Numerical Computation of Equilateral Quantum Graph Spectra

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Abstract

Under an equilateral quantum graph, we understand a metric graph equipped with a differential operator, suitable coupling conditions, and such that all edge lengths are equal. The differential operator of our interest is the negative second-order derivative, often called *Hamiltonian* acting on each edge. The spectrum of the quantum graph Γ is then understood as the spectrum of the differential operator acting on the metric graph. We are interested if there is any relation between the spectra of quantum graphs and discrete operators on the underlying combinatorial graph. And in fact, there exists a very useful and prominent relation to the spectrum of the combinatorial graph Laplacian matrix if we assume the graph to be equilateral. Yet, this relation only holds true for a specific part of the spectrum, which we will refer to as the vertex spectrum. The remaining non-vertex part of the spectrum needs separate consideration. By using a simple trick of inserting artificial vertices on the edges, we will see, that we can again reduce the non-vertex eigenvalues to a discrete problem and draw conclusions like specifying their multiplicity [AW]. In this context, we will speak of an *extended graph Laplacian system*.

It remains to develop an efficient method to compute the associated eigenfunctions. To do so, we only need the eigenvectors of the discrete graph Laplacian eigenvalue problem. In the case of non-vertex eigenfunctions, we present an improved method to calculate the required eigenvectors of the extended graph Laplacian system by using theoretically derived information on their values at the vertices of the quantum graph.

References

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