Multigrid methods based on high order tensor product B-Splines for the valuation of American options with stochastic volatility and their Greeks

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For the efficient numerical solution of elliptic variational inequalities on closed convex sets, multigrid methods based on piecewise linear basis function have been investigated over the past decades [5]. Essential to their success is the appropriate approximation of the constraint set on coarser grids which is based on function values for piecewise linear basis functions. On the other hand, there are a number of problems which profit from higher order approximations. Among these are problems of pricing American options, formulated as a parabolic free boundary value problem involving for example the Black-Scholes equation or the Heston-equation. The Heston equation is a parabolic partial differential equation depending on the underlying price and the volatility with a convection and diffusion term. We formulate the free boundary problem containing the Heston equation as a parabolic variational inequality and show the existence and uniqueness of the weak solution by using the result of [4].
In addition to computing the apriori unknown free boundary (the optimal exercise price of the option), of particular importance are accurate pointwise derivatives of the value of the stock option or volatility up to order two (the so-called Greek letters). We propose a monotone multigrid method for discretizations in terms of tensor product B-splines of arbitrary order and coincidental nodes in the interior in space and Crank-Nicolson in time to solve the parabolic asymmetric variational inequality on a closed convex set. To have a better approximation of the payoff function, which is only continuous in one point, by using high order B-splines, we let the nodes be coincidental in this specific point in the interior. Former works are about the monotone multigrid method with B-splines of high order without considering coincidental nodes and the application to the one dimensional Black-Scholes equation [2], or discretizing the free boundary problem (American option) or boundary value problem (European option) containing the two dimensional Heston-equation by finite difference- or finite element methods [1, 3, 6, 8]. We construct restriction operator and monotone coarse grid approximation for tensor product B-splines of order k with $k-1$ coincidental nodes in the interior. Additionally, a suitable smoother for the asymmetric discretised variational inequality is needed. Therefore an iterative method the so called projected Jacobi overrelaxation method is presented [7]. Finally, the method is applied to a discretized variational inequality, which is derived by the free boundary problem containing the Heston-equation. In particular, it is shown that a discretization of the asymmetric variational inequality based on tensor product B-splines of order four enables us to compute the second derivative of the value of the stock option pointwise to high precision.

References

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