On the best approximation of the infinitesimal generator of a contraction semigroup in a Hilbert space

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Let A be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space H. In 1971 T. Kato proved that

$$\|Ax\|^2 \le 2\|x\|\|A^2x\|, \qquad x \in \mathcal{D}(A^2),$$

where $\mathcal{D}(A^2)$ denotes the domain of the operator A^2. This inequality cannot be improved for all Hilbert spaces. The "worst case" is when $H = L_2(0,\infty)$ and A is the differentiation operator. Kato's inequality in this case is the well-known sharp inequality of Hardy, Littlewood and Pólya (1934)

$$\|f'\|^2 \le 2\|f\|\|f''\|$$

in the space $L_2(0,\infty)$ on the class of functions $f \in L_2(0,\infty)$ such that f' is locally absolutely continuous on $(0,\infty)$, and $f'' \in L_2(0,\infty)$. Inequalities of the above type in different spaces of functions also for higher derivatives are called Kolomogorov-type inequalities and have been extensively studied.

Inequalities of Kolmogorov-type are known to be closely connected with the so-called Stechkin problem on the best approximation of an unbounded linear operator by bounded linear ones with a prescribed norm. In this way Kato's inequality is naturally connected with the problem of the best approximation of the operator A by bounded linear operators in the space H on the class $Q_2 = \{x \in \mathcal{D}(A^2) : \|A^2x\| \le 1\}$ that can be formulated as follows: determine

$$E_N(A;Q_2) = \inf\{U(T) : \ T \in \mathcal{B}(N)\},$$

where $\mathcal{B}(N)$ is the set of linear bounded operators T with the norm $\|T\| \le N$ and

$$U(T) = \sup\{\|Ax - Tx\| : \ x \in Q_2\}$$

is the deviation of the operator T from the operator A on the class Q.

We give an upper estimate for the quantity $E_N(A;Q_2)$.

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