Finite Element Method for the Solution of Elliptic Partial Differential Equations on Graphs

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Abstract

The question of investigating partial differential equations (PDEs) on graphs arises in the context of an interdisciplinary research project of the prediction of protein propagation in the brain network of Alzheimer's Disease patients [WB]. In this context, graphs allow modelling the interconnected structure of the brain network, while PDEs, particularly diffusion equations, describe the protein propagation. As a first attempt to approximate such PDEs on graphs, we will in this work focus on second order elliptic PDEs and study numerical methods for the solution of systems arising from a finite element discretization.

In order to formulate PDEs on graphs, we explain the network structure with the help of metric graphs. Metric graphs use an edgewise parameterization of the graph such that differential operators can be defined on the graph. Additionally for the well-posedness of the PDE on a graph, we require Neumann-Kirchhoff conditions, a kind of flow conservation property, on all vertices.

We discretize the metric graph with a finite element method, as described by Mario Arioli and Michele Benzi [AB]. The discretization of the metric graph can be interpreted as an extended graph with additional vertices, so called internal vertices. We then choose a hat function basis on the extended graph. This motivates the characterisation of the system that arises from the weak formulation of the PDE as

$$\begin{pmatrix} \mathbf{H}_{\mathcal{E}\mathcal{E}} & \mathbf{H}_{\mathcal{E}\mathcal{V}} \\ \mathbf{H}_{\mathcal{V}\mathcal{E}} & \mathbf{H}_{\mathcal{V}\mathcal{V}} \end{pmatrix} \mathbf{u} = \mathbf{f},$$

where $\mathbf{u}$ is the coefficient vector of the solution of the PDE written in the hat function basis. Each submatrix corresponds to different adjacency of hat functions on the extended graph. Thus their size increase with more internal vertices on each of the edges. This is especially important for the matrix $\mathbf{H}_{\mathcal{E}\mathcal{E}}$, because it is a block-diagonal matrix, with each blocks size proportional to the number of internal vertices. Consequently a fine discretization leads to a large system of equations and high computational cost.

In comparison to a domain decomposition approach with suitable PCG-solver proposed in [AB], we use a multigrid method to reduce the size of the matrix, which needs to be solved directly, to a minimum level. The idea of multigrid methods is to iteratively solve a given system of equations with a *coarse grid correction.* We make use of a hierarchical discretization of internal vertices and use suitable intergrid operators in order to transform matrices and vectors from one level to another. We show first numerical results of the convergence rate of the multigrid method on test problems.

References

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