# The Transformation $f(x) \to f(\cos(x_1,),\ldots,\cos(x_n))$ on Function Spaces of Dominating Mixed Smoothness

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**Abstract**

Imagine folding a strip of paper into a zig-zag-like shape. If we are careful during the folding, then the paper strip will end up with sharp edges. It describes a surface which we might intuitively call *rough* or *non-smooth*. Next grab both ends of the strip and bend them towards each other, turning the strip into a star-like or cog-like shape. Did the paper strip become *more rough* during this process? Or did it perhaps become *smoother*? Going purely by intuition we might conclude that no, although the strip might be a little bent and stretched the overall *roughness* or *smoothness* of the strip did not significantly change.

It is one aim of the present work to confirm our intuition, that is, to show that a specific operation of periodisation does not completely destroy the smoothness properties of functions. More precisely, we study the composition operator

$$T_{\cos} : f(x_1,\ldots,x_n) \to f(\cos(x_1),\ldots,\cos(x_n))$$

taking functions from the euclidean space $\mathbb{R}^n$ to the torus $\mathbb{T}^n$. Composition operators on function spaces have been well-studied in the context of *changes of variable*. The literature knows numerous results for composition operators induced by bijective functions. Changes of variable with non-bijective functions are more difficult to handle (see for example [1] for results in this direction).

The operator above is clearly induced by a non-bijective function, making its analysis slightly complicated at times. Nonetheless, using techniques from [3, 4, 5, 6], we show that $T_{\cos}$ is a bounded operator on certain Besov and Triebel-Lizorkin spaces of dominating mixed smoothness. In more detail, we obtain the result that

$$T_{\cos} : S^r_{p,q}A(\mathbb{R}^n) \to S^r_{p,q}A(\mathbb{T}^n)$$

is bounded if

- $A = F$ and $r > 1$, $1 < p < \infty$, $1 < q \leq \infty$,
- or if $A = B$ and $r > 1/p$, $1 < p < \infty$, $0 < q \leq \infty$.

In the setting $A = B$ we also present some results for the limiting cases. Lastly, we also draw a connection between the Sobolev spaces $H^s([-1,1])$ and spaces $L^{2,s}_{-\frac{1}{2},-\frac{1}{2}}([-1,1])$ of Sobolev-type related to Chebyshev polynomials as they are considered in [2].

# References


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