Dynamic Symplectic Spinor Quantization: Relativistic Aspects of Computer Tomography

Walter J. Schempp
Lehrstuhl für Mathematik I
Universität Siegen
57068 Siegen, Germany
schempp@mathematik.uni-siegen.de

To a physicist, the 20th century begins on November 8, 1895 with Wilhelm Conrad Röntgen's totally unexpected discovery of X–rays. The technique of X–ray computer tomography was the predecessor of magnetic resonance tomography which is a non–invasive biomedical diagnostic imaging modality. The mathematizations of these diagnostic imaging modalities needs consistency with quantum and relativity theory. The reductive dual pair of real Lie groups

$$(\mathrm{Mp}(1,\mathbb{R}), \mathrm{PSO}(1,3,\mathbb{R}))$$

which are mutual centralizers in the symplectic Lie group $\mathrm{Sp}(2,\mathbb{R}) \hookrightarrow \mathrm{GL}(4,\mathbb{R})$. The dual pairing consists of the metaplectic group $\mathrm{Mp}(1,\mathbb{R}) = \widetilde{\mathrm{SL}}(2,\mathbb{R})$ acting as commutator group on the unitary dual of the Heisenberg unipotent Lie group, and the semi–simple Lorentz–Möbius Lie group $\mathrm{PSO}(1,3,\mathbb{R}) \cong \mathrm{PSL}(2,\mathbb{C})$ of orientation–preserving hyperbolic isometries acting on the universal covering group $\mathrm{Spin}(3,\mathbb{R})$ of the rotation group $\mathrm{SO}(3,\mathbb{R})$. Then the synchronization $\mathrm{Mp}(1,\mathbb{R}) \times \mathrm{PSO}(1,3,\mathbb{R})$ module action indicates mathematically the turn forwards and backwards from quantum field theory to the theory of relativity by completely determining the spectral decomposition of each other component.

The paper provides insight into the intrinsic electromagnetic quantum and relativistic symmetries associated with the highly efficient clinical modalities of X–ray computer tomography and magnetic resonance tomography by referring to the mathematical methodology of dynamic symplectic spinor quantization and the basic control mechanisms of deformation quantization. Due to the smooth line bundle of dynamic symplectic spinor quantization with the punctured complex plane $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$ as its typical one–dimensional complex fiber and the meromorphic differential 1–form $\alpha \in \Omega^1_{\mathbb{C}^{\times}}(\mathbb{P}_1(\mathbb{C}))$ on the compact Riemann surface of topological genus zero, a Kepplerian perspective to the relativistic effects on Global Navigation Satellite Systems such as Navstar GPS and its subsequent high precision Global Positioning System (GPS) Block Satellites is given.

Globally, the current situation in the theory of Global Navigation Satellite Systems is almost analogous to the following one: A century after Johannes Keppler, the astrophysicists were still

Figure 1: Dynamics of a symplectic spinor: Mathematically the dynamic symplectic spinor quantization process is performed by a smooth line bundle $(\Lambda_{\mathbb{C}}^{\times}, \alpha)$ over the complex projective line $\mathbb{P}_1(\mathbb{C}) \cong \mathbb{S}_2$ with the punctured complex plane $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$ as its typical one–dimensional complex fiber and the meromorphic differential 1–form $\alpha \in \Omega^1_{\mathbb{C}^{\times}}(\mathbb{P}_1(\mathbb{C}))$ on the compact Riemann surface of genus zero. Due to the synchronization $\mathrm{Mp}(1, \mathbb{R}) \times \mathrm{PSO}(1, 3, \mathbb{R})$ module action, the transitions from quantum field theory to the Schwarzschild metric and even to the Kerr metric of black hole cosmology via the Minkowski affine time–space metric of signature (1,3) are spectral steps to the software correction of the relativistic effects of the GPS constellation.

using the Kepplerian laws of planetary motion to correct the Ptolemaic cosmology of epicycles. Similarly, a century after Albert Einstein and although the relativistic effects are known to be not only important, but absolutely crucial for Global Positioning Systems to become operative, the community of GPS manufacturers still uses the classical Newtonian gravitation theory and correct their receiver software by relativistic effects instead of starting with quantum field theory and the theory of relativity rigorously right away. In the case of such a procedure, the coarse–acquisition or C/A code for the efficient transfer of transmission coordinate times from the orbiting components of the GPS to terrestrial receivers adopts its natural Lorentzian light–cone framework.

To improve GPS spatial precision so that terrestrial receiver positions can be determined with an uncertainty of only a centimeter, one has to account for all temporal relativistic effects down to a few hundredths of a nanosecond. But the second–order Doppler shift of an orbiting atomic clock, if it were not taken into account by the software control, would cause an error this large to build up in less than half a second. An effect of comparable size is contributed by the gravitational blueshift, which results when an atomic clock or a photon moves to lower altitude. If these relativistic effects were not corrected for by software, satellite atomic clock errors building up in just one day would cause inertial navigation errors of more than 11 km, quickly rendering the navigation system completely useless.