &= 0 \qquad \text{for } t \geq 0, \\ u(x,0) &= g(x) \quad \text{for } x \geq 0, \quad g \in X, \end{aligned}$$

where $a, b, c > 0$ and $c < \frac{b^2}{2a} < 1$. Then the solution semigroup is chaotic.

Proof. First we define the operators

$$\begin{aligned} A_1 f &= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \qquad \text{on } D(A_1) = \{f \in H^2([0,\infty)) : \; f(0) = 0\}, \\ B_1 f &= \frac{\mathrm{d}f}{\mathrm{d}x} \qquad \text{on } D(B_1) = \{f \in H^1([0,\infty))\}, \\ A &= aA_1 + bB_1 + cI. \end{aligned}$$

Then A_1 is the generator of a contractive semigroup, B_1 is dissipative, $D(A_1) \subset D(B_1)$ and for all $f \in D(A_1)$

$$\|bB_1 f\| \leq \frac{b}{\sqrt{2a}}(\|aA_1 f\| + \|f\|) \tag{3.5}$$

holds. Now Proposition 3.14, the assumption $\frac{b}{\sqrt{2a}} < 1$ and (3.5) imply that $aA_1 + bB_1$ is the generator of a contractive semigroup. By Proposition 3.13 this means that A generates a strongly continuous semigroup. To show that this semigroup is chaotic, we check the conditions of the (DSW)-criterion, Theorem 3.20.

We choose $U \subset \mathbb{C}$ to be the set

$$U = \left\{ \lambda \in \mathbb{C} : \; \left| \lambda - \left(c - \frac{b^2}{4a} \right) \right| < \frac{b^2}{4a}, \; \mathrm{Im}(\lambda) \neq 0 \text{ if } \mathrm{Re}(\lambda) \leq c - \frac{b^2}{4a} \right\}.$$

Then U has nonempty intersection with the imaginary axis because $c < \frac{b^2}{2a}$.

Now we find appropriate functions for $\lambda \in U$ and prove that they are eigenfunctions with eigenvalues λ. Let f^λ be defined as

$$f^\lambda(x) = \exp\left(-\frac{b}{2a}x \right) \sin\left(x\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \right).$$

By differentiation we see that

$$a\frac{\mathrm{d}^2 f^\lambda}{\mathrm{d}x^2} + b\frac{\mathrm{d}f^\lambda}{\mathrm{d}x} + cf^\lambda = \lambda f^\lambda, \quad f^\lambda(0) = 0,$$

and we can estimate

$$\begin{aligned} |f^\lambda(x)| &\leq \exp\left(-\frac{b}{2a}x \right) \exp\left(x \left| \sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \right| \right) \\ &\leq \exp\left[\frac{x}{\sqrt{a}} \left(-\sqrt{\frac{b^2}{4a}} + \sqrt{\left| \lambda - \left(c - \frac{b^2}{4a} \right) \right|} \right) \right]. \end{aligned}$$

For $\lambda \in U$ the exponent in the last term is negative, so $f^\lambda \in X$. Similarly we can show $\frac{\mathrm{d}^2 f^\lambda}{\mathrm{d}x^2} \in X$, and thus $f^\lambda \in D(A)$. This shows, that the functions f^λ are eigenvectors of A to eigenvalues $\lambda \in U$.

Let F_{f_*} be defined as

$$F_{f_*}(\lambda) = \langle f_*, f^\lambda \rangle,$$

for $\lambda \in U$ and $f_* \in X'$. By Proposition 2.19 we know $X' = X$. Let

$$\psi(x) = \begin{cases} \exp\left(-\frac{b}{2a}x\right) f_*(x) & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then we can write F_{f_*} using the Fourier transform Ψ of ψ as

$$F_{f_*}(\lambda) = \frac{1}{2i}\left[\Psi\left(-\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right) - \Psi\left(\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}}\right)\right].$$

The integrals of the Fourier transform converge absolutely, and the square root is analytic for $\lambda \in U$, it is possible to show that F_{f_*} depends analytically on $\lambda \in U$. Thus, if F_{f_*} vanishes on U, we deduce

$$\Psi(\mu) = \Psi(-\mu), \quad \text{for } \mu \in \mathbb{R}$$

by analyticity. This means that ψ is an even function and since it vanishes on the negative half-line, it vanishes on $\mathbb{R}$. This implies that $f_* = 0$.

All conditions of Theorem 3.20 have been shown to hold, thus the semigroup generated by A is chaotic. □

CHAPTER 4

Operators on Graphs

In many applications the physical structure of the problem is defined by a strong dependence on only one spatial direction, e.g. wave propagation through very thin, quasi-one-dimensional structures like carbon nanotubes. Metric graphs offer an ideal simplified scheme to describe the physical properties of these cases and equipped with the appropriate differential operator become quantum graphs, which can be used to tackle these challenging problems.

In Section 2.1 we already mentioned metric and quantum graphs and now we will examine this concept further. The definitions and theorems follow [8] and [26].

4.1. Metric Graphs

First we need to define and describe the structure of a metric graph to some detail. The concept of metric graphs is very similar to that of weighted graphs or networks, and they can in fact be identified with each other, as described in [4, Section 2] and [26, Section 3.2].

Definition 4.1. A tuple $\mathcal{G} = (D, l)$, where $D = (V, E, \partial)$ is a digraph and $l : E \to (0, \infty]$, $l(e) = l_e$ is the *length function*, is called *metric graph*, if

(i) every arc $e \in E$ is assigned a coordinate $x_e \in [0, l_e]$ which increases in the direction of the edge,

(ii) $e, f \in E$ and $f = \overline{e}$ implies
- $l_f = l_e$ and
- $x_f = l_e - x_e$.

An edge $e \in E$ with infinite length $l_e = \infty$ is called *lead*, see also Definition 2.6.

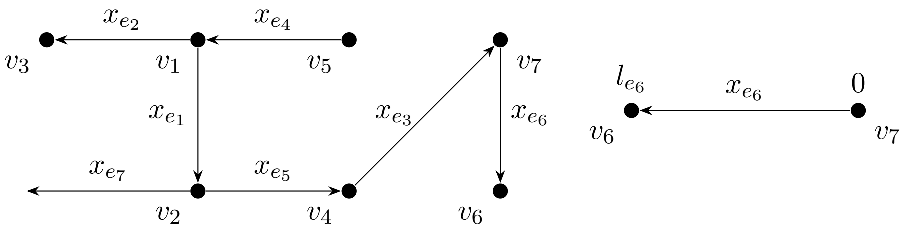
In a metric graph we do not only consider the vertices as points of the graph, but also every point on the edges. We will use the slight abuse of notation by writing $e^{\text{init}} \in V$ and $e^{\text{term}} \in V$ for the points $x_e = 0$ and $x_e = l_e$, respectively.

Example 4.2. Consider the metric graph $\mathcal{G}_b = (D_b, l_{\mathcal{G}_b})$, where $D_b = (V, E, \partial)$ is the digraph from Example 2.5 and

$$l_{\mathcal{G}_b} : E \to (0, \infty),\ l_{\mathcal{G}_b}(e_i) = l_{e_i},\ i \in \{1, 2, \ldots, 6\}.$$

Figure 3 illustrates the graph $\mathcal{G}_b$ and further shows how the coordinates x_{e_i} are defined along the edges e_i in the direction of the arc.

The term *metric* graph is justified, since it is possible to define a natural metric on the graph structure. We do this in three steps: First we introduce the concept of distance between two vertices.



(a) Representation of $\mathcal{G}_b$ with coordinates x_{e_i}. (b) Detail of edge e_6.

FIGURE 3. The metric graph $\mathcal{G}_b$.

Definition 4.3. Let $\mathcal{G} = (D, l)$ be a metric graph with $D = (V, E, \partial)$. The *distance* $\mathrm{dist}_{\mathcal{G}}(v_i, v_j)$ of two vertices $v_i, v_j \in V$ is defined by

$$\mathrm{dist}_{\mathcal{G}}(v_i, v_j) = \inf \left\{ \sum_{e \in \overline{E}} l(e) : \ P = (\overline{V}, \overline{E}, \overline{\partial}) \text{ is } v_i, v_j\text{-path in } U(D) \right\}.$$

Then we define the subdivision of a graph at a point on an edge.

Definition 4.4. Let $\mathcal{G} = (D, l)$ be a metric graph with $D = (V, E, \partial)$ and $x \in (0, l_{e_i})$ for some $e_i \in E$. The metric graph $\widetilde{\mathcal{G}} = (\widetilde{D}, \tilde{l})$ with set of vertices $\widetilde{V} = V \cup \{v_x\}$, set of edges $\widetilde{E} = (E \setminus \{e_i\}) \cup \{e_i', e_i''\}$, incidence map $\widetilde{\partial} : \widetilde{E} \to \widetilde{V}$ with

$$\widetilde{\partial}(e) = \begin{cases} \partial(e) & \text{for } e \notin \{e_i', e_i''\}, \\ (e_i^{\text{init}}, v_x) & \text{for } e = e_i', \\ (v_x, e_i^{\text{term}}) & \text{for } e = e_i'', \end{cases}$$

and length function $\tilde{l} : \widetilde{E} \to (0, \infty]$ with

$$\tilde{l}(e) = \begin{cases} l(e) & \text{for } e \notin \{e_i', e_i''\}, \\ x & \text{for } e = e_i', \\ l_{e_i} - x & \text{for } e = e_i'', \end{cases}$$

is called *subdivision of* $\mathcal{G}$ *at* x. If $x \in V$, then $\widetilde{\mathcal{G}} = \mathcal{G}$.

Figure 4 shows the result of subdividing an edge of a metric graph $\mathcal{G}$ at an arbitrary point x. Observe in particular the new vertex v_x and the new edges e_i', e_i'' with their respective coordinates $x_{e_i'}$, $x_{e_i''}$. By definition, we get $l_{e_i} = l_{e_i'} + l_{e_i''}$.

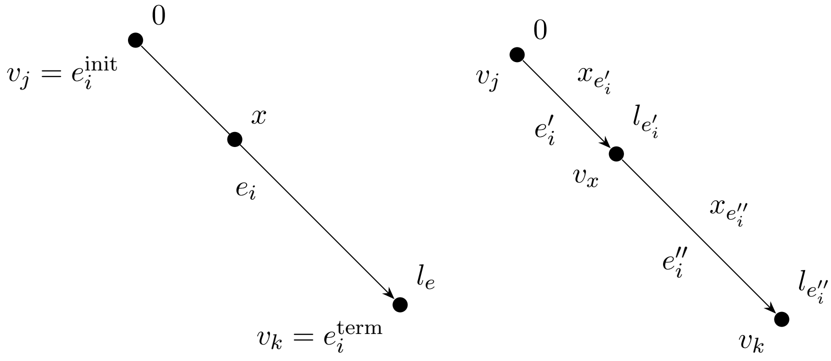
Finally we can extend the distance concept between vertices from Definition 4.3 to all points of a metric graph by considering the distance between vertices on a subdivision.

Definition 4.5. Let $\mathcal{G} = (D, l)$ be a metric graph and x, y two arbitrary points on $\mathcal{G}$. Let $\widetilde{\mathcal{G}}$ be the subdivision of $\mathcal{G}$ at x and y. The *distance* $\mathrm{dist}_{\mathcal{G}}(x, y)$ is defined by

$$\mathrm{dist}_{\mathcal{G}}(x, y) = \mathrm{dist}_{\widetilde{\mathcal{G}}}(v_x, v_y),$$

using the notation of Definition 4.4.

Verification that the function $\mathrm{dist}_{\mathcal{G}}(x, y)$ is indeed a metric is straight forward.



(a) Detail of $\mathcal{G}$ around x on e_i. (b) Detail of $\widetilde{\mathcal{G}}$ around v_x.

FIGURE 4. Subdivision of a metric graph $\mathcal{G}$ at point x.

Definition 4.6. A metric graph $\mathcal{G} = (D, l)$ is called *equilateral*, if l is constant.

Even though in some applications it can be useful, a metric graph in general is not embedded in an Euclidean space.

4.2. Function Spaces on Metric Graphs

By defining the coordinates x_e on a metric graph $\mathcal{G} = (D, l)$ it is possible to study not only discrete functions defined at the vertices, but also functions on the entire graph including the edges. For this purpose we introduce the following function spaces on $\mathcal{G}$.

Definition 4.7. Let $\mathcal{G} = (D, l)$ be a metric graph. For $p \in [1, \infty]$ denote by X^p the completion of the direct sum $\bigoplus_{e \in E} L^p(0, l_e)$ with respect to the norms

$$\|f\|_{L^p(\mathcal{G})}^p = \sum_{e \in E} \|f_e\|_{L^p(0,l_e)}^p = \sum_{e \in E} \int_0^{l_e} |f_e(x_e)|^p \,\mathrm{d}x_e \quad \text{for } p \in [1, \infty),$$
$$\|f\|_{L^\infty(\mathcal{G})} = \inf_{e \in E} \{c \in \mathbb{R} : |f_e(x_e)| \le c \text{ for a.e. } x_e \in (0, l_e)\},$$

see Definition 2.13. Then the space $L^p(\mathcal{G})$ is defined as

$$L^p(\mathcal{G}) = \left\{ f \in X^p : \|f\|_{L^p(\mathcal{G})} < \infty \right\}.$$

Definition 4.8. Let $\mathcal{G} = (D, l)$ be a metric graph and denote by X^2 the completion of the direct sum $\bigoplus_{e \in E} H^1(0, l_e)$ with respect to the norm

$$\|f\|_{H^1(\mathcal{G})}^2 = \sum_{e \in E} \|f_e\|_{H^1(0,l_e)}^2.$$

Then the space $H^1(\mathcal{G})$ is defined as

$$H^1(\mathcal{G}) = \left\{ f \in X^2 : f \text{ continuous on } \mathcal{G}, \|f\|_{H^1(\mathcal{G})} < \infty \right\}.$$

The condition for f to be continuous on $\mathcal{G}$ in the above definition means that for each vertex $v \in V$ the functions $f_e(v)$ assume the same value for all $e \in E_v$.

In order to examine problems of higher order, we need to introduce higher order Sobolev spaces on the metric graph $\mathcal{G}$, which are not defined in the natural way of $H^1(\mathcal{G})$, but without any continuity condition at the vertices.

Definition 4.9. For $k \in \mathbb{N}$, the Sobolev space $\widetilde{H}^k(\mathcal{G})$ is defined as the completion of $\bigoplus_{e\in E} H^k(0,l_e)$ with respect to the norm

$$\|f\|^2_{\widetilde{H}^k(\mathcal{G})} = \sum_{e\in E} \|f_e\|^2_{H^k(0,l_e)}.$$

A function f is an element of $\widetilde{H}^k(\mathcal{G})$, if $f_e \in H^k(0,l_e)$ and $\|f\|_{\widetilde{H}^k(\mathcal{G})} < \infty$. It is clear from the definition, that in the case $k = 0$ we get $\widetilde{H}^0(\mathcal{G}) = L^2(\mathcal{G})$.

According to [26, Section 3.2], the spaces $L^2(\mathcal{G})$ and $H^1(\mathcal{G})$, using the inner products defined by

$$(f,g)_{L^2(\mathcal{G})} = \sum_{e\in E} \int_0^{l_e} f_e(x_e)\overline{g_e(x_e)}\,\mathrm{d}x_e,$$
$$(f,g)_{H^1(\mathcal{G})} = (f,g)_{L^2(\mathcal{G})} + (f',g')_{L^2(\mathcal{G})},$$

are Hilbert spaces, and $H^1(\mathcal{G})$ is a dense subspace of $L^2(\mathcal{G})$.

4.3. Quantum Graphs

After defining metric graphs and the corresponding function spaces on them, we are now able to introduce the main object of interest.

Definition 4.10. A *quantum graph* $\Gamma = (\mathcal{G}, A, VC)$ is a triple of a metric graph $\mathcal{G}$, a linear operator A in an appropriate function space, whose domain $D(A)$ is defined by a set of vertex conditions VC.

Typical examples of vertex conditions are given in Example 4.13.

Remark 4.11. The operator A can be an arbitrary linear operator. Most commonly analyzed in the field of quantum graphs is the operator acting as the negative second derivative $-\frac{\mathrm{d}^2}{\mathrm{d}x_e^2}$ on each edge. Because of its frequent use in literature and applications, we will mainly consider this operator in the rest of this section.

We continue by establishing additional notations. Let $f = (f_e)_{e\in E} \in \widetilde{H}^2(\mathcal{G})$ be a function defined on some metric graph $\mathcal{G}$. For a vertex $v \in V$ the number $d_v = |E_v|$ is called *degree* of v. The vector

$$F(v) = (f_e(v))^T_{e\in E_v}$$

then consists of the values that f takes at v on the edges incident to v and the vector

$$F'(v) = (f'_e(v))^T_{e\in E_v}$$

of the values of the first weak derivative. For continuous functions, e.g. from the space $H^1(\Omega)$, the components of $F(v)$ will be identical. By Assumption 2.10(iii), these vectors have a finite number of components.

In order to treat second order differential operators on the edges, exactly two boundary conditions are required for every edge, which means that at each vertex $v \in V$ we need d_v conditions. For functions in $H^2(\Omega)$, the value of the function and of the first derivative can be considered. This means, that we can write all possible homogeneous conditions for a $H^2(\Omega)$ function in the form

$$A_v F(v) + B_v F'(v) = 0, \tag{4.1}$$

where $A_v, B_v \in \mathbb{C}^{d_v \times d_v}$. The matrix $(A_v B_v) \in \mathbb{C}^{d_v \times 2d_v}$ needs to have full rank d_v to guarantee the necessary number of independent conditions at each vertex.

The vertex conditions have a crucial influence on the properties of the operator on a quantum graph.

Theorem 4.12 ([8, Theorem 1.4.4])**.** *Let $\mathcal{G}$ be a finite metric graph and consider the operator A acting as the negative second derivative $-\frac{\mathrm{d}^2}{\mathrm{d}x_e^2}$ on each edge, with domain of functions in $\widetilde{H}^2(\mathcal{G})$ that satisfy vertex conditions specified in terms of function and derivative values. The operator A is self-adjoint if and only if the vertex conditions can be written in the form:*

(i) *For every $v \in V$ two $d_v \times d_v$ matrices A_v, B_v exist, so that the $d_v \times 2d_v$ matrix $(A_v B_v)$ has maximal rank,*
(ii) *the matrix $A_v B_v^*$ is self-adjoint,*
(iii) *the boundary values of f satisfy $A_v F(v) + B_v F'(v) = 0$.*

Proof. The proof can be found in [8, Theorem 1.4.4]. □

Example 4.13. We will now give some examples of vertex conditions on quantum graphs.

(i) *δ-type condition*: The condition at a vertex $v \in V$ reads as

$$\begin{cases} f \text{ is continuous at } v, \\ \sum_{e \in E_v} f'_e(v) = \alpha_v f(v), \end{cases}$$

where α_v is a fixed parameter at every vertex. Using the notation from Theorem 4.12, this condition can be expressed by the matrices

$$A_v = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & 1 & -1 \\ -\alpha_v & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad B_v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & 1 \end{pmatrix}.$$

We clearly see that $A_v B_v^*$ is self-adjoint if and only if $\alpha_v \in \mathbb{R}$, which would by Theorem 4.12 result in a self-adjoint quantum graph operator. If $\alpha_v \in \mathbb{C}$ is allowed, we call it a *generalized δ-type condition*.

(ii) *Kirchhoff-Neumann condition*: This condition is the special case of the δ-type condition for the parameter value $\alpha_v = 0$, which implies e.g. conservation of the current in the vertices when considering electrical networks.

The name is reminiscent of this mentioned application, where Gustav Kirchhoff was a significant contributor.

(iii) *Dirichlet condition*: Here the function vanishes at the vertices:

$$\begin{cases} f \text{ is continuous at } v, \\ f(v) = 0. \end{cases}$$

The Dirichlet condition is a decoupling condition, as it removes the link between the edges connected by the vertex. It can also be seen as a special case of the δ-type condition, for the case $\alpha_v \to \infty$.

Example 4.14. To better visualize the concept, consider the quantum graph $\Gamma_b = (\mathcal{G}_b, A, VC)$, where $\mathcal{G}_b = (D_b, l_{\mathcal{G}_b})$ is the metric graph from Example 4.2 and the operator A is defined as the second negative derivative on each edge $-\frac{\mathrm{d}^2}{\mathrm{d}x_e^2}$ with vertex conditions of Kirchhoff-Neumann-type. Then by Theorem 4.12 the quantum graph operator is self-adjoint. Figure 5 shows the structure of the graph around the vertex v_1 with the applied vertex conditions.

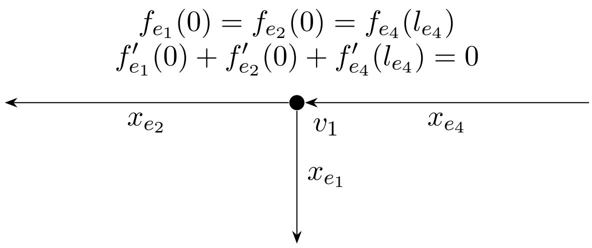


FIGURE 5. Kirchhoff-Neumann vertex conditions at v_1.

For an infinite graph, there need to be additional assumptions in order to guarantee the self-adjointness of the graph operator, which are given in the next theorem.

Theorem 4.15. *Let $\Gamma(\mathcal{G}, A, VC)$ be a quantum graph that satisfies the conditions from Theorem 4.12 and additionally the following assumptions:*

(i) *The length of the edges is uniformly bounded from below, i.e. $l_0 \in \mathbb{R}$ exists, so that $0 < l_0 \leq l_e$ for all $e \in E$.*

(ii) *The vertex conditions are given in the form (4.1), and the estimate*

$$\|B_v^{(-1)} A_v Q_v\| \leq S < \infty$$

holds uniformly for all vertices, where Q_v denotes the orthogonal projection on the range $R(B_v^*)$ of the adjoint of B_v, $B_v^{(-1)}$ is the inverse of B_v with domain $R(B_v^*)$, and the norm is the operator norm induced by the l^2-norm.

Then the operator A is self-adjoint.

PROOF. For the proof we refer to [8, Theorem 1.4.19] □

The vertex conditions described above are all considered local, so no interrelation between distinct vertices is allowed. This is not necessary though, since the topological information of a graph is contained entirely in the vertex conditions, and non-local conditions can be localized by identifying all vertices of a given graph. This turns it into a graph with only one vertex, where each edge is bent into a loop. The type of the vertex conditions does not necessarily stay the same after this procedure, see [8, Section 1.4.6].

Finally to conclude this section and to get a clearer understanding of quantum graphs, we give another example and explicitly calculate the spectrum of the graph, i.e. the spectrum of the operator with the applied vertex conditions.

Example 4.16 ([8, Example 1.4.3])**.** Consider the metric graph $\mathcal{G}^n = ((V, E^n, \partial), l^n)$ where $n \in \mathbb{N}$, $V = \{w, v_1, \dots, v_n\}$, $E^n = \{e_1, \dots, e_n\}$, $\partial(e_i) = (v_i, w)$ and $l^n(e_i) = l_{e_i} \in (0, \infty)$ for $i \in \{1, \dots, n\}$. The graph $\mathcal{G}^n$ is essentially a star graph with all arcs directed to the central vertex, as shown in Figure 6. We complement the graph $\mathcal{G}^n$ with the operator acting as the negative second

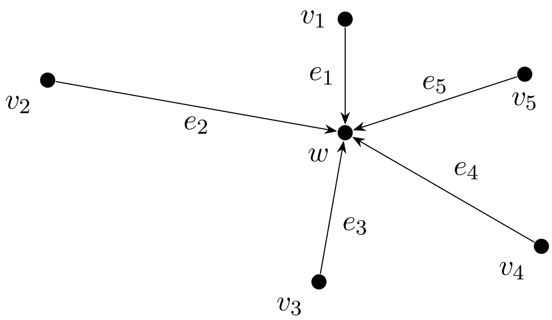


FIGURE 6. Exemplary depiction of a metric graph $\mathcal{G}^5$.

derivative on each edge, which we can write as

$$A = -A_n = -\begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x_{e_1}^2} & & 0 \\ & \ddots & \\ 0 & & \frac{\mathrm{d}^2}{\mathrm{d}x_{e_n}^2} \end{pmatrix},$$

and Kirchhoff-Neumann vertex conditions. The operator A thus has the domain

$$D(A) = \{f \in \widetilde{H}^2(\mathcal{G}^n) : \ A_v F(v) + B_v F'(v) = 0 \text{ for all } v \in V\},$$

where $A_{v_i} = 0$ and $B_{v_i} = 1$ for $i \in \{1, \dots, n\}$ and A_w, B_w is defined as the matrices from Example 4.13(i) with $\alpha_w = 0$. We know from Theorem 4.12 that A is self-adjoint.

When we want to calculate the eigenvalues of the quantum graph operator, we need to solve the problem

$$Af = \lambda f,$$

which boils down to solving for every $e \in E^n$ the ordinary differential equation

$$-\frac{\mathrm{d}^2 f_e}{\mathrm{d}x_e^2} = \lambda f_e.$$

To this end, we adopt the convention of using the outgoing derivative at each vertex, as mentioned in [8, Section 1.4]. Using the Kirchhoff-Neumann conditions at the outer vertices, we get the solutions

$$f_e(x_e) = a_e \cos\left(\sqrt{\lambda} x_e\right)$$

on each edge. The outgoing derivative at the central vertex then becomes

$$f_e'(l_e) = a_e \sin\left(\sqrt{\lambda} l_e\right)$$

and we get the expressions

$$\begin{cases} a_{e_1} \cos\left(\sqrt{\lambda} l_{e_1}\right) = \ldots = a_{e_n} \cos\left(\sqrt{\lambda} l_{e_n}\right) = C, \\ \sum_{i=1}^n a_{e_i} \sin\left(\sqrt{\lambda} l_{e_i}\right) = 0, \end{cases}$$

for the vertex conditions at the central vertex. To keep the example short we do not worry about the case $C = 0$ and assume $C \neq 0$. We see by dividing the second equation by C, that λ is an eigenvalue, if

$$\sum_{i=1}^{n} \tan\left(\sqrt{\lambda} l_{e_i}\right) = 0. \tag{4.2}$$

To make the example more tangible, we consider the explicit case of $\mathcal{G}^3$ with $l_{e_i} = i$ for $i \in \{1, 2, 3\}$, as depicted in Figure 7. Calculating all eigenvalues in this case is a hard task in itself, so we restrict our observations to some eigenvalues which can be seen directly from the above conditions.

We immediately see that $\lambda = 0$ is an eigenvalue and that it leads to eigenvectors which are constant on all edges. The values $\lambda = (2k\pi)^2$, $k \in \mathbb{Z}$, are also obviously eigenvalues of this graph, which result in eigenvectors that are cosine functions on each arc with increasing frequency as $|k|$ is incremented.

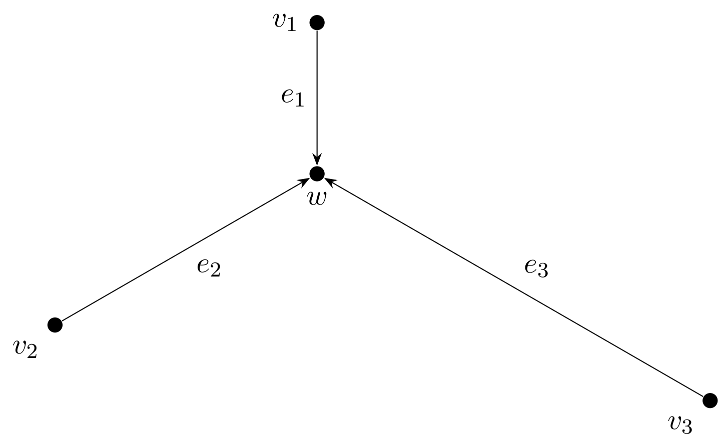


FIGURE 7. The metric graph $\mathcal{G}^3$, where $l^3(e_i) = i$ for $i \in \{1, 2, 3\}$.

CHAPTER 5

Chaotic C_0-Semigroups on Quantum Graphs

We now combine the concepts that we introduced separately in Section 3.3 and Chapter 4. We first give some conditions for the existence of chaotic C_0-semigroups on quantum graphs in Section 5.2. Then in Section 5.3, we construct a model problem, on the basis of which we further study chaoticity of the generated semigroups in Section 5.4.

5.1. Quantum Graphs and Abstract Cauchy Problems

The synthesis of the fields considered in the previous two chapters is all but trivial. In this section, we will clarify what we mean when speaking of the application of semigroup methods on quantum graphs.

We recall from Chapter 3, that the setting where operator semigroups naturally appear are abstract Cauchy problems of the type

$$\text{(ACP)} \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) \quad \text{for } t \geq 0, \\ u(0) = f, \end{cases}$$

where, if A is generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and if $f \in D(A)$, the mapping

$$t \mapsto u(t) = T(t)f$$

represents the classical solutions for (ACP).

In order to use this setting in the context of quantum graphs, we need to consider problems on metric graphs, which result in or can be converted into problems of the type (ACP). For this section we will use the examples of advection and diffusion type problems, which in their simplest variant take the forms

$$\frac{\partial u_e}{\partial t}(t, x_e) = \frac{\partial u_e}{\partial x_e}(t, x_e), \quad t \geq 0, \quad x_e \in (0, l_e), \quad e \in E, \text{ and} \tag{5.1}$$

$$\frac{\partial u_e}{\partial t}(t, x_e) = \frac{\partial^2 u_e}{\partial x_e^2}(t, x_e), \quad t \geq 0, \quad x_e \in (0, l_e), \quad e \in E, \tag{5.2}$$

on a metric graph $\mathcal{G}$, equipped with appropriate vertex conditions VC and initial conditions $u_e(0, x_e) = f(x_e)$. In the setting of the function spaces defined in Section 4.2, we can interpret these problems as (ACP) with the operator A for advection and diffusion respectively defined by

$$A^a f = \begin{pmatrix} \frac{\mathrm{d}f_{e_1}}{\mathrm{d}x_{e_1}} & 0 & \cdots & \cdots \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \frac{\mathrm{d}f_{e_i}}{\mathrm{d}x_{e_i}} & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}, \quad A^d f = \begin{pmatrix} \frac{\mathrm{d}^2 f_{e_1}}{\mathrm{d}x_{e_1}^2} & 0 & \cdots & \cdots \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \frac{\mathrm{d}^2 f_{e_i}}{\mathrm{d}x_{e_i}^2} & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix},$$

with domains

$$D(A^a) = \{f \in \widetilde{H}^1(\mathcal{G}) : \; f \text{ satisfies } VC\},$$

and

$$D(A^d) = \{f \in \widetilde{H}^2(\mathcal{G}) : \; f \text{ satisfies } VC\}.$$

At the same time, these problems can be seen from the viewpoint of Chapter 4, as the quantum graph $\Gamma = (\mathcal{G}, A, VC)$, with the operator A defined as above.

When we talk about C_0-semigroups on quantum graphs, we mean the strongly continuous semigroup that is generated by the quantum graph operator, in view of a problem similar to the type (5.1) or (5.2) on each edge of a metric graph. Of course the generator property of the quantum graph operator needs to be verified in every individual case, i.e. for different vertex conditions the operator may or may not be a generator.

5.2. General Considerations

We briefly examine the kind of quantum graph that is analyzed in most of the literature covering the subject, see e.g. [8], [22] and [23]. The main focus in these publications lies on quantum graphs, where the vertex conditions result in a self-adjoint linear operator. We described a characterization of these kinds of vertex conditions in Theorem 4.12 and 4.15.

Theorem 5.1. *Let X be a Hilbert space and A a self-adjoint linear operator in X. Then A does not generate a chaotic C_0-semigroup.*

Proof. Assume that A is generator of a strongly continuous semigroup, otherwise we are already done. From the final remark in Section 2.3 we know that the spectrum of a self-adjoint operator is real, therefore so is the point spectrum, i.e.

$$\sigma_p(A) \subset \mathbb{R}. \tag{5.3}$$

Assume A generates a chaotic C_0-semigroup, then by Theorem 3.22

$$|\sigma_p(A) \cap i\mathbb{R}| > n$$

holds for all $n \in \mathbb{N}$. Using (5.3) we get the estimate

$$|\sigma_p(A) \cap i\mathbb{R}| \leq |\mathbb{R} \cap i\mathbb{R}| = 1,$$

which contradicts the assumption. Thus A does not generate a chaotic semigroup. □

The theorem implies that when studying quantum graphs, where the vertex conditions result in a self-adjoint quantum graph operator, chaotic semigroups can not occur. This means that all the standard vertex conditions for finite metric graphs equipped with the standard negative second derivative operator on each edge, such as Dirichlet-, Kirchhoff-Neumann- and δ-conditions from Example 4.13 can be ruled out, if we are searching for generators of chaotic semigroups.

Another category of problems where chaotic semigroups can not occur, is the case of finite dimensional spaces X. This is an immediate consequense of Theorem 3.24(ii), as every linear operator in $\mathbb{C}^n$, $n \in \mathbb{N}$, has non-empty point spectrum. This may be relevant in the context of the finite element method for quantum graphs that was introduced in a recent publication, see [3].

5.3. A Model Problem

For the next part of our investigation, we introduce a model problem of diffusion-advection-reaction type, which we will use for our further examination of chaotic C_0-semigroups on quantum graphs.

Consider the metric graph $\mathcal{G}_n = (D_n, l)$, where $D_n = (\{v\}, E_n, \partial)$, $E_n = \{e_i : i \in \{1, \dots, n\}\}$ for a fixed $n \in \mathbb{N}$ and $\partial(e_i) = (v, \infty)$ for $i \in \{1, \dots, n\}$. The graph $\mathcal{G}_n$ is essentially a star graph with n leads, as shown in Figure 8 for the case $n = 5$. Let $X = L^2(\mathcal{G}_n) = L^2([0,\infty))^n$ and let A be the linear operator defined by the problem

$$\begin{aligned} &\frac{\partial u_e}{\partial t}(t,x_e) = a_e \frac{\partial^2 u_e}{\partial x_e^2}(t,x_e) + b_e \frac{\partial u_e}{\partial x_e}(t,x_e) + c_e u_e(t,x_e), \\ &u_e(t,0) \quad = 0 \qquad\qquad \text{for } t \geq 0, \\ &u_e(0,x_e) \quad = g_e(x_e) \qquad \text{for } x_e \geq 0, \quad g_e \in L^2([0,\infty)), \end{aligned} \tag{5.4}$$

on each edge $e \in E_n$ in the sense of Section 5.1, where for each $e \in E_n$ the edgewise constant parameters a_e, b_e, c_e satisfy the conditions $a_e, b_e, c_e > 0$, $c_e < 1$ and

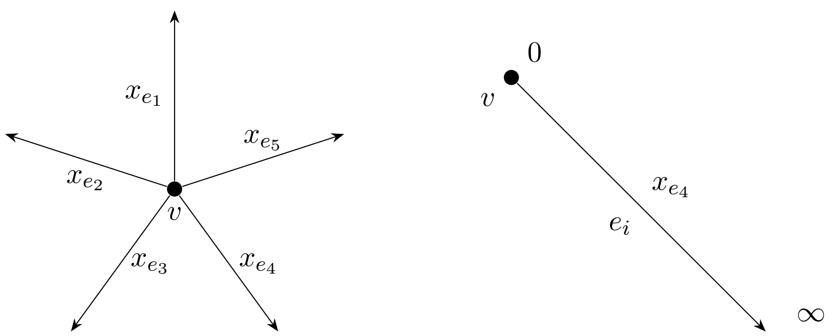
$$c_e < \frac{b_e^2}{2a_e}, \quad \frac{b_e^2}{a_e} < 1. \tag{5.5}$$

For $n \in \mathbb{N}$ we define the operators

$$A_n = \begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x_{e_1}^2} & & 0 \\ & \ddots & \\ 0 & & \frac{\mathrm{d}^2}{\mathrm{d}x_{e_n}^2} \end{pmatrix}, \quad B_n = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x_{e_1}} & & 0 \\ & \ddots & \\ 0 & & \frac{\mathrm{d}}{\mathrm{d}x_{e_n}} \end{pmatrix},$$

with domains

$$\begin{aligned} D(A_n) &= \{f \in \widetilde{H}^2(\mathcal{G}_n) : f_e(0) = 0 \text{ for all } e \in E_n\}, \\ D(B_n) &= \widetilde{H}^1(\mathcal{G}_n). \end{aligned}$$



(a) The metric graph $\mathcal{G}_5$ with five leads. (b) Detail of the edge e_4.

FIGURE 8. Depiction of the metric graph $\mathcal{G}_5$.

Note that for $n = 1$ we get the operators A_1 and B_1 from Example 3.26. We can now describe the quantum graph operator A by

$$\begin{aligned} A &= \operatorname{diag}(a_{e_i})A_n + \operatorname{diag}(b_{e_i})B_n + \operatorname{diag}(c_{e_i})I, \\ D(A) &= D(A_n), \end{aligned}$$

where $\operatorname{diag}(d_i)$ is defined as

$$\operatorname{diag}(d_i) = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}.$$

5.4. Semigroups for the Model Problem

We will now analyze different variations of the model problem, starting with the original version, and later changing parameters and vertex conditions to investigate the effects on the generated semigroup. Table 1 offers a short overview of the different variations of the model problem that we consider in this section.

TABLE 1. Variations of the model problem.

PDE	Vertex condition	Semigroup	Reference
second order ($a_e \neq 0$)	Dirichlet	chaotic	Theorem 5.2
	generalized δ-type	chaotic	Theorem 5.5
first order ($a_e = 0$)	Dirichlet	not chaotic	Theorem 5.3
	Kirchhoff-Neumann	not applicable	Remark 5.4
	δ-type	not applicable	Remark 5.4
	continuity	not generated	Remark 5.4
	mass conservation	not chaotic	Theorem 5.6

Theorem 5.2. *Consider the model problem. Then A generates a chaotic C_0-semigroup on X.*

PROOF. We claim that B_n is dissipative: By Definition 2.15(iii) we need to show that $\|(\lambda I - B_n)f\|_{L^2(\mathcal{G}_n)} \geq \lambda \|f\|_{L^2(\mathcal{G}_n)}$ for all $\lambda > 0$ and $f \in D(B_n)$. Using the definition of the L^2-norm on a metric graph from Definition 4.7 and the knowledge that B_1 is dissipative from Example 3.26, we get

$$\begin{aligned} \|(\lambda I - B_n)f\|_{L^2(\mathcal{G}_n)} &= \sqrt{\sum_{e \in E_n} \|(\lambda I - B_1)f_e\|^2_{L^2([0,\infty))}} \\ &\geq \sqrt{\sum_{e \in E_n} \lambda^2 \|f_e\|^2_{L^2([0,\infty))}} \\ &= \lambda \|f\|_{L^2(\mathcal{G}_n)}, \end{aligned} \tag{5.6}$$

where $f \in \widetilde{H}^1(\mathcal{G}_n) = D(B_n)$, which proves our claim.

As described in Example 4.13(iii), the imposed Dirichlet condition at the vertex v is a decoupling condition, so from the generator property in the one dimensional case, see Example 3.26, follows the correlating property in this metric graph setting, i.e. A_n generates a contractive strongly continuous semigroup, see also e.g. [14, 22].

We also see from the definitions that $D(A_n) \subset D(B_n)$ and, using (3.5) and Young's inequality, that for all $f \in D(A_n)$ the estimate

$$\begin{aligned}
\|\mathrm{diag}(b_{e_i})B_n f\|_{L^2(\mathcal{G}_n)} &= \sqrt{\sum_{e\in E_n} \|b_e B_1 f_e\|^2} \\
&\leq \sqrt{\sum_{e\in E_n} \frac{b_e^2}{2a_e} \left(\|a_e A_1 f_e\| + \|f_e\|\right)^2} \\
&\leq \sqrt{r_m \sum_{e\in E_n} \left(\|a_e A_1 f_e\|^2 + \|f_e\|^2\right)} \\
&\leq \sqrt{r_m}\left(\sqrt{\sum_{e\in E_n} \|a_e A_1 f_e\|^2} + \sqrt{\sum_{e\in E_n} \|f_e\|^2}\right) \\
&= \sqrt{r_m}\left(\|\mathrm{diag}(a_{e_i})A_n f\|_{L^2(\mathcal{G}_n)} + \|f\|_{L^2(\mathcal{G}_n)}\right)
\end{aligned} \tag{5.7}$$

holds, where we set $r_m = \max_{e\in E_n}\left\{\frac{b_e^2}{a_e}\right\}$. From (5.5) we get $\sqrt{r_m} < 1$ and thus by Proposition 3.14 the operator $\mathrm{diag}(a_{e_i})A_n + \mathrm{diag}(b_{e_i})B_n$ generates a contractive strongly continuous semigroup, which, using Proposition 3.13, implies that A generates a C_0-semigroup.

We will now check the conditions of the (DSW)-criterion from Theorem 3.20. Let U be the set

$$U = \bigcap_{e\in E_n} U_e,$$

where

$$U_e = \left\{\lambda \in \mathbb{C} : \left|\lambda - \left(c_e - \frac{b_e^2}{4a_e}\right)\right| < \frac{b_e^2}{4a_e},\ \mathrm{Im}(\lambda) \neq 0 \text{ if } \mathrm{Re}(\lambda) \leq c_e - \frac{b_e^2}{4a_e}\right\}.$$

We know that $U \cap i\mathbb{R} \neq \emptyset$, because by assumption $c_e < \frac{b_e^2}{2a_e}$ for all $e \in E_n$. For $\lambda \in U$ define the function

$$f^\lambda(x) = \begin{pmatrix} f_{e_1}^\lambda(x_{e_1}) \\ \vdots \\ f_{e_n}^\lambda(x_{e_n}) \end{pmatrix} = \begin{pmatrix} \exp\left(-\frac{b_{e_1}}{2a_{e_1}}x_{e_1}\right)\sin\left(x_{e_1}\sqrt{\frac{c_{e_1}-\lambda}{a_{e_1}} - \frac{b_{e_1}^2}{4a_{e_1}^2}}\right) \\ \vdots \\ \exp\left(-\frac{b_{e_n}}{2a_{e_n}}x_{e_n}\right)\sin\left(x_{e_n}\sqrt{\frac{c_{e_n}-\lambda}{a_{e_n}} - \frac{b_{e_n}^2}{4a_{e_n}^2}}\right) \end{pmatrix}.$$

By differentiation we can calculate the identity

$$\begin{aligned}
Af^\lambda &= \mathrm{diag}(a_{e_i})A_n f^\lambda + \mathrm{diag}(b_{e_i})B_n f^\lambda + \mathrm{diag}(c_{e_i})f^\lambda \\
&= \begin{pmatrix} a_{e_1}\frac{\mathrm{d}^2 f_{e_1}^\lambda}{\mathrm{d}x_{e_1}^2} \\ \vdots \\ a_{e_n}\frac{\mathrm{d}^2 f_{e_n}^\lambda}{\mathrm{d}x_{e_n}^2} \end{pmatrix} + \begin{pmatrix} b_{e_1}\frac{\mathrm{d} f_{e_1}^\lambda}{\mathrm{d}x_{e_1}} \\ \vdots \\ b_{e_n}\frac{\mathrm{d} f_{e_n}^\lambda}{\mathrm{d}x_{e_n}} \end{pmatrix} + \begin{pmatrix} c_{e_1} f_{e_1}^\lambda \\ \vdots \\ c_{e_n} f_{e_n}^\lambda \end{pmatrix} = \lambda f^\lambda,
\end{aligned} \tag{5.8}$$

and the validity of the boundary condition

$$f^\lambda(0) = \begin{pmatrix} f^\lambda_{e_1}(0) \\ \vdots \\ f^\lambda_{e_n}(0) \end{pmatrix} = 0$$

is obvious. Since for all $e \in E_n$ the estimate

$$|f^\lambda_e(x_e)| \le \exp\left[\frac{x_e}{\sqrt{a_e}}\left(-\sqrt{\frac{b_e^2}{4a_e}} + \sqrt{\left|\lambda - \left(c_e - \frac{b_e^2}{4a_e}\right)\right|}\right)\right]$$

holds and the exponent of the exponential function is negative for $\lambda \in U$, we can conclude that

$$\|f^\lambda\|_{L^2(\mathcal{G}_n)} = \left(\sum_{e\in E_n} \|f^\lambda_e\|^2_{L^2([0,\infty))}\right)^{\frac{1}{2}} < \infty,$$

and thus $f^\lambda \in X$. In the same way we can show that $A_n f^\lambda \in X$, so that $f^\lambda \in D(A)$. Combined with (5.8) this means that f^λ and λ are eigenvectors and eigenvalues of A, respectively.

From Proposition 2.19 and the definition of the space $L^2(\mathcal{G}_n)$ we can deduce that $X' = X$. Now choosing $f_* \in X'$, we define the map $F_{f_*} : U \to \mathbb{C}$, $\lambda \mapsto \langle f_*, f^\lambda\rangle$. Then let the function ψ be defined as

$$\psi(x) = \begin{pmatrix} \psi_{e_1}(x_{e_1}) \\ \vdots \\ \psi_{e_n}(x_{e_n}) \end{pmatrix},$$

where

$$\psi_e(x_e) = \begin{cases} \exp\left(-\frac{b_e}{2a_e}x_e\right) f_{*e}(x_e) & \text{if } x_e \ge 0, \\ 0 & \text{if } x_e < 0. \end{cases}$$

Using the Fourier transforms Ψ_e of ψ_e we can write the function F_{f_*} as

$$F_{f_*}(\lambda) = \frac{1}{2i}\sum_{e\in E_n}\left[\Psi_e\left(-\sqrt{\frac{c_e-\lambda}{a_e} - \frac{b_e^2}{4a_e^2}}\right) - \Psi_e\left(\sqrt{\frac{c_e-\lambda}{a_e} - \frac{b_e^2}{4a_e^2}}\right)\right].$$

Since the integrals involved in the Fourier transforms converge absolutely, and the square root is analytic for $\lambda \in U$, it is possible to show that $F_{f_*}(\lambda)$ depends analytically on $\lambda \in U$.

We can deduce by analyticity that $\Psi_e(\mu) = \Psi_e(-\mu)$ for all $\mu \in \mathbb{R}$ if F_{f_*} vanishes on the set U, and thus see that ψ_e is an even function. Since $\psi_e = 0$ on the negative half line, this means that $\psi_e \equiv 0$ which implies $f_{*e} = 0$ for all $e \in E_n$, which means $f_* = 0$.

This concludes the proof of the conditions for Theorem 3.20, and thus shows that the semigroup generated by the linear operator A is chaotic. □

The theorem is in fact an adaption of the one-dimensional case of a chaotic semigroup from Example 3.26, set on the metric graph $\mathcal{G}_n$ with a Dirichlet vertex condition at the vertex v. This type of vertex condition, as described in Example 4.13, is a decoupling condition, because the link between the edges is neglected. This reduces the value of the presented example of a chaotic semigroup on a quantum graph to some degree.

In order to extend our understanding of the model problem, we now examine some alterations of the model problem. First we consider the case $a_e = 0$, so that no second order derivatives appear.

Theorem 5.3. *Consider the model problem where $a_e = 0$, for all $e \in E_n$, and condition (5.5) is dismissed. Then A does not generate a chaotic C_0-semigroup on X.*

PROOF. Assume that A is the generator of a strongly continuous semigroup, otherwise we are already finished. We now show that the necessary spectral condition for the generator of a chaotic semigroup,

$$|\sigma_p(A) \cap i\mathbb{R}| > n \quad \text{for all } n \in \mathbb{N}, \tag{5.9}$$

from Theorem 3.22 is not met. Let $A = \operatorname{diag}(b_{e_i})B_n + \operatorname{diag}(c_{e_i})I$ with adapted domain

$$D(A) = D(B_n) = \{f \in H^1(\mathcal{G}_n) : \ f(0) = 0\}$$

to incorporate the vertex condition. In order to find the eigenvalues of the linear operator A, we need to consider the eigenvalue problem

$$\begin{aligned} Af = \lambda f, \quad f \in D(A) \quad &\Leftrightarrow \quad \operatorname{diag}(b_{e_i})B_n f = (\lambda I - \operatorname{diag}(c_{e_i})I)f \\ &\Leftrightarrow \quad \frac{\mathrm{d}f_e}{\mathrm{d}x_e} = \frac{\lambda - c_e}{b_e} f_e, \quad e \in E_n, \end{aligned} \tag{5.10}$$

For these ordinary differential equations we get the general solutions

$$f_e(x_e) = d_e \exp\left(\frac{\lambda - c_e}{b_e} x_e\right), \quad e \in E_n,$$

where d_e is an arbitrary constant. Using the boundary condition $f_e(0) = 0$ on each edge, we see that (5.10) has only the trivial solution. This means that condition (5.9) can not hold, thus A does not generate a chaotic strongly continuous semigroup on X. □

Remark 5.4. While in the case of Theorem 5.2 the Dirichlet vertex condition was essential in translating the one-dimensional example of a chaotic semigroup to the quantum graph, it is now responsible for the lack of eigenvalues of the quantum graph operator.

Due to their definition, the Kirchhoff-Neumann or δ-type condition are not applicable in this setting with a first order differential operator. Dropping the Dirichlet condition $f(0) = 0$ and leaving just the continuity condition in the vertex v would result in only $n - 1$ boundary conditions for the first order operator, which thus can not be the generator of a strongly continuous semigroup.

For the next case, we study the original problem with a different, not decoupling kind of vertex condition.

Theorem 5.5. *Consider the model problem with the Dirichlet vertex condition replaced by a generalized δ-type condition with $\alpha_v \in \mathbb{C} \setminus \bigcup_{e \in E_n} V_e$,* *where*

$$V_e = \left\{ z \in \mathbb{C} : \; z = \frac{n}{\sqrt{a_e}} \left[-\sqrt{\frac{b_e^2}{4a_e}} - \sqrt{\lambda - \left(c_e - \frac{b_e^2}{4a_e}\right)} \right], \lambda \in U_e \right\},$$

with U_e *defined by*

$$U_e = \left\{ \lambda \in \mathbb{C} : \; \left| \lambda - \left(c_e - \frac{b_e^2}{4a_e}\right) \right| < \frac{b_e^2}{4a_e}, \; \mathrm{Im}(\lambda) \neq 0 \text{ if } \mathrm{Re}(\lambda) \leq c_e - \frac{b_e^2}{4a_e} \right\}.$$

Then A *generates a chaotic* C_0*-semigroup on* X*.*

Proof. As before, we prove the conditions of the (DSW)-criterion. First note that the domain of A_n has changed to

$$D(A_n) = \{ f \in \widetilde{H}^2(\mathcal{G}_n) : \; A_v F(v) + B_v F'(v) = 0 \},$$

due to the new vertex condition. The $n \times n$-matrices A_v and B_v take the form of the matrices introduced in Example 4.13(i).

The proof of the generator property of A is analogous to Theorem 5.2: A_n generates a contractive semigroup, see e.g. [20], B_n is dissipative by (5.6) and due to (5.7) Proposition 3.14 can be applied. Finally Proposition 3.13 shows that A generates a strongly continuous semigroup.

For the following part of the proof we consider the set $U = \bigcap_{e \in E_n} U_e$. Now consider the functions

$$f^\lambda = \begin{pmatrix} f_{e_1}^\lambda \\ \vdots \\ f_{e_n}^\lambda \end{pmatrix},$$

where for all $e \in E_n$ the functions f_e^λ are defined as

$$f_e^\lambda(x_e) = \exp\left(-\frac{b_e}{2a_e} x_e\right) \left[\left(\frac{C_e}{\Lambda_e} + 1\right) \exp(\Lambda_e x_e) + \left(1 - \frac{C_e}{\Lambda_e}\right) \exp(-\Lambda_e x_e) \right],$$

with

$$\Lambda_e = \sqrt{\frac{b_e^2}{4a_e^2} + \frac{\lambda - c_e}{a_e}} \quad \text{and} \quad C_e = \frac{2\alpha_v a_e + n b_e}{2n a_e}.$$

We can see that $\Lambda_e \neq 0$ for $\lambda \in U$. Since by assumption $c_e < \frac{b_e^2}{2a_e}$, we know that $U \cap i\mathbb{R} \neq \emptyset$. We now show that the functions f^λ are eigenvectors of A to the eigenvalues λ.

By differentiation and some technical calculations, we can verify the identity

$$A f^\lambda = \mathrm{diag}(a_{e_i}) A_n f^\lambda + \mathrm{diag}(b_{e_i}) B_n f^\lambda + \mathrm{diag}(c_{e_i}) f^\lambda = \lambda f^\lambda,$$

and we also immediately see that

$$f_{e_i}^\lambda(0) = f_{e_j}^\lambda(0)$$

holds for every choice of $i, j \in \{1, \ldots, n\}$, thus satisfying continuity in the vertex v. By another calculation, we prove that

$$\sum_{e \in E_n} \frac{\mathrm{d} f_e^\lambda}{\mathrm{d} x_e}(0) = \sum_{e \in E_n} \left(2C_e - \frac{b_e}{a_e}\right) = \sum_{e \in E_n} \frac{2\alpha_v}{n} = \alpha_v f^\lambda(0)$$

holds, where $f^\lambda(0)$ is the, due to continuity unique, value of f^λ at the vertex. Now let

$$d_e^m = \max\left\{\left|\frac{C_e}{\Lambda_e}+1\right|, \left|1-\frac{C_e}{\Lambda_e}\right|\right\},$$

then we can estimate for each $e \in E_n$

$$\begin{aligned} |f_e^\lambda(x_e)| &\le d_e^m \exp\left(-\frac{b_e}{2a_e}x_e\right)|(\exp(\Lambda_e x_e)+\exp(-\Lambda_e x_e))| \\ &= 2d_e^m \exp\left(-\frac{b_e}{2a_e}x_e\right)\left|\cos\left(\sqrt{\frac{c_e-\lambda}{a_e}-\frac{b_e^2}{4a_e^2}}x_e\right)\right| \\ &\le 2d_e^m \exp\left(-\frac{b_e}{2a_e}x_e\right)\exp\left(\left|\sqrt{\frac{c_e-\lambda}{a_e}-\frac{b_e^2}{4a_e^2}}\right|x_e\right) \\ &\le 2d_e^m \exp\left[\frac{x_e}{\sqrt{a_e}}\left(-\sqrt{\frac{b_e^2}{4a_e}}+\sqrt{\left|\lambda-\left(c_e-\frac{b_e^2}{4a_e}\right)\right|}\right)\right], \end{aligned}$$

which by the same argument as in the proof of Theorem 5.2 shows that $f^\lambda \in L^2(\mathcal{G}_n)$ for $\lambda \in U$, and similarly $A_n f^\lambda \in L^2(\mathcal{G}_n)$, so that $f^\lambda \in D(A)$. This concludes the verification of the eigenvector property of the functions f^λ.

It remains to show that for every $f_* \in X' = X$, the map $F_{f_*} : U \to \mathbb{C}$, $\lambda \mapsto \langle f_*, f^\lambda\rangle$, is analytic and F_{f_*} does not vanish on U unless $f_* = 0$. We again define the function ψ edgewise by

$$\psi_e(x_e) = \begin{cases} \exp\left(-\frac{b_e}{2a_e}x_e\right) f_{*e}(x_e) & \text{if } x_e \ge 0, \\ 0 & \text{if } x_e < 0. \end{cases}$$

Then we can write the function F_{f_*}, again using the Fourier transform Ψ_e of ψ_e, as

$$F_{f_*}(\lambda) = \sum_{e\in E_n}\left[\left(\frac{C_e}{\Lambda_e}+1\right)\Psi_e(i\Lambda_e)+\left(1-\frac{C_e}{\Lambda_e}\right)\Psi_e(-i\Lambda_e)\right],$$

and analogous to the proof of Theorem 5.2 we see that F_{f_*} is analytic for $\lambda \in U$. Again similar to the proof of Theorem 5.2, if F_{f_*} vanishes on U, we conclude that

$$\Psi_e(\mu) = -\frac{\Lambda_e - C_e}{\Lambda_e + C_e}\Psi_e(-\mu)$$

for all $e \in E_n$, and by this ψ_e is, for $\alpha_v \in V$, an odd or respectively even function up to a multiplicative constant for $\Lambda_e > C_e$ and $\Lambda_e < C_e$. For $\alpha_v \in V$, we would get $\Lambda_e + C_e = 0$ for a $\lambda \in U$ for at least one $e \in E_n$.

Since $\psi_e = 0$ on the interval $(-\infty, 0)$, each component of ψ vanishes on $\mathbb{R}$ and thus $f_* = 0$. $\square$

Finally we want to discuss another variation of the first order case of the model problem. Therefore we introduce a new kind of vertex condition for the case of a transport operator on a metric graph.

We first adapt the problem by changing the following parameters: Set $n \ge 2$, $n > k \in \mathbb{N}$, $a_e = 0$ for all $e \in E_n$, dismiss condition (5.5) and set $b_{e_i} < 0$ for $i \in \{1,\dots,k\}$, while for $i \in \{k+1,\dots,n\}$ still $b_{e_i} > 0$ holds.

The problem is now similar to the case examined in [26, Section 4.5] where instead of positively and negatively incident edges at the vertex in combination with strictly positive coefficients to account for inflow and outflow, we only have negatively incident edges, but allow a reverse direction of transport using coefficients of the first order term with different signs.

This convention results in a direction of transport towards the central vertex on all edges where b_e is positive, and away from the vertex where the coefficient is negative. This is visualized in Figure 9 for $k = 2$ and the graph $\mathcal{G}_5$.

We then impose the conditions

$$\begin{aligned} f_{e_1}(v) &= \frac{1}{k} \sum_{i=k+1}^{n} f_{e_i}(v), \\ &\vdots \\ f_{e_k}(v) &= \frac{1}{k} \sum_{i=k+1}^{n} f_{e_i}(v), \end{aligned} \tag{5.11}$$

for the k edges with direction of transport away from the vertex, which represent conservation of mass at the central vertex: mass that is transported on an edge towards v is split uniformly on the outgoing edges.

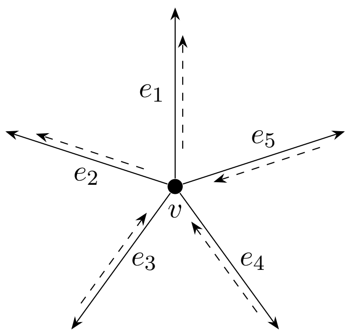


FIGURE 9. The graph $\mathcal{G}_5$ for the transport case of the model problem with $k = 2$, dashed arrows indicate the direction of transport.

Theorem 5.6. *Consider the model problem with $n \geq 2$, $n > k \in \mathbb{N}$, $a_e = 0$ for all $e \in E_n$, $b_{e_i} < 0$ for $i \in \{1, \ldots, k\}$, $b_{e_i} > 0$ for $i \in \{k+1, \ldots, n\}$, and vertex condition (5.11). Then A generates a strongly continuous semigroup that is not chaotic.*

PROOF. The assumed changes of the model problem result in the operator $A = \operatorname{diag}(b_{e_i})B_n + \operatorname{diag}(c_{e_i})I$ with domain

$$D(A) = D(B_n) = \{f \in \widetilde{H}^1(\mathcal{G}_n) : \ f \text{ satisfies } (5.11)\}.$$

Observe that, as mentioned above, we are considering the case of a transport problem similar to the problem in [26, Section 4.5]. By [26, Remark 4.63(1)] in combination with [19] we conclude that A generates a strongly continuous semigroup.

Now consider the eigenvalue problem of the operator A, which is of the same form as (5.10) with general solutions

$$f_e(x_e) = d_e \exp\left(\frac{\lambda - c_e}{b_e} x_e\right)$$

of the resulting ordinary differential equations, where d_e is an arbitrary constant. In order for $f \in X$ to hold, f_e needs to be a L^2-function for each $e \in E_n$, which translates to the condition

$$\frac{\mathrm{Re}(\lambda) - c_e}{b_e} < 0.$$

Keeping in mind that by definition $b_{e_1}, \ldots, b_{e_k} < 0$ and $b_{e_{k+1}}, \ldots, b_{e_n} > 0$, we get the conditions

$$\mathrm{Re}(\lambda) > c_{e_1} \quad \text{and} \quad \mathrm{Re}(\lambda) < c_{e_n}$$

for the functions f_{e_1} and f_{e_n}, respectively. Since by assumption $c_e > 0$ for all $e \in E_n$, we see that in the possibly nonempty intersection of these conditions λ always has positive real part, and thus by Theorem 3.22 the generated semigroup is not chaotic. □

CHAPTER 6

Conclusion and Future Research

The focus of this thesis was to introduce the reader to chaoticity of strongly continuous semigroups in the sense of Devaney, specified in his book [12], and its application on quantum graphs. To this end, we first had to establish the appropriate mathematical framework using mainly graph and operator theory, where we stated several results from literature. We then gave an overview of criteria for showing or disproving if a given operator generates a chaotic semigroup, relying mainly on [11], which we then used in the final chapter to examine chaoticity of semigroups generated by quantum graph operators.

In detail, we showed that in the cases of the in literature most frequently used quantum graphs, where the vertex conditions result in a self-adjoint quantum graph operator, chaoticity of the generated semigroup is impossible. On the other hand, we were able to show that for a more general definition of quantum graphs, chaotic solution semigroups of certain problems on metric graphs do exist. For the used model problem of diffusion-advection-reaction type on a general star graph with n leads we showed chaoticity of the generated semigroup for certain parameter ranges in the case of Dirichlet and generalized δ-type vertex conditions. Although this is not a general result, it shows that the possibility of solution semigroups on quantum graphs being chaotic needs to be considered in applications.

It seems there are no published research articles concerned with chaotic semigroups on quantum graphs. Taking the results from this thesis as a starting point, the next step for further research in this area could be investigating the existence of chaotic semigroups on more complicated graph structures than the star graph we used. One possible option would be a spiderweb graph, where concentric cycles are added to the star structure.

In Chapter 5 we used the sufficient (DSW)-criterion and its generalizations to prove and the necessary criteria summarized in Remark 3.25 to disprove whether a given operator generates a chaotic strongly continuous semigroup. A still open research item, though out of the direct focus of this text and beyond the scope of a master's thesis, is a full characterization of chaotic strongly continuous semigroups through the spectral properties of its generator.

Nomenclature

$\bigoplus_{i\in I} X_i$ ..... direct sum of $\{X_i\}_{i\in I}$, page 7.
$\prod_{i\in I} X_i$ ...... Cartesian product of $\{X_i\}_{i\in I}$, page 7.
$\langle\cdot,\cdot\rangle$ .... canonical duality pairing, page 21.
$(\cdot,\cdot)_X$ .... inner product of space X, page 7.
$\|\cdot\|_X$ .... norm of space X, page 7.
$(T(t))_{t\geq 0}$ ..... C_0-semigroup, page 13.
A^* .... adjoint operator, page 11.
$\mathcal{B}(X)$ .... space of bounded linear operators in X, page 9.
$\mathcal{B}(X_1, X_2)$ .... space of bounded linear operators from X_1 in X_2, page 9.
$C^k(\Omega)$ .... space of k-times continuously differentiable functions, page 7.
∂ .... incidence map, page 3.
D .... directed graph, page 4.
$D(A)$ .... domain of A, page 8.
$\mathrm{dist}_{\mathcal{G}}(x, y)$ ... distance function on a metric graph $\mathcal{G}$, page 28.
E .... set of edges, page 3.
e^{init} .... initial vertex of arc e, page 4.
e^{term} .... terminal vertex of arc e, page 4.
E_v .... set of edges incident to v, page 5.
$\mathcal{G}$ .... metric graph, page 27.
G .... graph, page 3.
Γ .... quantum graph, page 30.
$\widetilde{H}^k(\mathcal{G})$ .... Sobolev spaces on metric graph $\mathcal{G}$, page 30.
$H^1(\mathcal{G})$ .... Sobolev space on metric graph $\mathcal{G}$, page 29.
$H^k(\Omega)$ .... Sobolev Hilbert spaces, page 7.
l .... length function of a metric graph, page 27.
$L^p(\mathcal{G})$ .... Lebesgue spaces on metric graph $\mathcal{G}$, page 29.
$L^p(\Omega)$ .... Lebesgue spaces, page 6.
ω_0 .... growth bound, page 14.
P .... v, w-Path, page 5.
$\rho(A)$ .... resolvent set of A, page 10.
$R(\cdot, A)$ .... resolvent of A, page 10.
$R(A)$ .... range of A, page 8.
$\sigma(A)$ .... spectrum of A, page 10.
$s(A)$ .... spectral bound of A, page 10.
$\sigma_c(A)$ .... continuous spectrum of A, page 10.
$\sigma_p(A)$ .... point spectrum of A, page 10.
$\sigma_r(A)$ .... residual spectrum of A, page 10.
$U(D)$ .... underlying graph, page 6.
V .... set of vertices, page 3.

$W^{k,p}(\Omega)$ ..... Sobolev spaces, page 7.
X' .... dual space of X, page 9.
X_p .... set of periodic points, page 19.
ξ_f .... orbit map of f by $(T(t))_{t\geq 0}$, page 13.
$\xi^*_{f_*}$ .... orbit map of f_* by dual semigroup $(T^*(t))_{t\geq 0}$, page 23.

List of Figures

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Erklärung

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Ich erkläre, dass ich die vorliegende Abschlussarbeit mit dem Thema

On the Existence of Chaotic C_0-Semigroups on Quantum Graphs

selbstständig und ohne unzulässige Inanspruchnahme Dritter verfasst habe. Ich habe dabei nur die angegebenen Quellen und Hilfsmittel verwendet und die aus diesen wörtlich, inhaltlich oder sinngemäß entnommenen Stellen als solche den wissenschaftlichen Anforderungen entsprechend kenntlich gemacht. Die Versicherung selbstständiger Arbeit gilt auch für Zeichnungen, Skizzen oder graphische Darstellungen. Die Arbeit wurde bisher in gleicher oder ähnlicher Form weder derselben noch einer anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht. Mit der Abgabe der elektronischen Fassung der endgültigen Version der Arbeit nehme ich zur Kenntnis, dass diese mit Hilfe eines Plagiatserkennungsdienstes auf enthaltene Plagiate überprüft und ausschließlich für Prüfungszwecke gespeichert wird.

Munich, February 1, 2019