Admissible Representations for Continuous Computations

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Diplom-Informatiker Matthias Schröder

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angenommen aufgrund der Berichte von

Prof. Dr. K. Weihrauch, Hagen
Prof. Dr. D. Spreen, Siegen

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Chapter 1

Introduction

The traditional tool for computations on real numbers is floating point arithmetic. Real numbers are represented by finite words consisting of a mantissa and an exponent, both of fixed length. Since only approximations of the reals numbers are stored, floating point arithmetic does not provide a reliable computational model for the real numbers. There are several examples of seemingly harmless arithmetical problems for which floating point arithmetic produces surprisingly crude results far away from the correct solutions. Consider for instance the following system of linear equations:

\[
\begin{align*}
40157959.0 \cdot x + 67108865.0 \cdot y &= 1 \\
67108864.5 \cdot x + 112147127.0 \cdot y &= 0.
\end{align*}
\]

Applying the well–known formula

\[
x = \frac{b_1 \cdot a_{2,2} - b_2 \cdot a_{1,2}}{a_{1,1} \cdot a_{2,2} - a_{2,1} \cdot a_{1,2}},
\]

\[
y = \frac{a_{1,1} \cdot b_2 - a_{2,1} \cdot b_1}{a_{1,1} \cdot a_{2,2} - a_{2,1} \cdot a_{1,2}},
\]

floating point arithmetic with double precision variables (IEEE standard 754, 53bit mantissa) computes the values \( \tilde{x} = 112147127, \tilde{y} = -67108864.5 \) as the solution to this system of linear equations. However, the correct solution is the double of these values, namely \( x = 224294254, \ y = -134217729 \). Note that only five simple operations produce a completely wrong result, although the coefficients can be represented accurately by double precision variables\(^1\). Of course, increasing the size of the mantissa does not help substantially, because other systems of linear equations remain unsolvable\(^2\).

Floating point arithmetic does not even satisfy the associative law for multiplication. The logistic function

\[
x_{i+1} := 3.75 \cdot x_i \cdot (1.0 - x_i), \ x_0 := 0.5
\]

(cf. [Kul96, Mü01]) demonstrates this fact convincingly. Depending on the order of evaluation of the product \( 3.75 \cdot x_i \cdot (1.0 - x_i) \), floating point arithmetic using double precision variables produces the values \( 0.8358 \ldots, 0.5887 \ldots, 0.3097 \ldots \) for \( x_{100} \)\(^3\).

\(^1\) The reason for this error is that the result of the multiplication of \( a_{2,1} = 67108864.5 \) and \( a_{1,2} = 67108865 \) is rounded to the next even number, whereas \( a_{1,1} = 40157959 \) and \( a_{2,2} = 112147127 \) are chosen in such a way that their product is equal to the next odd number to \( a_{2,1} \cdot a_{1,2} \). Hence 1.0 is the computed value for the determinant \( a_{1,1} \cdot a_{2,2} - a_{2,1} \cdot a_{1,2} \), whereas the correct value is 0.5.

\(^2\) A system of linear equations causing similar problems, when variables with 64bit mantissa are used, is given by \( 228366445 \cdot x + 4294967297 \cdot y = 1, 2147483648.5 \cdot x + 40388473189 \cdot y = 0 \).

\(^3\) The correct value of \( x_{100} \) is roughly \( 0.888293992 \ldots \) computed by N. Müller’s iRRAM (cf. [Mü01]).
Chapter 1. Introduction

One possibility to avoid the unreliability of floating point arithmetic is to use rational arithmetic. On rational numbers, reliable computations are possible (in principle). Since the set \( \mathbb{Q} \) of rational numbers is countable, the rational numbers (in contrast to the reals) can be represented unambiguously by finite words (i.e. \( \mathbb{Q} \) has a notation). One simply stores any rational \( q \) by a pair \((z,n)\) of coprime integers with \( q = \frac{z}{n} \) and \( n > 0 \) and computes on these pairs in the obvious way. However, the example of the logistic function shows that rational arithmetic is not feasible: the space for storing the rational \( x_i \) roughly doubles in each iteration step of the logistic function, so that even the computation of the simple sequence \((x_i)_i\) needs exponential time.

Hence an approximative computational model on \( \mathbb{R} \) which enables to compute on the reals with arbitrary precision is necessary. Type-2 Theory of Effectivity (TTE) provides such a computational model (cf. [Wei00]). The basic idea of the TTE approach, which is considered in this thesis, is to represent the real numbers by infinite sequences of symbols (\( \omega \)-words) of a finite or countably infinite alphabet \( \Sigma \). Such a naming function, which formally is a partial surjection from \( \Sigma^\omega \) onto \( \mathbb{R} \), is called a representation. The actual computation is performed by a digital computer on the names of an appropriate representation \( \delta \). Roughly speaking, a real-valued function \( f : \mathbb{R} \to \mathbb{R} \) is defined to be computable w.r.t. \( \delta \) iff, for every name \( p \) of any argument \( x \), every finite prefix of a name of the result \( f(x) \) can be computed from some finite prefix \( p \) (cf. Subsection 2.1.4).

Some well-known examples of representations of \( \mathbb{R} \) are the decimal representation and the binary representation and called a real number.

The definition of the notion of a computable real number (cf. [Tur36]). He used the binary representation of \( \mathbb{R} \) and called a real number \( x \) computable iff there is a Turing machine computing the binary expansion of \( x \). However, the binary representation and the decimal representation are inappropriate for defining a reasonable notion of a computable real-valued function. Turing has already noticed this unsuitability implicitly (cf. [Tur37]). Actually, real multiplication by 3 turns out to be uncomputable w.r.t. the decimal representation and w.r.t. the decimal representation. Consider the \( \omega \)-word \( p = 0.\overline{3333} \ldots \) which is a \( \rho_{10} \)-name of \( \frac{1}{3} \). No finite prefix of \( p \) allows us to detect whether the number \( x \) represented by \( p \) satisfies \( x \leq \frac{1}{3} \) or \( x \geq \frac{1}{3} \), because each prefix of \( p \) can be extended to a name of a number less than \( \frac{1}{3} \) as well as to a name of a number greater than \( \frac{1}{3} \). But this information is necessary for computing the first symbol of any name of the number \( 1 = 3 \cdot \frac{1}{3} \). Hence multiplication by 3 fails to be computable w.r.t. the decimal representation. The reason for this unexpected flaw is not a recursion-theoretical one, but a topological one: the decimal representation does not provide any name for the number 1 reflecting the fact that 1 can be approximated simultaneously from below and above. By contrast, the topological property of admissibility, being the central notion investigated in this thesis, avoids such problems. Admissible
representations yield appropriate encodings of the represented elements. Fortunately, there are representations of \( \mathbb{R} \) which induce a reasonable computational model on the real numbers, for instance, the signed-digit representation using an additional “negative” digit with value \(-1\) (cf. [Avi61]). This representation turns out to be admissible (cf. Example 2.3.8), i.e., it is topologically well-formed. Basic functions like addition, multiplication, sine, etc. are computable w.r.t. that representation.

The representation approach enables to introduce computability on every set \( X \) with cardinality of the continuum: these sets are exactly those which are equipped with a representation \( \delta \) (i.e. a surjective function \( \delta : \subseteq \Sigma^\omega \to X \)). A function \( f : X \to X \) is defined to be computable with respect to \( \delta \) iff there exists a computable function \( g \) on \( \Sigma^\omega \) translating every \( \delta \)-name of an argument \( x \) into a \( \delta \)-name of the result \( f(x) \). A function \( g : \subseteq \Sigma^\omega \to \Sigma^\omega \) is called computable iff, for \( p \in \text{dom}(g) \), every finite prefix of \( f(p) \) can be computed from some finite prefix of the argument \( p \). The precise definitions are given in Section 2.1.

Any finite prefix of a \( \delta \)-name \( p \) can be viewed as an approximation of the represented element \( \delta(p) \). Therefore the computability notion provided by the TTE approach is an approximative one: while reading more and more precise information of the input, the computer produces more and more accurate approximations of the result. Thus it is important to investigate the approximation structure induced by a representation on its represented set. In mathematics, topological spaces are commonly used to study approximations (cf. [Eng89, Smy92, Wil70]). A more general concept than topological spaces are limit spaces (cf. Subsection 2.2.2 and [Hyl79, MS02]). However, topological spaces as well as limit spaces are too special to describe the approximation structure induced by representations appropriately. Therefore we introduce in Section 2.2 the new notion of a weak limit space. Every representation \( \delta \) of a set \( X \) induces a convergence relation on \( X \) which satisfies the axioms of weak limit spaces. This weak limit space generated by \( \delta \) describes appropriately the approximation structure induced by \( \delta \).

The natural continuity notion for functions between weak limit spaces is sequential continuity. We show that sequential continuity rather than topological continuity is the appropriate continuity notion in computable analysis. Every relatively computable function is sequentially continuous between the weak limit spaces generated by the involved representations, but it is not necessarily topologically continuous w.r.t. the final topologies induced by the same representations (cf. Subsections 2.2.4 and 2.4.5). Note that in analysis one is usually more interested in convergence of sequences than in the underlying topology. Thus it is adequate to consider the corresponding notion of sequential continuity rather than topological continuity.

Admissible representations are well-behaved in a topological sense. They are defined to be the maximal ones among those representations which generate the same weak limit space; maximality of a representation \( \delta \) means that every representation generating the same weak limit space as \( \delta \) can be translated to \( \delta \) by a continuous translator. Thus admissible representations of a given space \( \mathfrak{X} \) are the ones which encode in the best way the approximative structure of \( \mathfrak{X} \). C. Kreitz and K. Weihrauch were the first to introduce admissible representations of countably based \( T_0 \)-spaces (cf. [KW85]). As one of the basic results of this thesis, we show that admissible representations exist for a much larger class of spaces including many topological spaces that do not have a countable base and many weak limit spaces that are not limit spaces (cf. Section 2.3).
Chapter 1. Introduction

The question arises whether every weak limit space generated by an arbitrary (non-admissible) representation has an admissible, i.e. well-behaved, representation. Actually the answer is no. However, if we consider multirepresentations rather than single-valued representations, then the answer turns to be yes. Multirepresentations, formally being surjective correspondences between \( \Sigma^\omega \) and the represented set, allow their names to represent more than one element at the same time. In Section 2.4 we introduce multirepresentations and define relative computability with respect to multirepresentations as well as admissibility of multirepresentations.

In Chapter 3 we characterize the class of spaces which are equipped with an admissible (multi-) representation. In particular, we prove the interesting result that the category \( \text{AdmSeq} \) of sequential spaces with admissible multirepresentation is equal to the category of topological quotients of countably based spaces. This result establishes \( \text{AdmSeq} \) as a very natural class of topological spaces. Moreover, \( \text{AdmSeq} \) turns out to be equal to the category \( \text{PQ} \) of \( \omega \)-projecting quotients of countably based spaces, which has been introduced by M. Menni and A. Simpson. The category \( \text{AdmWeakLim} \) of admissibly representable weak limit spaces is characterized in a similar way.

Chapter 4 is devoted to constructions on weak limit spaces and topological spaces. Many of these constructions are shown to map admissibly representable spaces to admissibly representable spaces. In particular, we prove that the categories \( \text{AdmWeakLim} \) and \( \text{AdmSeq} \) are Cartesian closed. This means that the TTE approach is applicable to higher–type computations. Moreover, both categories have countable limits as well as countable colimits. Hence \( \text{AdmWeakLim} \) and \( \text{AdmSeq} \) form very natural classes of spaces. These results imply that our computability model can be applied easily to several non–metric, yet important spaces used in analysis, for example, the space of distributions. Distributions play an important role in the field of differential equations. The space of distributions as well as many function spaces do not have a countable base and hence cannot be handled by the classical notion of admissibility. We also investigate admissible multirepresentations of hyperspaces. Hyperspaces seem to be an appropriate way to deal with multi–valued functions (correspondences).

In Section 4.3 we concentrate on computability aspects. We prove the important result that any function that is computable w.r.t. arbitrary multirepresentations is also computable w.r.t. admissible ones. The proof is done by introducing operators which transform multirepresentations to admissible ones while preserving relative computability of functions. These operators are also used to define the notion of effectively admissible multirepresentations. Effectively admissible multirepresentations have nice computability properties.

Some of the results presented in this thesis have already been published, cf. [Sch01, Sch02a, Sch02b, Sch02c].

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1.1 Preliminaries

We denote by \( \mathbb{N} \) the set of natural numbers including 0, by \( \mathbb{N} \) the set of naturals, by \( \mathbb{Q} \) the set of rationals, and by \( \mathbb{R} \) the set of real numbers.

The quantifier \( (\forall \infty n) \) means for almost all numbers \( n \) and \( (\exists \infty n) \) means for infinitely many numbers \( n \), more precisely, for a subset \( M \subseteq \mathbb{N} \)

\[
(\forall \infty n \in M) S(n) \text{ abbreviates } (\exists n_0 \in \mathbb{N})(\forall n \in \{ i \in M \mid i \geq n_0 \}) S(n) \text{ and }
(\exists \infty n \in M) S(n) \text{ abbreviates } (\forall n_0 \in \mathbb{N})(\exists n \in \{ i \in M \mid i \geq n_0 \}) S(n).
\]

A correspondence \( F \subseteq X \to Y \) from the source \( X \) to the target \( Y \) is a triple \( F = (X, Y, \mathcal{G}(F)) \), where \( \mathcal{G}(F) \), the graph of \( F \), is a relation \( \mathcal{G}(F) \subseteq X \times Y \). The inverse of \( F \), i.e. the correspondence \( (Y, X, \{(y, x) \mid (x, y) \in \mathcal{G}(F)\}) \), is denoted by \( F^{-1} \).

For subsets \( A \subseteq X \) and \( B \subseteq Y \), we define the image of \( A \) under \( F \) by \( F[A] := \{ y \in Y \mid (\exists x \in A)(x, y) \in \mathcal{G}(F) \} \), the preimage of \( B \) under \( F \) by \( F^{-1}[B] := \{ x \in X \mid (\exists y \in B)(x, y) \in \mathcal{G}(F) \} \), the domain of \( F \) by \( \text{dom}(F) := F^{-1}[Y] \) and the range of \( F \) by \( \text{range}(F) := F[X] \).

We define the composition \( G \circ F \) of two correspondences \( F \subseteq X \to Y \) and \( G \subseteq Y \to Z \) to be the triple \( (X, Z, \{(x, z) \mid (\exists y \in F[x]) z \in G[y]\}) \).

The notation \( f : \subseteq X \to Y \) indicates that \( f \) is a partial function from \( X \) to \( Y \), i.e., \( f \) is a correspondence \( (X, Y, \mathcal{G}(f)) \) such that \( f[x] \) contains at most one element for every \( x \in X \), which is denoted by \( f(x) \), if it exists. We write \( f(x) = \text{div} \), if \( x \notin \text{dom}(f) \).

The notation \( f : X \to Y \) indicates that \( f \) is a total function from \( X \) to \( Y \), i.e., \( f \) is a correspondence \( (X, Y, \mathcal{G}(f)) \) such that, for every \( x \in X \), \( f[x] \) contains exactly one element.

1.1.1 Alphabets

As alphabets, we use either finite ones containing the symbols 0 and 1 or the sets \( \mathbb{N} \) or \( \mathbb{Z} \). We assume that every used alphabet \( \Sigma \) comes equipped with an injective standard surjection (numbering) \( \nu_{\Sigma} : \subseteq \mathbb{N} \to \Sigma \) which in the finite case has the domain \( \text{dom}(\nu_{\Sigma}) = \{0, \ldots, \text{card}(\Sigma) - 1\} \), where \( \text{card}(\Sigma) \) denotes the cardinality of \( \Sigma \), and which in the case \( \Sigma \in \{\mathbb{N}, \mathbb{Z}\} \) is defined by

\[
\nu_{\mathbb{N}}(n) := n \quad \text{and} \quad \nu_{\mathbb{Z}}(n) := \begin{cases} n + \frac{1}{2} & \text{if } n \text{ is odd} \\ n - \frac{1}{2} & \text{otherwise} \end{cases}
\]

for all \( n \in \mathbb{N} \).

1.1.2 Words and \( \omega \)-Words

Let \( \Sigma \) be an alphabet. We denote by \( \Sigma^* \) the sets of (finite) words over \( \Sigma \) and by \( \Sigma^\omega \) the set of infinite words (\( \omega \)-words) over \( \Sigma \), i.e. the set \( \Sigma^\omega = \{ p \mid p : \mathbb{N} \to \Sigma \} \).
For \( a \in \Sigma, u, v \in \Sigma^*, W \subseteq \Sigma^*, p, p' \in \Sigma^\omega, \) and \( n \in \mathbb{N} \) we denote by

\[
\begin{aligned}
\lg(u) & \quad \text{the length of the word } u; \\
u(n) & \quad \text{the } (n+1)-\text{st symbol of } u \text{ for } n < \lg(u), \\
a^n & \quad \text{the word consisting of } n \text{ symbols } a; \\
a^\omega & \quad \text{a sequence } q \text{ with } q(i) = a \text{ for every } i \in \mathbb{N}; \\
p^{<n} & \quad \text{the prefix of length } n \text{ of } p, \text{ i.e. the word } p(0)p(1)\ldots p(n-1); \\
p^{\leq n} & \quad p \text{ itself}; \\
p^{>n} & \quad \text{the sequence } p(n+1), p(n+2), \ldots, \text{ i.e. } p = p^{<n}:p^{>n}; \\
p^\omega & \quad p \text{ itself}; \\
up, u::p & \quad \text{the sequence } q \text{ with prefix } u \text{ followed by } p, \\
 & \quad \text{i.e. } q^{<\lg(u)} = u \text{ and } q(i + \lg(u)) = p(i) \text{ for all } i \in \mathbb{N}; \\
\underbrace{p^{\leq n}}_{u} & \quad \text{the word } p(0)p(1)\ldots p(n); \\
\underbrace{p^{>n}}_{u} & \quad \text{the sequence } p(n+1), p(n+2), \ldots, \text{ i.e. } p = p^{<n}:p^{>n}; \\
\sqsubseteq & \quad \text{the prefix–relation on } \Sigma^* \cup \Sigma^\omega, \text{ i.e. } u \sqsubseteq v : \iff (\exists w \in \Sigma^*) uw = v, \\
 & \quad u \sqsubseteq p : \iff (\exists j \in \mathbb{N}) u = p^{<j}, \text{ not } p \sqsubseteq u, p \sqsubseteq p' : \iff p = p'; \\
\uSigma^\omega & \quad \text{the set } \{ q \in \Sigma^\omega | u \sqsubseteq q \}; \\
W\Sigma^\omega & \quad \text{the set } \{ q \in \Sigma^\omega | (\exists w \in W) w \sqsubseteq q \}.
\end{aligned}
\]

### 1.1.3 Computability on Discrete Sets via Numberings

We assume that the reader is familiar with the classical definition of recursive (=computable) number functions \( g : \subseteq \mathbb{N}^k \to \mathbb{N} \) and of recursive (=computable) word functions \( \lambda : \subseteq (\Sigma^*)^k \to \Sigma^* \) over a finite alphabet \( \Sigma \). Computability can be transferred to other countable sets by numberings. A numbering of a set \( X \) is surjective function \( \nu : \subseteq \mathbb{N} \to X \). If \( \nu_i \) is a numbering of the set \( X_i \) for every \( i \in \{1 \ldots k+1\} \), then we say that a function \( f : \subseteq X_1 \times \ldots \times X_k \to X_{k+1} \) is \((\nu_1, \ldots, \nu_{k+1})\)-computable or computable with respect to \( \nu_1, \ldots, \nu_{k+1} \) iff there is a recursive number function \( g : \subseteq \mathbb{N}^k \to \mathbb{N} \) such that

\[
\nu_{k+1}(g(n_1, \ldots, n_k)) = f(\nu_1(n_1), \ldots, \nu_k(n_k)) \tag{1.2}
\]

holds for all \((n_1, \ldots, n_k) \in \text{dom}(\nu_1) \times \ldots \times \text{dom}(\nu_k)\) with \((\nu_1(n_1), \ldots, \nu_k(n_k)) \in \text{dom}(f)\).

This notion of computability is called relative computability w.r.t. numberings.

As the standard numbering of the set \( \mathbb{N}^2 \), we will use Cantor’s bijective and recursive pairing function \( \langle \cdot, \cdot \rangle_2 : \mathbb{N}^2 \to \mathbb{N} \) defined by \( \langle x, y \rangle_2 := y + \sum_{i=1}^{x+y} i \). For \( k > 2 \), we define the bijection \( \langle \ldots, \cdot \rangle_k : \mathbb{N}^k \to \mathbb{N} \) inductively by \( \langle x_1, \ldots, x_k \rangle_k := \langle \langle x_1, \ldots, x_{k-1} \rangle_{k-1}, x_k \rangle_2 \). Usually, we will write \( \langle \ldots, \cdot \rangle \) for \( \langle \ldots, \cdot \rangle_k \), if the arity \( k \) is clear from the context. These tupling functions as well as the projections \( \pi_{k,i} : \mathbb{N} \to \mathbb{N} \) of their inverses (satisfying \( \langle \pi_{k,i}(n), \ldots, \pi_{k,k}(n) \rangle_k = n \)) are computable.

For the set \( \mathbb{Q} \) we will use as the standard numbering \( \nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q} \) the one defined by \( \nu_{\mathbb{Q}}(z, n) := \frac{\nu_2(z)}{n+1} \) for all \( z, n \in \mathbb{N} \).
As the bijective standard numbering $\nu_{\Sigma^*}$ of $\Sigma^*$, where $\Sigma$ is a finite alphabet, we take $\nu_{\Sigma^*} : \mathbb{N} \rightarrow \Sigma^*$ defined by

$$\nu_{\Sigma^*} \left( \sum_{j=0}^{l-1} \text{card}(\Sigma)^j \cdot (i_j + 1) \right) := \nu_{\Sigma}(i_{l-1})\nu_{\Sigma}(i_{l-2}) \ldots \nu_{\Sigma}(i_1)\nu_{\Sigma}(i_0)$$

for $l \in \mathbb{N}$ and $i_0, \ldots, i_{l-1} \in \{0, \ldots, \text{card}(\Sigma) - 1\}$. Moreover, as the bijective standard numberings of $\mathbb{N}^*$ and $\mathbb{Z}^*$, we will use $\nu_{\mathbb{N}^*} : \mathbb{N} \rightarrow \mathbb{N}^*$ and $\nu_{\mathbb{Z}^*} : \mathbb{N} \rightarrow \mathbb{Z}^*$ defined by

$$\nu_{\mathbb{N}^*}(0) := \varepsilon \quad \text{and} \quad \nu_{\mathbb{N}^*}(1 + \langle \langle a_1, \ldots, a_l \rangle, l - 1 \rangle_2) := a_1 \ldots a_l$$

$$\nu_{\mathbb{Z}^*}(0) := \varepsilon \quad \text{and} \quad \nu_{\mathbb{Z}^*}(1 + \langle \langle a_1, \ldots, a_l \rangle, l - 1 \rangle_2) := \nu_{\mathbb{Z}^*}(a_1) \ldots \nu_{\mathbb{Z}^*}(a_l)$$

for all $l \geq 1$ and $a_1, \ldots, a_l \in \mathbb{N}$.

We define computability of word functions $h : \Sigma_1^* \times \ldots \times \Sigma_k^* \rightarrow \Sigma_{k+1}^*$ as relative computability w.r.t. these standard numberings (i.e. as $(\nu_{\Sigma_1^*}, \ldots, \nu_{\Sigma_k^*})$–computability). It belongs to the folklore of recursion theory that, for a finite alphabet $\Sigma$, a total function $h : (\Sigma^*)^k \rightarrow \Sigma^*$ is $(\nu_{\Sigma^*}, \ldots, \nu_{\Sigma^*})$–computable if and only if it is a recursive word function.

### 1.1.4 Category Theory

A good reference to category theory is [AL91]. We will introduce most notions of category theory at the place where they are used for the first time.
Chapter 1. Introduction
Chapter 2

Admissibility

This chapter is devoted to the definition of admissible representations and multirepresentations. We start in Section 2.1 with a short introduction to Type–2 Theory of Effectivity, our computational model. In Section 2.2 we investigate the approximation structure induced by a representation on the represented set. As an appropriate tool for that we introduce the class of weak limit spaces. Section 2.3 motivates and defines admissibility of representations. Moreover, we compare our notion of admissibility with the standard, more restricted notion of admissibility. In Section 2.4 we consider the generalized concept of multirepresentations and define notions like relative computability and admissibility for them generalizing the corresponding notions for representations.

2.1 Basics of the Type–2 Theory of Effectivity

This section gives a short introduction to the basic concepts of the Type–2 Theory of Effectivity (TTE). Further information can be found in [Wei00, KW85, Wei87, Wei95].

2.1.1 Representations

In classical recursion theory, computability on a countable set $X$ is usually defined by lifting the classical notion of computability of number functions $g : \subseteq \mathbb{N} \to \mathbb{N}$ to $X$ via an appropriate numbering of $X$. A numbering $\nu$ of a set $X$ is a surjection $\nu : \subseteq \mathbb{N} \to X$ providing a “name” for every element of $X$. Alternatively, one uses notations $\nu : \subseteq \Sigma^* \to X$ and the recursive word functions as the basic computability notion.

In the case of an uncountable set $X$ this approach fails, because the large cardinality excludes the existence of a numbering or a notation. The basic idea of the Type–2 Theory is to encode the objects of $X$ by $\omega$–words (infinite words) over a given alphabet $\Sigma$, i.e. by elements of the set $\Sigma^\omega = \{p \mid p : \mathbb{N} \to \Sigma\}$, and to compute on these names. The corresponding partial surjection $\delta : \subseteq \Sigma^\omega \to X$ mapping every name $p \in \text{dom}(\delta)$ to the encoded element $\delta(p)$ is called a representation of $X$.

Definition 2.1.1 (Representation)
A representation of a set $X$ over an alphabet $\Gamma$ is a surjection $\delta : \subseteq \Gamma^\omega \to X$.
If $\delta$ is a representation of $X$, we call the pair $(X, \delta)$ a represented space.
Chapter 2. Admissibility

Example 2.1.2 (The decimal representation)
A popular example of a representation of the set of real numbers is the usual decimal representation \( \rho_10 \) over the alphabet \( \Sigma_{\text{dec}} := \{0, \ldots, 9, *, -\} \) defined by

\[
\begin{align*}
\rho_10(a_{-k} \ldots a_0* a_1a_2\ldots) &= + \sum_{i=-k}^{\infty} 10^{-i} \cdot a_i \quad \text{and} \\
\rho_10(-a_{-k} \ldots a_0* a_1a_2\ldots) &= - \sum_{i=-k}^{\infty} 10^{-i} \cdot a_i
\end{align*}
\]

for all \( k \in \mathbb{N} \) and all digits \( a_{-k}, a_{-k+1}, \ldots \in \{0, \ldots, 9\} \).

2.1.2 Computability of \( \omega \)--Word Functions

The usual notion of computability for functions on \( \Sigma^\omega \) can be defined either via computable monotone word functions or by Type–2 machines. We extend this notion slightly and define also computability for functions \( g : \subseteq \Sigma_1^\omega \times \ldots \times \Sigma_k^\omega \rightarrow \Sigma_{k+1}^\omega \) where the alphabets \( \Sigma_1, \ldots, \Sigma_k \) are not necessarily equal and are equipped with standard numberings \( \nu_{\Sigma_1}, \ldots, \nu_{\Sigma_k} \subseteq \mathbb{N} \rightarrow \Sigma_k \) (cf. Subsection 1.1.1).

A function \( \lambda : \Sigma_1^* \times \ldots \times \Sigma_k^* \rightarrow \Sigma_{k+1}^* \) is called monotone iff \( (u_1 \subseteq v_1) \land \ldots \land (u_k \subseteq v_k) \) implies \( \lambda(u_1, \ldots, u_k) \subseteq \lambda(v_1, \ldots, v_k) \) for all words \( u_1, v_1 \in \Sigma_1^*, \ldots, u_k, v_k \in \Sigma_k^* \). If \( \lambda : \Sigma_1^* \times \ldots \times \Sigma_k^* \rightarrow \Sigma_{k+1}^\omega \) is monotone, then we denote by \( \lambda^\omega \) the partial function \( \lambda^\omega : \subseteq \Sigma_1^\omega \times \ldots \times \Sigma_k^\omega \rightarrow \Sigma_{k+1}^\omega \) defined by

\[
don(\lambda^\omega) := \{ (p_1, \ldots, p_k) \mid \lambda(p_1^\omega, \ldots, p_k^\omega) \mid n \in \mathbb{N} \}
\]

and

\[
\lambda^\omega(p_1, \ldots, p_k) := \text{the unique } q \in \Sigma_{k+1}^\omega \text{ with } (\forall n \in \mathbb{N}) \lambda(p_1^\omega, \ldots, p_k^\omega) \subseteq q = \sup \{ \lambda(p_1^\omega, \ldots, p_k^\omega) \mid n \in \mathbb{N} \}
\]

for all \( (p_1, \ldots, p_k) \in \text{dom}(\lambda^\omega) \).

A function \( g : \subseteq \Sigma_1^\omega \times \ldots \times \Sigma_k^\omega \rightarrow \Sigma_{k+1}^\omega \) is called computable iff there is a computable monotone word function \( \lambda : \Sigma_1^* \times \ldots \times \Sigma_k^* \rightarrow \Sigma_{k+1}^* \) such that \( \lambda \) approximates \( g \), i.e., \( \lambda^\omega = g \) holds. Computability of the word function \( \lambda \) is defined by means of the canonical effective standard numberings \( \nu_{\Sigma_1^*}, \ldots, \nu_{\Sigma_k^*} : \mathbb{N} \rightarrow \Sigma_k^* \) (cf. Subsection 1.1.3).

The set of computable \( \omega \)--word functions is closed under composition (cf. [Wei00]).

Lemma 2.1.3 (Composition preserves computability)

Let \( g_0 : \Gamma_1^\omega \times \ldots \times \Gamma_k^\omega \rightarrow \Gamma_{k+1}^\omega \) and \( g_i : \subseteq \Sigma_1^\omega \times \ldots \times \Sigma_l^\omega \rightarrow \Gamma_i^\omega \) be computable functions for \( i \in \{1, \ldots, k\} \). Then the composition \( g' := g_0 \circ (g_1 \times \ldots \times g_k) \) is computable.

Proof: There are computable word functions \( \lambda_0, \ldots, \lambda_k \) such that \( \lambda_i^\omega = g_i \) holds for every \( i \in \{0, \ldots, k\} \) and \( \lambda_0 \) satisfies additionally

\[
\lg (\lambda_0(w_1, \ldots, w_k)) \leq \min \{ \lg(w_1), \ldots, \lg(w_k) \}
\]
2.1 Basics of the Type–2 Theory of Effectivity

for all \((w_1, \ldots, w_k) \in \Gamma_1^* \times \ldots \times \Gamma_k^*\). The composition \(\lambda' := \lambda_0 \circ (\lambda_1 \times \ldots \times \lambda_k)\) is computable and can be easily proved to approximate \(g'\). Note that \(\lambda'\) would approximate an extension of \(g'\) without the above additional condition on \(\lambda_0\). However, in many cases it suffices to consider a computable extension of \(g'\).

By definition, every prefix of the result \(g(p_1, \ldots, p_k)\) of a computable function \(g : \subseteq \Sigma_1^\omega \times \ldots \times \Sigma_k^\omega \rightarrow \Sigma_{k+1}^\omega\) only depends on a finite prefix of each of the arguments \(p_1, \ldots, p_k\). This finiteness property means that \(g\) is topologically continuous (and thus sequentially continuous) with respect to the Cantor space topologies \(\tau_{\Sigma_1^\omega}, \ldots, \tau_{\Sigma_{k+1}^\omega}\) defined by

\[\tau_{\Sigma_i^\omega} := \{W \Sigma_i^\omega | W \subseteq \Sigma_i^*\}\]

The converse is not true; it is well–known that there are topologically continuous functions \(f : \Sigma_1^\omega \rightarrow \Sigma_1^\omega\) which are not computable. The notions of a topology and of a topologically continuous function are introduced in Subsection 2.2.2.

2.1.3 Type–2 Machines

The computable functions \(g : \subseteq (\Sigma^\omega)^k \rightarrow \Sigma^\omega\) are exactly those functions which are computed by Type–2 machines over \(\Sigma\) (cf. [Wei00] for finite alphabets and [Wei87] for the case \(\Sigma = \mathbb{N}\)). Briefly, a Type–2 machine is a usual Turing machine with changed semantics. It has \(k\) one–way input tapes, several work tapes and a single one–way output tape. The function \(\Gamma_M : \subseteq (\Sigma^\omega)^k \rightarrow \Sigma^\omega\) computed by a Type–2 machine \(M\) is defined by

\[
\Gamma_M(p_1, \ldots, p_k) := \begin{cases} 
q & \text{if } M \text{ with input } p_1, \ldots, p_k \text{ writes step by step } \in \Sigma^\omega \text{ onto the output tape} \\
\text{div} & \text{if } M \text{ with input } p_1, \ldots, p_k \text{ does not write infinitely many symbols onto the output tape.}
\end{cases}
\]

Since a Type–2 machine can never change a symbol already written onto the output tape (remember that it is an one–way–tape), each prefix of the output only depends on a prefix of each input. This finiteness property guarantees that Type–2 machines, although they formally handle infinite objects, yield a practical computational model: one can simulate a Type–2 machine \(M\) by an ordinary computer which computes from a natural number \(n\) and appropriate prefixes of the input of \(M\) the first \(n\) digits of the output of \(M\). In general, such a simulation would not be possible, if Type–2 machine were allowed to change symbols already written onto the output tape.

2.1.4 Realizability, Relative Continuity, and Relative Computability

As already mentioned, computability for functions between represented spaces is defined by transferring the computability notion on \(\omega\)–word functions via the considered representations (cf. [Wei00]).

Let \(\delta_i : \subseteq \Sigma_i^\omega \rightarrow X_i\) be a representation of a set \(X_i\) for \(i \in \{1, \ldots, k+1\}\). We say that a function \(g : \subseteq \Sigma_1^\omega \times \ldots \times \Sigma_k^\omega \rightarrow \Sigma_{k+1}^\omega\) is a \((\delta_1, \ldots, \delta_{k+1})\)–realization of a function
Chapter 2. Admissibility

2.1.5 Computable \( \omega \)-Words and Relatively Computable Elements

An \( \omega \)-word \( q \in \Sigma^\omega \) is called computable iff \( q \), viewed as a function from \( \mathbb{N} \) to \( \Sigma \), is \( (\nu_0, \nu_\omega) \)-computable. This condition is equivalent to the condition that the constant function \( c_q : \Sigma^\omega \to \Sigma^\omega \) defined by \( c_q(p) = q \) is computable.

For a given representation \( \delta : \subseteq \Sigma^\omega \to X \), an element \( x \in X \) is called \( \delta \)–computable iff \( x \) has a computable \( \delta \)–name \( p \in \Sigma^\omega \).

2.1.6 Continuous and Computable Reducibility

For defining appropriate classes of representations with suitable effectivity properties, the notions of continuous and of computable reducibility turn out to be useful.

Let \( \delta_1 : \subseteq \Sigma_1^\omega \to X_1 \) and \( \delta_2 : \subseteq \Sigma_2^\omega \to X_2 \) be functions. We say that a function \( g : \subseteq \Sigma_1^\omega \to \Sigma_2^\omega \) reduces or translates \( \delta_1 \) to \( \delta_2 \) iff \( \delta_1(p) = \delta_2(g(p)) \) holds for every \( p \in \text{dom}(\delta_1) \).

The representation \( \delta_1 \) is called continuously reducible (translatable) to \( \delta_2 \) (\( \delta_1 \leq_t \delta_2 \) for short) iff there is a continuous\(^1\) function \( g : \subseteq \Sigma_1^\omega \to \Sigma_2^\omega \) translating \( \delta_1 \) to \( \delta_2 \). Since restrictions of continuous functions from \( \Sigma_1^\omega \) to \( \Sigma_2^\omega \) are continuous, \( \delta_1 \leq_t \delta_2 \) is equivalent to the existence of a continuous function \( g : \subseteq \Sigma_1^\omega \to \Sigma_2^\omega \) with \( \delta_1 = \delta_2 \circ g \).

\(^1\)with respect to the Cantor topologies on \( \Sigma_1^\omega, \ldots, \Sigma_{k+1}^\omega \) (cf. Subsections 2.1.2, 2.2.2)

\(^2\)with respect to the Cantor topologies on \( \Sigma_1^\omega \) and \( \Sigma_2^\omega \) (cf. Subsection 2.1.2, 2.2.2)
2.1 Basics of the Type–2 Theory of Effectivity

The representation $\delta_1$ is called computably reducible (translatable) to $\delta_2$ ($\delta_1 \leq_{ct} \delta_2$ for short) iff there is a computable function $g : \subseteq \Sigma_1^* \rightarrow \Sigma_2^*$ translating $\delta_1$ to $\delta_2$. Clearly, $\delta_1 \leq_{ct} \delta_2$ implies $\delta_1 \leq_t \delta_2$.

Continuous equivalence and computable equivalence of $\delta_1$ and $\delta_2$ are defined by $\delta_1 \equiv_t \delta_2 :\iff (\delta_1 \leq_t \delta_2 \land \delta_2 \leq_t \delta_1)$ and $\delta_1 \equiv_{ct} \delta_2 :\iff (\delta_1 \leq_{ct} \delta_2 \land \delta_2 \leq_{ct} \delta_1)$, respectively. The relations $\leq_t$, $\leq_{ct}$, $\equiv_t$, and $\equiv_{ct}$ are transitive. Computably equivalent representations induce the same class of relatively computable functions (cf. [Wei00], Lemma 2.4.4).

2.1.7 Some Basic Representations

For $k \geq 1$ we define the bijections $\langle \cdot, \ldots, \cdot \rangle_{(\Sigma^\omega)^k} : (\Sigma^\omega)^k \rightarrow \Sigma^\omega$ and $\langle \cdot, \ldots, \cdot \rangle_{(\Sigma^\omega)^n} : (\Sigma^\omega)^n \rightarrow \Sigma^\omega$ by

$$\langle p_0, p_1, p_2, \ldots \rangle_{(\Sigma^\omega)^n}(\langle i, j \rangle_2) := p_i(j)$$

for all $p_0, p_1, p_2, \ldots \in \Sigma^\omega$ and $i, j \in \mathbb{N}$, where $\langle \cdot, \cdot \rangle_2$ denotes Cantor’s pairing function (see Subsection 1.1.3). For the sake of simplicity, we will usually omit the indices $(\Sigma^\omega)^k$ and $(\Sigma^\omega)^n$. The inverses $\rho_{(\Sigma^\omega)^k} := \left(\langle \cdot, \ldots, \cdot \rangle_{(\Sigma^\omega)^k}\right)^{-1}$ and $\rho_{(\Sigma^\omega)^n} := \left(\langle \cdot, \ldots, \cdot \rangle_{(\Sigma^\omega)^n}\right)^{-1}$ are obviously representations of $(\Sigma^\omega)^k$ and $(\Sigma^\omega)^n$. We denote the projections of $\rho_{(\Sigma^\omega)^k}$ and $\rho_{(\Sigma^\omega)^n}$ to one component by, respectively, $\pi_{k,i}$ and $\pi_{\infty,j}$ for $i \in \{1, \ldots, k\}$ and $j \in \mathbb{N}$. They, as well as $\rho_{(\Sigma^\omega)^k}$ and the function $(r, p) \mapsto \pi_{\infty,r(0)}(p)$, are computable and hence continuous.

For alphabets $\Sigma$ equipped with an injective standard numbering $\nu_\Sigma : \subseteq \mathbb{N} \rightarrow \Sigma$ according to Subsection 1.1.1 we define the injective representations $\varrho_{\Sigma^\omega} : \subseteq \mathbb{N}^\omega \rightarrow \Sigma^\omega$ and $\vartheta_{\Sigma^\omega} : \subseteq \Sigma^\omega \rightarrow \mathbb{N}^\omega$ by

$$\text{dom}(\varrho_{\Sigma^\omega}) := \{p \in \mathbb{N}^\omega \mid (\forall n \in \mathbb{N}) p(n) \in \text{dom}(\nu_\Sigma)\}, \varrho_{\Sigma^\omega}(p)(n) := \nu_\Sigma(p(n))$$

and

$$\text{dom}(\vartheta_{\Sigma^\omega}) := \{0^{a_0}10^{a_1}1\ldots \mid a_0, a_1, \ldots \in \mathbb{N}\}, \vartheta_{\Sigma^\omega}(0^{a_0}10^{a_1}1\ldots)(n) := a_n.$$ 

Their inverses are given by

$$\varrho_{\Sigma^\omega}^{-1}(p) = \nu_\Sigma^{-1}(p(0))\nu_\Sigma^{-1}(p(1))\nu_\Sigma^{-1}(p(2))\ldots$$

and

$$\vartheta_{\Sigma^\omega}^{-1}(q) = 0^{q(0)}10^{q(1)}10^{q(2)}1\ldots$$

for all $p \in \Sigma^\omega$ and $q \in \mathbb{N}^\omega$. All these representations are continuous with respect to the corresponding Cantor topologies, because a finite prefix of the image under each of these representations only depends on a finite prefix of the considered argument.

**Lemma 2.1.4**

Let $\delta : \subseteq \Sigma^\omega \rightarrow X$ and $\gamma : \subseteq \mathbb{N}^\omega \rightarrow X$ be representations of $X$.

Then $\delta \equiv_{ct} \delta \circ \varrho_{\Sigma^\omega}$ and $\gamma \equiv_{ct} \gamma \circ \vartheta_{\Sigma^\omega}$. 
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Proof:
One easily verifies that the representations θσω⁻¹, ϑσω⁻¹ :⊆ Nω → Σω and θωσ⁻¹, ϑωσ⁻¹ :⊆ Σω → Nω are not only continuous, but also computable ω-word functions in the sense of Subsection 2.1.2. Each of the four computable reducibilities is realized by one of these functions.

2.2 The Approximation Structure Induced by Representations

The definition of relative computability w.r.t. representations (cf. Subsection 2.1.4) provides an approximative computability notion. Roughly speaking, a function is defined to be computable w.r.t. given representations, if we are able to compute more and more precise approximations of the result while reading more and more information about the argument encoded in the actual name of the argument. Therefore it is important to analyse the kind of approximation implicitly used by a representation to encode the objects.

The usual tool to describe the way a representation δ :⊆ Σω → X approximates the elements of X is the final topology of δ. It is defined by

\[ \tau_δ := \{ O ⊆ X \mid (\exists U ∈ \tau_{Σ^ω}) \delta^{-1}[O] = U \cap dom(δ) \} \]

(2.3)

\[ = \{ O ⊆ X \mid (\forall p ∈ δ^{-1}[O])(\exists w ∈ Σ^*) (w ⊑ p \text{ and } δ[wΣ^ω] ⊆ O) \} . \]  (2.4)

This topology contains every finitely observable property (cf. [Smy92]), namely every property O ⊆ X that can be obtained from each name of each element satisfying this property by observing some finite prefix of that name.

It turns out that in some cases the final topology τδ is too inaccurate to describe the approximation structure induced by δ properly. A counterexample is the decimal representation ρ₁₀ of the real numbers (cf. Example 2.1.2). Its final topology is the usual Euclidean topology τR (cf. [Wei00, Theorem 4.1.13]). However, each name p of any number d in D := \{ z·10^{-i} \mid z ∈ Z, i ∈ N \} supplies an additional information about d, namely, depending on p, either that the number represented by p is in (−∞; d] or that it is in [d; ∞). This finitely observable information cannot be described as a set and is not grasped by the final topology of ρ₁₀. The well-known non–computability of multiplication by 3 w.r.t. ρ₁₀ is caused by the fact that the finitely observable information provided by ρ₁₀ about elements x ∈ 1₃ D \ D does not suffice to produce a name of 3x ∈ D. This “discontinuity” of x ↦ 3x is not revealed by the final topology of ρ₁₀.

Instead of the final topology of δ, we will consider the final convergence relation \( -_{δ} \subseteq X^N \times X \) induced by δ on X (cf. Subsection 2.2.4). This means that we equip the set X with a convergence relation assigning to sequences \((y_n)_n\) some elements \(x \in X\) considered to be the limits of \((y_n)_n\) (cf. Subsection 2.2.1). It turns out that multiplication by 3 is actually discontinuous w.r.t. the final convergence relation \( -_{ρ₁₀} \) (cf. Example 2.2.5). The corresponding continuity notion is sequential continuity (cf. Subsection 2.2.1). Relative computability implies sequential continuity w.r.t. the final convergence relations of the involved representations (cf. Lemma 2.2.3).
In general, \( \to_s \) is neither induced by any topology nor equal to the convergence relation of any limit space. In order to describe \( \to_s \) adequately, we introduce the notion of weak limit spaces (cf. Subsection 2.2.3). By Proposition 2.2.4 they form a very natural class of spaces including sequential topological spaces and limit spaces. We repeat the definitions of these well-known spaces in Subsections 2.2.6 and 2.2.2.

In Subsection 2.2.6, we compare the category \( \text{WeakLim} \) of weak limit spaces with the category \( \text{Lim} \) of limit spaces and the category \( \text{Seq} \) of sequential topological spaces. With every weak limit space \( X \), we associate in a straightforward way a limit space \( L(X) \) and a sequential topological space \( T(X) \) and prove the basic property of the functors \( L \) and \( T \) to preserve sequential continuity of multivariate functions.

### 2.2.1 Convergence Relations and Sequentially Continuous Functions

Let \( X \) be a set. For a sequence \( y : \mathbb{N} \to X \), we write \((y_n)_{n \in \mathbb{N}}\) or \((y_n)_n\). We also call any function \( y : \mathbb{N} \to X \), where \( \mathbb{N} = \mathbb{N} \cup \{\infty\} \), a (generalized) sequence and denote it by \((y_n)_{n \in \mathbb{N}}\) or by \((y_n)_{n \leq \infty}\).

We call any relation \( X \subseteq X^N \times X \) a convergence relation on \( X \) and the pair \( X = (X, \to_X) \) a sequential convergence space. We interpret \((y_n)_n \to_X x\) as convergence of \((y_n)_n\) to the point \( x \): we say that a sequence \((y_n)_n \in X^N\) converges to \( x \in X \) w.r.t. \( \to_X \) (or in the space \( X \)) iff \((y_n)_n \to_X x\) holds. If a sequence \((x_n)_n\) converges to \( x_\infty \) in \( X \), then we call \((x_n)_{n \leq \infty}\) a convergent sequence of \( X \), whereas \((x_n)_n\) is said to be a converging sequence of \( X \) if there is some element in \( X \) to which \((x_n)_n\) converges in \( X \).

Let \( M \) be a subset of \( X \). We define the subspace convergence relation \( \to_{X|M} \) on \( M \) inherited from \( X \) by \( \to_{X|M} := \to_X \cap (M^N \times M) \). A sequential convergence space \( Y = (Y, \to_Y) \) is called a subspace of \( X \) if \( Y \subseteq X \) and \( \to_Y = \to_X|_Y \) hold.

Let \( X_i = (X_i, \to_{X_i}) \) be a sequential convergence space for every \( i \in \mathbb{N} \). On the Cartesian product \( \prod_{i \in \mathbb{N}} X_i \), we define a convergence relation \( \to_{\bigotimes_{i \in \mathbb{N}} X_i} \) by

\[
(y_n)_n \to_{\bigotimes_{i \in \mathbb{N}} Y_i} \iff (\forall i \in \mathbb{N}) (pr_i(y_n))_n \to_{X_i} pr_i(y_n)
\]

for all sequences \((y_n)_{n \in \mathbb{N}}\) in \( \prod_{i \in \mathbb{N}} X_i \), where \( pr_j : \prod_{i \in \mathbb{N}} X_i \to X_j \) denotes the \( j \)-th projection function defined by \( pr_j(x_0, x_1, \ldots) := x_j \). We denote the product space \( \bigotimes_{i \in \mathbb{N}} X_i \) by \( \bigotimes_{i \in \mathbb{N}} X_i \). For \( k \geq 2 \), the finite product space \((X_1 \otimes \cdots \otimes X_k)\) is defined accordingly. We denote its convergence relation by \( \to_{X_1 \otimes \cdots \otimes X_k} \). For short, we will often write \( X_1 \otimes \cdots \otimes X_k \) instead of \((X_1 \otimes \cdots \otimes X_k)\) and \( \to_{X_1 \otimes \cdots \otimes X_k} \) instead of \( \to_{X_1 \otimes \cdots \otimes X_k} \).

Now we introduce the notion of sequential continuity. Let \( \mathcal{Y} = (Y, \to_Y) \) be a further sequential convergence space. Then a function \( f : X \to Y \) is defined to be continuous with respect to \( \to_X \) and \( \to_Y \) if \( f \) preserves convergent sequences, i.e., \((f(x_n))_n\) converges to \( f(x_\infty) \) w.r.t. \( \to_Y \), whenever \( \{x_n | n \in \mathbb{N}\} \subseteq \text{dom}(f) \) holds and \((x_n)_n\) converges to \( x_\infty \) w.r.t. \( \to_X \). In this case we also say that \( f \) is \((\to_X, \to_Y)\)-continuous or that \( f \) is a sequentially continuous function between the spaces \( X \) and \( Y \). Sequential continuity of multivariate functions is defined accordingly. Hence a function \( g : X_1 \times \cdots \times X_k \to Y \) is \((\to_{X_1}, \ldots, \to_{X_k})\)-continuous if and only if \( g \) is \((\to_{X_1}, \cdots, \to_{X_k}, \to_Y)\)-continuous. Obviously, composition of functions preserves sequential continuity.
2.2.2 Topological Spaces, Metric Spaces, Limit Spaces

In this subsection, we shortly recall the well-known concepts of topological spaces, metric spaces and limit spaces (cf. [Eng89, Fra65, Hyl79, MS02, Smy92, Wil70]).

Topological Spaces

A topological space \( X \) is a pair \((X, \tau_X)\), where \( X \) is a set and \( \tau_X \) is a family of subsets of \( X \) (called opens) which contains \( \emptyset \) and \( X \) and is closed under finite intersections and arbitrary unions. A base \( B \) of \( X \) is a family of open subsets of \( X \) such that every open set is an arbitrary union of certain sets in \( B \), whereas a subbase of \( X \) is a family \( B \) of subsets of \( X \) such that the family of all finite intersections of sets in \( B \) forms a base of \( X \). Topological spaces having a countable base are called countably based. The complements of open sets are called closed.

A neighbourhood of an element \( x \in X \) is a subset \( M \subseteq X \) with \( x \in \text{Int}(M) \), where \( \text{Int}(M) \) denotes the interior of \( M \) defined by \( \text{Int}(M) := \bigcup\{O \in \tau_X \mid O \subseteq M\} \), and a neighbourhood base of \( x \) is a family \( B \) of neighbourhoods of \( x \) such that every neighbourhood of \( x \) contains some \( B \in B \) as a subset. The closure \( \text{Cl}_\tau(M) \) of \( M \) is defined by \( \text{Cl}_\tau(M) := X \setminus \text{Int}(X \setminus M) \). A topological space is called first–countable iff every element has a countable neighbourhood base.

The space \( X = (X, \tau_X) \) is called a \( T_0 \)-space iff \( \{O \in \tau_X \mid x \in O\} = \{O \in \tau_X \mid y \in O\} \) implies \( x = y \). It is said to be a \( T_1 \)-space iff \( \{x\} \) is closed for every point \( x \in X \). It is called a Hausdorff space \((T_2\text{–space})\) iff for two points \( x, y \in X \) with \( x \neq y \) there are disjoint open sets \( U, V \in \tau_X \) with \( x \in U \) and \( y \in V \).

The topological space \( X = (X, \tau_X) \) induces a convergence relation on \( X \) denoted by \( \rightarrow_\tau \) or \( \rightarrow_{\tau_X} \). It is defined by saying that a sequence \( (x_n) \) converges to a point \( x_\infty \in X \) iff \( (x_n) \) is eventually in \( O \) for all open sets \( O \in \tau_X \) containing \( x_\infty \). A sequence \( (y_n) \) is eventually in a set \( M \) iff \((\forall n \in \Bbb N)y_n \in M\) holds. By \( \mathcal{W}(X) \) we denote the sequential convergence space \( \mathcal{W}(X) := (X, \rightarrow_\tau) \). For the sake of simplicity, if \( op \) is an operator defined on (a subset of) the sequential convergence spaces, then we will often write \( op(X) \) instead of \( op(\mathcal{W}(X)) \), if no confusion may occur. For example, if \( \mathcal{Y} = (Y, \tau_\mathcal{Y}) \) is a further topological space, then \( \mathcal{X} \otimes \mathcal{Y} \) denotes the sequential convergence space \( \mathcal{W}(X) \otimes \mathcal{W}(Y) \).

A subset \( K \subseteq X \) is called compact in \( X \) iff every open cover of \( K \) contains a finite subcover of \( K \) (cf. [Wil70]). (Note that some authors call subsets with this Heine–Borel–property quasi–compact, whereas they require of compact subsets to form a quasi–compact Hausdorff subspace of \( X \).) An open cover of a subset \( K \subseteq X \) is a family \( \mathcal{O} \) of open sets such that \( K \subseteq \bigcup \mathcal{O} \). A subset \( K \subseteq X \) is called countably compact in \( X \) iff every countable open cover of \( K \) contains a finite subcover of \( K \) (cf. [Wil70]). For every convergent sequence \( (x_n)_{n \leq \infty} \) the set \( K := \{x_n \mid n \in \Bbb N\} \) is compact, because, whenever \( \mathcal{O} \) is an open cover of \( K \), every set \( O \in \mathcal{O} \) with \( x_\infty \in O \) contains \( x_n \) for almost all \( n \in \Bbb N \).

Let \( \mathcal{Y} = (Y, \tau_\mathcal{Y}) \) be a further topological space. A partial function \( f : \subseteq X \rightarrow Y \) is called topologically continuous with respect to \( \tau_X \) and \( \tau_\mathcal{Y} \) or \((\tau_X, \tau_\mathcal{Y})\)–continuous iff for every open set \( V \in \tau_\mathcal{Y} \) there is an open set \( U \in \tau_X \) with \( f^{-1}(V) = U \cap \text{dom}(f) \).

Note that topological continuity implies sequential continuity (i.e. continuity w.r.t. the convergence relations induced by the topologies), whereas the converse is only true
2.2 The Approximation Structure Induced by Representations

in special cases, for example if \( X \) is a countably based space or if \( X \) is a sequential topological space and \( f \) is total (cf. [Eng89, Prop. 1.6.15]). In the following, continuity will always mean sequential continuity and not necessarily topological continuity. By \( \text{Top} \) we denote the category of topological spaces and of total topologically continuous functions (cf. [AL91]).

Now let \( f : \subseteq X \to Y \) be surjective. The final topology (or quotient topology) \( \tau_{X,f} \) induced on \( Y \) by \( f \) is defined by \( \tau_{X,f} := \{ V \subseteq Y \mid (\exists U \in \tau_X) f^{-1}(V) = U \cap \text{dom}(f) \} \).

We call \( f \) a Top–quotient function between \( X \) and \( Y \) iff \( \tau_{X,f} \) is equal to \( \tau_Y \). The space \( (Y, \tau_Y) \) is called a topological quotient of \( X \) iff there is a total function \( q : X \to Y \) such that the final topology \( \tau_{X,q} \) induced on \( Y \) by \( q \) is equal to \( \tau_Y \).

On the set \( \overline{\mathbb{N}} \) we use the topology \( \tau_{\overline{\mathbb{N}}} := \{ O \subseteq \overline{\mathbb{N}} \mid \infty \in O \Rightarrow (\forall n \in \mathbb{N}) n \in O \} \) which is induced by the countable base \( \{ \{ n \}, \{ n, \ldots, \infty \} \mid n \in \mathbb{N} \} \), on \( \mathbb{N} \) the discrete topology \( \tau_{\mathbb{N}} := 2^\mathbb{N} := \{ M \mid M \subseteq \mathbb{N} \} \), on the reals the usual Euclidean topology \( \tau_{\mathbb{R}} \) induced by the countable base \( \{ (q, q') \mid q, q' \in \mathbb{Q} \} \) of all open rational intervals, and on the set \( \mathbb{C} \) of complex numbers the product topology \( \tau_{\mathbb{C}} := \tau_{\mathbb{R}} \otimes \tau_{\mathbb{R}} \). The Cantor space topology \( \tau_{\Sigma^\omega} := \{ W \subseteq \Sigma^* \mid W \subseteq \Sigma^* \} \) on \( \Sigma^\omega \) has \( \{ w \Sigma^\omega \mid w \in \Sigma^\omega \} \) as a countable base, and the Baire space topology \( \tau_{\text{Baire}} := \{ W \subseteq \mathbb{N}^\omega \mid W \subseteq \mathbb{N}^\omega \} \) on \( \mathbb{N}^\omega \) has \( \{ w \mathbb{N}^\omega \mid w \in \mathbb{N}^\omega \} \) as a countable base. Hence, topological continuity and sequential continuity coincide for partial functions having subsets of \( \overline{\mathbb{N}}, \mathbb{N}, \mathbb{R}, \mathbb{C}, \Sigma^\omega \) or \( \mathbb{N}^\omega \) as their domains and topological spaces as their targets. We denote the convergence relations of these topologies by \( \rightarrow_{\overline{\mathbb{N}}}, \rightarrow_{\mathbb{N}}, \rightarrow_{\mathbb{R}}, \rightarrow_{\mathbb{C}}, \rightarrow_{\Sigma^\omega}, \) and \( \rightarrow_{\mathbb{N}^\omega} \), respectively. In the following, we will always assume them to be the used convergence relations on these sets, unless we specify different ones.

**Metric Spaces**

A metric space is a pair \( \mathfrak{X} = (X, d) \), where \( X \) is a set and the function \( d : X \times X \to \mathbb{R} \) called metric satisfies \( d(x, y) = 0 \iff x = y \), \( d(x, y) = d(y, x) \), and \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in X \). For \( x \in X \) and \( r \in \mathbb{R} \), \( B(x; r) \) denotes the ball \( \{ y \in X \mid d(x, y) < r \} \). The topology \( \tau_d \) induced by the metric \( d \) is defined as the topology induced by the base \( \{ B(x; q) \mid x \in X, q \in \mathbb{Q} \} \). The convergence relation of \( \mathfrak{X} \), denoted by \( \rightarrow_{\mathfrak{X}} \) or \( \rightarrow_d \), is defined as the convergence relation induced by the topology \( \tau_d \). By \( \mathcal{W}(\mathfrak{X}) \) we denote the sequential convergence space \( \mathcal{W}(\mathfrak{X}) := (X, \rightarrow_{\mathfrak{X}}) \).

**Limit Spaces**

A limit space is a sequential convergence space \( \mathfrak{X} = (X, \rightarrow_{\mathfrak{X}}) \) such that the convergence relation \( \rightarrow_{\mathfrak{X}} \) satisfies the following three axioms (cf. [Hyli79, MS02]):

1. \( (L1) \) \( (x)_n \rightarrow_{\mathfrak{X}} x; \)
2. \( (L2) \) \( (y_n)_n \rightarrow_{\mathfrak{X}} x \) implies \( (y_{\varphi(n)})_n \rightarrow_{\mathfrak{X}} x \) for every strictly increasing function \( \varphi : \mathbb{N} \to \mathbb{N}; \)
3. \( (L3) \) if for every strictly increasing function \( \varphi : \mathbb{N} \to \mathbb{N} \) there is some strictly increasing function \( \psi : \mathbb{N} \to \mathbb{N} \) with \( (y_{\varphi(\psi(n))})_n \rightarrow_{\mathfrak{X}} x, \) then \( (y_n)_n \rightarrow_{\mathfrak{X}} x \)

(where \( x \in X \) and \( (y_n)_n \in X^{\mathbb{N}} \)). By \( \text{Lim} \) we denote the category whose objects are the limit spaces and whose morphisms are the total sequentially continuous functions.
between them. Limit spaces satisfying the following axioms (L0) and (L4) are called $\mathcal{L}^*$–spaces, limit spaces satisfying (L0) are called $\mathcal{L}^*$–spaces (cf. [Kis60, Dud64, Kur66, Eng89]):

(L0) if $(y_n)_n \to_x x_1$ and $(y_n)_n \to_x x_2$, then $x_1 = x_2$;

(L4) if $(z_i)_i \to_x y_i$ for every $i \in \mathbb{N}$ and $(y_i)_i \to_x x$, then there are functions $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ with $(z_{\varphi(n)}, \psi(n))_n \to_x x$.

If Axiom (L0) is satisfied and $(x_n)_n$ converges, then $\lim_{n \to \infty} x_n$ denotes the unique limit of $(x_n)_n$. It is well-known and easy to prove that for every topological space $\mathcal{Y} = (Y, \tau_\mathcal{Y})$ and for every metric space $\mathcal{Z} = (Z, d)$ the sequential convergence spaces $\mathcal{W}(\mathcal{Y}) = (Y, \tau_{\mathcal{Y}})$ and $\mathcal{W}(\mathcal{Z}) = (Z, \tau_d)$ are limit spaces. Therefore we will treat any topological space and any metric space as a limit space. For the sake of simplicity, we will often write $\text{op}(\mathcal{Y})$ instead of $\text{op}(\mathcal{W}(\mathcal{Y}))$ and $\text{op}(\mathcal{Z})$ instead of $\text{op}(\mathcal{W}(\mathcal{Z}))$, if $\text{op}$ is an operator defined on (weak) limit spaces.

The convergence relation $\to_x$ of a limit space $\mathcal{X} = (X, \to_x)$ need not be induced by any topology (cf. Example 2.3.16). However, if $\mathcal{X}$ is an $\mathcal{L}^*$–space, then there is a topology inducing $\to_x$ (cf. [Kis60, Hyl79]). We will give an effective version of this theorem in Proposition 4.3.21.

2.2.3 Weak Limit Spaces

Now we introduce the new class of weak limit spaces. We define a pair $\mathcal{X} = (X, \to_x)$ to be a weak limit space iff $\mathcal{X}$ is a sequential convergence space (cf. Subsection 2.2.1) such that Axioms (L1), (L5), and (L6) are satisfied:

(L1) $(x)_n \to_x x$;

(L5) if $(x_n)_{n \leq \infty}$ is a convergent sequence of $\mathcal{X}$ and $(k_n)_{n \leq \infty}$ is a convergent sequence of $(\mathbb{N}, -\frac{1}{n})$, then $(x_{k_n})_n$ converges to $x_{k_\infty}$ in $\mathcal{X}$;

(L6) $(y_{n+1})_n \to_x x$ and $y_0 \in X$ imply $(y_n)_n \to_x x$.

Axiom (L5) can be reformulated by saying that the function $x : \mathbb{N} \to X$ is $(\to_{\mathbb{N}}, -\frac{1}{n})$–continuous, whenever $(x_n)_{n \leq \infty}$ is a convergent sequence of $\mathcal{X}$.

By WeakLim we denote the category whose objects are the weak limit spaces and whose morphisms are the total sequentially continuous functions between them. It is easy to prove that the class of weak limit spaces is closed under finite and under countable product as defined in Subsection 2.2.1. As $\mathcal{X} \otimes \mathcal{Y}$ forms a categorical product of weak limit spaces $\mathcal{X}$ and $\mathcal{Y}$, WeakLim is a “Cartesian category” (cf. [AL91] or Subsection 4.1.4). The class of weak limit spaces satisfying additionally Axiom (L0) (i.e., every converging sequence has only one limit) turns out to be exactly the class of $\mathcal{L}^*$–spaces defined in [Dud64]. From Axiom (L5) it follows that weak limit spaces fulfil Axiom (L2), whereas the following “zipping” Axiom (L7) is not satisfied by all weak limit spaces:

(L7) $(y_{2n})_n \to_x x$ and $(y_{2n+1})_n \to_x x$ imply $(y_n)_n \to_x x$.

A counterexample can be found in Example 2.2.5. Hence not every weak limit space is a limit space, because Axioms (L2) and (L3) imply Axiom (L7). By the next proposition, weak limit spaces are a generalization of limit spaces and thus of topological spaces.
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Proposition 2.2.1 Every limit space is a weak limit space.

Proof:
Let \( X = (X, \rightarrow_X) \) be a limit space.

Axiom (L5): Let \((x_n)_{n<\infty}\) be a convergent sequence of \(X\), and let \((k_n)_{n<\infty}\) be a convergent sequence of \((\mathbb{N}, \rightarrow)\). We show that every subsequence of \((x_{k_n})_n\) has a subsequence converging to \(x_{k_\infty}\) and apply Axiom (L3). Let \( \varphi : \mathbb{N} \rightarrow \mathbb{N} \) be strictly increasing.

If \( k_{\varphi(n)} = k_\infty \) for infinitely many \( n \in \mathbb{N} \), then there is a strictly increasing function \( \psi : \mathbb{N} \rightarrow \mathbb{N} \) with \( \text{range}(\psi) = \{n \in \mathbb{N} \mid k_{\varphi(n)} = k_\infty\} \). By Axiom (L1), \((x_{k_{\varphi(n)}})_n\) converges to \(x_{k_\infty}\).

Otherwise we have \( k_{\varphi(n)} \neq k_\infty \) for almost all \( n \in \mathbb{N} \) and therefore \( k_\infty = \infty \). Thus, for all \( m \in \mathbb{N} \) there is some \( n_m \in \mathbb{N} \) such that \( m \leq k_{\varphi(n)} < \infty \) holds for all \( n \geq n_m \).

Hence we can define \( \psi : \mathbb{N} \cup \{-1\} \rightarrow \mathbb{N} \cup \{-1\} \) recursively by \( \psi(-1) := -1 \) and

\[
\psi(i) := \min\{n > \psi(i-1) \mid k_{\varphi(i-1)} < k_{\varphi(n)} < \infty\}.
\]

Then \( \psi \) and \( i \mapsto k_{\varphi(i)} \) are strictly increasing. Thus \((x_{k_{\varphi(n)}})_n\) converges to \(x_\infty\) by Axiom (L2).

From Axiom (L3) we conclude that \((x_{k_n})_n\) converges to \(x_{k_\infty}\).

Axiom (L6): Let \((y_n)_n\) be a sequence in \(X\) such that \((y_{n+1})_n\) converges to some \(x \in X\).

For every strictly increasing function \( \varphi : \mathbb{N} \rightarrow \mathbb{N} \) the sequence \((y_{\varphi(n)})_n\), where \( \psi : \mathbb{N} \rightarrow \mathbb{N} \) is defined by \( \psi(n) := n + 1 \), is a subsequence of \((y_{n+1})_n\). Thus \((y_{\varphi(n)})_n\) converges to \(x\) by Axiom (L2). From Axiom (L3) we conclude \((y_n)_n \rightarrow_X x\).

We say that a weak limit space \( X = (X, \rightarrow_X) \) satisfies the \( T_0 \)-property iff for all \( x, y \in X \) with \( x \neq y \) we have \( (x)_n \not\rightarrow_X y \) or \( (y)_n \not\rightarrow_X x \). It is said to have the \( T_1 \)-property iff \( (x)_n \rightarrow_X y \) and \( (y)_n \rightarrow_X x \) implies \( x = y \) for all \( x, y \in X \). It is easy to see that these definitions are consistent with the usual definitions of topological \( T_0 \)-spaces and topological \( T_1 \)-spaces (cf. Subsection 2.2.2), i.e., a topological space \( \mathcal{X} \) is a topological \( T_0 \)-space (\( T_1 \)-space) if and only if the weak limit space \( \mathcal{W}(\mathcal{X}) \) has the \( T_0 \)-property (the \( T_1 \)-property).

The next technical lemma yields two characterizing properties of the weak limit spaces which we will use later. Further characterizations of the class of weak limit spaces are given by Propositions 2.2.4 and 3.1.3.

Lemma 2.2.2 (A technical characterization of weak limit spaces)
Let \( X = (X, \rightarrow_X) \) be a sequential convergence space. Then \( X \) is a weak limit space if and only if Axiom (L1) and the two following properties are satisfied:

1. For every convergent sequence \((x_n)_{n<\infty}\) of \(X\) the function \( \phi : \Sigma^\omega \rightarrow X \) defined by \( \phi(p) := x_{\min(\{\infty\} \cup \{n \in \mathbb{N} \mid p(n) \neq 0\})} \) is continuous w.r.t. \( \rightarrow_X \) and \( \rightarrow_{\Sigma^\omega} \).

2. For every \( (\rightarrow_{\Sigma^\omega}, \rightarrow_X) \)-continuous function \( \delta : \Sigma^\omega \rightarrow X \), every \( p \in \text{dom}(\delta) \) and every sequence \((z_n)_n\) that does not converge to \(\delta(p)\) in \(X\) there is some \( k \in \mathbb{N} \) such that \( z_n \notin \delta[p<k\Sigma^\omega] \) holds for infinitely many \( n \).
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2.2.4 Final Convergence Relations and WeakLim–Quotients

Let $\mathcal{X} = (X, \to_x)$ be a weak limit space, and let $q : \subseteq X \to Y$ be a surjection onto a set $Y$. The function $q$ induces on the set $Y$ a convergence relation $\to_{x,q}$ defined by

$$(yn)_n \to_{x,q} y_\infty \iff \exists(x_n)_{n \leq \infty} ((x_n)_n \to_x x_\infty \wedge (q(x_n))_{n \leq \infty} = (yn)_{n \leq \infty} \quad (2.5)$$

If $\delta : \subseteq Y \to Y$ is a representation, we will write $\to_q$ instead of $\to_{(\delta \circ q, \delta \circ \wedge), \delta'}$. It is easy to see that $\to_{x,q}$ is the finest\(^3\) (smallest) convergence relation on $Y$ for which $q$ is sequentially continuous, i.e., $q$ is $(\to_{x,q})$–continuous and, for every convergence

\(^3\)Given two convergence relations $\to_\gamma$ and $\to_\gamma$ on $X$, we call $\to_\gamma$ finer than $\to_\gamma$ if $\subseteq \subseteq$ holds, if the topology $\tau_1$ is finer than $\tau_2$ (i.e., $\tau_1 \supseteq \tau_2$). Another reason for using this term is that $\to_\gamma \subseteq \to_\gamma$ holds, if the topology $\tau_1$ is finer than $\tau_2$ (i.e., $\tau_1 \supseteq \tau_2$). The fact that for the convergence relations $\to_\gamma$ and $\to_\gamma$, induced by representations $\delta, \gamma$ with $\delta \subseteq \gamma$ (i.e., $\delta$ gives finer information than $\gamma$) we have $\to_\gamma \subseteq \to_\gamma$ (cf. Lemma 2.4.11(4)).
relation $\rightsquigarrow$ on $Y$ such that $q$ is $\left(\rightsquigarrow_X,\rightsquigarrow\right)$-continuous, $(y_n)_n \rightsquigarrow_{X,q} y_\infty$ implies $(y_n)_n \rightsquigarrow y_\infty$.

Since this construction of convergence relations generalizes the topological concept of final topologies\(^4\), we call $\rightsquigarrow_{X,q}$ the final convergence relation induced by $q$ on $Y$. We say that $q$ is a WeakLim-quotient function between $X$ and a weak limit space $Y = (Y, \rightsquigarrow_Y)$ iff $\rightsquigarrow_{X,q}$ is equal to $\rightsquigarrow_Y$. If $q$ is total, then we call $(Y, \rightsquigarrow_{X,q})$ the WeakLim-quotient\(^5\) of $X$ generated by $q$. One can show that the regular epimorphisms\(^6\) in the category WeakLim are exactly those morphisms whose underlying total functions are WeakLim-quotient mappings.

Relative computability implies sequential continuity with respect to the final convergence relations induced by the considered representations.

**Lemma 2.2.3 (Relative computability implies sequential continuity)**

For $i \in \{1, \ldots, k+1\}$, let $\delta_i : \Sigma_i^\omega \rightarrow X_i$ be a representation. Let $f : \subseteq X_1 \times \ldots \times X_k \rightarrow X_{k+1}$ be a function that is relatively continuous or computable w.r.t. the representations $\delta_1, \ldots, \delta_{k+1}$. Then $f$ is sequentially continuous w.r.t. the final convergence relations $\rightarrow_{q_1}, \ldots, \rightarrow_{q_{k+1}}$ and $\rightarrow_{q_{k+1}}$.

**Proof:**

There is a continuous function $g : \subseteq \Sigma_1^\omega \times \ldots \times \Sigma_k^\omega \rightarrow \Sigma_{k+1}^\omega$ realizing $f$ w.r.t. $\delta_1, \ldots, \delta_{k+1}$.

For $i \in \{1, \ldots, k\}$ we denote by $pr_i$ the $i$-th projection function from $X_1 \times \ldots \times X_k$ onto $X_i$. Let $(x_n)_{n \leq \infty}$ be a sequence in $\text{dom}(f)$ such that $(pr_i(x_n))_n$ converges to $pr_i(x_\infty)$ w.r.t. $\rightarrow_{q_i}$ for every $i \in \{1, \ldots, k\}$. Then, for every $i \in \{1, \ldots, k\}$, there is a convergent sequence $(p_i(n))_{n \leq \infty}$ with $(\delta_i(p_i(n)))_{n \leq \infty} = (pr_i(x_n))_{n \leq \infty}$. It follows

$$f(x_n) = f(\delta_1(p_1(n)), \ldots, \delta_k(p_k(n))) = \delta_{k+1}(g(p_1(n), \ldots, p_k(n)))$$

for all $n \in \mathbb{N}$. By continuity of $g$, the sequence $(g(p_1(n), \ldots, p_k(n)))_n$ converges to $g(p_1, \infty, \ldots, p_k, \infty)$. Thus $(f(x_n))_n$ converges to $f(x_\infty)$ with respect to $\rightarrow_{q_{k+1}}$. We conclude that $f$ is a sequentially continuous function between the WeakLim-quotient spaces $(X_1, \rightarrow_{q_1}), \ldots, (X_k, \rightarrow_{q_k})$, and $(X_{k+1}, \rightarrow_{q_{k+1}})$ generated by $\delta_1, \ldots, \delta_{k+1}$.

Note that relatively computable multivariate functions need not be topologically continuous w.r.t. the final topologies of the used representations (cf. Subsection 2.4.8). Moreover, a partial $\left(\delta_1, \delta_2\right)$-computable function $f : \subseteq X_1 \rightarrow X_2$ need neither be topologically continuous w.r.t. the final topologies $\tau_{\delta_1}$ and $\tau_{\delta_2}$ nor sequentially continuous w.r.t. $\tau_{\delta_1}$ and $\tau_{\delta_2}$ (cf. Example 2.4.33). Thus it is useful to consider not only topological spaces but also weak limit spaces.

The next proposition yields an important characterization of the class of weak limit spaces justifying their definition. It generalizes Franklin’s theorem stating that the sequential topological spaces are exactly the topological quotients of metric spaces (cf. [Fra65]).

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\(^4\)The final topology induced by a function on its target is the finest (largest) topology such that the function is topologically continuous.

\(^5\)Since the morphisms in WeakLim are total functions, we only call the quotient spaces generated by total functions WeakLim-quotients.

\(^6\)A morphism $q : X \rightarrow Y$ in a category $C$ is called a regular epimorphism iff $q$ forms a coequalizer (cf. Subsection 4.5.5) of some pair of parallel morphisms $f, g : A \rightarrow X$ in $C$. 
Proposition 2.2.4 (Characterization of weak limit spaces)

(1) If $\mathcal{X} = (X, \rightarrow_x)$ is a weak limit space and $q : \subseteq X \rightarrow Y$ is a surjection onto a set $Y$, then the quotient space $(Y, \rightarrow_{x,q})$ is a weak limit space.

(2) Every weak limit space is a $\text{WeakLim}$–quotient of a metric space.

(3) The class of weak limit spaces is equal to the class of the $\text{WeakLim}$–quotients of topological spaces.

Proof:

(1) Let $\mathcal{X} = (X, \rightarrow_x)$ be a weak limit space, and let $q : \subseteq X \rightarrow Y$ be a surjection onto a set $Y$.

Axiom (L1): The surjectivity of $q$ implies $(y)_n \rightarrow_{x,q} y$ for every $y \in Y$.

Axiom (L5): Let $(y_n)_n \leq \infty$ be a convergent sequence of $\mathcal{Y} = (Y, \rightarrow_{x,q})$, and let $(k_n)_n \leq \infty$ be a convergent sequence of $(\mathbb{N}, \rightarrow_q)$. There is a convergent sequence $(x_n)_n \leq \infty$ in $\mathcal{X}$ with $(q(x_n))_n \leq \infty = (y_n)_n \leq \infty$. Since $\mathcal{X}$ satisfies Axiom (L5), $(x_n)_n$ converges to $x_\infty$ in $\mathcal{X}$ implying that $(y_n)_n$ converges to $y_\infty$ in $\mathcal{Y}$ by definition of $\rightarrow_{x,q}$. Hence Axiom (L5) is satisfied.

Axiom (L6): Let $(y_n)_n \leq \infty$ be a generalized sequence such that $(y_{n+1})_n$ converges to $y_\infty$ w.r.t. $\rightarrow_{x,q}$. By surjectivity of $q$ and definition of $\rightarrow_{x,q}$, there is a generalized sequence $(x_n)_n \leq \infty$ with $(q(x_n))_n \leq \infty = (y_n)_n \leq \infty$ such that $(x_{n+1})_n$ converges to $x_\infty$. Since $\mathcal{X}$ satisfies Axiom (L6), $(x_n)_n$ converges to $x_\infty$ in $\mathcal{X}$ implying that $(y_n)_n$ converges to $y_\infty$ in $\mathcal{Y}$ by definition of $\rightarrow_{x,q}$.

(2) Let $\mathcal{X} = (X, \rightarrow_x)$ be a weak limit space. As the underlying set of the metric space to be constructed we choose $M := \mathbb{N} \times \rightarrow_x$ and as the metric $d : M \times M \rightarrow \mathbb{R}$ we use

$$d((a, ((x_i)_i), (b, ((y_j)_j), (y_\infty)_i))) := \left\{ \begin{array}{ll} |(\frac{1}{2})^a - (\frac{1}{2})^b| & \text{if } (\forall i \in \mathbb{N}) x_i = y_i \\ 2 & \text{otherwise.} \end{array} \right.$$ 

We define the function $q : M \rightarrow X$ by $q((a, ((x_i)_i), (y_\infty)_i)) := x_a$ and show $\rightarrow_{\mathfrak{M},q} = \rightarrow_x$, where $\mathfrak{M}$ denotes the weak limit space $(M, \rightarrow_q)$.

$\rightarrow_{\mathfrak{M},q} \subseteq \rightarrow_x$: Let $(x_n)_n$ converge to $x_\infty$ with respect to $\rightarrow_{\mathfrak{M},q}$. Then there is a sequence $(z_n)_n \leq \infty$ in $M$ with

$$\lim_{n \rightarrow \infty} d(z_n, z_\infty) = 0 \quad \text{and} \quad (\forall n \in \mathbb{N}) q(z_n) = x_n.$$ 

For every $n \in \mathbb{N}$ there is a convergent sequence $(y_{n,i})_i \leq \infty$ in $\mathcal{X}$ and some $a_n \in \mathbb{N}$ with $z_n = (a_n, ((y_{n,i})_i), (y_{n,\infty})_i)$. Since $\lim_{n \rightarrow \infty} d(z_n, z_\infty) = 0$, there is some $n_0$ such that $(y_{n,i})_i \leq \infty = (y_{n,i})_i \leq \infty$ for all $n \geq n_0$. As $(a_{n+n_0})_n$ converges to $a_\infty$ in $\mathbb{N}$ and $\mathcal{X}$ satisfies Axiom (L5), it follows that the sequence

$$(x_{n+n_0})_n = (y_{n+n_0}, a_{n+n_0})_n = (y_{\infty, a_{n+n_0}})_n$$

converges to $x_\infty = y_{\infty, a_\infty}$ in $\mathcal{X}$. Axiom (L6) implies $(x_n)_n \rightarrow_x x_\infty$.

\footnote{We assume $((\frac{1}{2})^\infty := 0.$}
−\mathfrak{q}_\rho \supseteq \rightarrow \chi: \text{ Let } (x_n)_n \text{ converge to } x_\infty \text{ w.r.t. } \rightarrow \chi. \text{ For every } n \in \mathbb{N} \text{ we define } z_n := (n, ((x_i)_i, x_\infty)), \text{ thus } q(z_n) = x_n. \text{ Since } d(z_n, z_\infty) = \frac{1}{2^n}, \text{ the sequence } (z_n)_n \text{ converges to } z_\infty. \text{ Thus } (x_n)_n \text{ converges to } x_\infty \text{ w.r.t. } \rightarrow \mathfrak{q}_\rho \text{ via the convergent sequence } (z_n)_{n \leq \infty}.

(3) This follows from (1) and (2).

\begin{proof}
Proposition 2.2.4(1) implies that for every representation \( \delta \subseteq \Sigma^\omega \rightarrow X \) the generated quotient space \((X, \rightarrow_\delta)\) is a weak limit space. In general, \((X, \rightarrow_\delta)\) fails to be a limit space. Thus \( \rightarrow_\delta \) need not be equal to the convergence relation induced by the final topology \( \tau_\delta \), because the convergence relations induced by topologies satisfy the axioms of limit spaces.

Example 2.2.5 (A weak limit space that is not a limit space)
As a counterexample, we consider the decimal representation \( \rho_{10} \) of the reals (cf. Example 2.1.2). The sequences \((1 - \left( \frac{1}{10} \right)^{n+1})_n \) and \((1 + \left( \frac{1}{10} \right)^{n+1})_n \) converge both to 1 in \((\mathbb{R}, \rightarrow_\rho_{10})\), because \((0 \ast 9^{n+1} 0^\omega)_n \) converges to \(0 \ast 9^\omega\) and \((1 \ast 0^n 10^\omega)_n \) converges to \(1 \ast 0^\omega\). However, the sequence \(\frac{9}{10}, \frac{11}{10}, \frac{99}{100}, \frac{101}{100}, \ldots\) does not converge to 1 w.r.t. \(\rightarrow_{\rho_{10}}\), because all names of numbers in the interval \((0; 1)\) start with the prefix \(0\ast\), so that we cannot find any converging sequence \((p_n)_n\) with \((\rho_{10}(p_n))_n = \left( \frac{9}{10}, \frac{11}{10}, \frac{99}{100}, \frac{101}{100}, \ldots \right)\). Hence \((\mathbb{R}, \rightarrow_{\rho_{10}})\) does not satisfy the zipping Axiom (L7) fulfilled in limit spaces. Thus the weak limit space \((\mathbb{R}, \rightarrow_{\rho_{10}})\) cannot be a limit space.

Note that the sequence \((\frac{9}{10}, \frac{11}{10}, \frac{99}{100}, \frac{101}{100}, \ldots)\) converges to \(\frac{1}{3}\) in the space \((\mathbb{R}, \rightarrow_{\rho_{10}})\): the corresponding sequence \((q_n)_n\) of names converging to \(0 \ast 3^\omega\) is given by

\[ q_{2n} = 0 \ast 3^{n+1} 0^\omega \quad \text{and} \quad q_{2n+1} = 0 \ast 3^{n+1} 6^\omega. \]

This demonstrates that indeed multiplication by 3 is not continuous w.r.t. \(\rightarrow_{\rho_{10}}\). Therefore \(x \mapsto 3x\) is not \((\rho_{10}, \rho_{10})\)-computable by Lemma 2.2.3.
\end{proof}

We summarize some basic facts about the approximation structure induced by a representation on the represented set.

Proposition 2.2.6 (The weak limit space generated by a representation)
Let \( \delta \subseteq \Sigma^\omega \rightarrow X \) be a representation of a set \( X \). Then:

(1) The space \((X, \rightarrow_\delta)\) is a weak limit space, but not necessarily a limit space.

(2) The representation \( \delta \) is a WeakLim-quotient function and thus a continuous representation of the weak limit space \((X, \rightarrow_\delta)\).

(3) Let \( \sim \) be any convergence relation on \( X \). Then \( \delta \) is \((\rightarrow_\omega, \sim)\)-continuous if and only if \((x_n)_n \rightarrow_\delta x_\infty \) implies \((x_n)_n \sim x_\infty \).
2.2.5 Sequentially Compact, Sequentially Open and Sequentially Closed Sets

For sequential convergence spaces satisfying the Axioms (L1) and (L2) there are natural notions of “compact”, “open” and “closed” sets (cf. [Bir36, Dud64], see also [Fra65, Eng89, MS02]).

Let $\mathfrak{X} = (X, \rightarrow_{\mathfrak{X}})$ be a weak limit space, thus Axioms (L1) and (L2) are satisfied. A subset $K \subseteq X$ is called sequentially compact in $\mathfrak{X}$ iff every sequence $(x_n)_n$ in $K$ has a converging subsequence that converges to an element of $K$, i.e., there are a strictly increasing function $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ and some $x_\infty \in K$ with $(x_{\varphi(n)})_n \rightarrow_{\mathfrak{X}} x_\infty$. In countably based topological spaces and in metric spaces a set $K$ is sequentially compact if and only if it is compact (cf. [Wil70]).

If $(y_n)_{n \leq \infty}$ is a convergent sequence of $\mathfrak{X}$, then the set $K := \{y_n \mid n \leq \infty\}$ is sequentially compact in $\mathfrak{X}$. For the proof, let $(x_m)_m$ be a sequence in $K$. If the set $\{x_m \mid m \in \mathbb{N}\}$ is finite, then $(x_m)_m$ has a constant subsequence which by Axiom (L1) converges to its only element. Otherwise we can define $\varphi, \psi : \mathbb{N} \cup \{-1\} \rightarrow \mathbb{N} \cup \{-1\}$ recursively by $\varphi(-1) := \psi(-1) := -1$,

$$
\varphi(k) := \min \{m \in \mathbb{N} \mid x_m \notin \{x_i, y_j, y_\infty \mid i \leq \varphi(k-1), j \leq \psi(k-1)\}\}
$$

and

$$
\psi(k) := \min \{n \in \mathbb{N} \mid y_n = x_{\varphi(k)}\}.
$$

Then $\varphi$ and $\psi$ are strictly increasing, $(x_{\varphi(k)})_k$ is equal to $(y_{\psi(k)})_k$, and $(x_{\varphi(k)})_k$ converges to $y_\infty \in K$ by Axiom (L2).

We call a subset $O$ of $X$ sequentially open in $\mathfrak{X}$ iff every sequence $(y_n)_n$ that converges to an element of $O$ is eventually in $O$. The complements of sequentially open sets are called sequentially closed in $\mathfrak{X}$. Axioms (L1) and (L2) imply that sequentially closed sets are characterized by the property that they contain all limits of their converging sequences. We denote by $\text{seq}(\rightarrow_{\mathfrak{X}})$ or seq($\mathfrak{X}$) the family of all sequentially open sets of $\mathfrak{X}$, and by $\text{sclo}(\mathfrak{X})$ the family of all sequentially closed sets of $\mathfrak{X}$.

2.2.6 Associated Topological Spaces and Associated Limit Spaces

Let $\mathfrak{X} = (X, \rightarrow_{\mathfrak{X}})$ be a weak limit space. Clearly, the family seq($\mathfrak{X}$) of all sequentially open sets is a topology on $X$ (cf. [Dud64]). We called it the associated topology of $\mathfrak{X}$. By $T(\mathfrak{X})$ we denote the associated topological space $(X, \text{seq}(\mathfrak{X}))$ and by $\rightarrow_{T(\mathfrak{X})}$ its convergence relation. If $\rightarrow_{\mathfrak{X}}$ is equal to $\rightarrow_{T(\mathfrak{X})}$, then we say that the weak limit space $\mathfrak{X}$ is a topological (weak) limit space. By Lemma 2.2.7(3), $\mathfrak{X}$ is a topological weak limit space if and only if there is a topology $\tau$ on $X$ with $\rightarrow_{\tau} = \rightarrow_{\mathfrak{X}}$.

If $\mathfrak{A} = (A, \tau_{\mathfrak{A}})$ is a topological space, then we denote by seq($\tau_{\mathfrak{A}}$) the topology seq($\rightarrow_{\mathfrak{A}}$). We call $(A, \text{seq}(\tau_{\mathfrak{A}}))$ the sequentialization of $\mathfrak{A}$. Clearly, we have

$$
(A, \text{seq}(\tau_{\mathfrak{A}})) = (A, \text{seq}(\mathfrak{A})) = (A, \text{seq}(\rightarrow_{\mathfrak{A}})) = T((A, \rightarrow_{\mathfrak{A}})) = T(\mathcal{W}(\mathfrak{A})).
$$

For short, we will sometimes write $T(\mathfrak{A})$ for the sequentialization $(A, \text{seq}(\tau_{\mathfrak{A}}))$ of $\mathfrak{A}$.

We show some properties of associated topologies (cf. [Dud64]).
Lemma 2.2.7 (Properties of the operators seq and \( T \))

Let \( X = (X, -x) \) be a weak limit space and \( \mathcal{A} = (A, \tau_\mathcal{A}) \) be a topological space.

1. For all sequences \( (x_n)_{n \leq \infty} \) of \( X \), \( (x_n)_n \rightarrow_x x_\infty \) implies \( (x_n)_n \rightarrow_{\tau(x)} x_\infty \), i.e. \(-x \subseteq -_{\tau(x)} \), thus \(-x \) is finer than or equal to \(-_{\tau(x)} \).

2. The topology \( \tau_\mathcal{A} \) is a subset of \( \text{seq}(\tau_\mathcal{A}) \), i.e. \( \tau_\mathcal{A} \subseteq \text{seq}(\tau_\mathcal{A}) \).

3. The convergence relation \( \rightarrow_{\tau(\mathcal{A})} \) is equal to \( \rightarrow_\mathcal{A} \), i.e. \( \mathcal{W}(T(\mathcal{W}(\mathcal{A}))) = \mathcal{W}(\mathcal{A}) \).

4. The topology \( \text{seq}(\rightarrow_{\tau(x)}) \) is equal to \( \text{seq}(-x) \), i.e. \( T(\mathcal{W}(\tau(x))) = \tau(x) \), which by convention can be written as \( T(\mathcal{W}(\tau(x))) = \tau(x) \).

5. The topology \( \text{seq}(\text{seq}(\tau_\mathcal{A}))) \) is equal to \( \text{seq}(\tau_\mathcal{A}) \), i.e. \( T(\mathcal{W}(\mathcal{W}(\mathcal{A}))) = \mathcal{W}(\tau(\mathcal{A})) \), which by convention can be written as \( T(\mathcal{W}(\mathcal{A})) = \tau(\mathcal{A}) \).

6. Let \( q : A \rightarrow X \) be surjective. If \( \mathcal{A} \) is countably based, then \( \text{seq}(\rightarrow_{\mathcal{A}, q}) \) is equal to the final topology \( \tau_{\mathcal{A}, q} = \{ V \subseteq X \mid (\exists U \in \tau_\mathcal{A}) q^{-1}[V] = U \cap \text{dom}(q) \} \) of \( q \).

Proof:

1. Suppose \((x_n)_n \rightarrow_x x_\infty \). Then \((x_n)_n \) is eventually in every sequentially open set containing \( x_\infty \). Thus \((x_n)_n \) converges to \( x_\infty \) w.r.t. the associated topology.

2. Let \( O \in \tau_\mathcal{A} \). By definition of the convergence relation of a topological space, every sequence that converges to an element in \( O \) is eventually in \( O \). Thus \( O \) is sequentially open.

3. Since every open set is sequentially open, \((x_n)_n \rightarrow_{\tau(\mathcal{A})} x_\infty \) implies \((x_n)_n \rightarrow_\mathcal{A} x_\infty \). Conversely, \((x_n)_n \rightarrow_\mathcal{A} x_\infty \) implies \((x_n)_n \rightarrow_{\tau(\mathcal{A})} x_\infty \) by (1).

4. “\( \subseteq \)” Let \( O \in \text{seq}(\rightarrow_{\tau(x)}) \). Let \((x_n)_{n \leq \infty} \) be a sequence with \((x_n)_n \rightarrow_x x_\infty \) and \( x_\infty \in O \). Then \((x_n)_n \rightarrow_{\tau(x)} x_\infty \) by (1). Since \( O \) is sequentially open in \( \tau(x) \), we have \( x_n \in O \) for almost all \( n \in \mathbb{N} \). Thus \( O \in \text{seq}(\rightarrow_x) \).

“\( \supseteq \)” Let \( O \in \text{seq}(\rightarrow_x) \). Let \((x_n)_{n \leq \infty} \) be a sequence with \((x_n)_n \rightarrow_{\tau(x)} x_\infty \) and \( x_\infty \in O \). Then \((\forall n \in \mathbb{N}) x_n \in O \) by definition of \( \rightarrow_{\tau(x)} \). Thus \( O \in \text{seq}(\rightarrow_{\tau(x)}) \).

The second statement follows from the fact that \( \text{seq}(\rightarrow_{\tau(x)}) \) is the topology of \( T(\mathcal{W}(\tau(x))) \) and \( \text{seq}(\rightarrow_x) \) is the one of \( \tau(x) \).

5. This can be deduced from (4) by considering \( \mathcal{W}(\mathcal{A}) \).

6. \( \tau_{\mathcal{A}, q} \subseteq \text{seq}(\rightarrow_{\mathcal{A}, q}) \):
Let \( V \in \tau_{\mathcal{A}, q} \). Let \((x_n)_{n \leq \infty} \) be a convergent sequence of \((X, \rightarrow_{\mathcal{A}, q}) \) with \( x_\infty \in V \). Choose \( U \in \tau_\mathcal{A} \) with \( q^{-1}[V] = U \cap \text{dom}(q) \) and a convergent sequence \((a_n)_{n \leq \infty} \) of \( \mathcal{A} \) with \((q(a_n))_{n \leq \infty} = (x_n)_{n \leq \infty} \). Since \( U \) is an open set containing \( x_\infty \), \((a_n)_n \) is eventually in \( U \). Thus we have \( x_n \in V \) for almost all \( n \in \mathbb{N} \). Hence \( V \in \text{seq}(\rightarrow_{\mathcal{A}, q}) \).
Chapter 2. Admissibility

Let \( \mathfrak{X} = (A, \tau_A) \) be a weak limit space. We define the open set \( U \in \tau_A \) by

\[
U := \bigcup \{ O \in \tau_A \mid O \cap \text{dom}(q) \subseteq q^{-1}[V] \}.
\]

Suppose for contradiction that there is some \( a \in q^{-1}[V] \setminus U \). Then for every \( n \in \mathbb{N} \) there is some \( b_n \in (\bigcap \{ \alpha_i \mid i \leq n, \ a \in \alpha_i \} \cap \text{dom}(q)) \setminus q^{-1}[V] \), because \( (\bigcap \{ \alpha_i \mid i \leq n, \ a \in \alpha_i \} \cap \text{dom}(q)) \) is an open set containing \( a \). Since \( (b_n)_n \) converges to \( a \) in \( \mathfrak{A} \), \( (q(b_n))_n \) converges to \( q(a) \) in \( (X, \rightarrow_{\mathfrak{A}, q}) \). Thus \( (q(b_n))_n \) is eventually in \( V \). This contradicts \( \{ b_n \mid n \in \mathbb{N} \} \cap q^{-1}[V] = \emptyset \).

We conclude \( q^{-1}[V] = U \cap \text{dom}(q) \). Hence \( V \in \tau_{A,q} \).

A topological space \( \mathfrak{X} = (A, \tau_A) \) is called a sequential space iff all sequentially open sets are open, i.e., \( \text{seq}(\rightarrow_{\mathfrak{A}}) \) is equal to \( \tau_A \). It is well-known that countably based spaces and metric spaces (in their roles as topological spaces) are sequential spaces (cf. [Eng89, Fra65]). Lemma 2.2.7(4) states that also the associated topological space \( T(\mathfrak{X}) \) of any weak limit space \( \mathfrak{X} \) is a sequential space.

By \( \text{Seq} \) we denote the category whose objects are the sequential spaces and whose morphisms are the total topologically continuous functions between them. Since topological continuity and sequential continuity are equivalent for total functions between sequential spaces (cf. [Eng89]), \( \text{Seq} \) is a full subcategory of \( \text{Lim} \) and of \( \text{WeakLim} \).

From Theorem 2.2.10(2) it follows that every continuous function between weak limit spaces \( \mathfrak{X} \) and \( \mathfrak{Y} \) is actually a topological continuous function between \( T(\mathfrak{X}) \) and \( T(\mathfrak{Y}) \). Thus \( T \) can be extended to a functor from \( \text{WeakLim} \) to \( \text{Seq} \) by defining \( T(f) := f \) for every morphism in \( \text{WeakLim} \). Similarly, \( W \) can be viewed as a functor from \( \text{Seq} \) to \( \text{WeakLim} \) mapping total topologically continuous function to themselves. From Lemma 2.2.7 and [AL91, Theorem 5.3.2] one can conclude that the functor \( T : \text{WeakLim} \rightarrow \text{Seq} \) is a left–adjoint to the functor \( W : \text{Seq} \rightarrow \text{WeakLim} \); for a weak limit space \( \mathfrak{X} = (X, \rightarrow_X) \) the unit \( \eta_X : X \rightarrow W(T(\mathfrak{X})) \) of the adjunction is simply the \( (\rightarrow_X, \rightarrow_{T(\mathfrak{X})}) \)-continuous identical function \( \text{id}_X \). Thus \( \text{Seq} \) is a full reflective subcategory of \( \text{WeakLim} \). From [Hyl79, Proposition 10.1] we know that \( \text{Seq} \) is as well a full reflective subcategory of \( \text{Lim} \). Hence the fact that \( \text{Seq} \) is a full reflective subcategory of \( \text{WeakLim} \) also follows from the result shown the next paragraph that \( \text{Lim} \) is in its turn a full reflective subcategory of \( \text{WeakLim} \).

With every weak limit space \( \mathfrak{X} = (X, \rightarrow_X) \), we associate a limit space \( L(\mathfrak{X}) := (X, \rightarrow_{L(\mathfrak{X})}) \) whose convergence relation \( \rightarrow_{L(\mathfrak{X})} \) is defined by:

\[
(x_n)_n \rightarrow_{L(\mathfrak{X})} x_\infty \iff \left( \forall \varphi : \mathbb{N} \rightarrow \mathbb{N} \text{ strictly increasing} \right) \left( \exists \psi : \mathbb{N} \rightarrow \mathbb{N} \text{ strictly increasing} \right) \ x_{\varphi(\psi(n))} \rightarrow_X x_\infty.
\]

We list some properties of associated limit spaces.

**Lemma 2.2.8 (Properties of the operator \( L \))**

Let \( \mathfrak{X} = (X, \rightarrow_X) \) be a weak limit space.
2.2 The Approximation Structure Induced by Representations

(1) For all sequences \((x_n)_{n \leq \infty}\) of \(X\), \((x_n)_n \rightarrow x_\infty\) implies \((x_n)_n \rightarrow_{\mathcal{L}(x)} x_\infty\).

(2) The space \(\mathcal{L}(X)\) is a limit space.

(3) If \((X, \rightsquigarrow)\) is a limit space with \(\rightarrow_X \subseteq \rightsquigarrow\), then \(\rightarrow_{\mathcal{L}(X)} \subseteq \rightsquigarrow\), i.e., \(\rightarrow_{\mathcal{L}(X)}\) is the finest (smallest) convergence relation on \(X\) containing \(\rightarrow_X\) and satisfying the axioms of limit spaces.

(4) If \(X\) is a limit space, then \(\mathcal{L}(X) = X\).

(5) The associated topological space \(T(\mathcal{L}(X))\) is equal to \(T(X)\).

Proof: We omit the straightforward proof.

The operator \(\mathcal{L}\) can be extended to a functor from \(\text{WeakLim}\) to \(\text{Lim}\) by letting \(\mathcal{L}\) map total sequentially continuous functions to themselves. Theorem 2.2.10(1) implies that every continuous function between weak limit spaces \(X\) and \(Y\) is a continuous one between the limit spaces \(\mathcal{L}(X)\) and \(\mathcal{L}(Y)\). From Lemma 2.2.8 and [AL91, Theorem 5.3.2] one can conclude that the functor \(\mathcal{L} : \text{WeakLim} \rightarrow \text{Lim}\) is a left–adjoint to the inclusion functor \(I : \text{Lim} \rightarrow \text{WeakLim}\). Similar to the topological case, for a weak limit space \(X = (X, \rightarrow_X)\) the unit \(\eta_X\) of the adjunction is the \((\rightarrow_X, \rightarrow_{\mathcal{L}(X)})\)–continuous identical function \(id_X\). Thus \(\text{Lim}\) is a full reflective subcategory of \(\text{WeakLim}\).

We now show the important property of the two operators \(\mathcal{L}\) and \(T\) to preserve sequential continuity of (total) multivariate functions. For the proof we apply the following very useful lemma.

**Lemma 2.2.9**

Let \(\mathfrak{X} = (X, \rightarrow_X), \mathfrak{Y} = (Y, \rightarrow_\mathfrak{Y}),\) and \(\mathfrak{Z} = (Z, \rightarrow_\mathfrak{Z})\) be weak limit spaces. Let \(f : X \times Y \rightarrow Z\) be a sequentially continuous function. Then for every sequentially compact set \(K\) of \(\mathfrak{Y}\) and every sequentially open set \(V\) of \(\mathfrak{Z}\) the set

\[U := \{ x \in X \mid (\forall y \in K) f(x, y) \in V \}\]

is sequentially open in \(\mathfrak{X}\).

Proof:

Suppose for contradiction that \(U\) is not sequentially open. Then there is a convergent sequence \((x_n)_{n \leq \infty}\) of \(X\) with \(x_\infty \in U\) and \((\forall n \in \mathbb{N}) x_n \notin U\). For every \(n \in \mathbb{N}\) there exists some \(y_n \in K\) with \(f(x_n, y_n) \notin V\). By sequential compactness of \(K\), there is a strictly increasing sequence \(\varphi : \mathbb{N} \rightarrow \mathbb{N}\) and some \(y_\infty \in K\) with \((y_{\varphi(n)})_n \rightarrow y_\infty\). By sequential continuity of \(f\), \((f(x_{\varphi(n)}, y_{\varphi(n)}))_n\) converges to \(f(x_\infty, y_\infty)\). Since \(f(x_\infty, y_\infty) \in V\) and \(f\) is sequentially continuous, there is some \(n_0 \in \mathbb{N}\) with \(f(x_{\varphi(n)}, y_{\varphi(n)}) \in V\) for all \(n \geq n_0\), a contradiction.
Theorem 2.2.10 (Sequential continuity is preserved by $\mathcal{L}$ and $T$)
Let $X_1 = (x_1, -x_1), \ldots, X_k = (x_k, -x_k)$, and $\mathcal{Y} = (Y, -\gamma)$ be weak limit spaces.

(1) Let $f : X_1 \times \ldots \times X_k \rightarrow Y$ be continuous w.r.t. $-x_1, \ldots, -x_k$, and $-\gamma$.
Then $f$ is continuous w.r.t. $-\mathcal{L}(X_1), \ldots, -\mathcal{L}(X_k)$, and $-\mathcal{L}(\mathcal{Y})$.

(2) Let $f : X_1 \times \ldots \times X_k \rightarrow Y$ be continuous w.r.t. $-x_1, \ldots, -x_k$, and $-\gamma$.
Then $f$ is continuous w.r.t. $-\mathcal{L}(X_1), \ldots, -\mathcal{L}(X_k)$, and $-\mathcal{L}(\mathcal{Y})$.

Proof:

(1) Let $(z_n)_{n<\infty}$ be a convergent sequence of $\mathcal{L}(X_1) \otimes \ldots \otimes \mathcal{L}(X_k)$ contained in $\text{dom}(f)$. Let $\psi : N \rightarrow N$ be strictly increasing. Let $pr_i$ denote the $i$-th projection function from $X_1 \times \ldots \times X_k$ onto $X_i$. Inductively, we construct strictly increasing functions $\psi_0, \ldots, \psi_k : N \rightarrow N$ as follows:

$i = 0$: Choose $\psi_0 := \varphi$.

$i - 1 < i$: Since $\psi_0 \circ \ldots \circ \psi_{i-1}$ is strictly increasing and $\langle pr_i(z_n) \rangle_{n<\infty}$ is a convergent sequence of $\mathcal{L}(X_i)$, we can choose a strictly increasing function $\psi_i : N \rightarrow N$ with $\langle pr_i(\psi_0 \circ \ldots \circ \psi_i(z_n)) \rangle_{n \rightarrow k}$ by virtue of the definition of $-\mathcal{L}(X_i)$.

Since $\langle pr_i(z_n(\psi_0 \circ \ldots \circ \psi_i(n(n)) \rangle_{n \rightarrow k}$ converges to $pr_i(z_\infty)$ in $X_i$ for every $i \in \{1, \ldots, k\}$, $(f(z_\psi(0 \circ \ldots \circ \psi_i(n)))_{n \rightarrow k}$ is a subsequence of $(f(z_\psi(n)))_{n \rightarrow k}$ that converges to $f(z_\infty)$ in $\mathcal{Y}$. This implies that $(f(z_n))_{n \rightarrow k}$ converges to $f(z_\infty)$ in $\mathcal{L}(\mathcal{Y})$.

(2) By induction, we show that, for every $i \in \{0, \ldots, k\}$, the function $f$ is continuous with respect to $-\mathcal{L}(X_1), \ldots, -\mathcal{L}(X_i), -x_{i+1}, \ldots, -x_k$, and $-\mathcal{L}(\mathcal{Y})$.

$i = 0$: Since we have $-\gamma \subseteq -\mathcal{L}(\mathcal{Y})$ by Lemma 2.2.7(1), $f$ is continuous with respect to $-x_1, \ldots, -x_k$ and $-\mathcal{L}(\mathcal{Y})$.

$i - 1 < i$: We define $h : X_1 \times (X_1 \times \ldots \times X_{i-1} \times x_{i+1} \times \ldots \times X_k) \rightarrow Y$ by $h(x_i, (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)) := f(x_1, \ldots, x_k)$. We denote by $-\mathcal{L}$ the convergence relation of the weak limit space

$$(X_1, -\mathcal{L}(X_1)) \otimes \ldots \otimes (X_{i-1}, -\mathcal{L}(X_{i-1})) \otimes X_{i+1} \otimes \ldots \otimes X_k.$$

Let $(x_n)_{n<\infty}$ and $(z_n)_{n<\infty}$ be sequences with $(x_n)_{n \rightarrow \gamma} x_\infty$ and $(z_n)_{n \rightarrow \gamma} z_\infty$. Let $V$ be a sequentially open set of $\mathcal{Y}$ containing $h(x_\infty, z_\infty)$. Since $h$ is $(-\mathcal{L}(X_1), \ldots, -\mathcal{L}(X_k))$-continuous by induction hypothesis, $(h(x_\infty, z_n))_{n \rightarrow \gamma}$ converges to $h(x_\infty, z_\infty)$ in $T(\mathcal{Y})$. Thus there is some $n_1$ with $\forall n \geq n_1$ $h(x_\infty, z_n) \in V$.

By Lemma 2.2.9 the set

$$U := \{ x \in X_1 \mid (\forall n \leq n \leq \infty) h(x, z_n) \in V \}$$

is sequentially open in $X_i$, because $h$ is $(-x_1, -\mathcal{L}(\mathcal{Y}))$-continuous, the set $\{ z_n \mid n_1 \leq n \leq \infty \}$ is sequentially compact in the above product space (as $(z_{n+1})_{n \rightarrow \gamma} z_\infty$), and $V$ is sequentially open in $T(\mathcal{Y})$ by Lemma 2.2.7(4). Since $x_\infty \in U$, there is some $n_2 \in \mathbb{N}$ with $\forall n \geq n_2 x_n \in U$. For all $n \geq \max\{ n_1, n_2 \}$ we have $h(x_n, z_n) \in V$. Thus $(h(x_n, z_n))_{n \rightarrow \gamma}$ converges to $h(x_\infty, z_\infty)$ w.r.t. $-\mathcal{L}(\mathcal{Y})$.

Hence $h$ is $(-\mathcal{L}(X_1), \ldots, -\mathcal{L}(X_k))$-continuous. This means that $f$ is continuous with respect to $-\mathcal{L}(X_1), \ldots, -\mathcal{L}(X_k), -x_{i+1}, \ldots, -x_k$, and $-\mathcal{L}(\mathcal{Y})$. 

✓
2.3 Admissible Representations

From this theorem one can deduce that the functors $\mathcal{L} : \text{WeakLim} \to \text{Lim}$ and $\mathcal{T} : \text{WeakLim} \to \text{Seq}$ preserve finite products. Let $\mathfrak{X} = (X, -_X)$ and $\mathfrak{Y} = (Y, -_Y)$ be weak limit spaces. By Theorem 2.2.10 the projections $\text{pr}_1 : X \times Y \to X$ and $\text{pr}_2 : X \times Y \to Y$ are, respectively, $(-\tau_{(X \otimes \mathfrak{Y})}, -\tau_{(X)})$-continuous and $(-\tau_{(X \otimes \mathfrak{Y})}, -\tau_{(\mathfrak{Y})})$-continuous. Let $f : Z \to X$ and $g : Z \to Y$ be sequentially continuous functions from a sequential topological space $\mathfrak{Z}$ to, respectively, $\mathcal{T}(\mathfrak{X})$ and $\mathcal{T}(\mathfrak{Y})$. Then the function $h : Z \to X \times Y$ with $h(z) := (f(z), g(z))$ is the only function satisfying $f = \text{pr}_1 \circ h$ and $g = \text{pr}_2 \circ h$. Clearly, $h$ is $(-3, -\tau_{(X)} \otimes -\tau_{(\mathfrak{Y})})$-continuous. From Theorem 2.2.10, applied to $\mathfrak{X}_1 := \mathfrak{X}, \mathfrak{X}_2 := \mathfrak{Y}, \mathfrak{Y}_1 := \mathfrak{X} \otimes \mathfrak{Y}$, and $f := \text{id}_{X \times Y}$, we obtain $-\tau_{(X)} \otimes -\tau_{(\mathfrak{Y})} \subseteq -\tau_{(X \otimes \mathfrak{Y})}$, hence $h$ is a sequentially continuous function from $\mathfrak{Z}$ to $\mathcal{T}(\mathfrak{X} \otimes \mathfrak{Y})$. Therefore $\mathcal{T}(\mathfrak{X} \otimes \mathfrak{Y})$ and the projections $\text{pr}_1, \text{pr}_2$ form a categorical product for $\mathcal{T}(\mathfrak{X})$ and $\mathcal{T}(\mathfrak{Y})$ in Seq. In a similar way, the functor $\mathcal{L}$ can be shown to preserve finite products.

Furthermore, one can prove that $\mathcal{L}$ preserves equalizers, whereas $\mathcal{T}$ does not.

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Of course, not every function is computable with respect to all representations of its source and its target. For example, not every number function is computable w.r.t. the representation $\delta_\mathbb{N} : \mathbb{N}^\omega \to \mathbb{N}$ defined by $\delta_\mathbb{N}(p) := p(0)$, because the $(\delta_\mathbb{N}, \theta_\mathbb{N})$-computability of a total number function implies its computability in the classical sense.

For two given representations $\delta_X$ and $\delta_Y$ of sets $X$ and $Y$ the natural question arises which functions are computable with respect to these representation. Conversely, we want to know how to construct appropriate representations of $X$ and $Y$ that admit relative computability of a given class of functions and thus have certain prescribed effectivity properties.

Since relative computability implies relative continuity (cf. Subsection 2.1.4), it turns out to be useful to consider two different kinds of effectivity properties, the topological ones and the “purely” computable ones. Many representations fail to be appropriate simply by having unsuitable topological properties preventing some interesting functions from being at least relatively continuous. The standard example is the decimal representation $\rho_{10}$ of $\mathbb{R}$ (cf. Subsection 2.1.1) and real addition which is not relatively continuous and hence not computable with respect to $\rho_{10}$.

Therefore we consider relative continuity and define a class of representations called admissible representations which have nice topological properties. This notion of admissibility is applicable to representations of weak limit spaces (cf. Subsection 2.2.3) and depends on the corresponding convergence relation. It generalizes the notion of admissibility in [KW85, Wei00] which is only defined for countably based $T_0$-spaces.

The goal of the definition of admissibility is to guarantee that every sequentially continuous function is relatively continuous w.r.t. admissible representations. In Subsection 2.3.3 we show that our definition of admissibility (cf. Subsection 2.3.1) leads to equivalence of relative continuity w.r.t. admissible representations and sequential continuity. In Subsection 2.3.2 we compare our notion of admissibility with the previous one from [Wei00, KW85] for countably based $T_0$-spaces.
2.3.1 The Definition of Admissible Representations

Let $\mathcal{X} = (X, \rightarrow_\mathcal{X})$ be a weak limit space. According to Section 2.2, any representation $\delta$ of $X$ induces a convergence relation $\rightarrow_\mathcal{X}$ on the set $X$ describing the approximation structure which is implicitly used by $\delta$ to encode the elements of $X$. It seems to be natural to demand from an admissible representation $\delta$ to induce exactly the convergence relation $\rightarrow_\mathcal{X}$ of $X$ (see Subsection 2.2.4).

Example 2.3.9 shows that it does not suffice to use WeakLim–quotient representations in order to guarantee relative continuity of all sequentially continuous functions. Hence a stronger notion is necessary. The idea is suggested by the following observation: if a representation $\delta : \subseteq \Sigma^\omega \rightarrow X$ satisfies our demand, then every continuous function $f : \subseteq \Sigma^\omega \rightarrow X$ is relatively continuous with respect to $\text{id}_{\Sigma^\omega}$ and $\delta$. This means that $f$ is continuously translatable (reducible) to $\delta$ (cf. Subsection 2.1.6). We say that $\delta$ has the universal property iff every continuous function $f : \subseteq \Sigma^\omega \rightarrow X$ is continuously translatable to $\delta$. It turns out that representations that are continuous and satisfy the universal property actually admit relative continuity of all sequentially continuous functions (cf. Theorem 2.3.18).

These considerations lead to the following definition of admissibility.

**Definition 2.3.1 (Admissible representations)**

Let $\mathcal{X} = (X, \rightarrow_\mathcal{X})$ be a weak limit space, and let $\delta : \subseteq \Sigma^\omega \rightarrow X$ be a representation of $X$.

1. We call $\delta$ admissible with respect to $\rightarrow_\mathcal{X}$ or $\rightarrow_\mathcal{X}$–admissible iff $\delta$ is sequentially continuous and every sequentially continuous function $\phi : \subseteq \Sigma^\omega \rightarrow X$ is continuously translatable to $\delta$ (i.e. $\phi \leq_t \delta$).

2. We call $\delta$ an admissible representation of the space $\mathcal{X}$ iff $\delta$ is admissible with respect to $\rightarrow_\mathcal{X}$.

3. We call $\delta$ an admissible representation of the set $X$ iff $\delta$ is an admissible representation of the generated weak limit space $(X, \rightarrow_\mathcal{X})$.

4. Let $\tau$ be a topology on $X$. We call $\delta$ admissible with respect to $\tau$ or $\tau$–admissible iff $\delta$ is admissible w.r.t. to the convergence relation $\rightarrow_\tau$ induced by the topology $\tau$.

By the convention to treat topological spaces as limit spaces (cf. Subsection 2.2.2), Definition 2.3.1 implicitly defines $\delta$ to be an admissible representation of a topological space $(X, \tau)$ iff $\delta$ is an admissible one of the limit space $(X, \rightarrow_\mathcal{X})$. Note that exactly in this case the representation $\delta$ is $\tau$–admissible.

Definition 2.3.1 does not explicitly require of an admissible representation of a weak limit space to induce the convergence relation of that space. Nevertheless, this property is satisfied by admissible representations. Thus a representation $\delta$ can only be admissible w.r.t. $\rightarrow_\mathcal{X}$. This fact justifies Definition 2.3.1(3).

**Proposition 2.3.2 (Quotient spaces generated by admissible represent.)**

Let $\delta : \subseteq \Sigma^\omega \rightarrow X$ be an admissible representation of a weak limit space $\mathcal{X} = (X, \rightarrow_\mathcal{X})$. Then $\delta$ is WeakLim–quotient function, i.e. $\rightarrow_\delta = \rightarrow_\mathcal{X}$. Furthermore, the final topology of $\delta$ is equal to the associated topology $\text{seq}(\mathcal{X})$ of $\mathcal{X}$.
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Proof:
Since $\delta$ is sequentially continuous, we have $(x_n)_n \rightarrow x_\infty \implies (x_n)_n \rightarrow x_\infty$.
Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $X$. By Lemma 2.2.2, the function $\phi : \Sigma^\omega \to X$ defined by

$$\phi(p) := x_{\min(\{\infty\} \cup \{n \in \mathbb{N} | p(n) \neq 0\})}$$

is sequentially continuous w.r.t. $\rightarrow_x$. The universality of $\delta$ yields a continuous function $g : \Sigma^\omega \to \Sigma^\omega$ with $\phi = \delta \circ g$. Define $q_\infty := g(0^\omega)$ and $q_n := g(0^n1^\omega)$ for every $n \in \mathbb{N}$. Then $(q_n)_n$ converges to $q_\infty$, because $g$ is continuous and $(0^n1^\omega)_n$ converges to $0^\omega$. Obviously, we have $(\delta(q_n))_{n \leq \infty} = (x_n)_{n \leq \infty}$. Thus $(x_n)_n$ converges to $x_\infty$ w.r.t. $\rightarrow_x$.

Hence $\delta$ is a WeakLim–quotient representation.

From Lemma 2.2.7 it follows that the final topology $\tau_5$ of $\delta$ is equal to $\text{seq}(X)$.

Admissible representations satisfy the following extended universal property.

Lemma 2.3.3 (Extended universal property)

Let $\delta : \subseteq \Sigma^\omega \to X$ be a representation of a weak limit space $X = (X, \rightarrow_x)$ such that the universal property is satisfied. Furthermore let $\Gamma_1, \ldots, \Gamma_k$ be alphabets.

Then for every sequentially continuous function $\phi : \subseteq \Gamma^\omega_1 \times \ldots \times \Gamma^\omega_k \to X$ there is a continuous function $g : \subseteq \Gamma^\omega_1 \times \ldots \times \Gamma^\omega_k \to \Sigma^\omega$ with $\phi = \delta \circ g$.

Proof:
The function $\phi' : \subseteq \Sigma^\omega \to X$ defined by

$$\phi'(p) := \phi(g_{\Gamma^\omega_1} \circ \vartheta_{\Sigma^\omega} \circ \pi_{k,1}(p), \ldots, g_{\Gamma^\omega_k} \circ \vartheta_{\Sigma^\omega} \circ \pi_{k,k}(p))$$

for all $p \in \Sigma^\omega$ is sequentially continuous (cf. Subsection 2.1.7). By the universal property, there is a continuous function $h : \subseteq \Sigma^\omega \to \Sigma^\omega$ with $\phi' = \delta \circ h$. We define $g : \subseteq \Gamma^\omega_1 \times \ldots \times \Gamma^\omega_k \to \Sigma^\omega$ by

$$g(q_1, \ldots, q_k) := \begin{cases} h(\vartheta_{\Sigma^\omega}^{-1}(g_{\Gamma^\omega_1}^{-1}(q_1)), \ldots, \vartheta_{\Sigma^\omega}^{-1}(g_{\Gamma^\omega_k}^{-1}(q_k))) & \text{if } (q_1, \ldots, q_k) \in \text{dom}(\phi) \\ \text{div} & \text{otherwise} \end{cases}$$

for all $(q_1, \ldots, q_k) \in \Gamma^\omega_1 \times \ldots \times \Gamma^\omega_k$. Since $h$, $(\cdot, \ldots, \cdot)$ and the inverses $\vartheta_{\Sigma^\omega}^{-1}, g_{\Gamma^\omega_1}^{-1}, \ldots, g_{\Gamma^\omega_k}^{-1}$ are sequentially continuous (cf. Subsection 2.1.7), $g$ is continuous. Obviously, $g$ satisfies $\phi = \delta \circ g$.

We prove that the admissible representations of a weak limit space $X = (X, \rightarrow_x)$ are exactly those representations which are maximal with respect to continuous reducibility among the sequentially continuous representations of $X$.

Proposition 2.3.4 (Maximality of admissible representations)

Let $\delta$ be a representation of a weak limit space $X = (X, \rightarrow_x)$.

Then $\delta$ is an admissible representation of $X$ if and only if $\delta$ is $\leq_{\text{max}}$-maximal in the class of all sequentially continuous representations of $X$. 

$\blacksquare$
Proof:

\(\rightarrow\): By admissibility, \(\delta\) is sequentially continuous. By Lemma 2.3.3, every sequentially continuous representation \(\zeta\) of \(\mathfrak{X}\) over any alphabet satisfies \(\zeta \leq \delta\). Thus \(\delta\) is \(\leq_t\)-maximal in the class of all sequentially continuous representations of \(\mathfrak{X}\).

\(\leftarrow\): Let \(\delta : \Sigma^\omega \to X\) be \(\leq_t\)-maximal in the class of all sequentially continuous representations of \(\mathfrak{X}\). Let \(\phi : \Sigma^\omega \to X\) be sequentially continuous. We define the function \(\zeta : \subseteq \Sigma^\omega \to X\) by \(\zeta(0:p) := \phi(p)\) and \(\zeta(1:p) := \delta(p)\). Then \(\zeta\) is a sequentially continuous representation of \(\mathfrak{X}\) and thus continuously translatable to \(\delta\) by some continuous function \(g : \subseteq \Sigma^\omega \to \Sigma^\omega\). The continuous function \(h : \subseteq \Sigma^\omega \to \Sigma^\omega\) defined by \(h(p) := g(0:p)\) translates \(\phi\) continuously to \(\delta\).

We obtain that the admissible representations of a weak limit space \(\mathfrak{X}\) form an equivalence class under the equivalence relation \(\equiv_t\) of continuous equivalence.

**Proposition 2.3.5 (Continuous equivalence preserves admissibility)**

Let \(\delta\) be an admissible representation of a weak limit space \(\mathfrak{X} = (X, \rightarrow_\omega)\), and let \(\zeta\) be any representation of \(X\).

Then \(\zeta\) is an admissible representation of \(\mathfrak{X}\) if and only if \(\zeta\) and \(\delta\) are continuously equivalent (i.e. \(\zeta \equiv_t \delta\)).

Proof:

\(\rightarrow\): By Lemma 2.3.3, \(\zeta \leq_t \delta\) follows from the continuity of \(\zeta\) and the universal property of \(\delta\), whereas \(\delta \leq_t \zeta\) follows from the continuity of \(\delta\) and the universal property of \(\zeta\).

\(\leftarrow\): Let \(\Sigma\) and \(\Gamma\) be the underlying alphabet of \(\delta\) and \(\zeta\), respectively. If \(\zeta \leq_t \delta\), then there is a continuous function \(h : \subseteq \Gamma^\omega \to \Sigma^\omega\) with \(\zeta = \delta \circ h\), hence \(\zeta\), being a composition of sequentially continuous functions, is sequentially continuous. By Lemma 2.3.3, we have \(\phi \leq_t \delta\) for every sequentially continuous function \(\phi : \subseteq \Gamma^\omega \to X\). The transitivity of \(\leq_t\) implies \(\phi \leq_1 \zeta\). Thus \(\zeta\) is an admissible representation of \(\mathfrak{X}\).

Now we will investigate some examples. First, we construct admissible representations of \((\mathbb{N}, \rightarrow_\omega)\) and its subspace \((\mathbb{N}, \rightarrow_\omega)\).

**Example 2.3.6 (Admissible representations of \((\mathbb{N}, \tau_\mathbb{N})\) and \((\mathbb{N}, \tau_\mathbb{N})\))**

We define \(\varphi_\mathbb{N} : \subseteq \mathbb{N}^\omega \to \overline{\mathbb{N}}\) by \(\varphi_\mathbb{N}(p) := \min \{\{\infty\} \cup \{n \in \mathbb{N} \mid p(n) \neq 0\}\}\). Obviously, \(\varphi_\mathbb{N}\) is continuous w.r.t. \(\rightarrow_\omega\) and \(\rightarrow_\mathbb{N}\).

Let \(\phi : \subseteq \mathbb{N}^\omega \to \overline{\mathbb{N}}\) be a sequentially continuous function. For every \(p \in \text{dom}(\phi)\) and \(n \in \mathbb{N}\) the number

\[m_{p,n} := \min \{i \in \mathbb{N} \mid \phi[p^{<i}\mathbb{N}^\omega] \subseteq \{n\} \text{ or } \phi[p^{<i}\mathbb{N}^\omega] \subseteq \mathbb{N} \setminus \{n\}\}\]

is smaller than \(\infty\), because the sets \(\{n\}\) and \(\mathbb{N} \setminus \{n\}\) are both open in \((\mathbb{N}, \tau_\mathbb{N})\) and \(\phi\) is topologically continuous (cf. Subsection 2.2.2). Thus we can define \(g : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega\) by \(\text{dom}(g) := \text{dom}(\phi)\) and

\[g(p)(n) := \begin{cases} 1 & \text{if } \phi[p^{<m_{p,n}\mathbb{N}^\omega}] \subseteq \{n\} \\ 0 & \text{otherwise} \end{cases}\]
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for all \( p \in \text{dom}(\phi) \) and \( n \in \mathbb{N} \). Then \( g \) is continuous, because each finite prefix of \( g(p) \)
only depends on a finite prefix of \( p \in \text{dom}(g) \); for a more precise proof let \((p_n)_{n \leq \infty} \) be a
convergent sequence of \((\mathbb{N}^\infty, \tau_{\mathbb{N}^\infty})\) contained in \( \text{dom}(g) \). Then for every \( n \in \mathbb{N} \) there
is some \( k_n \in \mathbb{N} \) with \((\forall k \geq k_n)(\forall i \leq n) p_k \in p^\infty_{m_{p_{\infty},i}} \mathbb{N}^\infty \). This implies \( m_{p_{\infty},i} = m_{p_{\infty},i} \) and
\((g(p_k))_{i=n} = (g(p_{\infty}))_{i=n} \) for all \( k \geq k_n \) and \( i \leq n \). Hence \( g \) is sequentially continuous.
Obviously we have \( g_{\mathbb{N}} \circ g = \phi \). Thus \( g \) translates \( \phi \) continuously to \( g_{\mathbb{N}} \). Hence \( g_{\mathbb{N}} \)
is an admissible representation of the limit space \((\mathbb{N}, \tau_{\mathbb{N}})\) and the topological space \((\mathbb{N}, \tau_{\mathbb{N}})\).

In a similar way, one can prove that \( g_{\mathbb{N}} : \mathbb{N}^\infty \to \mathbb{N} \) defined by \( g_{\mathbb{N}}(p) := p(0) \) is an
admissible representation of \((\mathbb{N}, \tau_{\mathbb{N}})\) and of \((\mathbb{N}, \tau_{\mathbb{N}})\).

The next example deals with the useful Sierpiński space.

**Example 2.3.7 (The Sierpiński space \( \mathcal{S}_i \))**
The Sierpiński space \( \mathcal{S}_i \) is defined to have the set \( \mathcal{S}_i := \{\bot, \top\} \) as its underlying set and
\( \tau_{\mathcal{S}_i} := \{\emptyset, \{\top\}, \{\bot, \top\}\} \) as its topology. We define the representation \( g_{\mathcal{S}_i} : \mathbb{N}^\infty \to \mathcal{S}_i \) by
\[
g_{\mathcal{S}_i}(p) := \begin{cases} 
\bot & \text{if } p = 0^\infty \\
\top & \text{otherwise}
\end{cases}
\]

Since obviously the convergence relation \( \rightarrow_{\mathcal{S}_i} \) of \( \mathcal{S}_i \) is given by
\[
(b_n)_n \rightarrow_{\mathcal{S}_i} b_\infty \iff (b_\infty = \bot \text{ or } (\forall \infty n \in \mathbb{N}) b_n = \top),
\]
\( g_{\mathcal{S}_i} \) is sequentially continuous. Now let \( \phi : \subseteq \mathbb{N}^\infty \to \mathcal{S}_i \) be sequentially continuous. The
function \( g : \mathbb{N}^\infty \to \mathbb{N}^\infty \) defined by
\[
g(p)(n) := \begin{cases} 
1 & \text{if } \phi[p^{<\infty} \mathbb{N}^\infty] \subseteq \{\top\} \\
0 & \text{otherwise}
\end{cases}
\]
is sequentially continuous, because \( g(p)(n) \) only depends on a finite prefix of \( p \), namely on \( p^{<\infty} \). One easily verifies that \( g \) translates \( \phi \) to \( g_{\mathcal{S}_i} \). Hence \( g_{\mathcal{S}_i} \) is an admissible
representation of \( \mathcal{S}_i \).

In Example 2.2.5 we have seen that the convergence relation \( \rightarrow_{\mathbb{R}_{10}} \) induced by the
decimal representation is not equal to the usual convergence relation \( \rightarrow_{\mathbb{R}} \) of the
Euclidian topology \( \tau_\mathbb{R} \) on \( \mathbb{R} \). Thus \( \mathbb{R}_{10} \) is not a WeakLim–quotient representation of the
Euclidean limit space \((\mathbb{R}, \tau_{\mathbb{R}})\). From Proposition 2.3.2 it follows that \( \rho_{10} \) is not an
admissible representation of \((\mathbb{R}, \tau_{\mathbb{R}})\). However, we will see in Example 4.3.19 that \( \rho_{10} \)
is an admissible representation of its WeakLim–quotient \((\mathbb{R}, \tau_{\mathbb{R}_{10}})\).

We introduce an admissible representation of the Euclidean space. It induces the
usual computability notion on \( \mathbb{R} \).

**Example 2.3.8 (Signed Digit Representation)**

We define the representations \( \rho_\mathbb{R} : \subseteq \mathbb{Z}^\infty \to \mathbb{R} \) and \( \varrho_\mathbb{R} : \subseteq \mathbb{N}^\infty \to \mathbb{R} \) of the real numbers by
\[
\rho_\mathbb{R}(p) := \sum_{i \in \mathbb{N}} p(i) \cdot \left(\frac{1}{2}\right)^i \quad \text{and} \quad \varrho_\mathbb{R} := \rho_\mathbb{R} \circ \varrho_{\mathbb{Z}^\infty}
\]
for \( p \in \text{dom}(\rho_R) := \{ q \in \mathbb{Z}^\omega \mid (\forall i \geq 1) q(i) \in \{-1, 0, 1\} \} \), where \( g_\mathbb{Z}^\omega \) is the representation of \( \mathbb{Z}^\omega \) from Subsection 2.1.7. The representation \( \rho_R \) is defined similarly as the signed digit representation \( \rho_{\text{sd}} \) of \( \mathbb{R} \) considered in [Wei00], except for using \( \mathbb{Z} \) as the alphabet instead of \( \{1, 0, 1, *\} \). Hence \( \rho_R \) and \( g_\mathbb{Z}^\omega \) are computably equivalent to \( \rho_{\text{sd}} \).

Thus the computational model induced by \( g_\mathbb{Z}^\omega \) on \( \mathbb{R} \) is very similar to the ones considered by other authors like A. Grzegorczyk in [Grz57], Ker-I Ko in [Ko91], S. Labhalla, H. Lombardi and E. Montai in [LLM01], M. Pour–El and J. Richards in [PER89] or V. Stoltenberg–Hansen and J.V. Tucker in [SHT99] (cf. [Wei00, Chapter 9]). From [Her99] we know that, except for computable equivalence, \( g_\mathbb{R} \) is the only representation which makes the structure \((\mathbb{R}, 0, 1, +, −, *, 1/\), \( \text{NormLim}, < \)) \( r \)-effectively categorical.

We prove directly that \( \rho_\mathbb{R} \) is an admissible representation of \((\mathbb{R}, −_e)\).

Continuity:
Let \( (p_n)_{n<\infty} \) a convergent sequence in \( \text{dom}(\rho_\mathbb{R}) \) and \( (x_n)_{n<\infty} \) in \( \mathbb{R} \) by

\[
\begin{align*}
\rho_\mathbb{R}([p_n]_{n<\infty}) &= [\rho_\mathbb{R}(p_n)]_{n<\infty}, \\
\rho_\mathbb{R}([p_n]_{n<\infty}) &= [\rho_\mathbb{R}(p_n)]_{n<\infty}, \\
\rho_\mathbb{R}([p_n]_{n<\infty}) &= [\rho_\mathbb{R}(p_n)]_{n<\infty}.
\end{align*}
\]

we have \( |x_n - x_\infty| \leq 2 \cdot (\frac{1}{2})^k \) for all \( n \geq n_k \). Thus \( (x_n)_n \xrightarrow{\infty} x_\infty \).

Surjectivity: Since every constant function \( \phi : \mathbb{Z}^\omega \rightarrow \mathbb{R} \) is sequentially continuous, the surjectivity follows from the universal property.

Universality:
Let \( \phi : \subseteq \mathbb{Z}^\omega \rightarrow \mathbb{R} \) be continuous with respect to \( −_e \). For every \( p \in \text{dom}(\phi) \) and every \( m \in \mathbb{N} \) we define \( k_{p,m} \in \mathbb{N} \) and \( x_{p,m} \in \mathbb{R} \) by

\[
\begin{align*}
k_{p,m} &:= \min \{ k \in \mathbb{N} \mid (\forall x, y \in \phi[p^{<k}\mathbb{Z}^\omega]) \ |x - y| \leq (\frac{1}{2})^{m+1} \}, \\
x_{p,m} &:= \frac{1}{2} \cdot (\inf(\phi[p^{<k_{p,m}}\mathbb{Z}^\omega]) + \sup(\phi[p^{<k_{p,m}}\mathbb{Z}^\omega]))
\end{align*}
\]

and \( z_{p,m} \in \mathbb{Z} \) recursively by

\[
\begin{align*}
z_{p,0} &:= |x_{p,0} + \frac{1}{2}| = \max \{ z \in \mathbb{Z} \mid z \leq x_{p,0} + \frac{1}{2} \} \quad \text{and, for } m > 0, \\
z_{p,m} &:= \begin{cases} 1 & \text{if } x_{p,m} - \sum_{i=0}^{m-1} z_{p,i} \cdot (\frac{1}{2})^i > + (\frac{1}{2})^{m+1} \\
-1 & \text{if } x_{p,m} - \sum_{i=0}^{m-1} z_{p,i} \cdot (\frac{1}{2})^i < - (\frac{1}{2})^{m+1} \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

We show inductively

\[
|x_{p,m} - \sum_{i=0}^{m} z_{p,i} \cdot (\frac{1}{2})^i| \leq (\frac{1}{2})^{m+1}.
\]

\( m = 0 \): This follows from \((\forall y \in \mathbb{R}) y - 1 < |y| \leq y. \)

\( m = 1 \rightarrow m. \)

For \( m > 0 \), we have by induction hypothesis

\[
\begin{align*}
 &|x_{p,m} - \sum_{i=0}^{m-1} z_{p,i} \cdot (\frac{1}{2})^i| \leq |x_{p,m} - x_{p,m-1}| + |x_{p,m-1} - \sum_{i=0}^{m-1} z_{p,i} \cdot (\frac{1}{2})^i| \\
 &\leq \frac{1}{2} \cdot (\sup(\phi[p^{<k_{p,m-1}}\mathbb{Z}^\omega]) - \inf(\phi[p^{<k_{p,m-1}}\mathbb{Z}^\omega])) + (\frac{1}{2})^{m} \\
 &\leq \frac{1}{2} \cdot (\frac{1}{2})^{m} + (\frac{1}{2})^{m} = \frac{3}{2} \cdot (\frac{1}{2})^{m}.
\end{align*}
\]

We consider the three different possible values of \( z_{p,m} \):

\(8\)For \( a, b \in \mathbb{R}, [a; b] \) denotes the interval \( \{ x \in \mathbb{R} \mid a \leq x \leq b \} \).
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We define both positive and negative digits. Thus the sequence $(q_n)_{n \in \mathbb{N}}$ is a convergent sequence of $(\mathbb{Z}, \leq \mathbb{N})$ for all $p \in dom(\phi)$ and $m \in \mathbb{N}$. Since $(\forall m \in \mathbb{N})(\forall p \in \mathbb{Z})|\phi(p) − x_{p,m}| \leq (\frac{1}{2})^{m+1}$, we obtain from Equation (2.6)

$$\phi(p) = \sum_{m=0}^{\infty} x_{p,m} = \rho_\mathbb{R}(g(p)).$$

Since $k_{p,m}, x_{p,m}$ and $z_{p,m}$ only depend on a finite prefix of $p$, namely on $p^{<k_{p,m}}$, $g$ is continuous. Thus $g$ translates $\phi$ continuously to $\rho_\mathbb{R}$.

Thus $\rho_\mathbb{R}$ is an admissible representation of $(\mathbb{R}, -_g)$ and of $(\mathbb{R}, \tau_\mathbb{R})$. Since the function $g_{\mathbb{Z}}$ and its inverse $\delta_{\mathbb{Z}}$ are sequentially continuous (cf. Subsection 2.1.7), also $g_{\mathbb{Z}}$ is an admissible representation of $(\mathbb{R}, -_g)$ by Proposition 2.3.5.

By modifying $\rho_\mathbb{R}$, we can construct a WeakLim–quotient representation of $(\mathbb{R}, -_g)$ which is not admissible.

**Example 2.3.9 (A WeakLim–quotient represent. that is not admissible)**

We define $\delta_{\mathbb{R}} : \mathbb{Z}^\omega \to \mathbb{R}$ by

$$dom(\delta_{\mathbb{R}}) := \{3::p | p \in \rho_\mathbb{R}^{-1}[\mathbb{R} \setminus \{0\}]\} \cup \{4::p | p \in dom(\rho_{\mathbb{R}}), (\forall i \in \mathbb{N})p(i) \geq 0\} \cup \{4::p | p \in dom(\rho_{\mathbb{R}}), (\forall i \in \mathbb{N})p(i) \leq 0\}.$$

$\delta_{\mathbb{R}}(3::p) := \rho_{\mathbb{R}}(p)$ and $\delta_{\mathbb{R}}(4::q) := \rho_{\mathbb{R}}(q)$ for all $p, q \in dom(\rho_{\mathbb{R}})$ with $3::p, 4::q \in dom(\delta_{\mathbb{R}})$. By sequential continuity of $\rho_{\mathbb{R}}, \delta_{\mathbb{R}}$ is sequentially continuous w.r.t. $-_g$.

Now let $(x_n)_{n \leq \infty}$ be a convergent sequence of $(\mathbb{R}, -_g)$.

Case $x_\infty \neq 0$: Since $\rho_{\mathbb{R}}$ is a WeakLim–quotient representation by Proposition 2.3.2, there is a convergent sequence $(p_n)_{n \leq \infty}$ in $\mathbb{Z}^\omega$ with $(\rho_{\mathbb{R}}(p_n))_{n \leq \infty} = (x_n)_{n \leq \infty}$. The sequence $(q_n)_{n \leq \infty}$ in $\mathbb{Z}^\omega$ defined by $q_n := 3::p_n$, if $x_n \neq 0$, and $q_n = 40^\omega$ otherwise, is a convergent sequence satisfying $(\delta_{\mathbb{R}}(q_n))_{n \leq \infty} = (x_n)_{n \leq \infty}$.

Case $x_\infty = 0$: Obviously, for every $n \in \mathbb{N}$ there exists a $\rho_{\mathbb{R}}$–name $p_n$ of $x_n$ satisfying $(\forall i \in \mathbb{N})p(i) \geq 0$ or $(\forall i \in \mathbb{N})p(i) \leq 0$. For every $m \in \mathbb{N}$ there is some $l_m \in \mathbb{N}$ with $(\forall n \geq l_m)|x_n| < (\frac{1}{2})^{m+1}$. Then $p_n^{<m} = 0^m$ for all $n \geq l_m$, because $p_n$ does not contain both positive and negative digits. Thus the sequence $(q_n)_{n \leq \infty} := (4::p_n)_{n \leq \infty}$ converges to $q_\infty := 40^\omega$. Clearly, we have $(\delta_{\mathbb{R}}(q_n))_{n \leq \infty} = (x_n)_{n \leq \infty}$. 

\[\]
Hence $\delta_{\text{wq}}$ is a WeakLim–quotient representation of $(\mathbb{R}, \rightarrow)$. 

We show that the sequentially continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) := x - 1$ is not relatively continuous w.r.t. to $\delta_{\text{wq}}$. 

Assume that there is a continuous function $g : \mathbb{Z}^\omega \rightarrow \mathbb{Z}^\omega$ realizing $f$. Then there is some $n \in \mathbb{N}$ with $g[310^n \mathbb{Z}^\omega] \subseteq 42\mathbb{Z}^\omega$, because $310^n$ is $\delta_{\text{wq}}$–name of 1 and the only $\delta_{\text{wq}}$–name of $f(1)$ is $40\mathbb{Z}^\omega$. Since $\delta_{\text{wq}}(310^n10\mathbb{Z}^\omega) = 1 + (\frac{1}{2})^n1$ and the only $\delta_{\text{wq}}$–names of $(\frac{1}{2})^n1$ in $42\mathbb{Z}^\omega$ are $400^n10\mathbb{Z}^\omega$ and $400^n01\mathbb{Z}^\omega$, we have either $g(310^n10\mathbb{Z}^\omega) = 400^n10\mathbb{Z}^\omega$ or $g(310^n10\mathbb{Z}^\omega) = 400^n01\mathbb{Z}^\omega$. By continuity of $g$, there is some $m \in \mathbb{N}$ such that $g[310^n10\mathbb{Z}^\omega] \subseteq 400^n1\mathbb{Z}^\omega$ or $g[310^n10\mathbb{Z}^\omega] \subseteq 400^n0\mathbb{Z}^\omega$, respectively. Since $\delta_{\text{wq}}[400^n1\mathbb{Z}^\omega] = [\frac{1}{2}+1; \frac{1}{2}]$ and $\delta_{\text{wq}}[400^n0\mathbb{Z}^\omega] = [0; (\frac{1}{2})^{n+1}]$, $g$ fails to transform either the $\delta_{\text{wq}}$–name$^9$ $310^n10\mathbb{Z}^\omega$ of $1 + (\frac{1}{2})^n1$ into a $\delta_{\text{wq}}$–name of $(\frac{1}{2})^n1 + (\frac{1}{2})^{n+1}$ or the $\delta_{\text{wq}}$–name $310^n01\mathbb{Z}^\omega$ of $1 + (\frac{1}{2})^n1 + (\frac{1}{2})^{n+1}$ into a $\delta_{\text{wq}}$–name of $(\frac{1}{2})^n1 + (\frac{1}{2})^{n+1}$. Thus, $f$ is not relatively continuous w.r.t. $\delta_{\text{wq}}$. This implies that the sequentially continuous representation $f \circ \delta_{\text{wq}}$ of $\mathbb{R}$ is not continuously translatable to $\delta_{\text{wq}}$. Hence, $\delta_{\text{wq}}$ is a WeakLim–quotient representation of $(\mathbb{R}, \rightarrow)$ which is not admissible.

The next example shows a countably based space which has a WeakLim–quotient representation, but no admissible representation.

**Example 2.3.10**

We equip the set $\{0, 1\}$ with the indiscrete topology $\{\emptyset, \{0, 1\}\}$. It induces the chaotic convergence relation $\rightarrow_{\text{ch}}$ on $\{0, 1\}$ declaring every sequence $(x_n)_{n \leq \infty}$ to be convergent. We show that the obviously continuous representation $\delta_{\text{ch}} : \mathbb{N}^\omega \rightarrow \{0, 1\}$ defined by

$$
\delta_{\text{ch}}(p) := \begin{cases} 
0 & \text{if } (\forall \overline{i} \in \mathbb{N}) p(\overline{i}) = 0 \\
1 & \text{otherwise}
\end{cases}
$$

is a WeakLim–quotient representation of $\mathcal{X} := (\{0, 1\}, \rightarrow_{\text{ch}})$. Let $(x_n)_{n \leq \infty}$ be a sequence in $\{0, 1\}$. Then the sequence $(p_n)_{n \leq \infty}$ defined by

$$
p_n := \begin{cases} 
0^\omega & \text{if } x_\infty = 0 \text{ and } x_n = 0 \\
0^n1^\omega & \text{if } x_\infty = 0 \text{ and } x_n = 1 \\
1^\omega & \text{if } x_\infty = 1 \text{ and } x_n = 1 \\
1^n0^\omega & \text{otherwise}
\end{cases}
$$

is a convergent sequence of $(\mathbb{N}^\omega, \rightarrow_{\text{wq}})$ and satisfies $(\forall n \in \mathbb{N}) \delta_{\text{ch}}(p_n) = x_n$. We obtain $(x_n)_{n \rightarrow_{\text{ch}} x_\infty}$, i.e. $\rightarrow_{\text{ch}} x = \rightarrow_{\text{ch}} x$. Since every sequence converges in $\mathcal{X}$ to 0 and to 1, $\mathcal{X}$ does not satisfy the $T_0$–property (cf. Subsection 2.2.3). By Proposition 2.3.13, $\mathcal{X}$ cannot have an admissible (single–valued) representation. Nevertheless, we will later see that $\mathcal{X}$ has an admissible “multirepresentation” (cf. Definition 2.4.1, Proposition 4.3.2).

With the same argument, the naive Cauchy representation $\rho_{\text{Ch}}$ is not an admissible representation.

---

$^9$We denote by 1 the digit $-1 \in \mathbb{Z}$. 
2.3 Admissible Representations

Example 2.3.11 (Non–admissibility of the naive Cauchy representation)
The naive Cauchy representation $\rho_{Cn} : \subseteq \mathbb{N}^\omega \rightarrow \mathbb{R}$ is defined by

$$\text{dom}(\rho_{Cn}) := \{q \in \mathbb{N}^\omega | \lim_{i \rightarrow \infty} \nu_q(q(i)) \text{ exists} \} \quad \text{and} \quad \rho_{Cn}(q) := \lim_{i \rightarrow \infty} \nu_q(q(i))$$

for $q \in \text{dom}(\rho_{Cn})$ (cf. [Wei00, Ex. 4.1.14]). Informally, a $\rho_{Cn}$–name of a real number $x$ yields a sequence of rationals converging to $x$.

We show that $\rightarrow_{\rho_{Cn}}$ is the chaotic convergence relation on $\mathbb{R}$ defining every sequence to converge to every element. Let $(x_n)_{n \leq \infty}$ be a generalized sequence of real numbers. For every $n \in \mathbb{N}$, there is some $q_n \in \text{dom}(\rho_{Cn})$ with $\rho_{Cn}(q_n) = x_n$. We define, for every $n \in \mathbb{N}$, the $\omega$–word $p_n$ by $p_n := q_n(0) \ldots q_n(n) : q_n$ and observe $\rho_{Cn}(p_n) = x_n$. Clearly, $(p_n)_n$ converges to $q$. This implies $(x_n)_n \rightarrow_{\rho_{Cn}} x_n$. Hence $\rightarrow_{\rho_{Cn}}$ is the chaotic convergence relation on $\mathbb{R}$.

Since every sequence in $(\mathbb{R}, \rightarrow_{\rho_{Cn}})$ converges to every element, $(\mathbb{R}, \rightarrow_{\rho_{Cn}})$ does not satisfy the $T_0$–property (cf. Subsection 2.2.3). Hence $\rho_{Cn}$ is not admissible by Proposition 2.3.13.

2.3.2 Comparison with the Standard Definition of Admissibility

In [Wei00], a representation $\gamma$ of a countably based $T_0$–space $\mathfrak{A} = (A, \tau_{\mathfrak{A}})$ is defined to be admissible with respect to $\tau_{\mathfrak{A}}$ iff $\gamma$ is continuously equivalent to a standard representation $\delta_{(A, \sigma, \nu)}$ of $\mathfrak{A}$ which is derived from a notation $\nu : \subseteq \Sigma^* \rightarrow \sigma$ of a subbase $\sigma$ of the space $\mathfrak{A}$. As a characterization, it is shown that a representation $\gamma : \subseteq \Sigma^\omega \rightarrow A$ is admissible w.r.t. $\tau_{\mathfrak{A}}$ iff $\gamma$ is topologically continuous$^{10}$ and every topologically continuous$^{10}$ representation $\zeta : \subseteq \Sigma^\omega \rightarrow A$ is continuously translatable to $\gamma$. The latter is equivalent to the fact that $\gamma$ is $\leq_1$–maximal in the class of all sequentially continuous representations, because topological continuity and sequential continuity are equivalent for partial functions between countably based spaces. From Proposition 2.3.4 we conclude that for representations of countably based $T_0$–spaces Definition 2.3.1 coincides with the standard definition of admissibility. Hence Definition 2.3.1 extends the standard definition of admissibility consistently.

Theorem 2.3.12 (Comparison of the two admissibility notions)

Let $\mathfrak{A} = (A, \tau_{\mathfrak{A}})$ be a countably based $T_0$–space.

Then a representation $\gamma : \subseteq \Sigma^\omega \rightarrow A$ is admissible w.r.t. $\tau_{\mathfrak{A}}$ in the standard sense if and only if $\gamma$ is admissible w.r.t. $\rightarrow_{\mathfrak{A}}$ in the sense of Definition 2.3.1.

Analogously to the standard case, the existence of an admissible representation of a weak limit space implies the $T_0$–property (cf. Proposition 2.3.13), i.e., $x$ is equal to $y$, whenever the constant sequences $(x)_n$ and $(y)_n$ both converge to $x$ and $y$ (see Subsection 2.2.3). On the other hand, there are weak limit spaces not satisfying Axiom (L3), non topological limit spaces, and non countably based topological spaces which all have admissible representations (cf. Examples 4.3.19, 2.3.16 and 2.3.15). However, first–countable topological spaces do not have any admissible representation, unless

---

$^{10}$with respect to $\tau_{\Sigma^\omega}$ and $\tau_{\mathfrak{A}}$
they are countably based (cf. Proposition 3.3.1). We conclude that Definition 2.3.1 extends the standard notion of admissibility strictly. On the other hand, the existence of an admissible representation implies the $T_0$-property.

**Proposition 2.3.13**

Every weak limit space that has an admissible representation satisfies the $T_0$-property.

**Proof:**

Let $\mathfrak{X} = (X, \to)$ be a weak limit space and $\delta : \subseteq \Sigma^\omega \to X$ be an admissible representation of $\mathfrak{X}$. Assume that there are points $x \neq y$ in $X$ such that the constant sequence $(x)_n$ converges to $y$ and $(y)_n$ converges to $x$. Axiom (L5) implies that every sequence that consists only of $x$ and $y$ converges both to $x$ and to $y$. Thus for every subset $A \subseteq \Sigma^\omega$ the function $\phi_A : \Sigma^\omega \to X$ defined by

$$\phi_A(p) := \begin{cases} x & \text{if } p \in A \\ y & \text{otherwise} \end{cases}$$

is sequentially continuous. By universality of $\delta$, there is a continuous function $g_A : \Sigma^\omega \to \Sigma^\omega$ with $\phi_A = \delta \circ g_A$. For all $B \neq A$ the function $g_B$ is not equal to $g_A$. This implies that the cardinality of the set $F_{\text{tot}}$ of all total continuous functions $f : \Sigma^\omega \to \Sigma^\omega$ is not smaller than the cardinality of the power set of $\Sigma^\omega$. This contradicts the existence of a representation of $F_{\text{tot}}$ (cf. Subsection 4.2.3) which implies that the cardinality of $F_{\text{tot}}$ is bounded by the cardinality of $\Sigma^\omega$.

Hence $X$ satisfies the $T_0$-property.

$\diamondsuit$

From [BrHe02] we know that an admissible representation of a countably based $T_0$-space has an open restriction. On the other hand, open mappings preserve the existence of a countable base (cf. [Eng89, Theorem 1.4.16]). We obtain:

**Lemma 2.3.14**

Let $\delta$ be an admissible representation of a topological space $\mathfrak{X}$. Then $\delta$ has an open restriction if and only if $\mathfrak{X}$ has a countable base.

The next example shows that there are non countably based topological spaces which have admissible representations.

**Example 2.3.15 (A non countably based top. space with admissible rep.)**

We equip the set $X := \{(\infty, \infty)\} \cup (\mathbb{N} \times \{\infty\}) \cup (\mathbb{N} \times \mathbb{N})$ with the $T_2$-topology

$$\tau := \left\{ O \subseteq X \ \big| \ ( (\infty, \infty) \in O \implies (\forall a \in \mathbb{N}) (a, \infty) \in O ) \ \text{and} \ (\forall a \in \mathbb{N}) ( (a, \infty) \in O \implies (\forall b \in \mathbb{N}) (a, b) \in O ) \right\} .$$

On the one hand, the limit operator $\lim_{n \to \infty}$ of the space $\mathfrak{X} := (X, \tau)$ obviously satisfies

$$\lim_{b \to \infty} (a, b) = (a, \infty) \ \text{and} \ \lim_{a \to \infty} \left( \lim_{b \to \infty} (a, b) \right) = \lim_{a \to \infty} (a, \infty) = (\infty, \infty), \quad (2.7)$$
2.3 Admissible Representations

but one the other hand no sequence in \( \mathbb{N}^2 \) converges to the point \((\infty, \infty)\) in this space. For the proof of the latter let \((y_n)_n\) be a sequence in \( \mathbb{N}^2 \). If there is some \( a \in \mathbb{N} \) with \( y_n \in \{a\} \times \mathbb{N} \) for infinitely many \( n \), then the set \( V_1 := X \setminus \{(a) \times \mathbb{N}\} \) is an open neighbourhood of \((\infty, \infty)\) such that \((y_n)_n\) is not eventually in \( V_1 \). Otherwise, for every \( a \in \mathbb{N} \) there is some \( b_a \) with \( y_n \notin \{(a, b) \mid b \geq b_a\} \) for all \( n \in \mathbb{N} \). Then \( U_2 := X \setminus \{(a, b) \mid a \in \mathbb{N}, \ b < b_a\} \) is an open neighbourhood of \((\infty, \infty)\) containing no element of \((y_n)_n\). Thus \((y_n)_n\) does not converge to \((\infty, \infty)\).

Suppose for contradiction that the point \((\infty, \infty)\) has a countable open neighbourhood base \( \{U_0, U_1, \ldots\} \). By definition of \( \tau \), for every \( n \in \mathbb{N} \) there is some \( y_n \in \mathbb{N}^2 \cap U_0 \cap \ldots \cap U_n \). Since for every open set \( O \) containing \((\infty, \infty)\) there is some \( k \) with \( U_k \subseteq O \), \((y_n)_n\) converges to \((\infty, \infty)\) contradicting the above observation. We conclude that \((X, \tau)\) is neither first-countable nor countably based.

We define the representation \( \delta : \subseteq \mathbb{N}^\omega \to X \) by

\[
\delta(0:0^\omega) := (\infty, \infty), \quad \delta((a + 1):0^\omega) := (a, \infty),
\]

and \( p \notin \text{dom}(\delta) \) for the “other” \( p \in \mathbb{N}^\omega \). From (2.7) it follows that \( \delta \) is continuous.

Let \( \phi : \subseteq \mathbb{N}^\omega \to X \) be continuous. Define \( X_\infty := \mathbb{N} \times \{\infty\} \) and \( X_a := \{a\} \times \mathbb{N} \) for \( a \in \mathbb{N} \).

Let \( p \in \text{dom}(\phi) \). There is some \( a \in \mathbb{N} \) with \( \phi(p) \in \{a\} \times \mathbb{N} \).

Suppose for contradiction that \( \phi[p^{< m}\mathbb{N}^\omega] \notin X_a \) holds for all \( m \in \mathbb{N} \). Then for every \( m \) there is some \( q_m \in p^{< m}\mathbb{N}^\omega \) with \( \phi(q_m) \notin X_a \). By continuity of \( \phi \), \( \phi(q_m) \) converges to \( \phi(p) \). This contradicts the fact that in the case \( a \in \mathbb{N} \) the set \( X_a \) is an open neighbourhood of \( \phi(p) \) and in case \( a = \infty \) the point \( \phi(p) \) is equal to \((\infty, \infty)\) which is not the limit of any sequence in \( X \setminus X_\infty \). Hence

\[
m_p := \min \left( \{\infty\} \cup \{i \in \mathbb{N} \mid (\exists j \in \mathbb{N}) \phi[p^{< i}\mathbb{N}^\omega] \subseteq X_j \} \right)
\]

is a natural number for all \( p \in \text{dom}(\phi) \). For every \( j \in \mathbb{N} \) we can prove in a similar way as in Example 2.3.6 that there is a continuous function \( g_j : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega \) with \( \phi(p) = \delta(g_j(p)) \) for all \( p \in \phi^{-1}[X_j] \). We define the function \( h : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega \) by

\[
h(p) := g_{\min \{j \in \mathbb{N} \mid \phi[p^{< m_p}\mathbb{N}^\omega] \subseteq X_j \}}(p)
\]

for all \( p \in \text{dom}(h) := \text{dom}(\phi) \). Since \( m_p \) only depends on a finite prefix of \( p \in \text{dom}(h) \), namely on \( p^{< m_p} \), \( h \) is continuous. Obviously, \( h \) translates \( \phi \) to \( \delta \). Thus \( \delta \) is an admissible representation of \( X \).

\[\square\]

We give an example of a non topological limit space that satisfies the \( T_1 \)–property and has an admissible representation.

Example 2.3.16 (A non topological limit space with admissible repres.)

We equip the set \( Z := \{(1, 0)\} \cup \{(2) \times \mathbb{N}\} \cup \{(3) \times \mathbb{N}\} \) with the convergence relation \( \to \) defined by

\[
(z_n)_n \to (1, 0) \iff (\forall m \in \mathbb{N})(\forall n \in \mathbb{N}) z_n \in \{(1, 0), (2, j) \mid j \geq m\}
\]

\[
(z_n)_n \to (2, a) \iff (\forall m \in \mathbb{N})(\forall n \in \mathbb{N}) z_n \in \{(2, a), (3, j) \mid j \geq m\}
\]

\[
(z_n)_n \to (3, b) \iff (\forall n \in \mathbb{N}) z_n = (3, b)
\]
for all sequences \((z_n)_n \in \mathbb{Z}^\mathbb{N}\) and \(a, b \in \mathbb{N}\). One can easily verify that \(\mathfrak{3} := (Z, \to_\mathfrak{3})\) is indeed a limit space and satisfies the \(T_1\)–property.

Let \(O \in \text{seq}(\to_\mathfrak{3})\) be a sequentially open set containing \((1, 0)\). Since \((2, a)_a\) converges to \((1, 0)\) in \(\mathfrak{3}\), there is some \(a_0 \in \mathbb{N}\) with \((2, a) \in O\) for all \(a \geq a_0\). As \((3, b)_b\) converges to \((2, a_0)\) in \(\mathfrak{3}\), the sequence \((3, b)_b\) is eventually in \(O\). Thus \((3, b)_b\) converges to \((1, 0)\) in the associated topological space \(T(\mathfrak{3})\), but it does not in the space \(\mathfrak{3}\). We conclude by Lemma 2.2.7(3) that \(\mathfrak{3}\) is not topological.

We define the representation \(\zeta : \subseteq \mathbb{N}^\omega \to \mathbb{Z}\) by

\[
\zeta(0::0^\omega) := (1, 0), \quad \zeta((a + 1)::0^\omega) := (2, a),
\]

\[
\zeta(00^\omega::1^\omega) := (2, a), \quad \zeta((a + 1)0^\omega::1^\omega) := (3, b)
\]

for all \(a, b \in \mathbb{N}\) and \(p \notin \text{dom}(\zeta)\) for the “other” \(p \in \mathbb{N}^\omega\). Similar to \(\delta\) in Example 2.3.15, \(\zeta\) can be proved to be an admissible representation of the limit space \(\mathfrak{3}\).

\[\heartsuit\]

2.3.3 Relative Continuity versus Sequential Continuity

Now we prove the promised equivalence of relative continuity with respect to admissible representations and sequential continuity (cf. Theorem 2.3.18). This theorem generalizes the Main Theorem in [Wei00] from countably based \(T_0\)–spaces to weak limit spaces.

The proof of Theorem 2.3.18 is divided into two parts. Each of them states sufficient properties of the considered representations for guaranteeing one of the two implications of the equivalence. The only–if–part is given by Lemma 2.2.3, and the if–part by the following Lemma.

Lemma 2.3.17 (Sequential continuity \(\rightarrow\) relative continuity)

For \(i \in \{1, \ldots, k\}\), let \(\delta_i\) be a sequentially continuous representation of a weak limit space \(\mathfrak{X}_i = (X_i, \to_{\mathfrak{X}_i})\). Let \(\gamma\) be a representation of a weak limit space \(\mathfrak{Y} = (Y, \to_{\mathfrak{Y}})\) satisfying the universal property.

Then every sequentially continuous function \(f : \subseteq X_1 \times \ldots \times X_k \to Y\) is relatively continuous w.r.t. the representations \(\delta_1, \ldots, \delta_k\), and \(\gamma\).

Proof:

For \(i \in \{1, \ldots, k\}\), let \(\Sigma_i\) be the underlying alphabet of \(\delta_i\), and let \(\Gamma\) be the one of \(\gamma\). As \(f\) and \(\delta_1, \ldots, \delta_k\) are sequentially continuous, the function \(\phi : \subseteq \Sigma_1^\omega \times \ldots \times \Sigma_k^\omega \to Y\) defined by \(\phi(p_1, \ldots, p_k) := f(\delta_1(p_1), \ldots, \delta_k(p_k))\) is sequentially continuous. By Lemma 2.3.3, there is a continuous function \(g : \subseteq \Sigma_1^\omega \times \ldots \times \Sigma_k^\omega \to \Gamma^\omega\) with

\[
\gamma(g(p_1, \ldots, p_k)) = \phi(p_1, \ldots, p_k) = f(\delta_1(p_1), \ldots, \delta_k(p_k))
\]

for all \((p_1, \ldots, p_k) \in \text{dom}(\phi)\). This implies that \(g\) realizes \(f\) w.r.t. \(\delta_1, \ldots, \delta_k\) and \(\gamma\).

\(\heartsuit\)

Propositions 2.3.2, Lemma 2.3.17 and Lemma 2.2.3 imply the Main Theorem for admissible representations of weak limit spaces.
Theorem 2.3.18 (Generalized Main Theorem)
For \( i \in \{1, \ldots, k+1\} \), let \( X_i = (X_i, \rightarrow_{X_i}) \) be a weak limit space and \( \delta_i \) be an admissible representation of \( X_i \).
Then a function \( f : \subseteq X_1 \times \ldots \times X_k \rightarrow X_{k+1} \) is relatively continuous with respect to \( \delta_1, \ldots, \delta_{k+1} \) if and only if \( f \) is sequentially continuous.

The Main Theorem in [Wei00] states that relative continuity w.r.t. admissible representations of countably based \( T_0 \)-spaces and topological continuity are equivalent. Since topological and sequential continuity coincide for partial multivariate functions between countably based spaces, Theorem 2.3.18 generalizes the Main Theorem in [Wei00].

In fact, we have proved a slightly stronger result: for guaranteeing equivalence of relative continuity and sequential continuity, it suffices that the representations of the source spaces are \textit{WeakLim}-quotient representations rather than admissible ones.

Proposition 2.3.19 (Sequential continuity vs. relative continuity)
For \( i \in \{1, \ldots, k\} \), let \( \delta_i \) be a \textit{WeakLim}-quotient representation of a weak limit space \( X_i = (X_i, \rightarrow_{X_i}) \). Let \( \gamma \) be an admissible representation of a weak limit space \( Y = (Y, \rightarrow_Y) \).
Then a function \( f : \subseteq X_1 \times \ldots \times X_k \rightarrow Y \) is relatively continuous with respect to \( \delta_1, \ldots, \delta_k \), and \( \gamma \) if and only if \( f \) is sequentially continuous.

The relationship between relative continuity on the one hand and sequential or topological continuity on the other hand is also discussed in Subsections 2.4.7 and 2.4.8.

2.4 Multirepresentations

By Proposition 2.3.13, admissible representations only exist for weak limit spaces satisfying the \( T_0 \)-property. In order to be able to compute on spaces which fail to be \( T_0 \), we introduce the concept of \textit{multirepresentations} and extend appropriately the notions of relative continuity, of relative computability, and of admissibility to multirepresentations (cf. Subsections 2.4.1, 2.4.2, 2.4.6). The class of spaces equipped with an admissible multirepresentation turns out to have nicer closure properties than the class of the spaces with admissible single–valued (functional) representations. Moreover, every weak limit space generated by a representation has an admissible multirepresentation, but not necessarily an admissible representation (cf. Corollary 4.3.3).

In Subsection 2.4.4 we introduce sequential continuity for correspondences and multirepresentations. The approximation structure induced by a multirepresentation on the represented set is discussed in Subsection 2.4.5. This is done by introducing three kinds of quotient spaces generated by a multirepresentation. As in the case of representations, it turns out that relative computability implies continuity w.r.t. the convergence relations of these quotient spaces (cf. Proposition 2.4.12). In Subsection 2.4.7 we generalize the Main Theorem 2.3.18 to multirepresentations. Some special aspects of admissible multirepresentations of topological spaces are discussed in Subsection 2.4.8.

2.4.1 The Definition of Multirepresentations
Informally, a multirepresentation (\textit{multi–valued representation}) of a set \( X \) is a “naming system” for \( X \) whose names, in contrast to those of single–valued representations,
may represent more than one object of \( X \). That means that we drop the property of ordinary representations of being functional and define a multirepresentation of a set \( X \) to be a surjective correspondence \( \delta : \subseteq \Sigma^\omega \rightarrow X \) (see Section 1.1). Hence, \( \delta \) is a triple \((\Sigma^\omega, X, G(\delta))\), where the graph \( G(\delta) \subseteq \Sigma^\omega \times X \) is a relation assigning to every \( p \in \Sigma^\omega \) the set
\[
\delta[p] = \{ x \in X \mid (p, x) \in G(\delta) \}
\]
of elements which are considered to be represented by \( p \). Surjectivity of \( \delta \) means \( X = \bigcup \{ \delta[p] \mid p \in \Sigma^\omega \} \), thus every element of \( X \) has some “name” \( p \in dom(\delta) \). By \( dom(\delta) \) we denote the domain of \( \delta \), i.e. the set \( \{ q \in \Sigma^\omega \mid (\exists y \in X)(q, y) \in G(\delta) \} \) of “names” of the multirepresentation \( \delta \).

**Definition 2.4.1 (Multirepresentation)**

A multirepresentation of a set \( X \) is a surjective correspondence \( \delta : \subseteq \Sigma^\omega \rightarrow X \), where \( \Sigma \) is an alphabet. If \( \delta \) is a multirepresentation of \( X \), we call the pair \((X, \delta)\) a represented space.

Multirepresentations encode information about the represented elements which does not necessarily suffice to identify them unambiguously. For example, the multirepresentation \( \gamma_{ub} : \mathbb{N}^\omega \rightarrow \mathbb{R} \) defined by
\[
\gamma_{ub}[p] := \{ x \in \mathbb{R} \mid x \leq p(0) \}
\]
yields as the only information an upper bound of the represented reals.

The following example presents a sequence of natural multirepresentations of \( \mathbb{R} \) from which the representation \( \varrho_\mathbb{R} \) from Example 2.3.8 can be derived.

**Example 2.4.2**

For \( m \in \mathbb{N} \), we define the multirepresentations \( \gamma_m : \subseteq \mathbb{Z}^\omega \rightarrow \mathbb{R} \) of the real numbers by
\[
\gamma_m[p] \ni x \iff |p(0) \cdot (\frac{1}{2})^m - x| \leq (\frac{1}{2})^m.
\]
Thus the only information given by a name \( p \in dom(\gamma_m) \) about any of the elements represented by \( p \) is a dyadic rational having at most distance \((\frac{1}{2})^m\) to each of these elements. It is easy to see that a sequence \((p_n)_n\) of \( \omega \)-words such that each \( p_i \) is a \( \gamma_i \)-name of a real number \( x \) determines \( x \) unambiguously. Thus the conjunction \((\bigwedge_{i \in \mathbb{N}} \gamma_i)\) (see Subsection 4.1.4) of the multirepresentations \( \gamma_0, \gamma_1, \ldots \) is a single–valued representation of \( \mathbb{R} \). This representation turns out to be computably equivalent (cf. Subsection 2.4.3) to the admissible representation \( \varrho_\mathbb{R} \) of the reals numbers from Example 2.3.8, because essentially the prefix \( q \leq^m \) of any name \( q \in dom(\varrho_\mathbb{R}) \) yields a dyadic rational number approximating \( \varrho_\mathbb{R}(p) \) with precision \((\frac{1}{2})^m\).

2.4.2 Realizability, Relative Continuity, and Relative Computability

Now we introduce realizability, relative continuity, and relative computability of functions with respect to multirepresentations in a similar way as in the case of representations (cf. Subsection 2.1.4). We then extend these notions from functions to correspondences.
For \( i \in \{1, \ldots, k+1\} \), let \( \delta_i : \Sigma^\omega_i \rightarrow X_i \) be a multirepresentation of a set \( X_i \). Let \( f : \subseteq X_1 \times \ldots \times X_k \rightarrow X_{k+1} \) be a function. We say that a function \( g : \subseteq \Sigma^\omega_1 \times \ldots \times \Sigma^\omega_k \rightarrow \Sigma^\omega_{k+1} \) is a \((\delta_1, \ldots, \delta_{k+1})\)-realization of \( f \) iff \( g \) transforms every name \( (p_1, \ldots, p_k) \) of an argument \((x_1, \ldots, x_k) \in \text{dom}(f)\) into a name of the image \( f(x_1, \ldots, x_k) \), i.e.,

\[
\delta_{k+1}[g(p_1, \ldots, p_k)] \ni f(x_1, \ldots, x_k)
\]

holds for all \((x_1, \ldots, x_k) \in \text{dom}(f)\) and all \( p_1, \ldots, p_k \) with \( \delta_1[p_1] \ni x_1, \ldots, \delta_k[p_k] \ni x_k \).

In this case we also say that \( g \) realizes \( f \) with respect to \( \delta_1, \ldots, \delta_{k+1} \). If \( \lambda : \subseteq \Sigma^\omega_1 \times \ldots \times \Sigma^\omega_k \rightarrow \Sigma^\omega_{k+1} \) is a \((\delta_1, \ldots, \delta_{k+1})\)-realization of \( f \) w.r.t. \( \delta_1, \ldots, \delta_{k+1} \), then we say that \( \lambda \) approximates \( f \) with respect to these multirepresentations.

Clearly, this definition generalizes the notion of realizability w.r.t. single-valued representations consistently (cf. Subsection 2.1.4). Note that \( \delta_{k+1}[g(p_1, \ldots, p_k)] \) may contain elements which are not the image of any argument in \( \delta_1[p_1] \times \ldots \times \delta_k[p_k] \).

We call \( f \) computable with respect to \( \delta_1, \ldots, \delta_{k+1} \) (or \( \delta_1, \ldots, \delta_{k+1} \)-computable) iff there is a computable function \( g : \subseteq \Sigma^\omega_1 \times \ldots \times \Sigma^\omega_k \rightarrow \Sigma^\omega_{k+1} \) realizing \( f \) w.r.t. \( \delta_1, \ldots, \delta_{k+1} \).

We call an element \( x \in X_1 \) computable with respect to \( \delta_1 \) or \( \delta_1 \)-computable iff it has a computable \( \delta_1 \)-name \( p \in \Sigma^\omega_1 \).

We call \( f \) (relatively) continuous with respect to \( \delta_1, \ldots, \delta_{k+1} \) (or \( \delta_1, \ldots, \delta_{k+1} \)-continuous) iff there is a continuous function \( g : \subseteq \Sigma^\omega_1 \times \ldots \times \Sigma^\omega_k \rightarrow \Sigma^\omega_{k+1} \) realizing \( f \) with respect to \( \delta_1, \ldots, \delta_{k+1} \). Since every computable \( \omega \)-word function is topologically continuous, relative computability implies relative continuity also in the case of multirepresentations.

Obviously, \( g \) is a \((\delta_1, \ldots, \delta_{k+1})\)-realization of the function \( f \) if and only if the following condition is fulfilled:

\[
(\forall p_1 \in \Sigma^\omega_1, \ldots, p_k \in \Sigma^\omega_k)((\delta_1[p_1] \times \ldots \times \delta_k[p_k]) \cap \text{dom}(f) \neq \emptyset \implies f[\delta_1[p_1] \times \ldots \times \delta_k[p_k]] \subseteq \delta_{k+1}[g(p_1, \ldots, p_k)])
\]

We use this condition to define realizability of correspondences and say that a function \( g : \subseteq \Sigma^\omega_1 \times \ldots \times \Sigma^\omega_k \rightarrow \Sigma^\omega_{k+1} \) realizes a correspondence \( F : \subseteq X_1 \times \ldots \times X_k \rightarrow X_{k+1} \) with respect to \( \delta_1, \ldots, \delta_{k+1} \) iff

\[
F[\delta_1[p_1] \times \ldots \times \delta_k[p_k]] \subseteq \delta_{k+1}[g(p_1, \ldots, p_k)]
\]

holds for all \( \omega \)-words \( p_1, \ldots, p_k \) with \( (\delta_1[p_1] \times \ldots \times \delta_k[p_k]) \cap \text{dom}(F) \neq \emptyset \). Informally, this means that \( g \) transforms every name of an argument \((x_1, \ldots, x_k)\) into a name \( q \) of all images of \((x_1, \ldots, x_k)\). Note that this name \( q \) may represent further elements which are not in the image of \((x_1, \ldots, x_k)\).

Relative computability and relative continuity of correspondences with respect to multirepresentations are defined analogously to the functional case. This definition of realizability of correspondence insures that Theorem 2.3.18 about the equivalence of relative continuity and sequential continuity can be extended not only from representations to multirepresentations, but also from realized functions to realized correspondences (cf. Theorem 2.4.25).
Example 2.4.3

We consider the multirepresentation \( \gamma_0 : \mathbb{Z}^\omega \rightarrow \mathbb{R} \) from Example 2.4.2.

For every \( c \in [-1;1] \), the function \( g : \mathbb{Z}^\omega \rightarrow \mathbb{Z}^\omega \) defined by \( g(p) := \omega^c \) is a \((\gamma_0, \gamma_0)\)-realization of the correspondence \( F_c : \mathbb{R} \rightarrow \mathbb{R} \) with \( F_c(x) := [-|c|; |c|] \) as well as of the constant real function \( x \mapsto c \).

Real addition by 1 is computable w.r.t. \( \gamma_0 \) via the function \( p \mapsto (p(0)+1) : \omega^\omega \). However, real addition by \( \frac{1}{3} \) is not relatively continuous w.r.t. \( \gamma_0 \), because on the one hand \( 0^\omega \) is a \( \gamma_0 \)-name of 0, but on the other there exists no \( \gamma_0 \)-name \( q \) of \( \frac{1}{3} \) (these are all contained in \( 0\mathbb{Z}^\omega \cup 1\mathbb{Z}^\omega \)) and no \( n \in \mathbb{N} \) satisfying

\[
\{ x + \frac{1}{3} | x \in \gamma_0[0^\infty\mathbb{Z}^\omega] \} \subseteq \gamma_0[q(0)\mathbb{Z}^\omega].
\]

Real addition is computable w.r.t. the multirepresentation \( \gamma_{ub} \) from Equation (2.8): a computable realizer is the function \( (p, q) \mapsto (p(0) + q(0)) : \omega^\omega \). Real subtraction, multiplication, and division fail to be \((\gamma_{ub}, \gamma_{ub}, \gamma_{ub})\)-continuous, because upper bounds of their arguments do not supply enough information to determine an upper bound of the result of these functions.

Another, more natural possibility to define relative computability of a correspondence \( F : X \rightrightarrows Y \) is the following: given two representations \( \delta \subseteq \Sigma^\omega \rightarrow X \) and \( \gamma \subseteq \Gamma^\omega \rightarrow Y \), we say that \( F \) is \emph{weakly computable} (\emph{weakly continuously realizable}) w.r.t. \( \delta \) and \( \gamma \) iff there is a computable (continuous) function \( g : \subseteq \Sigma^\omega \rightarrow \Gamma^\omega \) with

\[
(\forall p \in \text{dom}(\delta)) \gamma(g(p)) \in F[\delta(p)],
\]

i.e., \( g \) produces the name of one possible result of the “operation” \( F \) (cf. [Wei00, Definition 3.1.3]). However, there seems to be no natural notion of continuity of correspondences which characterizes those correspondences that are weakly continuously realizable w.r.t. admissible representations.

2.4.3 Continuous and Computable Reducibility

We extend the notions of continuous and computable reducibility (cf. Subsection 2.1.6) to multirepresentations in a straightforward way.

Let \( \delta_1 : \subseteq \Sigma_1^\omega \rightrightarrows X_1 \) and \( \delta_2 : \subseteq \Sigma_2^\omega \rightrightarrows X_2 \) be correspondences. We say that a function \( g : \subseteq \Sigma_1^\omega \rightarrow \Sigma_2^\omega \) translates or reduces \( \delta_1 \) to \( \delta_2 \) iff \( \delta_1[p] \subseteq \delta_2[g(p)] \) holds for every \( p \in \text{dom}(\delta_1) \). If \( \delta_1 \) and \( \delta_2 \) are multirepresentations, this means that \( g(p) \) must be a \( \delta_2 \)-name of at least all elements represented by \( p \) under \( \delta_1 \). If additionally \( X_1 \subseteq X_2 \) hold, then \( g \) translates \( \delta_1 \) to \( \delta_2 \) if and only if \( g \) is a \((\delta_1, \delta_2)\)-realization of the injection \( \iota : X_1 \rightarrow X_2 \) defined by \( \iota(x) := x \).

We call \( \delta_1 \) \emph{continuously reducible} (translatable) to \( \delta_2 \) \((\delta_1 \leq_c \delta_2 \) for short\) iff there is a continuous function \( g : \subseteq \Sigma_1^\omega \rightarrow \Sigma_2^\omega \) translating \( \delta_1 \) to \( \delta_2 \).

We call \( \delta_1 \) \emph{computably reducible} (translatable) to \( \delta_2 \) \((\delta_1 \leq \delta_2 \) for short\) iff there is a computable function \( g : \subseteq \Sigma_1^\omega \rightarrow \Sigma_2^\omega \) translating \( \delta_1 \) to \( \delta_2 \). Clearly, \( \delta_1 \leq_c \delta_2 \) implies \( \delta_1 \leq \delta_2 \).

Continuous and computable equivalence of \( \delta_1 \) and \( \delta_2 \) is defined by \( \delta_1 \equiv_c \delta_2 : \iff (\delta_1 \leq_c \delta_2 \land \delta_2 \leq_c \delta_1) \land \delta_1 \equiv \delta_2 : \iff (\delta_1 \leq \delta_2 \land \delta_2 \leq \delta_1) \), respectively.
2.4 Multirepresentations

These definitions of $\leq_{t}$, $\leq_{cp}$, $\equiv_{t}$, and $\equiv_{cp}$ are obviously consistent with those in Subsection 2.1.6 for single-valued representations. Since $h \circ g$ translates $\delta_1$ to the correspondence $\delta_3 \subseteq \Sigma_3^\omega \Rightarrow X_3$, if $g$ translates $\delta_1$ to $\delta_2$ and $h$ translates $\delta_2$ to $\delta_3$, these four relations are transitive.

Similar to the case of representations (cf. [Wei00]), continuously equivalent multirepresentations induce the same notion of relative continuity on the represented sets, whereas computably equivalent multirepresentations induce the same computability notion. Moreover, the relatively continuous and the relatively computable correspondences are closed under composition (as defined in Section 1.1).

**Lemma 2.4.4 (Closure properties of relative computability)**

*For every $i \in \mathbb{N}$ let $\delta_i \subseteq \Sigma_i^\omega \Rightarrow X_i$ and $\gamma_i \subseteq \Gamma_i^\omega \Rightarrow X_i$ be multirepresentations of a set $X_i$.*

1. If $\gamma_i \leq_{t} \delta_i$ for all $i \in \{1, \ldots, k\}$ and $\delta_{k+1} \leq_{t} \gamma_{k+1}$, then every $(\delta_1, \ldots, \delta_{k+1})$-continuous correspondence $F : \subseteq X_1 \times \ldots \times X_k \Rightarrow X_{k+1}$ is $(\gamma_1, \ldots, \gamma_{k+1})$-continuous.

2. If $\gamma_i \leq_{cp} \delta_i$ for all $i \in \{1, \ldots, k\}$ and $\delta_{k+1} \leq_{cp} \gamma_{k+1}$, then every $(\delta_1, \ldots, \delta_{k+1})$-computable correspondence $F : \subseteq X_1 \times \ldots \times X_k \Rightarrow X_{k+1}$ is $(\gamma_1, \ldots, \gamma_{k+1})$-computable.

3. Let $F_1 : \subseteq X_1 \Rightarrow X_2$ be a $(\delta_1, \delta_2)$-continuous correspondence and $F_2 : \subseteq X_2 \Rightarrow X_3$ be a $(\delta_2, \delta_3)$-continuous correspondence. Then the correspondence $F_2 \circ F_1$ is $(\delta_1, \delta_3)$-continuous.

4. Let $F_1 : \subseteq X_1 \Rightarrow X_2$ be a $(\delta_1, \delta_2)$-computable correspondence and $F_2 : \subseteq X_2 \Rightarrow X_3$ be a $(\delta_2, \delta_3)$-computable correspondence. Then the correspondence $F_2 \circ F_1$ is $(\delta_1, \delta_3)$-computable.

**Proof:**

1. For $i \in \{1, \ldots, k\}$, let $h_i : \subseteq \Gamma_i^\omega \rightarrow \Sigma_i^\omega$ be a continuous function translating $\gamma_i$ to $\delta_i$. Let $l : \subseteq \Sigma_{k+1}^\omega \rightarrow \Gamma_{k+1}^\omega$ be a continuous function translating $\delta_{k+1}$ to $\gamma_{k+1}$. Let $g : \subseteq \Sigma_{k+1}^\omega \times \ldots \times \Sigma_k^\omega \rightarrow \Sigma_{k+1}^\omega$ be a continuous function realizing the correspondence $F$ w.r.t. $\delta_1, \ldots, \delta_{k+1}$. We define $g' : \subseteq \Gamma_{k+1}^\omega \times \ldots \times \Gamma_{k+1}^\omega \rightarrow \Gamma_{k+1}^\omega$ by

$$g'(p_1, \ldots, p_k) := l(g(h_1(p_1), \ldots, h_k(p_k))).$$

Then the composition $g'$ is continuous. Let $(x_1, \ldots, x_k) \in \text{dom}(f)$, and let $p_1, \ldots, p_k$ be $\omega$-words with $(x_1, \ldots, x_k) \in \gamma_1[p_1] \times \ldots \times \gamma_k[p_k]$. Then $(x_1, \ldots, x_k) \in \delta_1[h_1(p_1)] \times \ldots \times \delta_k[h_k(p_k)]$. This implies

$$F[(x_1, \ldots, x_k)] \subseteq \delta_{k+1}[g(h_1(p_1), \ldots, h_k(p_k))]$$

$$\subseteq \gamma_{k+1}[l(g(h_1(p_1), \ldots, h_k(p_k)))] = \gamma_{k+1}[g'(p_1, \ldots, p_k)].$$

Hence $g'$ realizes $F$ with respect to $\gamma_1, \ldots, \gamma_{k+1}$.

2. Similar to (1) by using Lemma 2.1.3.
(3) Let \( g_1 : \Sigma_1^r \to \Sigma_2^q \) be a continuous function realizing \( F_1 \) w.r.t. \( \delta_1 \) and \( \delta_2 \), and let \( g_2 : \Sigma_2^q \to \Sigma_3^r \) be a continuous function realizing \( F_2 \) w.r.t. \( \delta_2 \) and \( \delta_3 \). Let \( p \in \text{dom}(F_2 \circ F_1 \circ \delta_1) \). Since \( F_1[\delta_1[p]] \neq \emptyset \), Equation (2.10) implies

\[
p \in \text{dom}(g_1) \text{ and } F_1[\delta_1[p]] \subseteq \delta_2[g_1(p)].
\]

Moreover, since \( F_2[\delta_2[g_1(p)]] \neq \emptyset \), we have

\[
g_1(p) \in \text{dom}(g_2) \text{ and } F_2[\delta_2[g_1(p)]] \subseteq \delta_3[g_2(g_1(p))].
\]

It follows

\[
F_2 \circ F_1[\delta_1[p]] = F_2[F_1[\delta_1[p]]] \subseteq F_2[\delta_2[g_1(p)]] \subseteq \delta_3[g_2(g_1(p))].
\]

Therefore the continuous function \( g_2 \circ g_1 \) realizes \( F_2 \circ F_1 \) w.r.t. \( \delta_1 \) and \( \delta_3 \).

(4) Similar to (3) by using Lemma 2.1.3.

\( \checkmark \)

2.4.4 Sequentially Continuous Correspondences

Now we introduce sequential continuity for correspondences between weak limit spaces. This notion of sequential continuity is defined in such a way that, analogously to the functional case (cf. Subsection 2.3.3), the equivalence of relative continuity and sequential continuity of functions is guaranteed by those multirepresentations that are \( \leq_1 \)-maximal (admissible) in the class of all multirepresentations being sequentially continuous in the extended sense.

**Definition 2.4.5 (Sequential continuity of correspondences)**

Let \( \mathcal{X} = (X, \rightarrow_X) \) and \( \mathcal{Y} = (Y, \rightarrow_Y) \) be weak limit spaces. A correspondence \( F : \subseteq X \rightrightarrows Y \) is defined to be (sequentially) continuous with respect to \( \rightarrow_X \) and \( \rightarrow_Y \) iff \( (x_n) \rightarrow_X x_\infty \) and \( (\forall n \in \mathbb{N}) y_n \in F[x_n] \) implies \( (y_n)_n \rightarrow_Y y_\infty \) for all sequences \( (x_n)_{n \leq \infty} \) in \( X \) and \( (y_n)_{n \leq \infty} \) in \( Y \). In this case we also say that \( F \) is \( (\rightarrow_X, \rightarrow_Y) \)-continuous and that \( F \) is a sequentially continuous correspondence between the spaces \( \mathcal{X} \) and \( \mathcal{Y} \). Sequential continuity of multivariate correspondences is defined accordingly.

\( \Box \)

Thus \( F : \subseteq X \rightrightarrows Y \) is a sequentially continuous correspondence between \( \mathcal{X} \) and \( \mathcal{Y} \) if and only if every sequence \( (y_n)_{n \leq \infty} \) that lies in the image of some convergent sequence \( (x_n)_{n \leq \infty} \) of \( \mathcal{X} \) is a convergent sequence of \( \mathcal{Y} \). Obviously, a function is sequentially continuous as a correspondence in the sense of Definition 2.4.5 if and only if it is sequentially continuous in the usual sense (cf. Subsection 2.2.1). Thus this definition generalizes the notion of sequential continuity for functions between weak limit spaces consistently.

Definition 2.4.5 yields a very strict notion of continuity. In particular, if \( F : \subseteq X \rightrightarrows Y \) is sequentially continuous, then, for \( x \in X \), every sequence of elements in \( F\{x\} \) converges
to every element in \( F[x] \). This implies that a lower and upper semi-continuous correspondence between two topological spaces need not be sequentially continuous w.r.t. the convergence relations induced by the two topologies. A counterexample is

\[
A := B := \{1, 2\}, \quad \tau_A := \{\emptyset, \{1, 2\}\}, \quad \tau_B := \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}
\]

and \( G : A \Rightarrow B \) defined by \( G[1] := G[2] := \{1, 2\} \). Conversely, every total sequentially continuous correspondence between two sequential topological spaces can be easily proved to be lower and upper semi-continuous. Moreover, sequential continuous correspondences having a \( T_0 \) weak limit space as codomain are necessarily single-valued.

**Lemma 2.4.6**

Let \( \mathfrak{X} = (X, \rightarrow_X) \) and \( \mathfrak{Y} = (Y, \rightarrow_Y) \) be weak limit spaces. Furthermore let \( F : \subseteq X \Rightarrow Y \) be a sequentially continuous correspondence.

1. Let \( x \in \text{dom}(F) \). Then every sequence \( (y_n)_n \) in \( F[x] \) converges to every element \( y_\infty \in F[x] \).
2. If \( \mathfrak{Y} \) has the \( T_0 \)-property, then \( F \) is a (partial) function.

**Proof:**

1. As the constant sequence \((x)_m\) converges to \( x \), every sequence \( (y_n)_n \in (F[x])^\mathbb{N} \) converges in \( \mathfrak{Y} \) to every element \( y_\infty \in F[x] \) by sequential continuity of \( F \).
2. Let \( x \in \text{dom}(F) \) and \( y_1, y_2 \in F[x] \). Since \( (x)_m \) converges to \( x \), the constant sequences \( (y_1)_n \) and \( (y_2)_n \) both converge to \( y_1 \) and to \( y_2 \) by sequential continuity of \( F \). By the \( T_0 \)-property of \( \mathfrak{Y} \), \( y_1 \) is equal to \( y_2 \).

We now show that composition of correspondences (cf. Section 1.1) preserves sequential continuity.

**Lemma 2.4.7** (Composition of corresp. preserves sequential continuity)

For \( i \in \{1, \ldots, k\} \), let \( F_i : \subseteq X_i \Rightarrow Y_i \) be a sequentially continuous correspondence between weak limit spaces \( \mathfrak{X}_i = (X_i, \rightarrow_{X_i}) \) and \( \mathfrak{Y}_i = (Y_i, \rightarrow_{Y_i}) \). Let \( \mathfrak{Z} = (Z, \rightarrow_Z) \) be a weak limit space and \( G : \subseteq Y_1 \times \ldots \times Y_k \Rightarrow Z \) be sequentially continuous.

Then the correspondence \( H := \subseteq X_1 \times \ldots \times X_k \Rightarrow Z \) defined by \( H[(x_1, \ldots, x_k)] := G[F_1[x_1] \times \ldots \times F_k[x_k]] \) is sequentially continuous.

**Proof:**

For \( i \in \{1, \ldots, k\} \), let \( (x_{i,n})_{n \leq \infty} \) be a convergent sequence of \( X_i \), and let \( (z_n)_{n \leq \infty} \in \prod_{n \in \mathbb{N}} H[(x_1, \ldots, x_k)] \). Then there are sequences \( (y_{1,n})_{n \leq \infty}, \ldots, (y_{k,n})_{n \leq \infty} \) with \( z_n \in G[(y_{1,n}, \ldots, y_{k,n})] \) and \( y_{i,n} \in F_i[x_{i,n}] \) for all \( n \in \mathbb{N} \) and \( i \in \{1, \ldots, k\} \). By sequential continuity of \( F_i \), \( (y_{i,n})_n \) converges to \( y_{i,\infty} \in \mathfrak{Y}_i \) for every \( i \in \{1, \ldots, k\} \). Hence \( (z_n)_n \) converges to \( z_\infty \in \mathfrak{Z} \) by sequential continuity of \( G \).

Similar to Lemma 2.2.2(2), one can prove the following property of continuous multirepresentations.

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11A correspondence \( G : A \Rightarrow B \) between two topological spaces \( \mathfrak{A} = (A, \tau_A) \) and \( \mathfrak{B} = (B, \tau_B) \) is called **lower semi-continuous** iff \( G^{-1}[O] \) is open in \( \mathfrak{A} \) for all open sets \( O \in \tau_B \). \( G \) is called **upper semi-continuous** iff \( G^{-1}[C] \) is closed in \( \mathfrak{A} \) for all closed sets \( C \subseteq B \).
Lemma 2.4.8
Let $\mathcal{X} = (X, \rightarrow_x)$ be a weak limit space, and let $\delta : \Sigma^\omega \rightharpoondown X$ be continuous w.r.t. $\rightarrow_{\Sigma^\omega}$ and $\rightarrow_x$. Then for every $p \in \text{dom}(\delta)$, every $x \in \delta[p]$ and every sequence $(z_n)_n$ that does not converge to $x$ in $\mathcal{X}$ there is some $k \in \mathbb{N}$ with $(\exists \infty n) z_n \notin \delta[p < k \Sigma^\omega]$.

Proof: Similar to the proof of Lemma 2.2.2(2).

Sequential continuity of a correspondence $F : \subseteq X_1 \times \ldots \times X_k \rightharpoondown Y$ can be characterized by sequential continuity of its associated function. The associated function $F^\mathcal{G} : \subseteq X_1 \times \ldots \times X_k \times Y \rightarrow Y$ of $F$ is defined by

$$\text{dom}(F^\mathcal{G}) := \{(x_1, \ldots, x_k, y) \in X_1 \times \ldots \times X_k \times Y \mid y \in F([x_1, \ldots, x_k])\}$$

and $F^\mathcal{G}(x_1, \ldots, x_k, y) := y$ for all $(x_1, \ldots, x_k, y) \in \text{dom}(F^\mathcal{G})$.

Lemma 2.4.9 (Characterization of sequential continuity)
For $i \in \{1, \ldots, k\}$, let $\mathcal{X}_i = (X_i, \rightarrow_{x_i})$ and $\mathcal{Y} = (Y, \rightarrow_y)$ be weak limit spaces.
Then a correspondence $F : \subseteq X_1 \times \ldots \times X_k \rightharpoondown Y$ is a sequentially continuous correspondence between $\mathcal{X}_1 \otimes \ldots \otimes \mathcal{X}_k$ and $\mathcal{Y}$ if and only if its associated function $F^\mathcal{G}$ is $(\rightarrow_{x_1}, \ldots, \rightarrow_{x_k}, \rightarrow_{ch}, \rightarrow_y)$-continuous, where $\rightarrow_{ch}$ denotes the chaotic convergence relation on $Y$ defined by $\rightarrow_{ch} := Y^N \times Y$.

Proof:
We restrict us to the case $k = 1$.

"$\Rightarrow$": Let $(x_n, y_n)_n \leq \infty$ be a convergent sequence of $\mathcal{X}_1 \otimes (Y, \rightarrow_{ch})$ that is contained in $\text{dom}(F^\mathcal{G})$. Then $(x_n)_n$ converges to $x_\infty$ in $\mathcal{X}_1$ by definition of $\mathcal{X}_1 \otimes (Y, \rightarrow_{ch})$. Since $F$ is a sequentially continuous correspondence between $\mathcal{X}_1$ and $\mathcal{Y}$, $(y_n)_n$ converges to $y_\infty$ in $\mathcal{Y}$. Thus $F^\mathcal{G}$ is $(\rightarrow_{x_1}, \rightarrow_{ch}, \rightarrow_y)$-continuous.

"$\Leftarrow$": Let $(x_n)_n \leq \infty$ be a convergent sequence of $\mathcal{X}_1$ and $(y_n)_n \leq \infty \in \prod_{n \in \mathbb{N}} F[x_n]$. Then the sequence $(x_n, y_n)_n \leq \infty$ is a convergent sequence of $\mathcal{X}_1 \otimes (Y, \rightarrow_{ch})$ and is contained in $\text{dom}(F^\mathcal{G})$. By continuity of $F^\mathcal{G}$, $(y_n)_n$ converges to $y_\infty$ in $\mathcal{Y}$. Thus $F$ is sequentially continuous.


\[2.4.5\hspace{1em}\text{Quotients Generated by Multirepresentations}\]

Now we define three kinds of quotient spaces generated by a multirepresentation $\delta$. They describe the approximation structure induced by $\delta$ on the represented set.

Let $\mathcal{X} = (X, \rightarrow_x)$ and $\mathcal{Y} = (Y, \rightarrow_y)$ be weak limit spaces. Let $Q : \subseteq X \rightharpoondown Y$ be a surjective correspondence between $X$ and $Y$. Like surjective functions (cf. Subsection 2.2.4), also the surjective correspondence $Q$ induces a final convergence relation on $Y$ with the property of being the finest (smallest) convergence relation on $Y$ making $Q$ sequentially continuous. This final convergence relation is equal to the convergence relation $\rightarrow_{x,Q}$ on $Y$ defined by

$$(y_n)_n -_{x,Q} y_\infty : \iff (\exists (x_n)_n \leq \infty)((x_n)_n -_x x_\infty \land (\forall n \in \mathbb{N}) Q[x_n] \ni y_n),$$
cf. Lemma 2.4.10(2). Clearly, this definition of \( -_{x,Q} \) is consistent with the definition of \( -_{x,q} \) for surjective functions \( q : \subseteq X \to Y \) in Subsection 2.2.4. We say that \( Q \) is a \textit{WeakLim–quotient} correspondence between the spaces \( \mathcal{X} \) and \( \mathcal{Y} \) iff \( -_{x,Q} \) is equal to \( -_{x,q} \). As in the functional case, the \textit{quotient space} \( (Y, -_{x,Q}) \) inherits from \( \mathcal{X} \) the property of being a weak limit space (cf. Lemma 2.4.10). If \( Q \) is a total surjective function, then \( (Y, -_{x,Q}) \) is called the \textit{WeakLim–quotient} of \( \mathcal{X} \) generated by \( Q \) (cf. Subsection 2.2.4).

We define the convergence relation \( \sim_{x,Q} \) induced by \( Q \) on \( Y \) by

\[
(y_n)_n \sim_{x,Q} y_\infty :\iff 
(\forall \varphi : \mathbb{N} \to \mathbb{N} \text{ strictly increasing})
(\exists \psi : \mathbb{N} \to \mathbb{N} \text{ strictly increasing}) \bigg( \exists (x_n)_{n \leq \infty} \in X^{\mathbb{N}} \bigg) \bigg( (x_n)_n \sim x_\infty \land (\forall n \in \mathbb{N}) \ y_{\varphi(\psi(n))} \in Q[x_n] \bigg).
\]

By Lemma 2.4.10(3), the \textit{quotient space} \( (Y, \sim_{x,Q}) \) is a limit space. Furthermore, \( \sim_{x,Q} \) is the finest (smallest) convergence relation on \( Y \) such that \( (Y, \sim_{x,Q}) \) is a limit space and \( Q \) is a sequentially continuous correspondence between \( \mathcal{X} \) and \( (Y, \sim_{x,Q}) \). We call \( Q : \subseteq X \to Y \) a \textit{Lim–quotient} correspondence between \( \mathcal{X} \) and \( \mathcal{Y} \) iff the convergence relation \( \sim_{x,Q} \) on \( Y \) defined is equal to \( -_{x,q} \). For total surjective functions \( q : X \to Y \), the quotient space \( (Y, \sim_{x,Q}) \) is called the \textit{Lim–quotient of} \( \mathcal{X} \) \textit{generated by} \( q \). The regular epimorphisms of the category \textit{Lim} are exactly those morphism whose underlying total functions are \textit{Lim–quotient} functions (cf. [MS02]).

Now let \( \tau_X \) and \( \tau_Y \) be topologies on \( X \) and \( Y \), respectively. Consistently with the definition for functions, we define the \textit{final topology} \( \tau_{\tau_X,Q} \) \textit{induced by} \( Q \) on \( Y \) by

\[
\tau_{\tau_X,Q} := \left\{ V \subseteq Y \mid (\exists U \in \tau_X)(Q^{-1}[V] \subseteq U \text{ and } Q[U] \subseteq V) \right\}.
\]

(2.12)

One can easily verify that \( \tau_{\tau_X,Q} \) is actually a topology on \( Y \). If \( \tau_{\tau_X,Q} \) is equal to \( \tau_Y \), then we call \( Q \) a \textit{Top–quotient} correspondence between the spaces \( (X, \tau_X) \) and \( (Y, \tau_Y) \).

**Lemma 2.4.10 (Quotient spaces generated by correspondences)**

Let \( \mathcal{X} = (X, -_x) \) and \( \mathcal{Y} = (Z, -_\gamma) \) be weak limit spaces. Let \( Q : \subseteq X \to Y \) be a surjective correspondence onto a set \( Y \).

1. The quotient space \( (Y, -_{x,Q}) \) is a weak limit space.
2. A correspondence \( F : \subseteq Y \to Z \) is \( (-_{x,Q}, -_\gamma) \)-continuous if and only if \( F \circ Q \) is \( (-_x, -_\gamma) \)-continuous.
3. The quotient space \( (Y, \sim_{x,Q}) \) is equal to the limit space \( \mathcal{L}(Y, -_{x,Q}) \).
4. If \( \mathcal{Y} \) is a limit space, then a correspondence \( F : \subseteq Y \to Z \) is \( (-_{x,Q}, -_\gamma) \)-continuous if and only if \( F \circ Q \) is \( (-_x, -_\gamma) \)-continuous.
5. If \( Q \) is total, then the final topology \( \tau_{\text{seq}(\mathcal{X}),Q} \) induced by \( Q \) is equal to \( \text{seq}(-_{x,Q}) \).
6. The implications \((y_n)_n -_{x,Q} y_\infty \Rightarrow (y_n)_n \sim_{x,Q} y_\infty \Rightarrow (y_n)_n -_{\text{seq}(\mathcal{X}),Q} y_\infty \) hold for all sequences \((y_n)_n \leq \infty \) in \( Y \).
Proof:

(1) The proof is similar to the one of Proposition 2.2.4(1).

(2) $$\Rightarrow$$: From the definition of $$\neg_x \, \, \neg_x$$ it follows that $$Q$$ is continuous with respect to $$\neg_x$$ and $$\neg_x$$. Thus $$F \circ Q$$ is sequentially continuous by Lemma 2.4.7.

$$\Leftarrow$$: Let $$(y_n)_n$$ converge to $$y_\infty$$ w.r.t. $$\neg_x \, \, \neg_x$$, and let $$(z_n)_n \in N$$ be a sequence with $$z_n \in F[y_n]$$ for all $$n \in N$$. There is a convergent sequence $$(x_n)_n \in N$$ with $$(y_n)_n \in \prod_{n \in N} Q[x_n]$$. Since $$F \circ Q$$ is $$(\neg_x, \neg_3)$$-continuous and $$(\forall n \in N) \, z_n \in (F \circ Q)[x_n]$$ holds, $$(z_n)_n \in N$$ is a convergent sequence of 3.

(3) Let $$(y_n)_n \leq N$$ be a generalized sequence in $$Y$$. Then we have

$$\iff (\exists \varphi : N \to N \text{ strictly increasing}) \, (\exists \psi : N \to N \text{ strictly increasing})
\iff (\exists (x_n)_n \leq X) \, ((x_n)_n \to x_\infty \wedge (\forall n \in N) \, y_{\varphi(n)} \in Q[x_n])
\iff (\exists \varphi : N \to N \text{ strictly increasing})
\iff (\exists (y_{\psi(n)})_n \to \neg_x \, \neg_x \circ \psi \circ (y_n)_n \to x_\infty \, \neg_x y_\infty)

This shows $$(Y, \, \neg_x \circ \psi \circ (y_n)_n \to x_\infty \, \neg_x y_\infty) = L(Y, \, x_\infty \, \neg_x y_\infty)$$.

(4) $$\Rightarrow$$: Since $$Q$$ is $$(\neg_x, \neg_x)$$-continuous by (2) and $$(\neg_x, \neg_x) \subseteq \neg_x$$ holds by (6), $$Q$$ is $$(\neg_x, \neg_x)$$-continuous. This implies by Lemma 2.4.7 that $$F \circ Q$$ is sequentially continuous w.r.t. $$\neg_x$$ and $$\neg_3$$.

$$\Leftarrow$$: Let $$(y_n)_n$$ converge to $$y_\infty$$ w.r.t. $$(\neg_x, \neg_x)$$, and let $$(z_n)_n \in N$$ be a sequence with $$z_n \in F[y_n]$$ for all $$n \in N$$. Let $$\varphi : N \to N$$ be strictly increasing. Then there is a convergent sequence $$(x_n)_n \leq X$$ of $$X$$ and a strictly increasing function $$\psi : N \to N$$ with $$(\forall n \in N) \, y_{\varphi(n)} \in Q[x_n]$$. Since $$F \circ Q$$ is sequentially continuous, $$(z_{\psi(n)})_n \leq N$$ is a convergent sequence of 3. This implies $$(z_n)_n \to \neg_3 z_\infty$$, because 3 satisfies Axiom (L3). Hence $$F$$ is continuous w.r.t. $$(\neg_x, \neg_x)$$ and $$\neg_3$$.

(5) $$\tau_{\text{seq}(X), Q} \subseteq \text{seq}(\neg_x, Q)$$: Let $$V \in \tau_{\text{seq}(X), Q}$$. Then there exists a set $$U \in \text{seq}(X)$$ with $$Q^{-1}[V] \subseteq U$$ and $$Q[U] \subseteq V$$. Let $$(y_n)_n \leq N$$ be a convergent sequence of $$(Y, \neg_x \circ \psi \circ (y_n)_n \to x_\infty \, \neg_x \circ \psi \circ (y_n)_n \to x_\infty \, \neg_x y_\infty)$$, i.e. $$V \in \text{seq}(\neg_x)$$. Then there is some $$y_\infty \in Q[x_\infty] \cap V$$. As $$Q$$ is total, there exists a sequence $$(y_n)_n$$ with $$(\forall n \in N) \, y_n \in Q[x_n]$$ and is sequentially open in $$U$$, i.e. $$(y_n)_n$$ converges to $$y_\infty \in V$$ w.r.t. $$(\neg_x, \neg_x), n \in N$$ such that $$(\forall n \in N) \, y_n \in V$$. It follows that $$(x_n)_n$$ is sequentially open in $$U$$. Hence $$U$$ is sequentially open. Let $$y \in Q[U]$$. Then there are some $$x \in Q^{-1}[y]$$ and some $$z \in Q[x]$$ with $$z \in V$$. As the constant sequence $$(y)_\mu$$ converges to $$z$$ w.r.t. $$(\neg_x, \neg_x)$$ and $$V$$ is sequentially open, $$y$$ is contained in $$V$$. Hence $$Q[U] \subseteq V$$. We conclude $$V \in \tau_{\text{seq}(X), Q}$$. 


(6) By Lemma 2.2.8 and by (4), we have $\rightarrow_{\mathcal{L}(Y, \rightarrow_{\mathcal{L}(X, Q)})} = \rightarrow_{\mathcal{L}(X, Q)}$.

Let $(y_n)_n$ be a sequence that converges to $y_\infty$ w.r.t. $\rightarrow_{\mathcal{L}(X, Q)}$. Let $V \in \tau_{\text{seq}(X), Q}$ be an open set with $y_\infty \in V$. Then there is some $U \in \text{seq}(X)$ with $Q^{-1}[V] \subseteq U$ and $Q[U] \subseteq V$.

Suppose for contradiction $y_n \notin V$ for infinitely many $n \in \mathbb{N}$. Then there is some strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ with $(\forall n \in \mathbb{N}) y_{\varphi(n)} \notin V$. By definition of $\rightarrow_{\mathcal{L}(X, Q)}$ there is a convergent sequence $(x_n)_{n \leq \infty}$ of $\mathcal{X}$ and a strictly increasing function $\psi : \mathbb{N} \to \mathbb{N}$ with $y_\infty \in Q[x_\infty]$ and $(\forall n \in \mathbb{N}) y_{\psi(n)} \in Q[x_n]$. Since $x_\infty \in Q^{-1}[V] \subseteq U$, there is some $n_0 \in \mathbb{N}$ with $(\forall n \geq n_0) x_n \in U$. It follows $y_n \in Q[U] \subseteq V$ for all $n \geq n_0$, a contradiction.

Hence $(y_n)_n$ is eventually in $V$. We conclude that $(y_n)_n$ converges to $y_\infty$ in $(Y, \tau_{\text{seq}(X), Q})$. Therefore $\rightarrow_{\mathcal{L}(Y, \rightarrow_{\mathcal{L}(X, Q)})}$.
Chapter 2. Admissibility

satisfies $\delta^{-1}[U] \subseteq O$ and $\delta[O] \subseteq U$. We conclude $U \in \tau_\delta$.

From Lemma 2.2.7(4) it follows that $\tau_\delta = \text{seq}(\rightarrow_\gamma)$ is a sequential topology.

(4) Let $g : \Sigma^* \rightarrow \Gamma^*$ be a continuous function translating $\delta$ to $\gamma$. Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $(Y, \rightarrow_\gamma)$. Then there is a convergent sequence of $(\Sigma^*, \rightarrow_\gamma)$ with $(\forall n \in \mathbb{N}) x_n \in \delta[p_n]$. As $(g(p_n))_{n \leq \infty}$ is a convergent sequence of $(\Gamma^*, \rightarrow_\gamma)$ by continuity of $g$ and since $x_n \in \gamma[g(p_n)]$ holds for all $n \in \mathbb{N}$, $(x_n)_n$ converges to $x_\infty$ in $(X, \rightarrow_\gamma)$. Hence $\rightarrow_\delta \subseteq \rightarrow_\gamma$.

Obviously, $\rightarrow_\delta \subseteq \rightarrow_\gamma$ implies $\rightarrow_{\mathcal{L}(\rightarrow_\delta)} \subseteq \rightarrow_{\mathcal{L}(\rightarrow_\gamma)}$. By Lemma 2.4.10(3) we obtain $\sim_\delta \subseteq \sim_\gamma$.

Let $O \in \text{seq}(\rightarrow_\gamma)$. Let $(x_n)_n$ be a sequence that converges in $(X, \rightarrow_\gamma)$ to some element $x_\infty \in O$. Since $\rightarrow_\gamma \subseteq \rightarrow_\delta$ holds, $(x_n)_n$ is eventually in $O$. Thus $O \in \text{seq}(\rightarrow_\delta)$ and $\text{seq}(\rightarrow_\delta) \subseteq \text{seq}(\rightarrow_\gamma)$. From (3) we conclude $\tau_\gamma \subseteq \tau_\delta$ which implies $\sim_{\mathcal{L}(\rightarrow_\delta)} \subseteq \sim_{\mathcal{L}(\rightarrow_\gamma)}$.

\[\square\]

Similar to the case of representations (cf. Lemma 2.2.3), relative computability with respect to multirepresentations implies sequential continuity with respect to the two kinds of final convergence relations induced by the involved multirepresentations. Moreover, relatively computable total functions are sequentially continuous functions between the generated topological spaces. Topological continuity of relatively computable functions w.r.t. the induced final topologies will be discussed in Subsection 2.4.8.

**Proposition 2.4.12 (Relative computability implies sequential continuity)**

For $i \in \{1, \ldots, k\}$, let $\delta_i : \Sigma_i^* \equiv X_i$ and $\gamma : \Gamma^* \equiv Y$ be multirepresentations.

Let $F : \equiv X_1 \times \ldots \times X_k \rightarrow Y$ be a correspondence which is relatively continuous or computable w.r.t. $\delta_1, \ldots, \delta_k$, and $\gamma$. Then:

1. The correspondence $F$ is sequentially continuous w.r.t. the convergence relations $\rightarrow_{\delta_1}, \ldots, \rightarrow_{\delta_k}$, and $\rightarrow_\gamma$.

2. The correspondence $F$ is sequentially continuous w.r.t. the convergence relations $\sim_{\delta_1}, \ldots, \sim_{\delta_k}$, and $\sim_\gamma$.

3. If $F$ is total and single–valued (i.e., $F$ is a total function), then $F$ is sequentially continuous w.r.t. the convergence relations $\rightarrow_{\delta_1}, \ldots, \rightarrow_{\delta_k}$, and $\rightarrow_\gamma$.

**Proof:**

1. Let $pr_i$ denote the $i$–th projection function from $X_1 \times \ldots \times X_k$ onto $X_i$. Let $(x_n)_{n \leq \infty}$ be a sequence in $\text{dom}(F)$ such that, for every $i \in \{1, \ldots, k\}$, $(pr_i(x_n))_n$ converges to $pr_i(x_\infty)$ in $(X_i, \rightarrow_{\delta_i})$. Let $(y_n)_{n \leq \infty} \in \prod_{i \in \mathbb{N}} F[x_n]$. For every $i \in \{1, \ldots, k\}$, there is a convergent sequence $(p_{i,n})_{n \leq \infty} \in \delta_i[p_{i,n}]$ for all $n \in \mathbb{N}$. There is a continuous function $g : \equiv \Sigma_1^* \times \ldots \times \Sigma_k^* \rightarrow \Gamma^*$ realizing $F$ w.r.t. $\delta_1, \ldots, \delta_k$, and $\gamma$. It follows

\[
y_n \in F[x_n] \subseteq F[\delta_1[p_{1,n}] \times \ldots \times \delta_k[p_{k,n}]] \subseteq \gamma[g(p_{1,n}, \ldots, p_{k,n})]
\]

holds for all $n \in \mathbb{N}$. By sequential continuity of $\gamma$ and $g$, the sequence $(y_n)_n$ converges to $y_\infty$ in $(Y, \rightarrow_\gamma)$. We conclude that $F$ is a sequentially continuous function between the spaces $(X_1, \rightarrow_{\delta_1}), \ldots, (X_k, \rightarrow_{\delta_k})$, and $(Y, \rightarrow_\gamma)$.
(2) By (1), $F$ is a sequentially continuous correspondence between the weak limit spaces $(X_1, \rightarrow_{\gamma_1}), \ldots, (X_k, \rightarrow_{\gamma_k})$, and $(Y, \rightarrow)$. By Lemma 2.4.9, the associated function $F^\gamma$ is continuous w.r.t. $\rightarrow_{\gamma_1}, \ldots, \rightarrow_{\gamma_k}, \rightarrow_{\gamma_k}, \rightarrow_{\gamma_k}$, and $\rightarrow$, where $\rightarrow_{\gamma_k}$ denotes the chaotic convergence relation on $Y$. Theorem 2.2.10 implies that $F^\gamma$ is a sequentially continuous function between the limit spaces $\mathcal{L}(X_1, \rightarrow_{\gamma_1}), \ldots, \mathcal{L}(X_k, \rightarrow_{\gamma_k})$, $\mathcal{L}(Y, \rightarrow_{\gamma_k})$, and $\mathcal{L}(Y, \rightarrow)$. By Lemma 2.4.11, we have $\mathcal{L}(X_i, \rightarrow_{\gamma_k}) = (X_i, \rightarrow_{\gamma_k})$ and $\mathcal{L}(Y, \rightarrow) = (Y, \rightarrow_{\gamma_k})$ for every $i \in \{1, \ldots, k\}$. Clearly, $\mathcal{L}(Y, \rightarrow_{\gamma_k}) = (Y, \rightarrow_{\gamma_k})$. Thus $F^\gamma$ is continuous w.r.t. $\rightarrow_{\gamma_1}, \ldots, \rightarrow_{\gamma_k}, \rightarrow_{\gamma_k}, \rightarrow_{\gamma_k}$ and $\rightarrow_{\gamma_k}$. We conclude by Lemma 2.4.9 that $F$ is sequentially continuous w.r.t. $\rightarrow_{\gamma_1}, \ldots, \rightarrow_{\gamma_k}$, and $\rightarrow_{\gamma_k}$.

(3) By (1), $F$ is a sequentially continuous function between the weak limit spaces $(X_1, \rightarrow_{\gamma_1}), \ldots, (X_k, \rightarrow_{\gamma_k}),$ and $(Y, \rightarrow)$. Theorem 2.2.10 implies that $F$ is a sequentially continuous function between $T(X_1, \rightarrow_{\gamma_1}), \ldots, T(X_k, \rightarrow_{\gamma_k}),$ and $T(Y, \rightarrow)$. By Lemma 2.4.11, we have $T(Y, \rightarrow) = (Y, \tau_\gamma)$ and $T(X_i, \rightarrow_{\gamma_k}) = (X_i, \tau_\gamma)$ for every $i \in \{1, \ldots, k\}$. Hence $F$ is sequentially continuous w.r.t. $\rightarrow_{\gamma_1}, \ldots, \rightarrow_{\gamma_k}, \rightarrow_{\gamma_k}$, and $\rightarrow_{\gamma_k}$.

Elements having a common name $p$ under a multirepresentation $\delta : \subseteq \Sigma^\omega \rightarrow X$ can not be distinguished from each other. The induced convergence relations $\rightarrow_\delta$ and $\sim_\delta$, reflect that fact by letting converge every sequence in $\delta[p]$ to every element contained in $\delta[p]$, and the final topology $\tau_\delta$ equips the elements in $\delta[p]$ with the same neighbourhoods. Thus none of the spaces $(X, \rightarrow_\gamma), (X, \sim_\gamma),$ and $(X, \tau_\delta)$ fulfill the $T_0$–property, if $\delta$ is not single–valued.

We now consider several examples.

**Example 2.4.13**

One easily verifies that the convergence relation $\rightarrow_{\gamma_m}$ induced by the multirepresentation $\gamma_m$ from Example 2.4.2 is given by

$$ (x_n)_n \rightarrow_{\gamma_m} x_\infty \iff (\exists z \in \mathbb{Z})(\forall^\infty n \in \overline{\mathbb{N}})|z \cdot (\frac{1}{2})^m - x_n| \leq (\frac{1}{2})^m. \quad (2.13) $$

Every sequence contained in an interval of length $(\frac{1}{2})^m$ converges to every number in this interval. Hence the weak limit space $(\mathbb{R}, \rightarrow_{\gamma_m})$ does not satisfy the $T_0$–property. Furthermore, $(\mathbb{R}, \rightarrow_{\gamma_m})$ is not a limit space, because on the one hand the constant sequences $(3(\frac{1}{2})^m + 1)_\mu$ and $(7(\frac{1}{2})^m + 1)_\mu$ both converge to $5(\frac{1}{2})^m + 1$, but on the other hand the alternating sequence $(3(\frac{1}{2})^m + 1, 7(\frac{1}{2})^m + 1, 3(\frac{1}{2})^m + 1, 7(\frac{1}{2})^m + 1, \ldots)$ does not. This contradicts Axiom (L3). Note that the two reals $3(\frac{1}{2})^m + 1, 7(\frac{1}{2})^m + 1$ have no common name, although both share a common name with $5(\frac{1}{2})^m + 1$.

The convergence relation $\sim_{\gamma_m}$ is given by

$$ (x_n)_n \sim_{\gamma_m} x_\infty \iff (\forall^\infty n \in \mathbb{N})x_n \in \left[\frac{x_{\infty} - 2^m}{2^m}; \frac{x_{\infty} - 2^m + 2}{2^m}\right]. $$

This can be shown by means of Equivalence (2.13) and Lemma 2.4.11(2).

Let $O \subseteq \text{seq}(\sim_{\gamma_m})$ and $x \in O$. As there are integers $z_1, z_2$ with $z_1 \cdot (\frac{1}{2})^m \in [x - (\frac{1}{2})^m; x]$ and $z_2 \cdot (\frac{1}{2})^m \in [x; x + (\frac{1}{2})^m]$, every element $y \in [x - (\frac{1}{2})^m; x + (\frac{1}{2})^m]$ satisfies $(y)_\mu \sim_{\gamma_m} x$. This implies $[x - (\frac{1}{2})^m; x + (\frac{1}{2})^m] \subseteq O$, because $O$ is sequentially open. By induction it
follows \((\forall n \in \mathbb{N}) [x - n \cdot (\frac{1}{2})^m; x + n \cdot (\frac{1}{2})^m] \subseteq O\). Thus \(O = \mathbb{R}\). We conclude by Lemma 2.4.11(3) that the final topology of \(\gamma_m\) is the indiscrete topology on \(\mathbb{R}\).

\[\blacksquare\]

**Example 2.4.14**

The convergence relation \(\rightarrow_{\text{ub}}\) induced by the multirepresentation \(\gamma_{\text{ub}}\) of \(\mathbb{R}\) from Equation (2.8) is given by

\[
(x_n)_n \rightarrow_{\text{ub}} x_\infty \iff (\exists b \in \mathbb{N})(\forall n \in \mathbb{N}) x_n \leq b.
\]

Thus every sequence \((x_n)_n\) converges in the generated quotient space \((\mathbb{R}, \rightarrow_{\text{ub}})\) either to every real number or to none. The space \((\mathbb{R}, \rightarrow_{\text{ub}})\) turns out to be a limit space, i.e. \(\sim_{\text{ub}} = \rightarrow_{\text{ub}}\) by Lemma 2.4.11. The associated topology of \((\mathbb{R}, \rightarrow_{\text{ub}})\) is the indiscrete topology on \(\mathbb{R}\), because every constant sequence converges to every real in \((\mathbb{R}, \rightarrow_{\text{ub}})\). Hence the final topology of \(\gamma_{\text{ub}}\) is the indiscrete topology by Lemma 2.4.11(3).

\[\blacksquare\]

**Example 2.4.15**

We consider the “lower” representations \(\varrho_{\prec, \varrho_{\prec^+}} : \subseteq \mathbb{N}^\omega \rightarrow \mathbb{R}\) of the real numbers defined by

\[
\varrho_{\prec}(p) = x :\iff x = \sup \{\nu_Q(p(i)) | i \in \mathbb{N}\},
\]

\[
\varrho_{\prec^+}(p) = x :\iff x = \varrho_{\prec}(p^{>0}) \land x \leq p(0)
\]

for all \(p \in \mathbb{N}^\omega\) and \(x \in \mathbb{R}\). The names of both representations yield a sequence of rational numbers which approximate the represented reals from below. As an additional information, a \(\varrho_{\prec^+}\)-name encodes some upper bound of the represented number.

We prove that the final topologies of both representations are equal to the lower topology \(\tau_{\prec} := \big\{ (x; \infty) | x \in \mathbb{R} \cup \{0, \mathbb{R}\} \big\} \) on the reals.

\(\tau_{\prec} \subseteq \tau_{\varrho_{\prec}}\): Let \(x \in \mathbb{R}\). We define the set \(W \subseteq \mathbb{N}^*\) by

\[
W := \{wa | w \in \mathbb{N}^*, a \in \mathbb{N}, \nu_Q(a) > x\}.
\]

One easily verifies \(\varrho_{\prec}^{-1}[(x; \infty)] = W \mathbb{N}^\omega \cap \text{dom}(\varrho_{\prec})\). Hence \((x; \infty) \in \tau_{\varrho_{\prec}}\). We obtain \(\tau_{\prec} \subseteq \tau_{\varrho_{\prec}}\).

\(\tau_{\varrho_{\prec}} \subseteq \tau_{\varrho_{\prec^+}}\): Obviously, \(\varrho_{\prec^+}\) can be continuously translated to \(\varrho_{\prec}\). Therefore \(\tau_{\varrho_{\prec}} \subseteq \tau_{\varrho_{\prec^+}}\) holds by Lemma 2.4.11(4).

\(\tau_{\varrho_{\prec^+}} \subseteq \tau_{\prec}\): For every \(y \in \mathbb{R}\) let \(p_y\) be a \(\varrho_{\prec}\)-name of \(y\).

Let \(O \in \tau_{\varrho_{\prec^+}}\) and \(x \in O\). There is some \(W \subseteq \mathbb{N}^*\) with \(\varrho_{\prec}^{-1}(O) = W \mathbb{N}^\omega \cap \text{dom}(\varrho_{\prec})\) such that the length of every word in \(W\) is at least \(2\).

We choose some \(q \in \mathbb{N}^\omega\) satisfying \(q(0) > x\) and \(x - (\frac{1}{2})^n < \nu_Q(q(n)) < x - (\frac{1}{2})^{n+1}\) for all \(n \geq 1\). Then \(\varrho_{\prec}(q) = x\). Hence there is some \(n_1 \geq 2\) with \(q^{<n_1} \in W\). In order to show that \(U := (\nu_Q(q(n_1 - 1)); \infty)\) is a subset of \(O\), let \(y \in U\).

Case \(y < x\): Then the \(\omega\)-word \(q' := q^{<n_1}::p_y\) is a \(\varrho_{\prec^+}\)-name of \(y\). Since \(q' \in W \mathbb{N}^\omega\), we have \(y \in O\).

Case \(y \geq x\): The \(\omega\)-word \(r := \max \{[y], 0\}::p_x\) is a \(\varrho_{\prec^+}\)-name of \(x\). Hence there is some \(n_2 \geq 2\) with \(r^{<n_2} \in W\). Since \(r(0) \geq y\), the \(\omega\)-word \(r' := r^{<n_2}::p_y\) satisfies \(\varrho_{\prec^+}(r') = y\). As \(r' \in W \mathbb{N}^\omega\), this implies \(y \in O\).
2.4 Multirepresentations

Hence we have $U \subseteq O$. Since $U \in \tau_<$, this implies that $O$ is a neighbourhood of $x$ w.r.t. the lower topology $\tau_<$. We conclude $O \in \tau_<$.

Although both representations have the same final topology, they do not induce the same convergence relation. In fact one can show that $\rightarrow_{\rho_<$} is induced by the lower topology $\tau_<$ and that $\rightarrow_{\rho_<$} satisfies

$$(x_n)_n \rightarrow_{\rho_<} x_\infty \iff (x_n)_n \rightarrow_{\tau_<} x_\infty \land (\exists b \in \mathbb{N})(\forall n \in \mathbb{N})x_n \leq b.$$  

Both representations turn out to be admissible. The admissibility of $\rho_<$ follows from Theorem 2.3.12 and the fact that $\rho_<$ is continuously equivalent to the “standard” representation $\rho_<$ from [Wei00, Definition 4.1.3] of the countably based space $(\mathbb{R}, \tau_<)$. The representation $\rho_{<*}$ is continuously equivalent to the “conjunction” of $\rho_<$ and the multirepresentation $\gamma_{ub}$ considered in Example 2.4.14. By Example 2.4.22, $\gamma_{ub}$ is admissible multirepresentation of the limit space $(\mathbb{R}, \rightarrow_{\gamma_{ub}})$ in the sense of Definition 2.4.16. By Proposition 4.1.7, conjunction of multirepresentations preserves admissibility. Hence $\rho_{<*}$ is admissible.

Since $\rightarrow_{\rho_{<*}}$ is not induced by the final topology of $\rho_{<*}$, which is equal to the associated topology $\text{seq}(\rightarrow_{\rho_{<*}})$ by Lemma 2.4.11(3), it follows that $(\mathbb{R}, \rightarrow_{\rho_{<*}})$ is not a topological weak limit space. Nevertheless, $(\mathbb{R}, \rightarrow_{\rho_{<*}})$ is a limit space. This fact demonstrates again that it is useful to consider not only topological spaces, but also (weak) limit spaces. The representation $\rho_{<*}$ has been used by V. Brattka to prove computable versions of the Uniform Boundedness Theorem (cf. [Bra02]).

2.4.6 Admissible Multirepresentations

Now we define admissibility of multirepresentations in such a way that sequentially continuous functions are relatively continuous with respect to admissible multirepresentations. We simply adapt the definition for the single–valued case (cf. Definition 2.3.1).

**Definition 2.4.16 (Admissible multirepresentations)**

Let $\mathcal{X} = (X, \rightarrow_X)$ be a weak limit space, and let $\delta : \Sigma^\omega \rightrightarrows X$ be a multirepresentation of $X$.

1. We call $\delta$ admissible with respect to $\rightarrow_X$ or $\rightarrow_X$–admissible iff $\delta$ is sequentially continuous and every sequentially continuous correspondence $\phi : \subseteq \Sigma^\omega \rightrightarrows X$ is continuously translatable (reducible) to $\delta$ (i.e. $\phi \leq_t \delta$).

2. We call $\delta$ an admissible multirepresentation of the space $\mathcal{X}$ iff $\delta$ is admissible with respect to $\rightarrow_X$.

3. We call $\delta$ an admissible multirepresentation of the set $X$ iff $\delta$ is an admissible representation of the generated weak limit space $(X, \rightarrow_\delta)$.

4. Let $\tau$ be a topology on $X$. Then we call $\delta$ admissible with respect to $\tau$ or $\tau$–admissible iff $\delta$ is admissible with respect to the convergence relation $\rightarrow_\tau$ induced by the topology $\tau$. 


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This definition extends Definition 2.3.1 consistently: If $\delta : \subseteq \Sigma^\omega \rightarrow X$ is a (single-valued) admissible representation of a weak limit space $\mathcal{X} = (X, \rightarrow_x)$ in the sense of Definition 2.3.1, then $\mathcal{X}$ has the $T_0$–property by Proposition 2.3.13, hence any sequentially continuous correspondence $\phi : \subseteq \Sigma^\omega \rightrightarrows X$ is single–valued by Lemma 2.4.6 and thus continuously translatable to $\delta$ by Definition 2.3.1. Proposition 2.3.13 and Lemma 2.4.6 imply furthermore:

Lemma 2.4.17
An admissible multirepresentation of a weak limit space $\mathcal{X}$ is single–valued if and only if $\mathcal{X}$ satisfies the $T_0$–property.

By $\text{AdmWeakLim}$ we denote the category whose objects are the weak limit spaces having an admissible multirepresentation and whose morphisms are the total sequentially continuous functions. $\text{AdmLim}$ denotes the full subcategory of $\text{AdmWeakLim}$ consisting of all limit spaces in $\text{AdmWeakLim}$. The category of sequential topological spaces with admissible multirepresentations and of total sequentially continuous (= topologically continuous) functions is denoted by $\text{AdmSeq}$.

Propositions 2.3.2, 2.3.4, 2.3.5 and Lemma 2.3.3 hold analogously for multirepresentations. Since the proofs are similar to the single–valued case, we omit most of them.

Proposition 2.4.18 (Quotient spaces generated by admissible multirep.)
Let $\delta : \subseteq \Sigma^\omega \rightrightarrows X$ be an admissible multirepresentation of a weak limit space $\mathcal{X} = (X, \rightarrow_x)$.

(1) $\delta$ is $\text{WeakLim}$–quotient multirepresentation of $\mathcal{X}$, i.e. $\rightarrow_y = \rightarrow_x$.

(2) If $\mathcal{X}$ is a limit space, then $\delta$ is $\text{Lim}$–quotient multirepresentation of $\mathcal{X}$, i.e. $\leadsto_z = \rightarrow_x$.

(3) The final topology $\tau_\delta$ of $\delta$ is equal to the associated topology $\text{seq}(\mathcal{X})$ of $\mathcal{X}$.

(4) The final topology $\tau_\gamma$ of an admissible multirepresentation $\gamma$ of a topological space $\mathcal{Y} = (Y, \tau_\mathcal{Y})$ is equal to $\text{seq}(\tau_\mathcal{Y})$.

Proof:
(1) Similar to Proposition 2.3.2.

(2) This follows from (1), Lemma 2.4.11(2), and Lemma 2.2.8.

(3) This follows from (1) and Lemma 2.4.11(3).

(4) Lemma 2.4.11(3) and (1) imply $\tau_\gamma = \text{seq}(\rightarrow_y) = \text{seq}(\rightarrow_{\mathcal{Y}}) = \text{seq}(\tau_\mathcal{Y})$.
2.4 Multirepresentations

Lemma 2.4.19 (Extended universal property)
Let \( \delta : \Sigma^\omega \rightrightarrows X \) be a multirepresentation of a weak limit space \( X = (X, \rightarrow_X) \) such that the universal property (i.e. \( \zeta \leq t \delta \) for all sequentially continuous correspondences \( \zeta : \subseteq \Sigma^\omega \rightrightarrows X \)) is satisfied. Furthermore let \( \Gamma_1, \ldots, \Gamma_k \) be alphabets. Then for every sequentially continuous correspondence \( \phi : \subseteq \Gamma_1^\omega \times \ldots \times \Gamma_k^\omega \rightrightarrows X \) there is a continuous function \( g : \subseteq \Gamma_1^\omega \times \ldots \times \Gamma_k^\omega \rightarrow \Sigma^\omega \) with \( \phi[(p_1, \ldots, p_k)] \subseteq \delta[g(p_1, \ldots, p_k)] \) for all \( (p_1, \ldots, p_k) \in \text{dom}(\phi) \).

Proposition 2.4.20 (Maximality of admissible multirepresentations)
Let \( \delta \) be a multirepresentation of a weak limit space \( X = (X, \rightarrow_X) \). Then \( \delta \) is an admissible multirepresentation of \( X \) if and only if \( \delta \) is \( \leq t \)-maximal in the class of all sequentially continuous multirepresentations of \( X \).

Proposition 2.4.21 (Continuous equivalence preserves admissibility)
Let \( \delta \) be an admissible multirepresentation of a weak limit space \( X = (X, \rightarrow_X) \) and \( \zeta \) be any multirepresentation of \( X \). Then \( \zeta \) is an admissible multirepresentation of \( X \) if and only if \( \zeta \) and \( \delta \) are continuously equivalent (i.e. \( \zeta \equiv t \delta \)).

As a first example, we show that the upper-bound multirepresentation \( \gamma_{ub} : \mathbb{N}^\omega \rightarrow \mathbb{R} \) from Equation (2.8) is admissible.

Example 2.4.22
Since \( \gamma_{ub} \) generates \( (\mathbb{R}, \rightarrow_{\gamma_{ub}}) \), \( \gamma_{ub} \) is a continuous multirepresentation of that space. Let \( \phi : \subseteq \mathbb{N}^\omega \rightrightarrows \mathbb{R} \) be \( (\rightarrow_{\gamma_{ub}}, \rightarrow_{\gamma_{ub}}) \)-continuous. Let \( p \in \text{dom}(\phi) \). Suppose for contradiction that for all \( n \in \mathbb{N} \) there is some \( y_n \in \phi[p^{<n}\mathbb{N}^\omega] \setminus (-\infty; n) \). By continuity of \( \phi \), the sequence \( (y_n)_n \) converges to every \( x \in \phi[p] \) w.r.t. \( \rightarrow_{\gamma_{ub}} \). By Example 2.4.14, there is some \( b \in \mathbb{N} \) with \( y_n \leq b \) for all \( n \in \mathbb{N} \). This contradicts that fact that \( (y_n)_n \) is unbounded. Hence we can define the function \( g : \subseteq \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) by

\[
g(q) := \begin{cases} 
\min \{ n \in \mathbb{N} | \phi[q^{<n}\mathbb{N}^\omega] \subseteq (-\infty; n) \} : 0^\omega & \text{if } q \in \text{dom}(\phi) \\
\text{div} & \text{otherwise}.
\end{cases}
\]

Since, for \( q \in \text{dom}(g) \), \( g(q)(0) \) only depends on a finite prefix of \( q \) (namely on \( q^{<g(q)(0)} \)), \( g \) is continuous. By the above consideration, \( g \) is well-defined. Obviously, \( g(p)(0) \) is an upper bound of \( \phi[p] \) for all \( p \in \text{dom}(\phi) \). Thus \( g \) translates \( \phi \) continuously to \( \gamma_{ub} \). We conclude that \( \gamma_{ub} \) is an admissible multirepresentation of \( (\mathbb{R}, \rightarrow_{\gamma_{ub}}) \).

We show that the multirepresentations \( \gamma_m \) of \( \mathbb{R} \) from Example 2.4.2 are admissible.
Example 2.4.23
Let $m \in \mathbb{N}$ and let $\phi : \subseteq \mathbb{Z}^\omega \rightarrow \mathbb{R}$ be a $(-\omega, -\gamma_m)$-continuous correspondence. Let $p \in dom(\phi)$ and $x \in \phi[p]$. There are at most three integers $z_0, z_1, z_2$ in the interval $[x \cdot 2^m - 1; x \cdot 2^m + 1]$.

Suppose for contradiction that for every $n$ there is some $y_n \in \phi[p^{<n}\mathbb{Z}^\omega] \setminus [(z_{n-3}\lceil \frac{n}{2} \rceil - 1) \cdot (\frac{1}{2})^m; (z_{n-3}\lceil \frac{n}{2} \rceil + 1) \cdot (\frac{1}{2})^m]$.

By sequential continuity of $\phi$, $(y_n)_n$ converges to $x$ w.r.t. $-\gamma_m$. By Example 2.4.13, there is some $n_0 \in \mathbb{N}$ and some $i \in \{0, 1, 2\}$ such that $|z_i \cdot (\frac{1}{2})^m - y_n| \leq (\frac{1}{2})^m$ for all $n \geq n_0$. This contradicts $y_{n_0+i} \notin [(z_i - 1) \cdot (\frac{1}{2})^m; (z_i + 1) \cdot (\frac{1}{2})^m]$.

Thus, the number

$$n_p := \min \{ n \in \mathbb{N} \mid (\exists z \in \mathbb{Z}) \phi[p^{<n}\mathbb{Z}^\omega] \subseteq [(z - 1) \cdot (\frac{1}{2})^m; (z + 1) \cdot (\frac{1}{2})^m] \}$$

exists. Hence we can define $g : \subseteq \mathbb{Z}^\omega \rightarrow \mathbb{Z}^\omega$ by

$$g(p)(0) := \min \{ z \in \mathbb{Z} \mid \phi[p^{<n}\mathbb{Z}^\omega] \subseteq [(z - 1) \cdot (\frac{1}{2})^m; (z + 1) \cdot (\frac{1}{2})^m] \}$$

and $g(p)(j) := 0$ for all $p \in \text{dom}(g) := \text{dom}(\phi)$ and $j > 0$. Then $g$ is continuous, because $g(p)(0)$ only depends on a finite prefix of $p \in \text{dom}(g)$, namely on $p^{<n_p}$. Since we have $\phi[p] \subseteq \phi[p^{<n_p}\mathbb{Z}^\omega] \subseteq [(g(p)(0) - 1) \cdot (\frac{1}{2})^m; (g(p)(0) + 1) \cdot (\frac{1}{2})^m] = \gamma_m[g(p)]$, $g$ translates $\phi$ continuously to $\gamma_m$. Thus $\gamma_m$ is an admissible multirepresentation of the space $(\mathbb{R}, -\gamma_m)$.

Any set $Y$ can be equipped with an admissible, but almost useless multirepresentation called the \textit{chaotic multirepresentation} of $Y$.

Example 2.4.24
We define the \textit{chaotic multirepresentation} $\gamma_{ch} : \subseteq \Sigma^\omega \rightarrow Y$ of a set $Y$ by $G(\gamma_{ch}) := \Sigma^\omega \times Y$, i.e., every $\omega$–word in $\Sigma^\omega$ is declared to be a name of every element. Of course, this multirepresentation does not encode any information about any element. The convergence relation induced by $\gamma_{ch}$ on $Y$ reflects this fact by being the \textit{chaotic convergence relation} $\rightarrow_{ch}$ on $Y$, which is induced by the indiscrete topology $\{\emptyset, Y\}$ on $Y$ and defines every generalized sequence $(y_n)_{n \leq \infty}$ to be convergent. Clearly, every multirepresentation of $Y$ is sequentially continuous w.r.t. the \textit{chaotic convergence relation} and is translatable to $\gamma_{ch}$ by the constant function $p \mapsto 0^\omega$. Thus $\gamma_{ch}$ is admissible.

Note that on the one hand every function $f : \subseteq Y \rightarrow Y$ is computable w.r.t. $\gamma_{ch}$, but on the other hand the only $(\gamma_{ch}, \varphi_\mathbb{R})$–continuous functions $h : Y \rightarrow \mathbb{R}$ are the constant ones.

2.4.7 Relative Continuity versus Sequential Continuity
As in the case of single–valued representations (cf. Subsection 2.3.3), admissibility of multirepresentations guarantees equivalence of relative continuity and sequential continuity. We show this equivalence not only for functions, but also for correspondences.
2.4 Multirepresentations

Theorem 2.4.25 (Main Theorem for Multirepresentations)
For every \( i \in \{1, \ldots, k+1\} \), let \( X_i = (X_i, \rightarrow_{x_i}) \) be a weak limit space and \( \delta_i \) be an admissible multirepresentation of \( X_i \).
Then a correspondence \( F : \subseteq X_1 \times \ldots \times X_k \Rightarrow X_{k+1} \) is relatively continuous with respect to \( \delta_1, \ldots, \delta_{k+1} \) if and only if \( F \) is sequentially continuous.

Theorem 2.4.25 follows from Propositions 2.4.18, 2.4.12 and the following generalization of Lemma 2.3.17.

Lemma 2.4.26 (Sequential continuity → relative continuity)
For every \( i \in \{1, \ldots, k\} \), let \( \delta_i \) be a sequentially continuous multirepresentation of a weak limit space \( X_i = (X_i, \rightarrow_{x_i}) \). Let \( \gamma \) be a multirepresentation of a weak limit space \( \mathcal{Y} = (Y, \rightarrow_{y}) \) satisfying the universal property.
Then every sequentially continuous correspondence \( F : \subseteq X_1 \times \ldots \times X_k \Rightarrow Y \) is relatively continuous w.r.t. the multirepresentations \( \delta_1, \ldots, \delta_k \), and \( \gamma \).

Proof:
For \( i \in \{1, \ldots, k\} \), let \( \Sigma_i \) be the underlying alphabet of \( \delta_i \) and let \( \Gamma \) be the one of \( \gamma \). Since \( F \) and \( \delta_1, \ldots, \delta_k \) are sequentially continuous, the correspondence \( \phi : \subseteq \Sigma_1 \times \ldots \times \Sigma_k \Rightarrow Y \)
defined by \( \phi([p_1, \ldots, p_k]) := F[\delta_1[p_1] \times \ldots \times \delta_k[p_k]] \) is sequentially continuous by Lemma 2.4.7. By Lemma 2.4.19, there is a continuous function \( g : \subseteq \Sigma_1 \times \ldots \times \Sigma_k \rightarrow \Gamma \) with
\[
\gamma[g(p_1, \ldots, p_k)] \supseteq \phi([p_1, \ldots, p_k]) = F[\delta_1[p_1] \times \ldots \times \delta_k[p_k]]
\]
for all \( (p_1, \ldots, p_k) \in \text{dom}(\phi) \). This implies that \( g \) realizes \( F \) w.r.t. \( \delta_1, \ldots, \delta_k \), and \( \gamma \) (cf. Condition (2.10) on page 47).

Proposition 2.4.12 and Lemma 2.4.26 imply the following slight generalization of Theorem 2.4.25.

Proposition 2.4.27 (Sequential continuity vs. relative continuity)
For every \( i \in \{1, \ldots, k\} \), let \( \delta_i \) be a WeakLim-quotient multirepresentation of a weak limit space \( X_i = (X_i, \rightarrow_{x_i}) \). Let \( \gamma \) be an admissible multirepresentation of a weak limit space \( \mathcal{Y} = (Y, \rightarrow_{y}) \).
Then a correspondence \( F : \subseteq X_1 \times \ldots \times X_k \Rightarrow Y \) is relatively continuous with respect to \( \delta_1, \ldots, \delta_k \), and \( \gamma \) if and only if \( F \) is sequentially continuous.

In the case that the target space is a limit space, it suffices to have Lim-quotient multirepresentations of the source spaces to guarantee equivalence of sequential continuity and relative continuity.

Proposition 2.4.28 (Sequential continuity vs. relative continuity)
For \( i \in \{1, \ldots, k\} \), let \( \delta_i \) be a Lim-quotient multirepresentation of a limit space \( X_i = (X_i, \rightarrow_{x_i}) \). Let \( \gamma \) be an admissible multirepresentation of a limit space \( \mathcal{Y} = (Y, \rightarrow_{y}) \).
Then a correspondence \( F : \subseteq X_1 \times \ldots \times X_k \Rightarrow Y \) is relatively continuous with respect to \( \delta_1, \ldots, \delta_k \), and \( \gamma \) if and only if \( F \) is sequentially continuous.
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Proof:

“⇐”: This follows from Lemma 2.4.26, because Lim-quotient multirepresentations are sequentially continuous.

“⇒”: By Proposition 2.4.12, F is a sequentially continuous correspondence between the limit space \((X_1, \sim_{\delta_1}), \ldots, (X_k, \sim_{\delta_k})\), and \((Y, \sim_{\gamma})\). By assumption we have \(X_i = (X_i, \sim_{\delta_i})\) for every \(i \in \{1, \ldots, k\}\). Proposition 2.4.18 implies \((Y, \sim_{\gamma}) = \mathcal{Y}\). Hence F is a sequentially continuous correspondence between \(X_1, \ldots, X_k\), and \(Y\).

✓

2.4.8 Admissible Multirepresentations of Topological Spaces

Whereas a multirepresentation \(\delta\) can only be admissible with respect to at most one convergence relation (cf. Propositions 2.3.2, 2.4.18), there might be several topologies with respect to which \(\delta\) is admissible. This is due to the fact that different topologies may induce the same convergence relation.

Example 2.4.29

We equip the set \(Y := \{(\infty, \infty)\} \cup (\mathbb{N} \times \mathbb{N})\) with two topologies. As the first one we choose the discrete topology on \(Y\), i.e. \(\tau_1 := 2^Y\), and as the second one the subspace topology inherited from the topological space \((X, \tau)\) considered in Example 2.3.15, i.e.:

\[
\tau_2 := \{ O \subseteq Y \mid (\infty, \infty) \in O \implies (\forall^\infty a \in \mathbb{N})(\forall^\infty b \in \mathbb{N})(a, b) \in O \}. 
\]

Since \(\{(\infty, \infty)\}\) is not contained in \(\tau_2\), the topologies \(\tau_1\) and \(\tau_2\) are different. No sequence contained in \(\mathbb{N}^2\) converges to \((\infty, \infty)\) in the subspace \((Y, \tau_2)\), because it does not in the space \((X, \tau)\) (cf. Example 2.3.15). Moreover, for each \((a, b) \in \mathbb{N}^2\) the singleton \(\{(a, b)\}\) is contained in \(\tau_2\). Thus every sequence that converges in \((Y, \tau_2)\) is eventually equal to its limit. This implies that the convergence relations induced by \(\tau_1\) and \(\tau_2\) are equal, i.e. \(W((X, \tau_1)) = W((X, \tau_2))\).

An obviously \(\tau_1\)-admissible and hence \(\tau_2\)-admissible representation \(\delta_Y : \mathbb{N}^\omega \to Y\) can be defined by

\[
\delta_Y(p) := \begin{cases} (p(1), p(2)) & \text{if } p(0) = 0 \\ (\infty, \infty) & \text{otherwise} \end{cases}
\]

for all \(p \in \mathbb{N}^\omega\).

\(\square\)

Let \(\delta\) be an admissible multirepresentation of a topological space \(\mathcal{X} = (X, \tau_\mathcal{X})\). From Proposition 2.4.18 we know that the final topology \(\tau_\delta\) of \(\delta\) (cf. Equation (2.12)) is equal to \(\text{seq}(\mathcal{X})\), the family of all sequentially open subsets of \(\mathcal{X}\) (cf. Subsection 2.2.5). By Lemma 2.2.7(4), \(\text{seq}(\mathcal{X})\) is a sequential topology. Thus an admissible multirepresentation \(\delta\) of \(\mathcal{X}\) is a \(\text{Top}\)-quotient multirepresentation of \(\mathcal{X}\) if and only if \(\mathcal{X}\) is a sequential topological space. The space \((X, \tau_2)\) in Example 2.4.29 is an example for a topological space that has an admissible representation which is not a \(\text{Top}\)-quotient function (cf. Subsection 2.2.2). On the other hand, an admissible multirepresentation of a limit space is necessarily a Lim-quotient multirepresentation by Proposition 2.4.18 and Lemma 2.4.11.
Since many interesting non sequential topological spaces used in mathematics can be equipped with an admissible representation, we do not require of an admissible representation of a topological space to induce the topology of that space. An example is the admissibly representable space \((D(\mathbb{R}), \tau_{LC})\) of test functions (cf. [BN73]). The usual topology \(\tau_{LC}\) on \(D(\mathbb{R})\) is locally convex, but not sequential, whereas its sequentialization \(\text{seq}(\tau_{LC})\) is not locally convex (cf. [BN73] or Example 4.5.21).

We prove the following lemma about topological spaces that have the same admissible multirepresentations.

**Lemma 2.4.30 (Topological spaces having the same admissible multirep.)**

Let \(\tau_1, \tau_2\) be two topologies on a set \(X\). In the following, \((a) \implies (b) \iff (c) \implies (d):\)

(a) There is a multirepresentation of \(X\) which is \(\tau_1\)-admissible and \(\tau_2\)-admissible.

(b) The topology \(\text{seq}(\tau_1)\) is equal to \(\text{seq}(\tau_2)\).

(c) The convergence relation induced by \(\tau_1\) is equal to the one induced by \(\tau_2\).

(d) A multirepresentation of \(X\) is \(\tau_1\)-admissible if and only if it is \(\tau_2\)-admissible.

**Proof:**

“(a) \implies (b)”: This follows from Proposition 2.4.18(4).

“(b) \implies (c)”: Lemma 2.2.7 implies \((\tau_1 = \text{seq}(\tau_1) = \text{seq}(\tau_2) = \text{seq}(\tau_2) = \tau_2)\).

“(c) \implies (b)”: By definition of \(\text{seq}(\tau_i)\), we have \(\text{seq}(\tau_1) = \text{seq}(\tau_2) = \text{seq}(\tau_2) = \text{seq}(\tau_2)\).

“(c) \implies (d)”: This follows directly from Definition 2.4.16.

As a corollary we obtain:

**Corollary 2.4.31**

A topological space \(X = (X, \tau_X)\) has an admissible multirepresentation if and only if its sequentialization \(\text{seq}(X) = (X, \text{seq}(\tau_X))\) has an admissible multirepresentation.

Propositions 2.4.27 and 2.4.28 can be generalized to \(\text{Top}\)-quotient multirepresentations in the case of total functions.

**Proposition 2.4.32 (Sequential continuity vs. relative continuity)**

For every \(i \in \{1, \ldots, k\}\), let \(\delta_i\) be a \(\text{Top}\)-quotient multirepresentation of a topological space \(X_i = (X_i, \tau_i)\). Let \(\gamma\) be an admissible multirepresentation of a topological space \(\bar{\mathcal{Q}} = (Y, \tau_{\bar{\mathcal{Q}}})\).

Then a total function \(f : X_1 \times \ldots \times X_k \to Y\) is relatively continuous with respect to \(\delta_1, \ldots, \delta_k\), and \(\gamma\) if and only if \(f\) is sequentially continuous.
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Proof:

“⇐”: For every \( i \in \{1, \ldots, k\} \) we have \( \delta_i \subseteq \tau(X_i, \sigma_{\delta_i}) = \tau_{\delta_i} = \tau_i \) by Lemmas 2.2.7 and 2.4.11, hence \( \delta_i \) is a sequentially continuous multirepresentation between \( (\mathbb{N}^\omega, \rightarrow^\omega) \) and \( X_i \). Thus \( f \) is relatively continuous w.r.t. \( \delta_1, \ldots, \delta_k \) by Lemma 2.4.26.

“⇒”: By Proposition 2.4.12, \( f \) is a sequentially continuous function between the topological spaces \( (X_1, \tau_{\delta_1}), \ldots, (X_k, \tau_{\delta_k}) \), and \( (Y, \tau_\gamma) \). By assumption we have \( \tau_{\delta_i} = \tau_i \) for every \( i \in \{1, \ldots, k\} \). From Proposition 2.4.18 we know \( \tau_\gamma = \text{seq}(\tau_Y) \), hence \( \rightarrow_{\tau_\gamma} \) is equal to \( \rightarrow_{\tau_Y} \) by Lemma 2.2.7(3). We conclude that \( f \) is a sequentially continuous function between the topological spaces \( X_1, \ldots, X_k \) and \( Y \).

Proposition 2.4.32 cannot be generalized to partial functions, as the following example shows.

Example 2.4.33

We consider the non topological limit space \( \mathfrak{3} = (Z, \rightarrow_\mathfrak{3}) \) and its admissible representation \( \zeta \) from Example 2.3.16. By Proposition 2.3.2, \( \zeta \) is a Top–quotient representation of the topological space \( T(\mathfrak{3}) \). Let \( \varrho_N : \mathbb{N}^\omega \rightarrow \mathbb{N} \) be the admissible representation of \( (\mathbb{N}, \tau_\mathbb{N}) \) from Example 2.3.6 with \( \varrho_N(p) = p(0) \). We define the partial function \( f : \subseteq Z \rightarrow \mathbb{N} \) by

\[
  f(z) := \begin{cases} 
    0 & \text{if } z = (1, 0) \\
    1 & \text{if } z \in \{3\} \times \mathbb{N} \\
    \text{div} & \text{otherwise}.
  \end{cases}
\]

Then \( f \) is computable and thus continuous w.r.t. \( \zeta \) and \( \varrho_N \), because the computable function \( g : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) defined by

\[
  g(p) := \begin{cases} 
    0^\omega & \text{if } p(0) = 0 \\
    1^\omega & \text{otherwise}
  \end{cases}
\]

realizes \( f \) w.r.t. these representations. But \( f \) is not a sequentially continuous function between the topological spaces \( T(\mathfrak{3}) \) and \( (\mathbb{N}, \tau_\mathbb{N}) \), because \( ((3, b))_b \) converges to \( (1, 0) \) in \( T(\mathfrak{3}) \) (cf. Example 2.3.16) and the image sequence \( (f(3, b))_b \) does not converge to \( f(1, 0) \) in \( (\mathbb{N}, \tau_\mathbb{N}) \).

Since topological continuity and sequential continuity for total functions between topological spaces coincide, if the domain space is sequential (cf. [Eng89, Prop. 1.6.15]), we obtain the following corollary from Proposition 2.4.32.

Proposition 2.4.34 (Topological continuity vs. relative continuity)

Let \( \delta \) be a Top–quotient multirepresentation of a topological space \( \mathfrak{X} = (X, \tau_\mathfrak{X}) \), and let \( \gamma \) be an admissible multirepresentation of a topological space \( \mathfrak{Y} = (Y, \tau_\mathfrak{Y}) \). Then a total function \( f : X \rightarrow Y \) is relatively continuous with respect to \( \delta \) and \( \gamma \) if and only if \( f \) is topologically continuous.

Proposition 2.4.34 cannot be generalized to multivariate functions, because the topological product of sequential spaces with admissible representations need not be sequential. A counterexample is the sequential space \( (D(\mathbb{R}), \tau_{\text{Lim}} \mathcal{C}_k(\mathbb{R})) \) of test functions: addition is sequentially continuous, but not topologically continuous w.r.t. \( \tau_{\text{Lim}} \mathcal{C}_k(\mathbb{R}) \), cf. Example 4.5.21.
Chapter 3

Characterization

We characterize in this chapter the class of spaces that have an admissible multirepresentation and compare this class with other concepts. In Section 3.1 the characterization is done by means of the concept of limit bases and pseudobases. Section 3.2 provides the fundamental result that the objects of the categories $\text{AdmSeq}$, $\text{AdmLim}$, $\text{AdmWeakLim}$ are exactly the corresponding quotients (topological quotient, $\text{Lim}$–quotient, $\text{WeakLim}$–quotient) of countably based topological spaces. In Section 3.3 we list various topological properties of topological spaces that are equipped with an admissible multirepresentation. Section 3.4 is devoted to an investigation of the relationship of these categories with the $\aleph_0$–spaces introduced by E. Michael, with the categories $\text{PQ}$ and $\text{PQL}$ introduced by M. Menni and A. Simpson, and with the equi-
logical spaces introduced by D.S. Scott.

3.1 Characterization of Admissibly Representable Spaces by Limit Bases and Pseudobases

In this section we characterize the class of weak limit spaces that have an admissible multirepresentation by means of countable limit bases and, in the case of topological spaces, by means of pseudobases. Limit bases and pseudobases are generalizations of topological bases. For many operators on weak limit spaces and on topological spaces there exist nice operators on limit bases (pseudobases) mapping limit bases (pseudobases) of the source spaces to limit bases (pseudobases) of the resulting spaces (cf. Lemmas 3.1.4, 3.1.5, 3.1.12, 3.1.13, Proposition 4.2.7, [Sch01], [Sch02a]). In Subsection 3.1.2 we prove that the admissibly representable weak limit spaces are exactly those weak limit spaces that have a countable limit base. Similarly, we show in Subsection 3.1.3 that a topological space has an admissible multirepresentation if and only if it has a countable pseudobase.

3.1.1 Limit Bases

Let $\mathcal{X} = (X, \to_X)$ be a sequential convergence space, and let $\mathcal{B}$ be a family of subsets of $X$. We call $\mathcal{B}$ a limit base of $\mathcal{X}$ iff for all elements $x \in X$ and all sequences $(y_n)_n, (z_n)_n$
in \( X \) there is an element \( B \in \mathcal{B} \) such that the following properties hold:

(i) \( x \in B \)

(ii) \((y_n)_n \to_x x \implies (\forall \infty n \in \mathbb{N}) y_n \in B\)

(iii) \((z_n)_n \not\to_x x \implies (\exists \infty n \in \mathbb{N}) z_n \notin B\).

If \((y_n)_n\) converges to \( x \) and \((z_n)_n\) does not, then we say that a set \( B \) with Properties (i), (ii), (iii) separates the sequences \((y_n)_n\) and \((z_n)_n\).

It is easy to see that every subbase \( \mathcal{B} \) of a topological space \((X, \tau)\) with \( \bigcup \mathcal{B} = X \) is a limit base of the corresponding limit space \((X, \to_X)\). Every superset of limit base of \( X \) is also a limit base of \( X \). This implies that, in contrast to topological bases, limit bases do not characterize the convergence relation \( \to_X \) unambiguously: the power set of \( X \) as well as the family of all countable subsets of \( X \) form limit bases of all weak limit spaces with underlying set \( X \) (cf. the proof of Proposition 3.1.3).

**Example 3.1.1**

We show that family \( \mathcal{B} \) of all closed intervals with rational endpoints is a limit base of the Euclidean limit space \((\mathbb{R}, \to_\mathbb{R})\). Let \((x_n)_{n \leq \infty}\) be a convergent sequence of real numbers, and let \((z_n)_{n \in \mathbb{N}}\) be a sequence that does not converge to \( x_\infty \). For \( k \in \mathbb{N} \), we define \( B_k \in 2^X \) by

\[
B_k := \{ (1,0), (2,j) \mid j \geq m \}, \quad \{ (2,a), (3,j) \mid j \geq m \}, \quad \{ (3,b) \} \mid a, b, m \in \mathbb{N} \}
\]

is a limit base of \( 3 \).

Interestingly, the weak limit spaces are characterized by the property of having a limit base.

**Proposition 3.1.3** (Characterization of weak limit spaces via limit bases)

A sequential convergence space \( \mathfrak{X} = (X, \to_\mathfrak{X}) \) is a weak limit space if and only if \( \mathfrak{X} \) has a limit base.

**Proof:**

\("\implies\): We show that the power set \( 2^X \) of \( X \) is a limit base of \( \mathfrak{X} \).

Let \((x_n)_{n \leq \infty}\) be a convergent sequence of \( \mathfrak{X} \), and let \((z_n)_{n \in \mathbb{N}}\) be a sequence that does not converge to \( x_\infty \). For \( k \in \mathbb{N} \), we define \( B_k \in 2^X \) by \( B_k := \{ x_\infty \} \cup \{ x_i \mid i \geq k \} \).

Suppose for contradiction \((\forall k \in \mathbb{N}) (\forall \infty n \in \mathbb{N}) z_n \in B_k \). Then there is a strictly increasing sequence \( l_0, l_1, \ldots \in \mathbb{N} \) with \((\forall k \in \mathbb{N}) \{ z_n \mid n \geq l_k \} \in B_k \). For every \( k \in \mathbb{N} \)
Lemma 3.1.4 (Countable limit bases for WeakLim–quotient spaces)

Let \( X = (X, \rightarrow_X) \) and \( Y = (Y, \rightarrow_Y) \) be weak limit spaces. Let \( B_X \) be a countable limit base of \( X \) and \( F : X \rightarrow Y \) be a WeakLim–quotient correspondence (i.e., \( F \) is surjective and \( \rightarrow_{F^{-1}} = \rightarrow_Y \)). Then the family

\[
B_Y := \{ F[B_0 \cap \ldots \cap B_k] \mid k \in \mathbb{N} \text{ and } \{ B_0, \ldots, B_k \} \subseteq B_X \}
\]

is a countable limit base of \( Y \).

Proof:

Let \((y_n)_{n \leq \infty}\) be a convergent sequence of \( Y \). As \( F \) is a WeakLim–quotient correspondence, there is a convergent sequence \((x_n)_{n \leq \infty}\) of \( X \) with \((\forall n \in \mathbb{N}) y_n \in F[x_n]\). We choose a sequence \( \beta_0, \beta_1, \beta_2, \ldots \) of sets with

\[
\{ \beta_k \mid k \in \mathbb{N} \} = \{ B \in B_X \mid x_\infty \in B \text{ and } (\forall \in \mathbb{N}) x_n \in B \}.
\]

Let \((z_n)_{n \in \mathbb{N}}\) be a sequence that does not converge to \( y_\infty \).

Suppose for contradiction that \((z_n)_{n \in \mathbb{N}}\) is eventually in \( F[\bigcap_{j=0}^{k} \beta_j] \) for every \( k \in \mathbb{N} \). Then...
there is a strictly increasing sequence \( l_0, l_1, l_2, \ldots \in \mathbb{N} \) with \( \{ z_n \mid n \geq l_k \} \subseteq F[\bigcap_{j=0}^{k} \beta_j] \) for all \( k \in \mathbb{N} \). Moreover, for every \( k \in \mathbb{N} \) and \( n \in \{ l_k, \ldots, l_{k+1} - 1 \} \) there is some \( a_n \in \bigcap_{j=0}^{k} \beta_j \) with \( z_n \in F[a_n] \). If \( (a_n + l_0) \) did not converge to \( x_\infty \), there would be some \( i \in \mathbb{N} \) with \( (\exists n \in \mathbb{N}) a_n \notin \beta_i \), because \( B_X \) is a limit base of \( X \), contradicting \( (\forall n \geq l_i) a_n \in \beta_i \). Hence \( (a_n + l_0) \rightarrow x_\infty \). Since \( F \) is sequentially continuous, \( (z_n + l_0) \) converges to \( y_\infty \) in \( \mathcal{Y} \). By Axiom (L6) this contradicts \( (z_n) \notin \mathcal{Y} \).

Therefore there is some \( k \in \mathbb{N} \) with \( (\exists n \in \mathbb{N}) z_n \notin F[\beta_0 \cap \cdots \cap \beta_k] \). As \( (y_n) \leq \infty \) is eventually in \( F[\beta_0 \cap \cdots \cap \beta_k] \), this shows that the countable set \( \mathcal{B}_\beta \) is a limit base of \( \mathcal{Y} \).

Note that a corresponding property of countable topological spaces and topologically quotient functions does not hold: topological quotients of countably based spaces need not be countably based (cf. [Eng89] or Example 2.3.15).

**Lemma 3.1.4** implies that every weak limit space \( X = (X, \rightarrow_X) \) that is equipped with a WeakLim-quotient or admissible multirepresentation \( \delta : \Sigma^\tau \Rightarrow X \) has a countable limit base. Indeed, \( \{ \delta[w\Sigma^\tau] \mid w \in \Sigma^\tau \} \) is a limit base of \( X \), because \( \{ w\Sigma^\tau \mid w \in \Sigma^\tau \} \cup \{ \emptyset \} \) is a countable base of \( (\Sigma^\tau, \tau_{\Sigma^\tau}) \) closed under finite intersection.

Since limit bases do not determine weak limit spaces unambiguously, it is impossible to characterize sequential continuity by means of arbitrary countable limit bases. However, the following lemma gives at least a necessary condition for sequential continuity of correspondences between weak limit spaces equipped with countable limit bases.

**Lemma 3.1.5** (Necessary condition on limit bases for seq. continuity)

Let \( X = (X, \rightarrow_X) \) and \( \mathcal{Y} = (Y, \rightarrow_Y) \) be weak limit spaces, and let \( \mathcal{B}_X \) and \( \mathcal{B}_\beta \) be countable limit bases of \( X \) and \( \mathcal{Y} \), respectively. Let \( F : X \Rightarrow Y \) be a sequentially continuous correspondence.

Then for every convergent sequence \( (x_n)_{n \leq \infty} \) of \( X \), every \( y \in F[x_\infty] \) and every sequence \( (z_n)_{n \in \mathbb{N}} \) which does not converge to \( y \) in \( \mathcal{Y} \) there are some \( k \in \mathbb{N} \) and sets \( A_0, \ldots, A_k \in \mathcal{B}_X \), \( B \in \mathcal{B}_\beta \) with

\[
(\forall n \in \mathbb{N}) \quad x_n \in A_0 \cap \cdots \cap A_k, \quad F[A_0 \cap \cdots \cap A_k] \subseteq B \quad \text{and} \quad (\exists n \in \mathbb{N}) \quad z_n \notin B.
\]

**Proof:**

As \( \mathcal{B}_X \) and \( \mathcal{B}_\beta \) are limit bases, there are sequences \( \alpha_0, \alpha_1, \alpha_2, \ldots \) and \( \beta_0, \beta_1, \beta_2, \ldots \) of subsets with

\[
\{ \alpha_i \mid i \in \mathbb{N} \} = \{ A \in \mathcal{B}_X \mid x_\infty \in A \and (\forall n \in \mathbb{N}) \ x_n \in A \} \quad \text{and} \quad \{ \beta_j \mid j \in \mathbb{N} \} = \{ B \in \mathcal{B}_\beta \mid y \in B \and (\exists n \in \mathbb{N}) \ z_n \notin B \}.
\]

Suppose for contradiction \( (\forall k, l \in \mathbb{N}) F[\bigcap_{i=0}^{l} \alpha_i] \notin \beta_l \). Then for every \( m \in \mathbb{N} \) there are \( a_m \in \bigcap_{i=0}^{m} \alpha_i \) and \( b_m \in F[a_m] \) with \( b_m \notin \beta_{m-\lfloor \sqrt{m} \rfloor} \). If \( (a_m)_m \) did not converge to \( x_\infty \), then there would be some \( i \in \mathbb{N} \) with \( a_m \notin \alpha_i \) for infinitely many \( m \), because \( B_X \) is a limit base of \( X \), contradicting \( (\forall m \geq i) a_m \in \alpha_i \). Hence \( (a_m)_m \rightarrow x_\infty \). By sequential continuity of \( F \), \( (b_m)_m \) converges to \( y \). Since \( \mathcal{B}_\beta \) is a limit base of \( \mathcal{Y} \), there are some \( j, m_0 \in \mathbb{N} \) with \( b_m \in \beta_j \) for all \( m \geq m_0 \). For \( m_1 := \lfloor (m_0 + j)^2 \rfloor \) we have on the one hand \( b_{m_1} \notin \beta_j \), as \( j = m_1 - \lfloor \sqrt{m_1} \rfloor \), but one the other hand \( b_{m_1} \in \beta_j \), since \( m_1 \geq m_0 \). This yields the contradiction.
Hence there are some k, l ∈ N with $F[α_0 \cap \ldots \cap α_k] \subseteq β_l$. Since $(x_n)_n$ is eventually in $α_0 \cap \ldots \cap α_k$, this concludes the proof.

3.1.2 Construction of an Admissible Multirepresentation from a Countable Limit Base

Let $X = (X, \rightarrow)$ be a weak limit space, and let $β : \mathbb{N} \rightarrow \mathcal{B}$ be a numbering of a countable limit base $\mathcal{B}$ of $X$. We construct an admissible multirepresentation $δ_{X, β} : \mathbb{N}^ω \rightrightarrows X$ of $X$ as follows: For $p \in \mathbb{N}^ω$ and $x \in X$ we define

$$δ_{X, β}[p] \ni x : \iff \begin{cases} En(p) \subseteq \{i \in \mathbb{N} \mid x \in β_i\} & \text{and} \\ (\forall (z_n)_n \in X^n) & (z_n)_n \nleftrightarrow \mathbb{N} \quad \text{for infinitely many} \quad \text{limit base elements containing} \quad x. \end{cases} \quad (3.1)$$

where $En(p) := \{p(j) - 1 \mid j \in \mathbb{N} \text{ and } p(j) > 0\}$. Thus a $δ_{X, β}$–name $p$ of $x$ lists “sufficiently many” limit base elements containing $x$. Note that in general $δ_{X, β}$ would not become an admissible multirepresentation of $X$, if all names of $x$ were demanded to list all limit base elements containing $x$, because limit base elements are not necessarily sequentially open.

Proposition 3.1.6 (Admissibility of $δ_{X, β}$)

Let $X = (X, \rightarrow)$ be a weak limit space, and let $β : \mathbb{N} \rightarrow \mathcal{B}$ be a numbering of a limit base $\mathcal{B}$ of $X$. Then the correspondence $δ_{X, β}$ defined by Equivalence (3.1) is an admissible multirepresentation of $X$.

Proof:

Surjectivity: Let $x \in X$. Every $p \in \mathbb{N}^ω$ with $\text{range}(p) = \{n + 1 \mid n \in \mathbb{N}, \ x \in β_n\}$ is a $δ_{X, β}$–name of $x$, because $\{β_0, β_1, \ldots\}$ is a limit base of $X$.

Continuity: Let $(p_n)_{n \leq \infty}$ be a convergent sequence in $\text{dom}(δ_{X, β})$, and let $(x_n)_{n \leq \infty}$ be a sequence with $x_n \in δ_{X, β}[p_n]$ for all $n \in \mathbb{N}$.

Suppose for contradiction that $(x_n)_n$ does not converge to $x_∞$. By definition of $δ_{X, β}$, there is some $i \in En(p_∞)$ with $x_n \notin β_i$ for infinitely many $n$. There are some $j, n_0 \in \mathbb{N}$ with $i = p_∞(j) - 1$ and $(∀ n \geq n_0) p_n(j) = p_∞(j) = i + 1$. This implies $x_n \in β_i$ for all $n \geq n_0$, a contradiction.

Hence $(x_n)_n$ converges to $x_∞$. Therefore $δ_{X, β}$ is sequentially continuous.

Universality: Let $φ : \subseteq \mathbb{N}^ω \rightrightarrows X$ be sequentially continuous. We define the function $g : \mathbb{N}^ω \rightarrow \mathbb{N}^ω$ for every $q \in \mathbb{N}^ω$ and $i, j \in \mathbb{N}$ by

$$g(q)((i, j)) := \begin{cases} i + 1 & \text{if } φ[q^{<j}\mathbb{N}^ω] \subseteq β_i \\ 0 & \text{otherwise.} \end{cases}$$

Then $g$ is continuous, because $g(q)((i, j))$ only depends on a finite prefix of $q$, namely on $q^{<j}$.

We show that $g$ translates $φ$ to $δ_{X, β}$. Let $p \in \mathbb{N}^ω$ and $x \in φ[p]$. Obviously we have $x \in β_i$ for all $i \in En(g(p))$. Let $(z_n)_n$ be a sequence that does not converge to $x$. By Lemma 3.1.5 there are words $u_0, \ldots, u_k \in \mathbb{N}^ω$ and $i \in \mathbb{N}$ with

$$p \in u_0\mathbb{N}^ω \cap \ldots \cap u_k\mathbb{N}^ω, \ φ[u_0\mathbb{N}^ω \cap \ldots \cap u_k\mathbb{N}^ω] \subseteq β_i \text{ and } (∃ν \in \mathbb{N}) z_n \notin β_i,$$
because \( \{ w \omega^w \mid w \in \omega^* \} \) is a base and therefore a limit base of \((\omega^\omega, \tau_{\omega^\omega})\). It follows \( \phi[p \omega^2 \omega^w] \subseteq \beta_i \) for \( j := \max \{ \lg(u_0), \ldots, \lg(u_k) \} \). Hence \( i \in En(g(p)) \). This implies that \( g(p) \) is a \( \delta_{\omega, \beta} \)–name of \( x \). Thus \( \phi \leq_t \delta_{\omega, \beta} \).

From Propositions 3.1.6, 2.4.18 and Lemma 3.1.4 we obtain the following characterizations of the weak limit spaces with admissible multirepresentations.

**Proposition 3.1.7 (Charac. of admissibly representable weak limit spaces)**

A weak limit space \( X = (X, \rightarrow_X) \) has an admissible multirepresentation, iff \( X \) has a countable limit base, and also iff \( X \) has a \( \text{WeakLim} \)–quotient multirepresentation.

Propositions 3.1.7, 2.3.13 and Lemma 2.4.6 imply

**Proposition 3.1.8 (Charac. of admissibly representable weak limit spaces)**

A weak limit space \( X = (X, \rightarrow_X) \) has an admissible representation if and only if \( X \) has a countable limit base and satisfies the \( T_0 \)–property.

### 3.1.3 Pseudobases of Topological Spaces

Let \( X = (X, \tau_X) \) be a topological space, and let \( B \) be a family of subsets of \( X \). We call \( B \) a pseudobase of \( X \) iff for every convergent sequence \((x_n)_{n \leq \infty}\) of \( X \) and every open set \( O \in \tau_X \) with \( x_\infty \in O \) there is a set \( B \in B \) and some \( n_0 \in \mathbb{N} \) with

\[
\{ x_\infty, x_n \mid n \geq n_0 \} \subseteq B \subseteq O.
\]

We call \( B \) a pseudosubbase of \( X \) iff the family

\[
\{ B_1 \cap \ldots \cap B_k \mid k \geq 1 \text{ and } \{ B_1, \ldots, B_k \} \subseteq B \}
\]

of all finite intersections of sets in \( B \) is a pseudobase of \( X \). Obviously, every base of the space \( X \) is a pseudobase of \( X \). Conversely, every pseudobase consisting of open sets is a base of \( X \). Like limit bases, pseudobases do not characterize topological spaces unambiguously, because the power set \( 2^X \) is a pseudobase of all topological spaces with underlying set \( X \).

**Example 3.1.9**

We show that the family \( B \) of all closed intervals with rational endpoints is a pseudobase of the Euclidean space \( (\mathbb{R}, \tau_\mathbb{R}) \) (cf. Example 3.1.1). Let \((x_n)_{n \leq \infty} \) be a convergent sequence of real numbers and \( O \in \tau_\mathbb{R} \) be an open set containing \( x_\infty \). Then there is some \( \varepsilon > 0 \) with \( (x_\infty - \varepsilon, x_\infty + \varepsilon) \subseteq O \). Choose two rationals \( q_1, q_2 \) with \( x_\infty - \varepsilon < q_1 < x_\infty < q_2 < x_\infty + \varepsilon \). Then \([q_1; q_2]\) is a closed interval in \( B \) with \((\forall n \in \mathbb{N}) \) \( x_n \in [q_1; q_2] \) and \([q_1; q_2] \subseteq O \). Hence \( B \) is a pseudobase of \( (\mathbb{R}, \tau_\mathbb{R}) \).

We prove the following relationship between pseudobases and limit bases.

**Lemma 3.1.10 (Pseudosubbases vs. limit bases)**

Let \( X = (X, \tau_X) \) be a topological space, and let \( B \) be a countable family of subsets of \( X \). Then \( B \) is a pseudosubbase of \( X \) if and only if \( B \) is a limit base of \( \mathcal{W}(X) \).
Lemma 3.1.12 (Countable pseudobases for topological quotients)

Let \( (x_n)_{n \leq \infty} \) be a convergent sequence of \( X \). Since \( X \) is an open set, there is some \( B \in \mathcal{B} \) with \( x_\infty \in B \) and \( x_n \in B \) for infinitely many \( n \).

Let \( (z_n)_n \) be a sequence that does not converge to \( x_\infty \). Then there is an open set \( O \) containing \( x_\infty \) such that \( (z_n)_n \) is not eventually in \( O \). Since \( \mathcal{B} \) is a pseudosubbase of \( X \), there are some \( k \geq 1 \), \( n_0 \in \mathbb{N} \) and sets \( B_1, \ldots, B_k \in \mathcal{B} \) with

\[ \{x_\infty, x_n \mid n \geq n_0\} \subseteq B_1 \cap \ldots \cap B_k \subseteq O. \]

As \( O \) does not contain \( z_n \) for infinitely many \( n \), there is some \( i \in \{1, \ldots, k\} \) with \((\exists \infty n \in \mathbb{N}) \ z_n \notin B_i \). This set \( B_i \in \mathcal{B} \) separates the sequences \( (x_n)_{n \leq \infty} \) and \( (z_n)_n \).

Hence \( \mathcal{B} \) is a limit base of \( X \).

Corollary 3.1.11 (Countable pseudobases and sequentialization)

Let \( X = (X, \tau_X) \) be a topological space, and let \( \mathcal{B} \) be a countable family of subsets of \( X \). Then \( \mathcal{B} \) is a pseudosubbase of \( X \) if and only if \( \mathcal{B} \) is a pseudosubbase of the sequentialization \( T(\mathfrak{X}) \).

We show that topological quotients of sequential topological spaces inherit the property of having a countable pseudobase (cp. Lemma 3.1.4).

Lemma 3.1.12 (Countable pseudobases for topological quotients)

Let \( \mathfrak{X} = (X, \tau_X) \) and \( \mathfrak{Y} = (Y, \tau_Y) \) be sequential spaces. Let \( \mathcal{A} \) be a countable pseudobase of \( \mathfrak{X} \), and let \( q : X \to Y \) be a Top-quotient function (i.e., \( q \) is surjective and \( \tau_{X,q} = \tau_Y \)).

Then the family

\[ \mathcal{B} := \left\{ \text{upset}(q[A_1 \cup \ldots \cup A_k]) \mid k \in \mathbb{N} \text{ and } \{A_1, \ldots, A_k\} \subseteq \mathcal{A} \right\}, \]

where \( \text{upset}(M) := \{y \in Y \mid \text{Cl}_{\mathcal{Y}}(\{y\}) \cap M \neq \emptyset\} \), is a countable pseudobase of \( \mathfrak{Y} \).
Let $K$ be a compact set in $\mathcal{Y}$, and let $V \in \tau_\mathcal{Y}$ be a non-empty open superset of $K$. As $q^{-1}[V]$ is open in $\mathcal{X}$ and non-empty, there is a sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ of sets in $\mathcal{A}$ with \( \{ \alpha_i \mid i \in \mathbb{N} \} = \{ A \in \mathcal{A} \mid A \subseteq q^{-1}[V] \} \). We prove that there is some $n_0 \in \mathbb{N}$ with

\[
\{ O \in \tau_\mathcal{Y} \mid q[\alpha_0 \cup \ldots \cup \alpha_{n_0}] \subseteq O \} \subseteq \{ O \in \tau_\mathcal{Y} \mid K \subseteq O \}.
\]

(3.2)

Suppose for contradiction that for all $n \in \mathbb{N}$ there exists an open set $O_n \in \tau_\mathcal{Y}$ such that $q[\alpha_0 \cup \ldots \cup \alpha_n] \subseteq O_n$ and $K \nsubseteq O_n$. We define the function $g : X \times \mathbb{N} \to \mathcal{S}_\mathcal{K}$ by

\[
g(x, n) := \begin{cases} 
\top & \text{if either } n = \infty \text{ and } x \in q^{-1}[V] \text{ or } n < \infty \text{ and } x \in q^{-1}[O_n] \\
\bot & \text{otherwise}
\end{cases}
\]

and show that $g$ is a sequentially continuous function between $\mathcal{X}$, $(\mathbb{N}, \tau_\mathcal{N})$, and the Sierpiński space $\mathcal{S}_\mathcal{K}$ (cf. Example 2.3.7). Let $(x_n)_{n \leq \infty}$ and $(m_n)_{n \leq \infty}$ be convergent sequences of, respectively, $\mathcal{X}$ and $(\mathbb{N}, \tau_\mathcal{N})$. Since every sequence in $\mathcal{S}_\mathcal{K}$ converges to $\bot$ and the sets $q^{-1}[O_n]$ are open in $\mathcal{X}$, the only interesting case is $g(x, m_\infty) = \top$ and $m_\infty = \infty$, hence $x_\infty \in q^{-1}[V]$. As $\mathcal{A}$ is a pseudobase of $\mathcal{X}$, there are some $i, n_1 \in \mathbb{N}$ with $\{x_1, \ldots, x_i\} \subseteq \alpha_i \subseteq q^{-1}[V]$.

Moreover, since $(m_n)_{n \leq \infty}$ converges to $\infty$, there is some $n_2 \in \mathbb{N}$ with $(\forall n \geq n_2) m_n \geq i$. For all $n \geq \max\{n_1, n_2\}$ we therefore have $x_n \in \alpha_i \subseteq q^{-1}[O_{m_n}]$ and $g(x_n, m_n) = \top$, because $q[\alpha_i] \subseteq q[\alpha_0 \cup \ldots \cup \alpha_{m_n}] \subseteq O_{m_n}$. Hence $g$ is sequentially continuous.

Let $m \in \mathbb{N}$. As $\{m, \ldots, \infty\}$ is sequentially compact in $(\mathbb{N}, \tau_\mathcal{N})$ and $\{\top\} \in \tau_\mathcal{S}_\mathcal{K}$, the set

\[
U_m := \{ x \in X \mid (\forall i \in \{m, \ldots, \infty\}) g(x, i) \in \{\top\} \}
\]

is sequentially open by Lemma 2.2.9. Thus $U_m$ is open, because $\mathcal{X}$ is sequential. Since $q$ is a Top-quotient function and

\[
q^{-1} \left( \bigcap_{i \geq m} O_i \cap V \right) = \bigcap_{i \geq m} q^{-1}[O_i] \cap q^{-1}[V] = U_m
\]

holds, the set $V_m := \bigcap_{i \geq m} O_i \cap V$ is open in $\mathcal{Y}$. For every $y \in K \subseteq V$ there is some $x_y \in q^{-1}[V]$ with $q(x_y) = y$. By sequential continuity of $g$, there exists some $m_y \in \mathbb{N}$ with $(\forall i \geq m_y) g(x_y, i) = \top$ implying $x_y \in U_{m_y}$ and $y \in V_{m_y}$. Therefore $K \subseteq \bigcup_{m \in \mathbb{N}} V_m$. By compactness of $K$, there is some $n_3 \in \mathbb{N}$ with $K \subseteq \bigcup_{m \leq n_3} V_m = V_{n_3} \subseteq O_{n_3}$. This contradicts $K \nsubseteq O_{n_3}$ and shows (3.2).

Let $y \in K$. If $y$ were not in upset($q[\alpha_0 \cup \ldots \cup \alpha_{n_0}]$), we would have $q[\alpha_0 \cup \ldots \cup \alpha_{n_0}] \subseteq Y \setminus \text{Cl}_{\mathcal{Y}}(\{y\}) \in \tau_\mathcal{Y}$ and therefore $K \subseteq Y \setminus \text{Cl}_{\mathcal{Y}}(\{y\})$ by (3.2), a contradiction.

Let $y \in \text{upset}(q[\alpha_0 \cup \ldots \cup \alpha_{n_0}])$. Then we have $\emptyset \neq \text{Cl}_{\mathcal{Y}}(\{y\}) \cap q[\alpha_0 \cup \ldots \cup \alpha_{n_0}] \subseteq \text{Cl}_{\mathcal{Y}}(\{y\}) \cap V$ which implies $y \in V$. We conclude $K \subseteq \text{upset}(q[\alpha_0 \cup \ldots \cup \alpha_{n_0}]) \subseteq V$.

Now let $(y_j)_{j \leq \infty}$ be a convergent sequence of $\mathcal{Y}$ with $y_\infty \in V$. Then there is some $j_0 \in \mathbb{N}$ with $C := \{y_{j_0}, \ldots, y_\infty\} \subseteq V$. By the above consideration there is some $B \in \mathcal{B}$ with $C \subseteq B \subseteq V$, because $C$ is compact. Hence $\mathcal{B}$ is a pseudobase of $\mathcal{Y}$.

\[
\checkmark
\]

Similar to Lemma 3.1.5, one can prove the following necessary condition for sequential continuity between topological spaces with countable pseudobases.

\footnote{Actually, we have shown that $\mathcal{B}$ is a $\mathcal{M}$-pseudobase of $\mathcal{Y}$ (cf. Subsection 3.4.1).}
Lemma 3.1.13 (Necessary condition on pseudobases for seq. continuity)

Let $\mathfrak{X} = (X, \tau_\mathfrak{X})$ and $\mathfrak{Y} = (Y, \tau_\mathfrak{Y})$ be topological spaces, and let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be countable pseudobases of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Let $F : \subseteq X \rightrightarrows Y$ be a sequentially continuous correspondence. Then for every convergent sequence $(x_n)_{n \leq \infty}$ of $\mathfrak{X}$, every $y \in F[x_\infty]$ and every open neighbourhood $O \in \tau_\mathfrak{Y}$ of $y$ there are some $k \in \mathbb{N}$ and sets $A_0, \ldots, A_k \in \mathcal{B}_X$, $B \in \mathcal{B}_Y$ with

$$(\forall n \in \mathbb{N}) \ x_n \in A_0 \cap \ldots \cap A_k \text{ and } F[A_0 \cap \ldots \cap A_k] \subseteq B \subseteq O.$$ 

From Proposition 3.1.7 and Lemma 3.1.10 we obtain the following characterization of the topological spaces that have an admissible multirepresentation by means of pseudobases.

Proposition 3.1.14 (Charac. of admissibly representable top. spaces)

A topological space $\mathfrak{X} = (X, \tau_\mathfrak{X})$ has an admissible multirepresentation if and only if $\mathfrak{X}$ has a countable pseudobase.

Propositions 3.1.14, 2.3.13, and Lemma 2.4.6 imply

Proposition 3.1.15 (Charac. of admissibly representable top. spaces)

A topological space $\mathfrak{X} = (X, \tau_\mathfrak{X})$ has an admissible representation if and only if $\mathfrak{X}$ has a countable pseudobase and satisfies the $T_0$–property.

3.2 Characterization of Admissibly Representable Spaces as Quotients of Countably Based Spaces

In this section we prove that the objects of $\text{AdmWeakLim}$ (of $\text{AdmLim}$, of $\text{AdmSeq}$) are exactly the $\text{WeakLim}$–quotients (the $\text{Lim}$–quotients, the topological quotients) of countably based spaces\(^{2}\).

Lemma 3.1.4 and Proposition 3.1.7 imply that the class of admissibly representable weak limit spaces are closed under forming $\text{WeakLim}$–quotients. In particular, all $\text{WeakLim}$–quotients of countably based spaces have an admissible multirepresentation. It turns out that the converse holds as well. Thus we obtain the following Characterization Theorem of the weak limit spaces having an admissible multirepresentation.

Theorem 3.2.1 (Characterization of the spaces in $\text{AdmWeakLim}$)

A weak limit space $\mathfrak{X} = (X, \rightarrow_\mathfrak{X})$ has an admissible multirepresentation if and only if $\mathfrak{X}$ is a $\text{WeakLim}$–quotient of a countably based topological space.

\(^{2}\)A weak limit space $\mathfrak{X} = (X, \rightarrow_\mathfrak{X})$ is called a $\text{WeakLim}$–quotient (a $\text{Lim}$–quotient) of a topological space $\mathfrak{A} = (A, \tau_\mathfrak{A})$ iff there is surjective function $q : A \rightarrow X$ with $\rightarrow_{\mathfrak{A} \circ q} = \rightarrow_\mathfrak{X}$ (with $\rightarrow_{\mathfrak{A} \circ q} = \rightarrow_\mathfrak{A}$), cf. Subsections 2.2.4 and 2.4.5. A topological space $\mathfrak{X} = (X, \tau_\mathfrak{X})$ is called a topological quotient of a topological space $\mathfrak{A} = (A, \tau_\mathfrak{A})$ iff there is surjective function $q : A \rightarrow X$ with $\tau_\mathfrak{A} \circ q = \tau_\mathfrak{X}$ (cf. Subsection 2.2.2).
Proof:

$\leftarrow\rightarrow$: Let $\mathfrak{A} = (A, \tau_\mathfrak{A})$ be a countably based topological space and $q : A \to X$ be a function with $\sim_{\mathfrak{A}, q} = \sim_X$. By Lemma 3.1.4, $\mathfrak{A}$ has a countable limit base and hence an admissible multirepresentation by Proposition 3.1.7.

$\rightarrow\leftarrow$: Let $\delta : \subseteq \mathbb{N}^\omega \Rightarrow X$ be an admissible multirepresentation of $\mathfrak{X}$. We equip the set $\mathcal{G}(\delta) = \{(p, x) \mid x \in \delta[p]\}$ with the topology

$$\tau_\mathfrak{A} := \{(U \times X) \cap \mathcal{G}(\delta) \mid U \in \tau_{\mathbb{N}^\omega}\}.$$  

The family $\{(\mathbb{N}^\omega \times X) \cap \mathcal{G}(\delta) \mid W \subseteq \mathbb{N}^\omega, W \ \text{finite}\}$ forms a countable base of $\mathfrak{A} := (\mathcal{G}(\delta), \tau_\mathfrak{A})$. Obviously, a sequence $(p_n, x_n)_{n \leq \infty}$ in $\mathcal{G}(\delta)$ is a convergent sequence of $\mathfrak{A}$ if and only if $(p_n)_n$ converges to $p_\infty$ in $(\mathbb{N}^\omega, \tau_{\mathbb{N}^\omega})$.

We define $q : \mathcal{G}(\delta) \to X$ by $q(p, x) := x$ for all $(p, x) \in \mathcal{G}(\delta)$ and show $\sim_{\mathfrak{A}, q} = \sim_X$. Let $(x_n)_{n \leq \infty}$ be a sequence in $X$. By Proposition 2.4.18, $\delta$ induces the convergence relation of $\mathfrak{X}$. Hence we have

$$(x_n)_n \sim_{\mathfrak{A}} x_\infty \iff (x_n)_n \sim_X x_\infty$$

$\iff (\exists (p_n)_{n \leq \infty}) ((p_n)_n \sim_{\mathbb{N}^\omega} p_\infty \land (\forall n \in \mathbb{N}) x_n \in \delta[p_n])$

$\iff (\exists (a_n)_{n \leq \infty}) ((\forall n \in \mathbb{N}) (p_n, x_n) \in \mathcal{G}(\delta)) \land (p_n, x_n)_n \sim_{\mathfrak{A}} (p_\infty, x_\infty))$

$\iff (\exists (a_n)_{n \leq \infty}) ((\forall n \in \mathbb{N}) (q(a_n) = x_n))$

$\iff (x_n)_n \sim_{\mathfrak{A}, q} x_\infty.$

Hence $\mathfrak{X}$ is the WeakLim–quotient of $\mathfrak{A}$ generated by $q$.

Since by Lemma 2.4.17 all admissible multirepresentations of weak limit spaces satisfying the $T_0$–property are single–valued and the Baire space $(\mathbb{N}^\omega, \tau_{\mathbb{N}^\omega})$ is metrizable by a separable metric, we obtain the following corollary.

**Corollary 3.2.2 (Characterization of the $T_0$–spaces in AdmWeakLim)**

A weak limit space $\mathfrak{X} = (X, \sim_X)$ satisfying the $T_0$–property has an admissible representation if and only if $\mathfrak{X}$ is a WeakLim–quotient of a separable metric space.

Note that by Proposition 2.2.4 the weak limit spaces are exactly the WeakLim–quotients of (possibly non–separable) metric spaces.

One can show that a countable limit base $\mathcal{B}$ of a weak limit space $\mathfrak{X}$ is also a limit base of the associated limit space $\mathcal{L}(\mathfrak{X})$, whenever $\mathcal{B}$ is closed under finite union. This fact implies the following Characterization Theorem, being an analogue to Theorem 3.2.1. However, we prove this theorem with the help of Proposition 4.5.1.

**Theorem 3.2.3 (Characterization of the spaces in AdmLim)**

A limit space $\mathfrak{X} = (X, \sim_X)$ has an admissible multirepresentation if and only if $\mathfrak{X}$ is a Limit–quotient of a countably based topological space.

Proof:

$\leftarrow\rightarrow$: Let $\mathfrak{A} = (A, \tau_\mathfrak{A})$ be a countably based topological space and $q : A \to X$ be a surjective function with $\sim_{\mathfrak{A}, q} = \sim_X$. By Proposition 3.1.14, $\mathfrak{A} = (A, \tau_\mathfrak{A})$ has an admissible multirepresentation. Proposition 4.5.1 implies that $\mathfrak{X}$ has an admissible multirepresentation as well.
3.2 Characterization as Quotients of Countably Based Spaces

"⇒": By Theorem 3.2.1, there is a countably based space \( \mathfrak{A} = (A, \tau_{\mathfrak{A}}) \) and a function \( q : A \to X \) with \( \sim_{\mathfrak{A}, q} = \sim_X \). By Lemmas 2.2.8 and 2.4.10 we have

\[
\mathfrak{X} = \mathcal{L}(\mathfrak{X}) = \mathcal{L}(X, \sim_{\mathfrak{A}, q}) = (X, \sim_{\mathfrak{A}, q}).
\]

Hence \( \mathfrak{X} \) is the \( \text{Lim} \)-quotient of \( \mathfrak{A} \) generated by \( q \).

From Lemma 3.1.12 we obtain the following characterization of the sequential spaces with admissible multirepresentation.

**Theorem 3.2.4 (Characterization of the spaces in \( \text{AdmSeq} \))**

1. A sequential topological space \( \mathfrak{X} = (X, \tau_{\mathfrak{X}}) \) has an admissible multirepresentation if and only if \( \mathfrak{X} \) is a topological quotient of a countably based topological space.

2. A sequential topological space \( \mathfrak{X} = (X, \tau_{\mathfrak{X}}) \) has an admissible representation if and only if \( \mathfrak{X} \) has the \( T_0 \)-property and is a topological quotient of a countably based \( T_0 \)-space.

3. A topological space \( \mathfrak{X} = (X, \tau_{\mathfrak{X}}) \) has an admissible multirepresentation if and only if its sequentialization \( T(\mathfrak{X}) = (X, \text{seq}(\tau_{\mathfrak{X}})) \) is a topological quotient of a countably based space.

**Proof:**

1. "\( \Leftarrow \)" : Let \( \mathfrak{A} = (A, \tau_{\mathfrak{A}}) \) be a countably based and hence sequential topological space and \( q : A \to X \) be a surjective function with \( \tau_{\mathfrak{A}, q} = \tau_X \). Since topological bases are pseudobases, \( \mathfrak{X} \) has a countable pseudobase by Lemma 3.1.12 and thus a countable limit base. By Proposition 3.1.6, \( \mathfrak{X} \) has an admissible multirepresentation.

   The original proof avoiding pseudobases embeds the space \( \mathcal{W}(\mathfrak{X}) \) into the function space \( \mathcal{C}(\mathcal{C}(\mathfrak{X}, \mathfrak{S}), \mathfrak{S}) \), cf. Proposition 4.5.1 and Subsections 4.2.1, 4.3.1.

   "\( \Rightarrow \)" : By Theorem 3.2.1, there is a countably based space \( \mathfrak{A} = (A, \tau_{\mathfrak{A}}) \) and a function \( q : A \to X \) with \( \sim_{\mathfrak{A}, q} = \sim_X \). Lemma 2.2.7(6) implies \( \tau_{\mathfrak{A}, q} = \text{seq}(\sim_{\mathfrak{A}, q}) = \text{seq}(\sim_X) = \text{seq}(\tau_X) = \tau_X \). Hence \( \mathfrak{X} \) is the topological quotient of \( \mathfrak{A} \) generated by \( q \).

2. "\( \Leftarrow \)" : This is a consequence of (1) and Lemma 2.4.17.

   "\( \Rightarrow \)" : Let \( \delta : \subseteq \mathbb{N}^\omega \to X \) be an admissible representation of \( \mathfrak{X} \). We equip the set \( A := \text{dom}(\delta) \) with the subspace topology \( \tau_A \) inherited from \( \tau_{\mathfrak{X}} \) and define \( q : A \to X \) by \( q(p) := \delta(p) \). Clearly, a sequence \( (p_n)_{n \leq \infty} \) in \( A \) is a convergent sequence of \( \mathfrak{A} := (A, \tau_{\mathfrak{A}}) \) if and only if \( (p_n) \) converges to \( p_\infty \) w.r.t. \( \tau_{\mathfrak{X}} \). By Proposition 2.3.2, we have \( \sim_A = \sim_X \) and hence \( \sim_{\mathfrak{A}, q} = \sim_X \). Lemma 2.2.7(6) implies \( \tau_{\mathfrak{A}, q} = \text{seq}(\sim_{\mathfrak{A}, q}) = \text{seq}(\sim_X) = \text{seq}(\tau_X) = \tau_X \). Hence \( \mathfrak{X} \) is a topological quotient of \( \mathfrak{A} \).

3. This follows from (1) and Corollary 2.4.31.
Let $\mathfrak{X} = (X, \tau_X)$ be a weak limit space with admissible multirepresentation. From Theorems 3.2.1, 3.2.4, and Lemma 2.2.7(6) it follows that the associated topological space $T(\mathfrak{X})$ has an admissible multirepresentation, too. If $B$ is a countable limit base of $\mathfrak{X}$ that is closed under finite intersection and under finite union, then the family

$$\{ \text{upset}(B) \mid B \in B \},$$

where $\text{upset}(B) := \{ x \in X \mid \text{Cls}_{T(\mathfrak{X})}(\{x\}) \cap B \neq \emptyset \}$,

can be shown to be a countable pseudobase of $T(\mathfrak{X})$. Conversely, a weak limit space $\mathfrak{Y} = (Y, \tau_Y)$ such that the associated topological space $T(\mathfrak{Y})$ has an admissible representation need not be in $\text{AdmWeakLim}$. A counterexample is given by $Y := \mathbb{R} \cup \{ \perp \}$ equipped with the convergence relation $\rightarrow_\mathfrak{Y}$ defined by

$$(y_n)_n \rightarrow_\mathfrak{Y} \perp :\iff \{y_n \mid n \in \mathbb{N} \} \text{ is finite},$$

$$(y_n)_n \rightarrow_\mathfrak{Y} z :\iff (\exists n_0 \in \mathbb{N}) \{ \{y_n \mid n \geq n_0 \} \subseteq \mathbb{R} \text{ and } (y_{n+n_0})_n \rightarrow_X z \}$$

for all sequence $(y_n)_n$ in $Y$ and $z \in \mathbb{R}$. We omit the details.

### 3.3 Some Topological Properties of Admissibly Representable Topological Spaces

We list some additional topological properties of topological spaces that have an admissible multirepresentation.

**Proposition 3.3.1 (Properties of admissibly representable top. spaces)**

Let $\mathfrak{X} = (X, \tau_X)$ be a topological space, and $\delta :\subseteq \mathbb{N}^\omega \Rightarrow X$ be an admissible multirepresentation of $\mathfrak{X}$.

1. If $\mathfrak{X}$ is first–countable, then $\mathfrak{X}$ is countably based.
2. The space $\mathfrak{X}$ has a countable dense set$^3$.
3. The space $\mathfrak{X}$ is a hereditarily Lindelöf space$^4$.
4. If $\mathfrak{X}$ is regular$^5$, then $\mathfrak{X}$ is normal$^6$ and perfectly normal$^6$.
5. A subset $K \subseteq X$ is compact if and only if $K$ is countably compact.

$^3$A subset $D$ of $X$ is called dense in $\mathfrak{X}$ if its closure $\text{Cls}(D)$ is equal to $X$.

$^4$A topological space $\mathfrak{Y} = (Y, \tau_Y)$ is called a Lindelöf space if every open cover $O$ of $Y$ has a countable subcover (cf. [Wil70]). Note that we do not require of Lindelöf spaces to be regular, as some authors do. The space $\mathfrak{Y}$ is called a hereditarily Lindelöf space if every subspace of $\mathfrak{Y}$ is Lindelöf.

$^5$A topological space $\mathfrak{Y} = (Y, \tau_Y)$ is called regular (or a $T_3$–space) if $\mathfrak{Y}$ is a $T_1$–space and for every $y \in Y$ and every open set $O$ containing $y$ there is an open set $U$ with $y \in U \subseteq \text{Cls}(U) \subseteq O$, where $\text{Cls}(U)$ denotes the topological closure of $U$ (cf. [Eng89, Wil70]).

$^6$A topological space $\mathfrak{Y}$ is called normal (or a $T_4$–space) if $\mathfrak{Y}$ is a $T_1$–space and for all disjoint closed sets $A, B$ in $\mathfrak{Y}$ there are disjoint open sets $U, V$ in $\mathfrak{Y}$ with $A \subseteq U$ and $B \subseteq V$. It is called perfectly normal (or a $T_{\text{b}}$–space) if it is a $T_1$–space and for all disjoint closed sets $A, B$ in $\mathfrak{Y}$ there is some topologically continuous function $f : Y \rightarrow \mathbb{R}$ with $f^{-1}\{0\} = A$ and $f^{-1}\{1\} = B$ (cf. [Eng89, Wil70]).
3.3 Topological Properties of Admissibly Representable Topological Spaces

Proof:

(0) At first we show that a correspondence $\phi : \subseteq \Sigma^w \rightarrow X$ is sequentially continuous if and only if $\delta$ satisfies

$$\forall O \in \tau_X (\forall p \in \phi^{-1}[O]) (\exists k \in \mathbb{N}) \phi[p^{<k}\Sigma^w] \subseteq O.$$ 

$\implies$ : Let $O \in \tau_X$ and $p \in \phi^{-1}[O]$. Suppose that for every $k \in \mathbb{N}$ there is some $y_k \in \phi[p^{<k}\Sigma^w] \setminus O$. By sequential continuity of $\phi$, $(y_k)_k$ converges to each element $x \in \phi[p] \cap O \neq \emptyset$ which implies that $(y_k)_k$ is eventually in $O$, a contradiction.

$\impliedby$ : Let $(p_n)_{n \leq \infty}$ be a convergent sequence in $dom(\phi)$, let $(x_n)_{n \leq \infty}$ be a sequence with $x_n \in \phi[p_n]$ for all $n \in \mathbb{N}$, and let $O$ be an open neighbourhood of $x_\infty$. Since $p_\infty \in \phi^{-1}[O]$, there is some $k \in \mathbb{N}$ with $\phi[p^{<k}\Sigma^w] \subseteq O$. Moreover, there is some $n_0 \in \mathbb{N}$ with $p_n \in p^{<k}\Sigma^w$ for all $n \geq n_0$. Hence $(x_n)_n$ is eventually in $O$. Therefore $(x_n)_n$ converges to $x_\infty$ in $X$.

(1) At first we show that for every element $x \in X$ there is a name $p_x \in \delta^{-1}[x]$ such that the family $\{ Int(\delta[p^{<n}\Sigma^w]) \mid m \in \mathbb{N} \}$ is an open neighbourhood base of $x$.

Let $\{ U_0, U_1, U_2 \ldots \}$ be a countable open neighbourhood base of $x$. We define $\phi : \subseteq \mathbb{N}^w \rightarrow X$ by

$$dom(\phi) := \{ 0^w \} \cup \{ 0^n1q \mid n \in \mathbb{N}, q \in dom(\delta), \delta[q] \subseteq \bigcap_{i=0}^n U_i \}$$

and

$$\phi[0^w] := \{ x \} \quad \text{and} \quad \phi[0^n1q] := \delta[q]$$

for all $n \in \mathbb{N}$ and $q \in dom(\delta)$ with $\delta[q] \subseteq \bigcap_{i=0}^n U_i$. For showing sequential continuity of $\phi$ with the help of Statement (0), let $O \in \tau_X$ and $p \in \phi^{-1}[O]$.

Case $p = 0^w$: Since $\{ x \} = \phi[0^w] \subseteq O$, there is some $i \in \mathbb{N}$ with $U_i \subseteq O$. By definition of $\phi$, we have $\phi[0^i\mathbb{N}^w] \subseteq U_i \subseteq O$.

Case $p \neq 0^w$: There are $n \in \mathbb{N}$ and $q \in dom(\delta)$ with $p = 0^n1q$. As $\delta$ is continuous by (0), there is some $k \in \mathbb{N}$ with $\delta[q^{<k}\mathbb{N}^w] \subseteq O$. It follows $\phi[p^{<n+1+k}\mathbb{N}^w] \subseteq O$.

Statement (0) implies that $\phi$ is sequentially continuous. Hence there is a continuous function $g : \subseteq \mathbb{N}^w \rightarrow \mathbb{N}^w$ with $\phi = \delta \circ g$. We define $p_x := g(0^w)$ and prove that $p_x$ has the desired property. Since $x \in \phi[0^w]$, $p_x$ is $\delta$–name of $x$. Let $V$ be an open set containing $x$. Since $\delta$ is continuous, there is some $m \in \mathbb{N}$ with $\delta[p^{<m}\mathbb{N}^w] \subseteq V$. By continuity of $g$, there is some $n \in \mathbb{N}$ with $g[0^n\mathbb{N}^w] \subseteq p^{<m}\mathbb{N}^w$. By surjectivity and by sequential continuity of $\delta$ and by Statement (0) we have $\bigcap_{i=0}^n U_i = \phi[0^i\mathbb{N}^w]$. It follows

$$x \in \bigcap_{i=0}^n U_i = \phi[0^i\mathbb{N}^w] \subseteq \delta[g[0^n\mathbb{N}^w]] \subseteq \delta[p^{<m}\mathbb{N}^w] \subseteq V.$$ 

Thus $x \in Int(\delta[p^{<m}\mathbb{N}^w]) \subseteq V$. Hence $p_x$ has the desired properties.

We conclude that every open set $O \in \tau_X$ satisfies

$$O = \bigcup \{ Int(\delta[w\mathbb{N}^w]) \mid w \in \mathbb{N}^* \text{ and } Int(\delta[w\mathbb{N}^w]) \subseteq O \},$$

because for every $x \in O$ there is some $m \in \mathbb{N}$ with $Int(\delta[p^{<m}\mathbb{N}^w]) \subseteq O$. Thus $B := \{ Int(\delta[w\mathbb{N}^w]) \mid w \in \mathbb{N}^* \}$ is a countable base of $\tau_X$. 
Let $\text{Proposition 3.3.2 (Admissibly representable Hausdorff spaces)}$

Admissibly representable Hausdorff spaces enjoy the following properties.

(1) Let $\mathcal{X} = (X, \tau_X)$ be a Hausdorff space with an admissible representation. There exists a countable subset $B \subseteq \tau_X$ such that, for every compact subset $K$ of $\mathcal{X}$, the family $\{B \cap K \mid B \in \mathcal{B}\}$ is a base of the compact subspace $(K, \tau_X|_K)$.

(2) For every $w \in \mathbb{N}^*$ with $\delta[w\mathbb{N}^*] \neq \emptyset$ we choose some element $\alpha_w \in \delta[w\mathbb{N}^*]$ and define

$$D := \{ \alpha_w \mid w \in \mathbb{N}^* \text{ and } \delta[w\mathbb{N}^*] \neq \emptyset \}.$$  

Let $x \in X$. By surjectivity of $\delta$, there is some $p \in \mathbb{N}^*$ with $x \in \delta[p]$. From the continuity of $\delta$ it follows that $(\alpha_{p^n})_n$ converges to $x$. This implies $\text{Cls}(D) = X$.

(3) Let $M \subseteq X$ and let $\mathcal{O} \subseteq \tau_X$ be an open cover of $M$ (i.e. $M \subseteq \bigcup \mathcal{O}$). We define

$$W := \{ w \in \mathbb{N}^* \mid (\exists O \in \mathcal{O}) \delta[w\mathbb{N}^*] \subseteq O \}$$

and choose for each $w \in W$ some open set $O_w \in \mathcal{O}$ with $\delta[w\mathbb{N}^*] \subseteq O_w$. For every $x \in M$ there is some $O \in \mathcal{O}$ with $x \in O$ and therefore some $w \in W$ with $x \in \delta[w\mathbb{N}^*] \subseteq O$ by surjectivity and continuity of $\delta$. Hence $M \subseteq \bigcup\{O_w \mid w \in W\}$, i.e., the countable family $\{O_w \mid w \in W\}$ is a countable subcover of $M$. We conclude that $\mathcal{X}$ is a hereditarily Lindelöf space.

(4) Since $\mathcal{X}$ is regular Lindelöf space, $\mathcal{X}$ is normal by [Wil70, Theorem 16.8]. Let $O$ be an open set. By Proposition 3.1.15, $\mathcal{X}$ has a countable pseudobase $\mathcal{B}$. We define the family $\mathcal{A}$ of closed sets by

$$\mathcal{A} := \{ \text{Cls}(B) \mid B \in \mathcal{B} \text{ and } \text{Cls}(B) \subseteq O \}.$$  

Then $\bigcup \mathcal{A} \subseteq O$. Let $x \in O$. Since $\mathcal{X}$ is regular, there is some open set $U$ with $x \in U \subseteq \text{Cls}(U) \subseteq O$. As $\mathcal{B}$ is pseudobase of $\mathcal{X}$, there is some $B \in \mathcal{B}$ with $x \in B \subseteq U$. Since $x \in \text{Cls}(B) \subseteq \text{Cls}(U) \subseteq O$, we have $\text{Cls}(B) \in \mathcal{A}$. Hence $\bigcup \mathcal{A} = O$. Thus every open set in the normal space $\mathcal{X}$ is an $F_\sigma$–set, hence every closed set is a $G_\delta$–set. By [Eng89, Theorem 1.5.19], $\mathcal{X}$ is perfectly normal.

(5) By (3), every subspace of $\mathcal{X}$ is a Lindelöf space. Clearly, a Lindelöf space is countably compact if and only if it is compact (cf. [Wil70]).

Admissibly representable Hausdorff spaces enjoy the following properties.

**Proposition 3.3.2 (Admissibly representable Hausdorff spaces)**

Let $\mathcal{X} = (X, \tau_X)$ be a Hausdorff space with an admissible representation.

1. There exists a countable subset $\mathcal{B} \subseteq \tau_X$ such that, for every compact subset $K$ of $\mathcal{X}$, the family $\{B \cap K \mid B \in \mathcal{B}\}$ is a base of the compact subspace $(K, \tau_X|_K)$.

2. If $\mathcal{X}$ is locally compact\(^7\), then $\mathcal{X}$ is countably based.\(^8\)

3. A subset $K \subseteq X$ is compact in $\mathcal{X}$, iff $K$ is sequentially compact in $\mathcal{X}$, and also iff $K$ is compact in the sequentialization $\mathcal{T}(\mathcal{X}) = (X, \text{seq}(\tau_X))$.

\(^7\)A topological space is called locally compact iff every point of the space has a neighbourhood base of compact sets (cf. [Smy92]).

\(^8\)Additionally, one can show by Lemma 3.4.1 that spaces in $\text{AdmSeq}$ that are locally compact or core–compact have a countable base. Compact Hausdorff spaces with admissible representation are countably based, because they are locally compact, whereas compact non–Hausdorff spaces in $\text{AdmSeq}$ do not necessarily have a countable base: they need not be locally compact.
3.3 Topological Properties of Admissibly Representable Topological Spaces

(4) The sequentialization $\mathcal{T}(\mathcal{X}) = (X, \text{seq}(\tau_{\mathcal{X}}))$ of $\mathcal{X}$ is equal to the Kelleyfication\footnote{The Kelleyfication of $\mathcal{X}$ is defined to be the topological space $k(\mathcal{X}) := (X, k(\tau_{\mathcal{X}}))$, where the topology $k(\tau_{\mathcal{X}})$ is given by $k(\tau_{\mathcal{X}}) := \{O \subseteq X \mid (\forall K \subseteq X, K \text{ compact})(\exists V \in \tau_{\mathcal{X}})K \cap V = K \cap O\}$.} $k(\mathcal{X}) = (X, k(\tau_{\mathcal{X}}))$ of $\mathcal{X}$.

Proof:

(1) We adapt the proof of Theorem 3.1.19 in [Eng89]. By Proposition 3.1.14, $\mathcal{X}$ has a countable pseudobase $\{\beta_0, \beta_1, \ldots\}$. For every pair $(i, j)$ in the set

$$J := \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid (\exists U, V \in \tau_{\mathcal{X}})(U \cap V = \emptyset, \beta_a \subseteq U, \beta_b \subseteq V)\}$$

we choose two open sets $U_{i,j}$ and $V_{i,j}$ satisfying

$$U_{i,j} \cap V_{i,j} = \emptyset, \beta_i \in U_{i,j} \text{ and } \beta_j \in V_{i,j}.$$ 

We define $\mathcal{B}$ to be the family of all finite intersections of sets in $\{U_{i,j} \mid (i, j) \in J\}$. Let $K$ be a compact subset of $\mathcal{X}$. Let $O \in \tau_{\mathcal{X}|K}$ and $x \in O$. There is some open set $O' \in \tau_{\mathcal{X}}$ with $O = O' \cap K$. Let $y \in K \setminus O$. Since $\mathcal{X}$ is a Hausdorff space, there are disjoint open sets $D, D' \in \tau_{\mathcal{X}}$ with $x \in D$ and $y \in D'$. As $\{\beta_0, \beta_1, \ldots\}$ is a pseudobase of $\mathcal{X}$, there exist two numbers $i_y, j_y \in \mathbb{N}$ with $x \in \beta_{i_y} \subseteq D$ and $y \in \beta_{j_y} \subseteq D'$. Since $(i_y, j_y) \in J$, the family $\{V_{i,y} \mid z \in K \setminus O \cup \{O'\}\}$ is an open cover of $K$. By compactness of $K$, there are elements $z_1, \ldots, z_m \in K \setminus O$ with

$$K \setminus O \subseteq V_{i_{z_1}, j_{z_1}} \cup \ldots \cup V_{i_{z_m}, j_{z_m}}.$$ 

Clearly, the sets $U := U_{i_{z_1}, j_{z_1}} \cap \ldots \cap U_{i_{z_m}, j_{z_m}} \in \mathcal{B}$ and $V := V_{i_{z_1}, j_{z_1}} \cup \ldots \cup V_{i_{z_m}, j_{z_m}}$ are disjoint. Hence we have

$$x \in K \cap U \subseteq K \setminus V \subseteq K \setminus (K \setminus O) = O.$$ 

Therefore $\{K \cap B \mid B \in \mathcal{B}\}$ is a base of $(K, \tau_{\mathcal{X}|K})$.

(2) Let $x \in \mathcal{X}$. There exists a compact neighbourhood $K$ of $x$. By (1) there is a countable subset $\mathcal{B} \subseteq \tau_{\mathcal{X}}$ such that $\{K \cap B \mid B \in \mathcal{B}\}$ is a base of $(K, \tau_{\mathcal{X}|K})$. Let $O \in \tau_{\mathcal{X}}$ be an open set containing $x$. Since $O \cap \text{Int}(K)$ is open in $(K, \tau_{\mathcal{X}|K})$, there is some $B \in \mathcal{B}$ with $x \in K \cap B \subseteq O \cap \text{Int}(K)$. It follows $x \in \text{Int}(K) \cap B \subseteq O$. Hence $\{\text{Int}(K) \cap B \mid B \in \mathcal{B}, x \in B\}$ is a countable neighbourhood base of $x$. We conclude that $\mathcal{X}$ is first–countable. Hence $\mathcal{X}$ is countably based by Proposition 3.3.1(1).

(3) (a) Let $K$ be compact in $\mathcal{X}$. Since the subspace $(K, \tau_{\mathcal{X}|K})$ has an admissible representation, namely the restriction $\delta|_K$, $(K, \tau_{\mathcal{X}|K})$ is countably based by (2). Since sequential compactness and compactness coincide in countably based spaces (cf. [Wil70, Ex. 17G + Theorem 16.9]), $K$ is sequentially compact.

(b) Let $K$ be sequentially compact in $\mathcal{X}$. Then $K$ is sequentially compact in $\mathcal{T}(\mathcal{X})$, because $\mathcal{X}$ and $\mathcal{T}(\mathcal{X})$ have the same convergent sequences (cf. Lemma 2.2.7). Since sequential compactness implies countable compactness (cf. [Wil70, Ex. 17G]), $K$ is countably compact in $\mathcal{T}(\mathcal{X})$. As $\delta$ is an admissible multirepresentation of $\mathcal{T}(\mathcal{X})$ by Lemma 2.4.30, $K$ is compact in $\mathcal{T}(\mathcal{X})$ by Proposition 3.3.1(5).
Chapter 3. Characterization

(c) Every set $K$ that is compact in $T(\mathfrak{X})$ is also compact in $\mathfrak{X}$, because the topology $\text{seq}(\tau_{\mathfrak{X}})$ contains $\tau_{\mathfrak{X}}$ as a subset (cf. Lemma 2.2.7).

(4) $k(\tau_{\mathfrak{X}}) \subseteq \text{seq}(\tau_{\mathfrak{X}})$: Let $O \in k(\tau_{\mathfrak{X}})$. Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $\mathfrak{X}$ with $x_\infty \in O$. Since the set $K := \{ x_n \mid n \in \mathbb{N} \}$ is compact in $\mathfrak{X}$ (cf. Subsection 2.2.2), there is an open set $V \in \tau_{\mathfrak{X}}$ with $K \cap V = K \cap O$. As $x_\infty \in V$, there is some $n_0 \in \mathbb{N}$ with $(\forall n \geq n_0) x_n \in V$. Therefore $(x_n)_{n}$ is eventually in $O$. We conclude that $O$ is sequentially open. Hence $k(\tau_{\mathfrak{X}}) \subseteq \text{seq}(\tau_{\mathfrak{X}})$.

$\text{seq}(\tau_{\mathfrak{X}}) \subseteq k(\tau_{\mathfrak{X}})$: Let $O \in \text{seq}(\tau_{\mathfrak{X}})$. Let $K$ be a compact subset of $\mathfrak{X}$ and $x \in K \cap O$. By (2), the subspace $(K, \tau_{\mathfrak{X}|K})$, being a compact admissibly representable Hausdorff space by Subsection 4.1.5, is countably based. Thus there is a sequence $U_{x,0}, U_{x,1}, \ldots$ of open neighbourhoods of $x$ in $\mathfrak{X}$ such that $\{ K \cap U_{x,i} \mid i \in \mathbb{N} \}$ forms a countable neighbourhood base of $x$ in $(K, \tau_{\mathfrak{X}|K})$. Suppose for contradiction $(\forall n \in \mathbb{N}) K \cap \bigcap_{i=0}^{n} U_{x,i} \not\subseteq K \cap O$. Then for every $n \in \mathbb{N}$ there is some $y_n \in K \cap \bigcap_{i=0}^{n} U_{x,i} \setminus O$. For every $V \in \tau_{\mathfrak{X}|K}$ with $x \in V$ there is some $j \in \mathbb{N}$ with $K \cap U_{x,j} \subseteq V$. It follows $y_n \in V$ for every $n \geq j$. Therefore $(y_n)_{n}$ converges to $x$ in $(K, \tau_{\mathfrak{X}|K})$ and thus in $\mathfrak{X}$. This implies that $(y_n)_{n}$ is eventually in the sequentially open set $O$, a contradiction. Therefore there is some $n_x \in \mathbb{N}$ with $K \cap \bigcap_{i=0}^{n_x} U_{x,i} \subseteq K \cap O$. We conclude

$$K \cap O = K \cap \bigcup_{x \in K \cap O} \bigcap_{i=0}^{n_x} U_{x,i}.$$ 

Hence $O \in k(\tau_{\mathfrak{X}})$ and $\text{seq}(\tau_{\mathfrak{X}}) \subseteq k(\tau_{\mathfrak{X}})$.

Moreover, one can prove that for an admissibly representable regular space $\mathfrak{X} = (X, \tau_{\mathfrak{X}})$ the family $B$ in Proposition 3.3.2(1) can be chosen in such a way that $B$ is a countable base of a regular space with underlying set $X$.

There are several interesting open problems on the properties of admissibly representable topological space. We formulate some of them:

(1) Is every compact subset of an arbitrary admissibly representable topological space sequentially compact?

(2) Is the kellyfication of an arbitrary admissibly representable topological space equal to the sequentialization?

(3) Does kellyfication of admissibly representable topological spaces preserve regularity?

(4) Does kellyfication of admissibly representable topological spaces preserve zero–dimensionality?

If the answer to the last question is yes, then the function space $\mathbb{R}^\mathbb{R}$ in our approach is equal to $\mathbb{R}^\mathbb{R}$ in M. Escardó’s functional programming approach $\text{RealPCF+}$ (cf. [BES02]).
3.4 Comparison with Other Concepts

In this section we investigate the relationship of the class of spaces that have an admissible multirepresentation with other concepts. The $\aleph_0$–spaces (cf. Subsection 3.4.1) have been defined in 1966 by E. Michael, apparently without any intention towards computability. The categories $\mathbb{P}Q$ and $\mathbb{P}Q_L$ (cf. Subsection 3.4.2) and also the category $\text{Equ}$ (cf. Subsection 3.4.3) have been introduced in order to describe topological aspects of computations on non–discrete sets. We prove that $\mathbb{P}Q$ is equal to $\text{AdmSeq}$ and that $\mathbb{P}Q_L$ is equal to $\text{AdmLim}$.

3.4.1 $\aleph_0$–Spaces versus Admissibly Representable Regular Spaces

In [Mi66], E. Michael introduced the class of $\aleph_0$–spaces consisting of certain regular topological spaces and proved that it has nice closure properties. We now show that this class is equal to the class of the regular topological spaces with admissible representation.

For the definition of $\aleph_0$–spaces, E. Michael also used a notion of pseudobase. Since his notion does not coincide exactly with our notion (cf. Subsection 3.1.3), we will use the term “$\mathcal{M}$–pseudobase” instead.

Let $(X, \tau_X)$ be a topological space. A family $B$ of subsets of $X$ is called a $\mathcal{M}$–pseudobase of $X$ iff for all compact sets $K$ of $X$ and all open sets $O \in \tau_X$ with $K \subseteq O$ there is some $B \in B$ with $K \subseteq B \subseteq O$. A topological space is called an $\aleph_0$–space iff it is regular and has a countable $\mathcal{M}$–pseudobase (cf. [Mi66]).

We prove the following relationship between pseudobases and $\mathcal{M}$–pseudobases.

**Lemma 3.4.1 (Pseudobases versus $\mathcal{M}$–pseudobases)**

Let $\mathcal{X} = (X, \tau_X)$ be a topological space.

1. Every $\mathcal{M}$–pseudobase of $X$ is also a pseudobase of $X$.
2. If $\mathcal{X}$ is sequential and has a countable pseudobase, then $\mathcal{X}$ has a countable $\mathcal{M}$–pseudobase.
3. If $\mathcal{X}$ is a Hausdorff space and has a countable pseudobase, then $\mathcal{X}$ has a countable $\mathcal{M}$–pseudobase.

**Proof:**

1. Let $B$ be a $\mathcal{M}$–pseudobase of $X$. Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $X$, and let $U \in \tau_X$ contain $x_\infty$. Then there is some $n_0 \in \mathbb{N}$ with $(\forall n \geq n_0) x_n \in U$. The set $K := \{x_\infty, x_n \mid n \geq n_0\} \subseteq U$ is compact in $\mathcal{X}$, because, whenever $O$ is an open cover of $K$ and $O \in \mathcal{O}$ contains $x_\infty$, $(x_n)_n$ is eventually in $O$. Since $B$ is a $\mathcal{M}$–pseudobase of $X$, there is a set $B \in B$ with $K \subseteq B \subseteq U$. We conclude that $B$ is a pseudobase of $X$.

2. Let $A$ be a countable pseudobase of $X$. An inspection of the proof of Lemma 3.1.12 in the case $\mathcal{Y} := \mathcal{X}$ and $q := id_X$ shows that we have actually proved that $B := \{\text{upset}(A_1 \cup \ldots \cup A_k) \mid k \in \mathbb{N} \text{ and } \{A_1, \ldots, A_k\} \subseteq A\}$, where $\text{upset}(M) := \{x \in X \mid \text{Cl}_{X}(\{x\}) \cap M \neq \emptyset\}$, is a $\mathcal{M}$–pseudobase of $\mathcal{X}$. Another proof is possible by using Proposition 4.4.9.
(3) This follows from (2), Proposition 3.3.2(3), and Corollary 3.1.11. Actually, with
the help of the fact that compactness and sequential compactness coincide in
Hausdorff spaces with admissible multirepresentations (cf. Proposition 3.3.2), one
can show that the closure under finite union of a countable pseudobase of \( \mathcal{X} \) is a
\( \mathcal{M} \)-pseudobase of the Hausdorff space \( \mathcal{X} \).

\[
\checkmark
\]

Lemma 3.4.1 and Proposition 3.1.14 imply the following characterization of regular
spaces with admissible representations.

**Proposition 3.4.2 (\( \aleph_0 \)-spaces versus admissibly representable spaces)**

A regular space \( \mathcal{X} = (X, \tau_X) \) has an admissible representation if and only if \( \mathcal{X} \) is an
\( \aleph_0 \)-space.

From this proposition and Example 12.4 in [Mi66] we obtain the interesting result
that a countable regular topological space need not necessarily have an admissible
representation. A counterexample is constructed by equipping the set of continuous
functions between the Cantor space and the discrete space \( \{0, 1\} \) with the
topology of pointwise convergence (cf. [Eng89, Wil70]) and by selecting an appropriate
countable subspace of that function space (cf. [Mi66, Ex. 12.4]).

**Lemma 3.4.3**

There exists a regular topological space with countable underlying set which does not
have any admissible representation.

This lemma implies that topological spaces that have a countable point–pseudobase
need not have an admissible multirepresentation or a countable pseudobase. A point–
pseudobase \( B \) of a topological space is a family of subsets such that for every open set
\( O \) and every point \( x \in O \) there is some \( B \in B \) with \( x \in B \subseteq O \) (cf. [Mi66]). Another
name for point–pseudobase is network (cf. [Eng89]).

Example 12.7 in [Mi66] and Proposition 3.4.2 yield the following lemma.

**Lemma 3.4.4**

There exists a regular topological space which is the topological quotient of an admissibly
representable regular space, but does not have any admissible representation.

However, the topological quotients of admissibly representable sequential spaces have
admissible multirepresentations (cf. Corollary 4.5.3 or Lemma 3.1.12).

### 3.4.2 The Categories \( \text{PQ}, \text{PQ}_L, \) and \( \text{PQ}_W \)

In [MS02], M. Menni and A. Simpson introduced the categories \( \text{PQ} \) and \( \text{PQ}_L \). In this
subsection we prove that these categories are equal to \( \text{AdmSeq} \) and \( \text{AdmLim} \), respectivly.
Furthermore we extend \( \text{PQ}_L \) straightforwardly to a category \( \text{PQ}_W \) consisting of
certain weak limit spaces and show that \( \text{PQ}_W \) is equal to \( \text{AdmWeakLim} \).

Let \( \mathcal{X} = (X, \to_\mathcal{X}) \) be a weak limit space and \( \mathfrak{A} = (A, \tau_A) \) be a countably based
topological space. A function \( q : A \to X \) is called \( \omega \)-projecting iff \( q \) is sequentially
continuous and for every countably based space \( \mathfrak{Z} = (Z, \tau_Z) \) and every continuous
function \( f : Z \to X \) there is a continuous function \( g : Z \to A \) with \( f = q \circ g \) (cf. [MS02]). In this situation \( \mathcal{X} \) is called an \( \omega \)-projecting quotient of \( \mathfrak{A} \). This name is
justified by the following lemma.
3.4 Comparison with Other Concepts

Lemma 3.4.5
Let \( q : A \to X \) be an \( \omega \)-projecting function between a countably based space \( A = (A, \tau_A) \) and a weak limit space \( \mathfrak{X} = (X, \to_X) \). Then \( \mathfrak{X} \) is the WeakLim-quotient of \( A \) generated by \( q \).

Proof:
Let \( (x_n)_{n \leq \infty} \) be a convergent sequence of \( \mathfrak{X} \). By Axiom (L5) the function \( n \mapsto x_n \) is continuous w.r.t. \( \to_X \) and \( \to_X \). Since \( \to_X \) is induced by a countably based topology, there is a continuous function \( g : \mathbb{N} \to A \) with \((\forall n \in \mathbb{N}) q(g(n)) = x_n \). As \((g(n))_{n \leq \infty} \) is a convergent sequence of \( A \), \((x_n)_{n \leq \infty} \) converges to \( x_\infty \) w.r.t. \( \to_{a,q} \). Thus \( \to_X \subseteq \to_{a,q} \). The continuity of \( q \) implies \( \to_{a,q} \subseteq \to_X \). Hence \( q \) is a WeakLim-quotient function.

The countably based space \( A \) can be viewed as a “representing” space, and \( q \) plays the role of a “representation”. We prove the following analogue of Theorem 2.3.18.

Lemma 3.4.6
Let \( \mathfrak{X} = (X, \to_X) \) and \( \mathfrak{Y} = (Y, \to_Y) \) be weak limit spaces, let \( A = (A, \tau_A) \) and \( B = (B, \tau_B) \) be countably based topological spaces, and let \( p : A \to X \) and \( q : B \to Y \) be \( \omega \)-projecting functions. Then a function \( f : X \to Y \) is sequentially continuous if and only if there is a continuous function \( g : A \to B \) with \( q \circ g = f \circ p \).

Proof:
“\( \Rightarrow \)” : Since \( f \circ p \) is a continuous function with a countably based domain and \( q \) is \( \omega \)-projecting, there is a continuous function \( g : A \to B \) with \( f \circ p = q \circ g \).

“\( \Leftarrow \)” : Let \( (x_n)_{n \leq \infty} \) be a convergent sequence of \( \mathfrak{X} \). By Lemma 3.4.5 there is a convergent sequence \( (a_n)_{n \leq \infty} \) of \( A \) with \((\forall n \in \mathbb{N}) p(a_n) = x_n \). It follows \( f(x_n) = f(p(a_n)) = q(g(a_n)) \) for all \( n \in \mathbb{N} \). As \( q \circ g \) is sequentially continuous, \((f(x_n))_n \) converges to \( f(x_\infty) \). Hence \( f \) is sequentially continuous.

M. Menni and A. Simpson define \( \text{PQL} \) (\( \text{PQ} \)) to be the category of limit spaces \( \mathfrak{X} = (X, \to_X) \) (of sequential topological spaces \( \mathfrak{X} = (X, \tau_X) \)) for which there are a countably based topological space \( A = (A, \tau_A) \) and an \( \omega \)-projecting function \( q : A \to X \) (cf. [MS02]). The morphisms of \( \text{PQL} \) and \( \text{PQ} \) are the sequentially continuous functions between the considered spaces.

It is natural to define \( \text{PW} \) as the category whose objects are the weak limit spaces that are \( \omega \)-projecting quotients of countably based topological spaces and whose morphisms are the sequentially continuous functions. Obviously, \( \text{PQL} \) and \( \text{PQ} \) are full subcategories of \( \text{PW} \).

A. Bauer has proved that every sequential \( T_0 \)–space is an object of \( \text{PQ} \) if and only if it has an admissible representation (cf. [Bau00, Bau01]). We show the following generalization of this result by extending A. Bauer’s proof.

Proposition 3.4.7
A weak limit space \( \mathfrak{X} = (X, \to_X) \) has an admissible multirepresentation if and only if \( \mathfrak{X} \) is an object of \( \text{PW} \).
Proof:

“$\Longleftrightarrow$”: There are a countably based space $\mathfrak{X} = (A, \tau_\mathfrak{A})$ and an $\omega$–projecting function $q : A \to X$. By Lemma 3.4.5, $q$ is a WeakLim–quotient mapping between $\mathfrak{A}$ and $\mathfrak{X}$. Lemma 3.1.4 implies that $\mathfrak{X}$ has a countable limit base. Hence $\mathfrak{X}$ has an admissible multirepresentation by Proposition 3.1.7.

“$\Rightarrow$”: By Proposition 3.1.7, $\mathfrak{X}$ has a countable limit base $\{\beta_0, \beta_1, \beta_2, \ldots\}$. We define the set $A \subseteq 2^\mathbb{N} \times X$ by

$$(P, x) \in A :\iff \begin{cases} P \subseteq \{i \in \mathbb{N} | x \in \beta_i\} \\
(\forall (z_n)_n \in X^\mathbb{N}) ((z_n)_n \not\in_x x \implies (\exists i \in P)(\exists n \in \mathbb{N}) z_n \notin \beta_i)
\end{cases}$$

for $P \subseteq \mathbb{N}$ and $x \in X$. Let $\tau_2^\mathfrak{A}$ be the Scott–topology on $2^\mathbb{N}$ which is induced by the countable subbase $\{(P \subseteq \mathbb{N} | m \in P) | m \in \mathbb{N}\} \cup \{2^\mathbb{N}\}$. We equip $A$ with the subspace topology $\tau_\mathfrak{A}$ inherited from the product topology of $\tau_2^\mathfrak{A}$ and the indiscrete topology $\{\emptyset, \{X\}\}$. Then $\mathfrak{A} := (A, \tau_\mathfrak{A})$ is countably based, because $\tau_2^\mathfrak{A}$ and $\{\emptyset, \{X\}\}$ have countable bases. We define $q : A \to X$ by $q(P, x) := x$ and show that $q$ is $\omega$–projecting.

Continuity: Let $(P_n, x_n)_{n \leq \infty}$ be a convergent sequence of $\mathfrak{A}$.

Suppose for contradiction that $(x_n)_n$ does not converge to $x_\infty$. By definition of $A$, there is some $i \in P_\infty$ with $x_n \notin \beta_i$ for infinitely many $n$. As $(P_n)_{n \leq \infty}$ is a convergent sequence of $(2^\mathbb{N}, \tau_2^\mathfrak{A})$, there is some $n_0 \in \mathbb{N}$ with $(\forall n \geq n_0) i \in P_n$. This implies $x_n \notin \beta_i$ for all $n \geq n_0$, a contradiction.

Hence $(q(P_n, x_n))_n$ converges to $q(P_\infty, x_\infty) = x_\infty$. Therefore $q$ is sequentially continuous.

$\omega$–Projectivity: Let $\mathfrak{Z} = (Z, \tau_3)$ be a countably based topological space, and let $f : Z \to X$ be a sequentially continuous function from $\mathfrak{Z}$ to $\mathfrak{X}$. We choose a countable base $\{T_{\omega}, T_{\omega_1}, T_{\omega_2}, \ldots\}$ of $\mathfrak{Z}$ and define the function $h : Z \to 2^\mathbb{N}$ by

$$h(z) := \{i \in \mathbb{N} | (\exists j \in \mathbb{N})(z \in \alpha_j \land f[\alpha_j] \subseteq \beta_i)\}$$

for all $z \in Z$. Then $h$ is topologically continuous w.r.t. $\tau_3$ and $\tau_2^\mathfrak{A}$, because for every $m \in \mathbb{N}$ the inverse image of the subbase set $\{P \subseteq \mathbb{N} | m \in P\}$ satisfies

$$h^{-1}\{P \subseteq \mathbb{N} | m \in P\} = \bigcup\{\alpha_j | j \in \mathbb{N} \land f[\alpha_j] \subseteq \beta_m\}$$

and is thus open in $\mathfrak{Z}$.

Let $z \in Z$. We show $(h(z), f(z)) \in A$. Clearly, $h(z) \subseteq \{i \in \mathbb{N} | f(z) \in \beta_i\}$. Let $(y_n)_n$ be a sequence in $X^\mathbb{N}$ that does not converge to $f(z)$ by contradiction. Let $y_n)_n$ be a sequence in $X^\mathbb{N}$ that does not converge to $f(z)$. By continuity of $f$ and by Lemma 3.1.5, there are some $i, k \in \mathbb{N}$ and $l_0, \ldots, l_k \in \mathbb{N}$ with

$$z \in \alpha_{l_0} \cap \ldots \cap \alpha_{l_k}, f[\alpha_{l_0} \cap \ldots \cap \alpha_{l_k}] \subseteq \beta_i$$

because $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ is a limit base of $\mathfrak{Z}$. Since $\alpha_{l_0} \cap \ldots \cap \alpha_{l_k}$ is an open set, there is some $j \in \mathbb{N}$ with $z \in \alpha_j \subseteq \alpha_{l_0} \cap \ldots \cap \alpha_{l_k}$. It follows $i \in h(z)$. This implies $(h(z), f(z)) \in A$.

Hence we can define $g : Z \to A$ by $g(z) := (h(z), f(z))$ for all $z \in Z$. Clearly, $f = q \circ g$. Since $f$ is continuous w.r.t. $\tau_3$ and $\tau_3$ and $\tau_2^\mathfrak{A}$, $g$ is $\tau_3$–continuous. This proves the $\omega$–projectivity of $q$.

We conclude that $\mathfrak{X}$ is an object of $\mathcal{PQ}_W$. 

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Since the objects of \( \mathcal{PQ}_L \) are exactly the limit spaces in \( \mathcal{PQ}_W \) and since the objects of \( \mathcal{PQ} \) are those sequential spaces \( \mathfrak{X} \) for which \( \mathcal{W}(\mathfrak{X}) \) is an object of \( \mathcal{PQ}_W \), Proposition 3.4.7 implies the following Characterization Theorem.

**Theorem 3.4.8**

\[ \mathcal{PQ}_W = \text{AdmWeakLim}, \mathcal{PQ}_L = \text{AdmLim}, \mathcal{PQ} = \text{AdmSeq}. \]

### 3.4.3 Relationship with the Category \( \mathcal{EQU} \)

The category \( \mathcal{EQU} \) of equilogical spaces and equivariant maps has been introduced by D.S. Scott. The benefit of \( \mathcal{EQU} \) is that it forms a Cartesian closed category that contains the non Cartesian closed category \( \mathcal{Top} \). The benefit of \( \mathcal{PQ} \) is that it contains the non Cartesian closed category \( \mathcal{Top} \).

D.S. Scott. The benefit of \( \mathcal{PQ} \) is that it contains the non Cartesian closed category \( \mathcal{Top} \).

Moreover, \( \mathcal{PQ} \) is a largest common subcategory of \( \mathcal{EQU} \) and \( \mathcal{Seq} \), and \( \mathcal{PQ}_L \) is a largest common subcategory of \( \mathcal{EQU} \) and \( \mathcal{Lim} \) (cf. [MS02]). We show that also \( \text{AdmWeakLim} = \mathcal{PQ}_W \) is a full subcategory of \( \mathcal{EQU} \). The corresponding inclusion functor \( \mathcal{I} : \mathcal{PQ}_W \to \mathcal{EQU} \) can be constructed as follows: For a weak limit space \( \mathfrak{X} = (X, \tau_\mathfrak{X}) \) we choose a countably based space \( \mathfrak{A}_\mathfrak{X} = (A_\mathfrak{X}, \tau_\mathfrak{X}) \) and an \( \omega \)-projecting function \( q_\mathfrak{X} : A_\mathfrak{X} \to X \), then we equip \( A_\mathfrak{X} \) with the equivalence relation \( \equiv \subseteq A_\mathfrak{X} \times A_\mathfrak{X} \) given by \( a_1 \equiv a_2 \iff q_\mathfrak{X}(a_1) = q_\mathfrak{X}(a_2) \) and define the equilogical space \( \mathcal{I}(\mathfrak{X}) \) by \( \mathcal{I}(\mathfrak{X}) := (\mathfrak{A}_\mathfrak{X}, \equiv) \). Moreover, for a continuous function \( f : X \to Y \) between weak limit spaces \( \mathfrak{X} = (X, \tau_\mathfrak{X}) \) and \( \mathfrak{Y} = (Y, \tau_\mathfrak{Y}) \) we choose some topologically continuous \( g : A_\mathfrak{X} \to A_\mathfrak{Y} \) with \( f \circ q_\mathfrak{X} = q_\mathfrak{Y} \circ g \) (cf. Lemma 3.4.6) and define the morphism \( \mathcal{I}(f) \) between \( \mathcal{I}(\mathfrak{X}) \) and \( \mathcal{I}(\mathfrak{Y}) \) as the topologically continuous function \( \mathcal{I}(f) : [A_\mathfrak{X}]_\equiv \to [A_\mathfrak{Y}]_\equiv \). Moreover, \( \mathcal{I}(f) \) satisfies \( \mathcal{I}(f_1 \circ f_2) = \mathcal{I}(f_1) \circ \mathcal{I}(f_2) \). Lemma 3.4.6 implies that \( \mathcal{I} \) restricted to \( \mathcal{C}(\mathfrak{X}, \mathfrak{Y}) \) is a bijection between \( \mathcal{C}(\mathfrak{X}, \mathfrak{Y}) \) and the set of morphisms between \( \mathcal{I}(\mathfrak{X}) \) and \( \mathcal{I}(\mathfrak{Y}) \), i.e., \( \mathcal{I} \) is full and faithful. Hence \( \mathcal{I} \) establishes \( \text{AdmWeakLim} = \mathcal{PQ}_W \) as a full subcategory of \( \mathcal{EQU} \).

\(^{10}\)We follow here the definition in [MS02]. In the original definition the topological space \( \mathfrak{Y} \) is required to be a \( T_0 \)-space.
Chapter 4
Constructions

In this chapter we investigate operators on spaces that preserve the existence of an admissible multirepresentation. We begin with initial constructions in Section 4.1. Examples for initial constructions are product and conjunction. Section 4.2 is devoted to exponentiation. In particular, we prove that the category of admissibly representable spaces is Cartesian closed. In Section 4.3 we introduce three operators on multirepresentations that transform multirepresentations into admissible ones while preserving relative computability of functions. This proves the fundamental result that the use of admissible multirepresentations does not decrease the class of relatively computable functions. In Section 4.4 admissible multirepresentations of hyperspaces are investigated. Section 4.5 is devoted to final constructions. Examples for final constructions are coproduct and disjunction as well as forming quotient spaces.

4.1 Initial Constructions

This section is devoted to the investigation of initial constructions. Generating a space $X$ by an initial construction means to equip the underlying $X$ with the coarsest (largest) convergence relation or with the coarsest (smallest) topology such that a given countable family of functions $h_i$ from the domain $X$ into given spaces $Y_i$ become continuous. In Subsections 4.1.1 and 4.1.2 we prove that this construction principle preserves admissible representability. Product and conjunction of spaces (cf. Subsection 4.1.4) as well as forming subspaces (cf. Subsection 4.1.5) are basic examples for initial constructions. In Subsection 4.1.6 we demonstrate that the categories $\text{AdmWeakLim}$, $\text{AdmLim}$, and $\text{AdmSeq}$ of admissibly representable spaces are closed under equalizers and countable limits (formed in $\text{WeakLim}$).

4.1.1 Initial Convergence Relations

Let $X$ be a set, and let $\mathfrak{Y}_i = (Y_i, \to_{\mathfrak{Y}_i})$ be a weak limit space for each $i \in \mathbb{N}$. Any sequence $(h_i)_i$ of functions $h_i : X \to Y_i$ induces a convergence relation on $X$ defined by

\[(x_n) \to^{(h_i)_i} x_\infty : \iff (\forall i \in \mathbb{N}) (\forall n) \left( (h_i(x_n))_n \to_{\mathfrak{Y}_i} h_i(x_\infty) \right).\]

\footnote{Given two convergence relations $\to_\gamma$ and $\to_\delta$ on $X$, we call $\to_\gamma$ coarser than $\to_\delta$ iff $\to_\gamma \supseteq \to_\delta$, see also Footnote 3 on page 24.}
Universality: Let \( \phi \) be the initial convergence relation generated by \((h_i)_i\) and \( (\mathfrak{g})_i \). One easily verifies that \( \rightarrow^{(h_i)_i} \) is the coarsest (largest) convergence relation on \( X \) such that each function \( h_i \) is sequentially continuous; this means that a convergence relation \( \rightsquigarrow \) on \( X \) is a subset of \( \rightarrow^{(h_i)_i} \) if and only if for every \( i \in \mathbb{N} \) the function \( h_i \) is \((\rightsquigarrow, \rightarrow^{(h_i)_i})\)-continuous. The class of weak limit spaces, the class of limit spaces and the class of topological weak limit spaces are closed under this operator\(^2\).

The initial convergence relation \( \rightarrow^{h_1,\ldots,h_k} \) generated by \( h_1, \ldots, h_k \) is defined by

\[
(x_n) \rightarrow^{h_1,\ldots,h_k} x_\infty \iff (\forall i \in \{1, \ldots, k\}) (h_i(x_n))_n \rightarrow^{\mathfrak{g}_i} h_i(x_\infty)
\]

and has analogous properties as \( \rightarrow^{(h_i)_i} \).

Proposition 4.1.1 shows how to construct admissible multirepresentations of the weak limit spaces \((X, \rightarrow^{(h_i)_i})\) and \((X, \rightarrow^{h_1,\ldots,h_k})\) from given admissible multirepresentations of the spaces \( (\mathfrak{g})_i \).

**Proposition 4.1.1 (Admissible multireps. for initial convergence relations)**

Let \( X \) be a set. For \( i \in \mathbb{N} \), let \( \gamma_i : \Sigma^\omega \rightrightarrows Y_i \) be an admissible multirepresentation of a weak limit space \( (\mathfrak{g})_i = (Y_i, \rightarrow^{\mathfrak{g}_i}) \) and \( h_i : X \rightarrow Y_i \) be a function. Define \( \delta^{(\infty)} : \subseteq \Sigma^\omega \rightrightarrows X \) and \( \delta^{(k)} : \subseteq \Sigma^\omega \rightrightarrows X \) by\(^3\)

\[
\delta^{(\infty)}[p] \ni x \iff (\forall i \in \mathbb{N}) h_i(x) \in \gamma_i[\pi_{\infty,i}(p)]
\]

and

\[
\delta^{(k)}[p] \ni x \iff (\forall i \in \{1, \ldots, k\}) h_i(x) \in \gamma_i[\pi_{k,i}(p)]
\]

for all \( p \in \Sigma^\omega \) and \( x \in X \).

Then \( \delta^{(\infty)} \) and \( \delta^{(k)} \) are admissible multirepresentations of \((X, \rightarrow^{(h_i)_i})\) and \((X, \rightarrow^{h_1,\ldots,h_k})\) respectively.

**Proof:**

We only show that \( \delta^{(\infty)} \) is an admissible multirepresentation of \((X, \rightarrow^{(h_i)_i})\); the admissibility of \( \delta^{(k)} \) follows analogously.

**Continuity:** Let \((p_n)_{n \leq \infty} \) be a convergent sequence in \( \text{dom}(\delta^{(\infty)}) \), and let \((x_n)_{n \leq \infty} \in \prod_{n \in \mathbb{N}} \delta^{(\infty)}[p_n] \). For every \( i \in \mathbb{N} \), \((h_i(x_n))_{n \leq \infty} \) is a convergent sequence of \( (\mathfrak{g})_i \), because we have \((h_i(x_n))_{n \leq \infty} \subseteq \prod_{n \in \mathbb{N}} \gamma_i[\pi_{\infty,i}(p_n)] \) and \( \gamma_i \circ \pi_{\infty,i} \) is sequentially continuous. Thus \((x_n)_n \) converges to \( x_\infty \) in \((X, \rightarrow^{(h_i)_i})\).

**Universality:** Let \( \phi : \subseteq \Sigma^\omega \rightrightarrows X \) be a sequentially continuous multirepresentation of \((X, \rightarrow^{(h_i)_i})\). Let \( i \in \mathbb{N} \). Then the correspondence \( \phi_i : \subseteq \Sigma^\omega \rightrightarrows X \) defined by \( \phi_i[p] := h_i[\phi[p]] \) is sequentially continuous, because \( h_i \) and \( \phi \) are sequentially continuous. By admissibility of \( \gamma_i \), there is a continuous function \( g_i : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) translating \( \phi_i \) to \( \gamma_i \). We define \( g : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) continuously by

\[
g(p) := \left\langle g_0(p), g_1(p), g_2(p), \ldots \right\rangle.
\]

For every \( p \in \text{dom}(\phi) \), \( x \in \phi[p] \) and \( i \in \mathbb{N} \) we have \( h_i(x) \in \phi_i[p] \subseteq \gamma_i[\pi_{\infty,i}(g(p))] = \gamma_i[\pi_{\infty,i}(g(p))] \), thus \( x \in \delta^{(\infty)}[g(p)] \) and \( \phi[p] \subseteq \delta^{(\infty)}[g(p)] \). Hence \( g \) translates \( \phi \) continuously to \( \delta^{(\infty)} \).

---

\(^2\)For topological weak limit spaces this follows from Lemma 4.1.5; the proofs for limit spaces and weak limit spaces are straightforward.

\(^3\)The computable functions \( \pi_{\infty,i} \) and \( \pi_{k,i} \) are defined in Subsection 2.1.7.
Note that the multirepresentations constructed in Proposition 4.1.1 need not be single-valued, if δ₀, δ₁, ..., are single-valued, for example in the case that each of the generating functions is constant.

As \( \rightarrow^{h_1, \ldots, h_k} \) is the coarsest (largest) convergence relation on \( X \) making the functions \( h_1, \ldots, h_k \) sequentially continuous, every function \( f : \subseteq Z \rightarrow X \) between any weak limit space \( Z = (Z, \rightarrow) \) and the space \( (X, \rightarrow^{h_1, \ldots, h_k}) \) is sequentially continuous if and only if the functions \( h_1 \circ f, \ldots, h_k \circ f \) are sequentially continuous. We show that relative computability of \( f \) can be characterized analogously.

**Proposition 4.1.2 (Computability properties of \( \delta^{(k)} \))**

Let the assumptions of Proposition 4.1.1 be satisfied. Let \( \zeta : \subseteq \Sigma^\omega \rightarrow Z \) be a multirepresentation of a set \( Z \).

Then a function \( f : \subseteq Z \rightarrow X \) is \((\zeta, \delta^{(k)})\)-computable if and only if \( h_i \circ f \) is \((\gamma_i, \delta^{(k)})\)-computable for every \( i \in \{1, \ldots, k\} \).

**Proof:**

\[ \implies \]: Let \( i \in \{1, \ldots, k\} \). Let \( l : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) be a computable function realizing \( f \) w.r.t. \( \zeta \) and \( \delta^{(k)} \). For every \( p \in \text{dom}(\zeta) \) and \( z \in \zeta[p] \) we have \( f(z) \in \delta^{(k)}[l(p)] \), hence \( h_i(f(z)) \in \gamma_i[\pi_{k,i}(l(p))] \). Thus \( \pi_{k,i} \circ l \) realizes \( h_i \circ f \) w.r.t. \( \zeta \) and \( \gamma_i \). By being a composition of computable functions, \( \pi_{k,i} \circ l \) is computable. Therefore \( h_i \circ f \) is \((\zeta, \gamma_i)\)-computable.

\[ \impliedby \]: For every \( i \in \{1, \ldots, k\} \) let \( g_i : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) be a computable function realizing \( h_i \circ f \) w.r.t. \( \zeta \) and \( \gamma_i \). We define \( l : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) by \( l(p) := \langle g_1(p), \ldots, g_k(p) \rangle \).

Clearly, \( l \) is computable. For every \( i \in \{1, \ldots, k\} \), \( q \in \text{dom}(\zeta) \), and \( z \in \zeta[q] \) we have \( h_i(f(z)) \in \gamma_i[g_i(q)] = \gamma_i[\pi_{k,i}(l(q))] \) and therefore \( f(z) \in \delta^{(k)}[l(q)] \). Hence \( l \) is a computable \((\zeta, \delta^{(k)})\)-realization of \( f \).

In order to extend Proposition 4.1.2 to the countably infinite case, we have to demand in the second statement of the equivalence “uniform” computability of the functions \( h_0 \circ f, h_1 \circ f, \ldots \).

**Proposition 4.1.3 (Computability properties of \( \delta^{(\infty)} \))**

Let the assumptions of Proposition 4.1.1 be satisfied. Let \( \zeta : \subseteq \Sigma^\omega \rightarrow Z \) be a multirepresentation of a set \( Z \).

Then a function \( f : \subseteq Z \rightarrow X \) is \((\zeta, \delta^{(\infty)})\)-computable if and only if there is a computable function \( g : \subseteq \mathbb{N}^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega \) such that for every \( i \in \mathbb{N} \) the function \( p \mapsto g(i::0^\omega, p) \) realizes \( h_i \circ f \) w.r.t. \( \zeta \) and \( \gamma_i \).

**Proof:**

\[ \impliedby \]: Let \( l : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) be defined by \( l(p) := \langle g(00^\omega, p), g(10^\omega, p), g(20^\omega, p), \ldots \rangle \).

Then \( l \) is computable and realizes \( f \) w.r.t. \( \zeta \) and \( \delta^{(\infty)} \) (cf. the proof of Proposition 4.1.2).

\[ \implies \]: Let \( l : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) be a computable \((\zeta, \delta^{(\infty)})\)-realization of \( f \). One easily verifies that the computable function \( g : \subseteq \mathbb{N}^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega \) defined by \( g(r, p) := \pi_{\infty,r(0)}(l(p)) \) has the required properties.
4.1.2 Initial Topologies

Let $X$ be a set, and let $\mathcal{Y}_i = (Y_i, \tau_i)$ be a topological space for each $i \in \mathbb{N}$. The *initial topology* $\tau_{(h_i)}$ generated on $X$ by a sequence $(h_i)_i$ of functions $h_i : X \to Y_i$ is defined as the topology induced by the subbase

$$\{ h_i^{-1}[U] \mid i \in \mathbb{N}, U \subseteq \tau_i \}.$$  

The initial topology $\tau_{(h_i)}$ is the coarsest (smallest) topology on $X$ for which each function $h_i$ topologically continuous (cf. [Eng89, Smy92]). The initial topology $\tau_{h_1,\ldots,h_k}$ generated by $h_1, \ldots, h_k$ is defined accordingly. The initial topological spaces $(X, \tau_{h_1,\ldots,h_k})$ and $(X, \tau_{h_1})$ inherit from the spaces $\mathcal{Y}_i$ the property of having an admissible multirepresentation.

**Proposition 4.1.4 (Admissible multireps. w.r.t. initial topologies)**

Let $X$ be a set. Let $\gamma_i : \Sigma^\omega \to \mathcal{Y}_i$ be an admissible multirepresentation of a topological space $\mathcal{Y}_i = (Y_i, \tau_i)$ and let $h_i : X \to Y_i$ be a function for $i \in \mathbb{N}$. Let $\delta^{(\infty)} : \subseteq \Sigma^\omega \to X$ and $\delta^{(k)} : \subseteq \Sigma^\omega \to X$ be defined as in Proposition 4.1.1. Then $\delta^{(\infty)}$ and $\delta^{(k)}$ are admissible multirepresentations of $(X, \tau_{(h_i)})$ and $(X, \tau_{h_1,\ldots,h_k})$, respectively.

This proposition follows from Proposition 4.1.1 and the following lemma.

**Lemma 4.1.5**

Let $X$ be a set. For $i \in \mathbb{N}$, let $\mathcal{Y}_i = (Y_i, \tau_i)$ be a topological space and $h_i : X \to Y_i$ be a function. Then:

1. The convergence relation $\rightarrow_{(h_i)}$ induced by the initial topology $\tau_{(h_i)}$ is equal to the initial convergence relation $\rightarrow_{(h_i)}$ generated by $(h_i)_i$ and $(\mathcal{Y}(\mathcal{Y}_i))_i$.

2. The convergence relation $\rightarrow_{h_1,\ldots,h_k}$ induced by the initial topology $\tau_{h_1,\ldots,h_k}$ is equal to the initial convergence relation $\rightarrow_{h_1,\ldots,h_k}$ generated by the functions $h_1, \ldots, h_k$ and the weak limit spaces $\mathcal{W}(\mathcal{Y}_1), \ldots, \mathcal{W}(\mathcal{Y}_k)$.

**Proof:**

We only show the first statement which implies the second one.

$\rightarrow_{(h_i)} \subseteq \rightarrow_{(h_i)}$: Let $(x_n)_n \subseteq X$ be a convergent sequence of $(X, \tau_{(h_i)})_i$.

Let $i \in \mathbb{N}$, and let $U$ be an open neighbourhood of $h_i(x_\infty)$. By definition, $h_i^{-1}[U]$ is open in $(X, \tau_{(h_i)})$. Thus the sequence $(x_n)_n$ is eventually in $h_i^{-1}[U]$. Hence we have $h_i(x_n) \in U$ for almost all $n$. Therefore, $(h_i(x_n))_n$ converges to $h_i(x_\infty)$ in $\mathcal{Y}_i$.

We conclude that $(x_n)_n$ converges to $x_\infty$ w.r.t. $\rightarrow_{(h_i)}$.

$\rightarrow_{(h_i)} \subseteq \rightarrow_{(h_i)}$: Let $(x_n)_n$ converges to $x_\infty$ w.r.t. $\rightarrow_{(h_i)}$.

Let $O \subseteq \tau_{(h_i)}$ with $x_\infty \in O$. There are $l \in \mathbb{N}$, $i_1, \ldots, i_l \in \mathbb{N}$ and open sets $U_1 \subseteq \tau_{i_1}, \ldots, U_l \subseteq \tau_{i_l}$ with $x_\infty \in h_{i_1}^{-1}[U_1] \cap \cdots \cap h_{i_l}^{-1}[U_l] \subseteq O$. For every $j \in \{1, \ldots, l\}$ there is some $n_j \in \mathbb{N}$ with $h_{i_j}(x_n) \in U_j$ for all $n \geq n_j$, because the sequence $(h_{i_j}(x_n))_n$ converges to $h_{i_j}(x_\infty)$. This implies $x_n \in O$ for all $n \geq \max\{n_1, \ldots, n_l\}$. Therefore $(x_n)_n$ converges to $x_\infty$ in $(X, \tau_{(h_i)})_i$.

$\checkmark$
4.1 Initial Constructions

4.1.3 Homomorphisms

Let $\mathfrak{X} = (X, \rightarrow_X)$ and $\mathfrak{Y} = (Y, \rightarrow_\gamma)$ be weak limit spaces. We call a function $h : X \rightarrow Y$ a homomorphism between $\mathfrak{X}$ and $\mathfrak{Y}$ iff the equivalence

$$(x_n)_n \rightarrow_X x_\infty \iff (h(x_n))_n \rightarrow_\gamma h(x_\infty)$$

holds for every sequence $(x_n)_{n \leq \infty}$ in $X$. In this case we also say that $h$ reflects convergent sequences between $\mathfrak{X}$ and $\mathfrak{Y}$. Since this means that $\rightarrow_X$ is equal to the convergence relation $\rightarrow^h$ generated by $h$ and $\mathfrak{Y}$, we obtain the following proposition:

Proposition 4.1.6 (Homomorphisms)

Let $h : X \rightarrow Y$ be a homomorphism between weak limit spaces $\mathfrak{X} = (X, \rightarrow_X)$ and $\mathfrak{Y} = (Y, \rightarrow_\gamma)$, and let $\gamma : \subseteq \Sigma^\omega \Rightarrow Y$ be an admissible multirepresentation of $\mathfrak{Y}$.

Then $^h h^{-1} \circ \gamma$ is an admissible multirepresentation of $\mathfrak{X}$.

Although Proposition 4.1.1 implies this proposition, we give the direct proof.

**Proof:**

**Continuity:** Let $(p_n)_{n \leq \infty}$ be a convergent sequence of $(\Sigma^\omega, \tau_{\Sigma^\omega})$, and let $(x_n)_{n \leq \infty} \in \prod_{n \in \mathbb{N}} h^{-1} \circ \gamma[p_n]$. By continuity of $\gamma$, $(h(x_n))_n$ converges to $h(x_\infty)$. Since $h$ reflects convergent sequences, $(x_n)_n$ converges to $x_\infty$.

**Universality:** Let $\phi : \subseteq \Sigma^\omega \Rightarrow X$ be continuous. By continuity of $h$, $h \circ \phi$ is continuous and thus continuously translatable to $\gamma$ by some continuous function $g : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$. Let $p \in \Sigma^\omega$ and $x \in \phi[p]$. Then $h(x) \in (h \circ \phi)[p]$ and thus $h(x) \in \gamma[g(p)]$. This implies $x \in (h^{-1} \circ \gamma)[g(p)]$. Hence $g$ also translates $\phi$ to $h^{-1} \circ \gamma$.

4.1.4 Product and Conjunction

Now we will investigate product and conjunction which are two very useful examples for initial constructions.

For each $i \in \mathbb{N}$ let $\mathfrak{X}_i = (X_i, \rightarrow_{X_i})$ be a weak limit space. The convergence relation of the product space $\bigotimes_{i \in \mathbb{N}} \mathfrak{X}_i$ is defined by

$$(x_n)_n \rightarrow_{\bigotimes_{i \in \mathbb{N}} \mathfrak{X}_i} x_\infty \iff (\forall i \in \mathbb{N}) (pr_i(x_n))_n \rightarrow_{X_i} pr_i(x_\infty)$$

(cf. Subsection 2.2.1). Clearly, it is equal to the initial convergence relation generated by the projections $pr_j : \prod_{i \in \mathbb{N}} X_i \rightarrow X_j$. The analogue holds for the finite product.

By the conjunction $\bigwedge_{i \in \mathbb{N}} \mathfrak{X}_i$ of the weak limit spaces $\mathfrak{X}_0, \mathfrak{X}_1, \ldots$ we mean the weak limit space $\bigotimes_{i \in \mathbb{N}} X_i, \bigwedge_{i \in \mathbb{N}} \rightarrow_{x_i})$ whose convergence relation is defined by

$$(x_n)_n \bigwedge_{i \in \mathbb{N}} \rightarrow_{x_i}) x_\infty \iff (\forall i \in \mathbb{N}) (x_n)_n \rightarrow_{X_i} x_\infty$$

for all sequences $(x_n)_{n \leq \infty}$ in $\bigcap_{i \in \mathbb{N}} X_i$. It is obvious that $(\bigwedge_{i \in \mathbb{N}} \rightarrow_{x_i})$ is generated by the injections $\iota_j : \bigcap_{i \in \mathbb{N}} X_i \rightarrow X_j, x \mapsto x$. The finite conjunction $\mathfrak{X}_1 \land \ldots \land \mathfrak{X}_k$ and its convergence relation $(\rightarrow_{x_1} \land \ldots \land \rightarrow_{x_k})$ are defined accordingly. One can easily verify that $\mathfrak{X}_1 \land \ldots \land \mathfrak{X}_k$ and $\bigwedge_{i \in \mathbb{N}} \mathfrak{X}_i$ are weak limit spaces.

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4The multirepresentation $h^{-1} \circ \gamma : \subseteq \Sigma^\omega \Rightarrow X$ maps $p \in \Sigma^\omega$ to $\{x \in X \mid h(x) \in \gamma[p]\}$.
Chapter 4. Constructions

Let $\delta_i : \subseteq \Sigma^w \Rightarrow X_i$ be a multirepresentation of $X_i$ for every $i \in \mathbb{N}$, and let $k \geq 1$. Analogously to [Wei00], we define the multirepresentations $(\bigotimes_{i \in \mathbb{N}} \delta_i) : \subseteq \Sigma^w \Rightarrow X_1 \times \ldots \times X_k$ and $(\bigimes_{i \in \mathbb{N}} \delta_i) : \subseteq \Sigma^w \Rightarrow \prod_{i \in \mathbb{N}} X_i$ of the Cartesian products $X_1 \times \ldots \times X_k$ and $\prod_{i \in \mathbb{N}} X_i$, respectively.

We easily verify that $X$ and $\delta$ multirepresentations are subbases of $\bigotimes_{i \in \mathbb{N}} X_i$ and $\bigimes_{i \in \mathbb{N}} X_i$, respectively. The finite Tychonoff topology defined to be the topology generated by the injections $\bigotimes_{i \in \mathbb{N}} X_i$ and $\bigimes_{i \in \mathbb{N}} X_i$.

From Proposition 4.1.1 we conclude that these operators on multirepresentations preserve admissibility.

**Proposition 4.1.7 (Admissible multireps. for products and conjunctions)**

For $i \in \mathbb{N}$, let $\delta_i : \subseteq \Sigma^w \Rightarrow X_i$ be an admissible multirepresentation of a weak limit space $X_i = (X_i, \rightarrow_{x_i})$.

1. $(\bigotimes_{i \in \mathbb{N}} \delta_i)$ is an admissible multirepresentation of $(X_1 \otimes \ldots \otimes X_k)$.
2. $(\bigimes_{i \in \mathbb{N}} \delta_i)$ is an admissible multirepresentation of $\bigimes_{i \in \mathbb{N}} X_i$.
3. $(\bigotimes_{i \in \mathbb{N}} \delta_i)$ is an admissible multirepresentation of $\bigotimes_{i \in \mathbb{N}} X_i$.
4. $(\bigimes_{i \in \mathbb{N}} \delta_i)$ is an admissible multirepresentation of $\bigimes_{i \in \mathbb{N}} X_i$.

For $i \in \mathbb{N}$ let $\mathcal{S}_i = (Z_i, \tau_i)$ be a topological space. The Tychonoff or product topology $\bigimes_{i \in \mathbb{N}} \tau_i$ is defined to be the topology on $\mathbb{N} \times Z_i$ generated by the projections $pr_j : \prod_{i \in \mathbb{N}} Z_i \rightarrow Z_j$ ([Eng89, Wil70]). The conjunction $\bigotimes_{i \in \mathbb{N}} \tau_i$ of the topologies $\tau_0, \tau_1, \ldots$ is defined to be the topology generated by the injections $\iota_j : \prod_{i \in \mathbb{N}} Z_i \rightarrow Z_j, z \mapsto z$. One easily verifies that

$$\{Z_0 \times \ldots \times Z_{j-1} \times O \times Z_{j+1} \times Z_{j+2} \times \ldots \mid j \in \mathbb{N}, O \in S_j\}$$

are subbases of $\bigimes_{i \in \mathbb{N}} \tau_i$ and $\bigotimes_{i \in \mathbb{N}} \tau_i$, if $S_0, S_1, S_2, \ldots$ are subbases of $\tau_0, \tau_1, \tau_2, \ldots$, respectively. The finite Tychonoff topology $\tau_1 \otimes \ldots \otimes \tau_k$ and the finite conjunction $\tau_1 \wedge \ldots \wedge \tau_k$ are defined accordingly. None of these topologies need be sequential, if $\tau_0, \tau_1, \ldots$ are sequential.

We obtain from Proposition 4.1.4 the following proposition.

5The projections $\pi_{\infty,i}$ and $\pi_{k,i}$ are defined in Subsection 2.1.7.
Proposition 4.1.8 (Topological product and conjunction)
For \( i \in \mathbb{N} \), let \( \delta_i \subseteq \Sigma^\omega \Rightarrow Z_i \) be an admissible multirepresentation of a topological space \( Z_i = (Z_i, \tau_i) \).

1. \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)\) is an admissible multirepresentation of \((\prod_{i=1}^k Z_i, \bigotimes_{i=1}^k \tau_i)\).

2. \((\bigotimes_{i \in \mathbb{N}} \delta_i)\) is an admissible multirepresentation of \((\prod_{i \in \mathbb{N}} Z_i, \bigotimes_{i \in \mathbb{N}} \tau_i)\).

3. \((\delta_1 \land \ldots \land \delta_k)\) is an admissible multirepresentation of \((Z_1 \cap \ldots \cap Z_k, \bigwedge_{i=1}^k \tau_i)\).

4. \((\bigwedge_{i \in \mathbb{N}} \delta_i)\) is an admissible multirepresentation of \((\bigcap_{i \in \mathbb{N}} Z_i, \bigwedge_{i \in \mathbb{N}} \tau_i)\).

Similar as in the case of single–valued representations, one can prove the following effectivity properties of the multirepresentations \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)\) and \((\delta_1 \land \ldots \land \delta_k)\) (cf. Proposition 4.1.2). They generalize the corresponding statements in [Wei00] for single–valued representations.

Lemma 4.1.9 (Effectivity properties of \( \boxtimes \) and \( \land \))
Let \( \delta_i, \gamma_i : \Sigma^\omega \Rightarrow X_i \) be multirepresentations of a weak limit space \( X_i = (X_i, \rightarrow_{X_i}) \) for \( i \in \{1, \ldots, k\} \). Let \( \zeta : \Sigma^\omega \Rightarrow Z \) be a multirepresentation of a set \( Z \).

1. A function \( f : \subseteq Z \to \prod_{i=1}^k X_i \) is \((\zeta, (\delta_1 \boxtimes \ldots \boxtimes \delta_k))\)–computable if and only if \( \text{pr}_i \circ f \) is \((\zeta, \delta_i)\)–computable for every \( i \in \mathbb{N} \).

2. A function \( f : \subseteq Z \to \bigcap_{i=1}^k X_i \) is \((\zeta, (\delta_1 \land \ldots \land \delta_k))\)–computable if and only if \( \iota_i \circ f \) is \((\zeta, \delta_i)\)–computable for every \( i \in \{1, \ldots, k\} \).

3. A correspondence \( F : \subseteq X_1 \times \ldots \times X_k \Rightarrow Z \) is \((\delta_1 \boxtimes \ldots \boxtimes \delta_k, \zeta)\)–computable if and only if \( F \) is \((\delta_1, \ldots, \delta_k, \zeta)\)–computable.

4. Statements (1), (2), (3) hold analogously for relative continuity.

5. If \( \delta_i \leq \text{cp} \gamma_i \) for every \( i \in \{1, \ldots, k\} \), then \((\delta_1 \boxtimes \ldots \boxtimes \delta_k) \leq \text{cp} (\gamma_1 \boxtimes \ldots \boxtimes \gamma_k)\) and \((\delta_1 \land \ldots \land \delta_k) \leq \text{cp} (\gamma_1 \land \ldots \land \gamma_k)\). The analogue holds for \( \leq_{\text{tr}} \).

Proof: We omit the straightforward proofs.

Let \( \mathcal{C} \) be a category, and let \( X \) and \( Y \) be objects of \( \mathcal{C} \). A categorical product of \( X \) and \( Y \) is a triple \((X \times_{\mathcal{C}} Y, p_{X,Y}, q_{X,Y})\), where \( X \times_{\mathcal{C}} Y \) is an object and \( p_{X,Y} : X \times_{\mathcal{C}} Y \to X \), \( q_{X,Y} : X \times_{\mathcal{C}} Y \to Y \) are morphisms in \( \mathcal{C} \) with the following property: for every object \( Z \) there is an operator \( \langle \cdot, \cdot \rangle^X_Y \) such that, for all morphisms \( f : Z \to X \), \( g : Z \to Y \), \( h : Z \to X \times_{\mathcal{C}} Y \), \( (f,g)^{X,Y} : f \circ p_{X,Y} \circ h, g \circ q_{X,Y} \circ h)^{X,Y} \) are morphisms between \( Z \) and \( X \times_{\mathcal{C}} Y \) with \( p_{X,Y} \circ (f,g)^{X,Y} = f, q_{X,Y} \circ (f,g)^{X,Y} = g \), and \( (p_{X,Y} \circ h, g_{X,Y} \circ h)^{X,Y} = h \). Hence \( p_{X,Y}, q_{X,Y} \) can be viewed as “projection morphisms” from the “product” of \( X \) and \( Y \) to, respectively, \( X \) and \( Y \). The category \( \mathcal{C} \) is called Cartesian iff all pairs of objects have a categorical product (cf. [AL91]).

Propositions 4.1.7, 4.1.8, Lemma 4.1.9, and Corollary 2.4.31 imply the following corollary. As the object of a categorical product in \( \textbf{Seq} \) of sequential spaces \( X = (X, \tau_X) \) and \( Y = (Y, \tau_Y) \), one takes the sequential space \( (X \times Y, \text{seq}(\tau_X \otimes \tau_Y)) \), i.e. the sequentialization of the ordinary topological product of \( X \) and \( Y \).
Corollary 4.1.10 The categories $\text{AdmWeakLim}$, $\text{AdmLim}$, and $\text{AdmSeq}$ are Cartesian.

4.1.5 Subspaces

Let $\mathcal{X} = (X, \rightarrow_X)$ be a weak limit space equipped with an admissible multirepresentation $\delta : \subseteq \Sigma^\omega \rightrightarrows X$ and let $\mathcal{M}$ be a subset of $X$. The subspace convergence relation $\rightarrow_{\mathcal{M}}$ induced by $\mathcal{X}$ on $\mathcal{M}$ is defined by $\rightarrow_{\mathcal{M}} := \rightarrow_X \cap (\mathcal{M}^\omega \times \mathcal{M})$. Clearly, $(\mathcal{M}, \rightarrow_{\mathcal{M}})$ is equal to the conjunction of $\mathcal{X}$ and the space $(\mathcal{M}, \rightarrow_{\mathcal{M}})$ equipped with the chaotic convergence relation $\rightarrow_{\mathcal{M}}$ on $\mathcal{M}$ (cf. Example 2.4.24). An admissible multirepresentation of $(\mathcal{M}, \rightarrow_{\mathcal{M}})$ is the restriction of $\delta$ to $\mathcal{M}$, i.e. the correspondence $\delta|^{\mathcal{M}} = (\Sigma^\omega, \mathcal{M}, \delta \cap (\Sigma^\omega \times \mathcal{M}))$: As a continuous translator from any $(\rightarrow_{\Sigma^\omega}, \rightarrow_{\mathcal{M}})$–continuous correspondence $\phi : \subseteq \Sigma^\omega \rightrightarrows \mathcal{M}$ to $\delta|^{\mathcal{M}}$ we can use any continuous function $g : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ translating the $(\rightarrow_{\Sigma^\omega}, \rightarrow)$–continuous correspondence $\phi|^{\mathcal{M}}$ to $\delta$.

The subspace topology induced by a topological space $(X, \tau_X)$ on a subset $\mathcal{M}$ of $X$ is defined by $\tau_{\mathcal{M}|^{\mathcal{M}}} := \{O \cap \mathcal{M} \mid O \in \tau_X\}$ (cf. [Eng89, Smy92, Wil70]). With the same proof as above, $\delta|^{\mathcal{M}}$ is an admissible multirepresentation of the topological subspace $(\mathcal{M}, \tau_{\mathcal{M}|^{\mathcal{M}}})$, whenever $\delta$ is an admissible multirepresentation of $(X, \tau_X)$.

4.1.6 Equalizers and Countable Limits

An equalizer of a pair of parallel morphisms $f, g : X \rightarrow Y$ in a category $\mathcal{C}$ is an object $E$ and a morphism $\iota : E \rightarrow X$ in $\mathcal{C}$ with $f \circ \iota = g \circ \iota$ such that for all objects $Z$ and all morphisms $h : Z \rightarrow X$ with $f \circ h = g \circ h$ there is exactly one morphism $k : Z \rightarrow E$ satisfying $\iota \circ k = h$, cf. [AL91].

Let $\mathcal{X} = (X, \rightarrow_X)$ and $\mathcal{Y} = (Y, \rightarrow_Y)$ be weak limit spaces and $f, g : X \rightarrow Y$ be two continuous functions from $\mathcal{X}$ to $\mathcal{Y}$. One easily verifies that an equalizer of the pair $(f, g)$ formed in $\text{WeakLim}$ or in $\text{Lim}$ is given by the subspace $(\mathcal{M}, \rightarrow_{\mathcal{M}})$, where $\mathcal{M} := \{x \in X \mid f(x) = g(x)\}$, together with the injection $\iota : \mathcal{M} \rightarrow X, x \mapsto x$.

The object of the equalizer formed in $\text{Seq}$ of two topologically continuous functions $f, g : X \rightarrow Y$ between sequential spaces $\mathcal{X} = (X, \tau_X)$ and $\mathcal{Y} = (Y, \tau_Y)$ is the sequentialization $(\mathcal{M}, \text{seq}(\tau_X|\mathcal{M}))$ of the subspace $(\mathcal{M}, \tau_{\mathcal{M}|\mathcal{M}})$, where $\mathcal{M} := \{x \in X \mid f(x) = g(x)\}$.

From Subsection 4.1.5 we obtain that equalizers of pairs of parallel morphism in $\text{AdmWeakLim}$ formed in $\text{WeakLim}$ are contained in $\text{AdmWeakLim}$. Thus $\text{AdmWeakLim}$ has equalizers for every pair of parallel morphisms, moreover the inclusion functor of $\text{AdmWeakLim}$ into $\text{WeakLim}$ preserves equalizers. The corresponding statements hold for $\text{AdmLim}$ and $\text{AdmSeq}$. As $\text{AdmWeakLim}$, $\text{AdmLim}$, and $\text{AdmSeq}$ have countable products by Propositions 4.1.7, 4.1.8 and Corollary 2.4.31, we can conclude by [AL91, Theorem 6.3.1] that these categories have all countable limits. Moreover, the inclusion functors of these categories into $\text{WeakLim}$ preserve countable limits.

4.2 Exponentiation

In this section we investigate function spaces and their admissible multirepresentations. We prove that the categories $\text{WeakLim}$, $\text{AdmWeakLim}$, $\text{AdmLim}$, and $\text{AdmSeq}$ are Cartesian closed (cf. Subsections 4.2.2 and 4.2.4). For this purpose we generalize the
well-known notion of continuous convergence of sequences of functions between limit spaces (or topological spaces) to sequences of functions between weak limit spaces (cf. Subsection 4.2.1). In Subsection 4.2.5, admissible multirepresentations of the set of partial sequentially continuous functions are studied. Here the concept of multirepresentations demonstrates its usefulness: for most spaces $\mathfrak{X}$ and $\mathfrak{Y}$ having an admissible single-valued representation, the corresponding admissible multirepresentation of the set of partial continuous functions between $\mathfrak{X}$ and $\mathfrak{Y}$ is not single-valued. In Subsection 4.2.6, we show that the categories $\text{AdmWeakLim}$ and $\text{AdmLim}$ are even locally Cartesian closed.

In this and the following sections we restrict ourselves to Baire space multirepresentations.

### 4.2.1 Continuous Convergence

Let $\mathfrak{X} = (X, \rightarrow_{\mathfrak{X}})$ and $\mathfrak{Y} = (Y, \rightarrow_{\mathfrak{Y}})$ be weak limit spaces. By $C(\mathfrak{X}, \mathfrak{Y})$ we denote the set of all total sequentially continuous functions between $\mathfrak{X}$ and $\mathfrak{Y}$. We equip $C(\mathfrak{X}, \mathfrak{Y})$ with the convergence relation of continuous convergence. It is defined by saying that a sequence $(f_n)_n$ of functions in $C(\mathfrak{X}, \mathfrak{Y})$ converges continuously to a function $f_\infty \in C(\mathfrak{X}, \mathfrak{Y})$ iff the transpose $f^t: \mathbb{N} \times X \rightarrow Y, f^t(n, x) := f_n(x)$ (4.1) is $(\rightarrow_{\mathfrak{X}}, \rightarrow_{\mathfrak{Y}})$-continuous. We denote this convergence relation by $\Rightarrow_{\mathfrak{X}, \mathfrak{Y}}$ or, for short, by $\Rightarrow$, and the function space $(C(\mathfrak{X}, \mathfrak{Y}), \Rightarrow_{\mathfrak{X}, \mathfrak{Y}})$ by $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$. Due to Proposition 4.2.1(1), $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ is a weak limit space. It turns out that $(f_n)_n$ converges continuously to $f_\infty$ if and only if the implication

\[
( (x_n) \rightarrow_{\mathfrak{X}} x_\infty \land (m_n) \rightarrow_{\mathfrak{Y}} \infty ) \Rightarrow ( (f_{m_n}(x_n))_n \rightarrow_{\mathfrak{Y}} f_\infty(x_\infty) )
\]

holds for all sequences $(x_n)_n \leq \infty$ in $X$ and $(m_n)_n$ in $\mathbb{N}$ (cf. Proof of Proposition 4.2.1). Moreover, in the case of limit spaces this definition of continuous convergence corresponds to the ordinary one (cf. [MS02, Hyl79]): we have

\[
(f_n)_n \Rightarrow f_\infty \iff (\forall (x_n)_n \leq \infty)( (x_n) \rightarrow_{\mathfrak{X}} x_\infty \Rightarrow (f_n(x_n))_n \rightarrow_{\mathfrak{Y}} f_\infty(x_\infty) )
\]

whenever the target space $\mathfrak{Y}$ is a limit space. We show this equivalence within the proof of the following proposition which classifies the type of the space $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ depending on the type of $\mathfrak{Y}$.

**Proposition 4.2.1**

Let $\mathfrak{X} = (X, \rightarrow_{\mathfrak{X}})$ and $\mathfrak{Y} = (Y, \rightarrow_{\mathfrak{Y}})$ be weak limit spaces.

1. The function space $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ is a weak limit space.
2. If $\mathfrak{Y}$ is a limit space, then $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ is a limit space.
3. If $\mathfrak{Y}$ is a topological weak limit space, then $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ is a topological weak limit space.
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Proof:

(1) At first we show that (4.2) implies continuous convergence; the converse follows from the continuity of the transpose $f^t$. Let $(f_n)_{n \leq \infty}$ be a sequence of $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ satisfying (4.2). Let $(m_n)_{n \leq \infty}$ and $(x_n)_{n \leq \infty}$ be convergent sequences of $(\mathbb{N}, \rightarrow_n)$ and $\mathfrak{X}$, respectively.

Case $m_\infty = \infty$: By (4.2), $(f^t(m_n, x_n))_{n \leq \infty}$ is a convergent sequence of $\mathfrak{Y}$.

Case $m_\infty \neq \infty$: Then there is some $n_0 \in \mathbb{N}$ with $(\forall n \geq n_0) m_n = m_\infty$. By Axiom (L5) for $\mathfrak{X}$ and by continuity of $f_{m_\infty}$, $(f^t(m_{i+n_0}, x_{i+n_0}))_{i}$ converges to $f^t(m_\infty, x_\infty)$. Since $\mathfrak{Y}$ satisfies Axiom (L6), the sequence $(f^t(m_n, x_n))_{i}$ converges to $(f^t(m_\infty, x_\infty))$. Hence $f^t$ is $(-\infty, -\infty)$-continuous.

We verify Axioms (L1), (L5), (L6).

Axiom (L5): Let $(f_n)_{n \leq \infty}$ and $(k_n)_{n \leq \infty}$ be convergent sequences of $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ and $(\mathbb{N}, \rightarrow_n)$, respectively. We have to show $(f_{k_n})_n \Rightarrow f_{k_\infty}$, i.e., that the function $(m, x) \mapsto f_{k_n}(x)$ is continuous. Let $(m_n)_{n \leq \infty}$ and $(x_n)_{n \leq \infty}$ be convergent sequences of $(\mathbb{N}, \rightarrow_n)$ and $\mathfrak{X}$, respectively. Then $(k_{m_n})_n$ converges to $k_{m_\infty}$, because $(\mathbb{N}, \rightarrow_n)$ is a weak limit space. Hence $(f^t(k_{m_n}, x_n))_{n}$ converges to $f^t(k_{m_\infty}, x_\infty)$ in $\mathfrak{Y}$, because the transpose $f^t$ is $(-\infty, -\infty)$-continuous. This implies $(f_{k_{m_n}}(x_n))_n \rightarrow f_{k_{m_\infty}}(x_\infty)$. Hence $(m, x) \mapsto f_{k_n}(x)$ is continuous. We conclude $(f_{k_n})_n \Rightarrow f_{k_\infty}$.

Axiom (L6): Let $(f_n)_{n \leq \infty}$. Let $(x_n)_{n \leq \infty}$ be a sequence of functions such that $(f_{n+1})_n$ converges continuously to some $f_{\infty} \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$. Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $\mathfrak{X}$, and let $(m_n)_{n \leq \infty}$ converge to $\infty$ in $(\mathbb{N}, \rightarrow_n)$. There is some $n_0 \in \mathbb{N}$ with $(\forall n \geq n_0) m_n \geq 1$. As $(m_{i+n_0} - 1)_i$ converges to $\infty$ and $(x_{i+n_0})_i$ converges to $x_\infty$ by Axiom (L5) for $\mathfrak{X}$, the sequence $(f_{m_{i+n_0}}(x_{i+n_0}))_i$ converges to $f_{\infty}(x_\infty)$ by continuity of the transpose $(m, x) \mapsto f_{m+1}(x)$. Since $\mathfrak{Y}$ satisfies Axiom (L6), $(f_{m_n}(x_n))_n$ converges to $f_{m_\infty}(x_\infty)$. We conclude that $(f_n)_{n \leq \infty}$ converges continuously to $f_{\infty}$ by (4.2).

(2) At first we show Equivalence (4.3). “$\Rightarrow$” is trivial; for the proof of “$\Leftarrow$” with the help of Condition (4.2), assume that $(f_n)_{n \leq \infty}$ satisfies the right hand side of Equivalence (4.3). Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $\mathfrak{X}$ and let $\xi : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $(\xi(n))_n$ converges to $\infty$. Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function.

Case 1: Let $\xi \varphi(n) = \infty$ for infinitely $n \in \mathbb{N}$.

Then there is a strictly increasing function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ with $(\forall n \in \mathbb{N}) \xi \varphi(n) = \infty$. By continuity of $f_{\infty}$, $(f_{\xi \varphi(n)}(x_{\varphi(n)}))_n$ converges to $f_{\infty}(x_\infty)$, because $(x_{\varphi(n)})_n$ converges to $x_\infty$ by Axiom (L5) for $\mathfrak{X}$.

Case 2: There is some $n_0 \in \mathbb{N}$ with $\xi \varphi(n) \neq \infty$ for all $n \geq n_0$.

Then we can define $\psi : \mathbb{N} \rightarrow \mathbb{N}$ inductively by $\psi(0) := n_0$ and

$$\psi(m + 1) := \min \left\{ i > n_0 \mid \xi \varphi(i) > \max \{ \xi \varphi(j) \mid \psi(0) \leq j \leq \psi(m) \} \right\},$$

where $i > n_0$ and $\xi \varphi(i) > \max \{ \xi \varphi(j) \mid \psi(0) \leq j \leq \psi(m) \}$. Then $\psi(m + 1)$ is well-defined and $\psi(m + 1) > n_0$. By induction, $\psi$ is strictly increasing and $\epsilon(\psi(m)) > n_0$ for all $\epsilon(\psi(m)) > n_0$. Then $(\forall n \in \mathbb{N}) \xi \varphi(n) \neq \infty$. By continuity of $f_{\infty}$, $(f_{\xi \varphi(n)}(x_{\varphi(n)}))_n$ converges to $f_{\infty}(x_\infty)$, because $(x_{\varphi(n)})_n$ converges to $x_\infty$ by Axiom (L5) for $\mathfrak{X}$.
4.2 Exponentiation

because \((\xi \varphi(n))_n\) converges to \(\infty\). Obviously, \(\psi\) and \(\xi \varphi \psi\) are strictly increasing and thus injective. We define \(\zeta : \mathbb{N} \to \mathbb{N}\) by

\[
\zeta(n) := \begin{cases} 
\varphi \psi((\xi \varphi)^{-1}(n)) & \text{if } n \in \text{range}(\xi \varphi \psi) \\
\infty & \text{otherwise.}
\end{cases}
\]

Then \((\zeta(n))_n\) converges to \(\infty\). Thus \((x_{\zeta(n)})_n\) converges to \(x_\infty\), because \(\mathfrak{X}\) satisfies Axiom (L5). Hence \((f_n(x_{\zeta(n)}))_n\) converges to \(f_\infty(x_\infty)\) by assumption. Since \((f_{\xi \varphi}(x_{\varphi \psi(n)}))_n\) is equal to \((f_{\xi \varphi}(x_{\zeta(\xi \varphi \psi(n))}))_n\) and thus forms a subsequence of \((f_n(x_{\zeta(n)}))_n\), this implies that \((f_{\xi \varphi}(x_{\varphi \psi(n)}))_n\) converges to \(f_\infty(x_\infty)\).

Therefore, in both cases \((f_{\xi \varphi}(x_{\varphi \psi}))(n)\) has a subsequence converging to \(f_\infty(x_\infty)\) implying that \((f_{\xi \varphi}(x_{\varphi \psi}(n)))_n\) converges to \(f_\infty(x_\infty)\), because \(\mathfrak{Y}\) satisfies Axiom (L3). Hence \((f_n)_{n \leq \infty}\) satisfies Condition (4.2). This proves Equivalence (4.3).

By Proposition 4.2.1(1), \(\mathcal{C}(\mathfrak{X}, \mathfrak{Y})\) satisfies Axiom (L1) and Axiom (L2). In order to show that Axiom (L3) holds as well, let \((f_n)_{n \leq \infty}\) be a non convergent sequence of \(\mathcal{C}(\mathfrak{X}, \mathfrak{Y})\). By Equivalence (4.3), there is a convergent sequence \((x_n)_{n \leq \infty}\) such that \((f_n(x_n))_n\) does not converge to \(f_\infty(x_\infty)\) in \(\mathfrak{Y}\). Since \(\mathfrak{Y}\) satisfies Axiom (L3), there is some strictly increasing function \(\varphi : \mathbb{N} \to \mathbb{N}\) such that no subsequence of \((f_{\varphi \psi}(x_{\varphi \psi}(n)))_n\) converges to \(f_\infty(x_\infty)\). For any strictly increasing function \(\psi : \mathbb{N} \to \mathbb{N}\), we have on the one hand \((x_{\varphi \psi(n)}(n)) \to x_\infty\) and on the other hand \((f_{\varphi \psi}(x_{\varphi \psi}(n)))_n \not\to f_\infty(x_\infty)\), hence \((f_{\varphi \psi}(n))_n\) does not converge continuously to \(f_\infty\). Therefore \(\mathcal{C}(\mathfrak{X}, \mathfrak{Y})\) fulfills Axiom (L3).

(3) This follows from Lemma 4.2.2.

\[
\square
\]

Let \(\mathfrak{Z} = (Z, \tau_3)\) be a topological space, and let \(\mathfrak{X} = (X, \to_X)\) be a weak limit space. We equip the set\(^6\) \(\mathcal{C}(\mathfrak{X}, \mathfrak{Z})\) with four topologies, the compact open topology \(\tau_{\mathfrak{X}, \mathfrak{Z}}^{\mathfrak{co}}\), the countably compact open topology \(\tau_{\mathfrak{X}, \mathfrak{Z}}^{\mathfrak{co}}\), the sequentially compact open topology \(\tau_{\mathfrak{X}, \mathfrak{Z}}^{\mathfrak{seq}}\), and the simple topology \(\tau_{\mathfrak{X}, \mathfrak{Z}}^{\mathfrak{simple}}\). They are defined by having, respectively, the sets

\[
\begin{align*}
\mathcal{F}(K, O) := & \{ K \text{ is compact in } (X, \text{seq}(X)) \text{ and } O \in \tau_3 \}, \\
\mathcal{F}(K, O) := & \{ K \text{ is countably compact in } (X, \text{seq}(X)) \text{ and } O \in \tau_3 \}, \\
\mathcal{F}(K, O) := & \{ K \text{ is sequentially compact in } \mathfrak{X} \text{ and } O \in \tau_3 \}, \text{ and} \\
\mathcal{F}(\{ x_n | n \in \mathbb{N} \}, O) := & \{ (x_n)_{n \in \mathbb{N}} \to x_\infty \text{ and } O \in \tau_3 \}
\end{align*}
\]

as a subbase (cf. [Wil70, Eng89, Smy92]), where \(\mathcal{F}(A, B)\) denotes the set

\[
\mathcal{F}(A, B) := \{ f \in \mathcal{C}(\mathfrak{X}, \mathfrak{Z}) | f(A) \subseteq B \}
\]

for every \(A \subseteq X\) and \(B \subseteq Z\). None of these topologies need be sequential, if \(\mathfrak{Z}\) is a sequential space and \(\mathfrak{X}\) is topological: a well–known counterexample is given by

\[^6\text{Remember that }\mathcal{C}(\mathfrak{X}, \mathfrak{Z}), \mathcal{C}(\mathfrak{X}, \mathfrak{Z}), \text{ and } \mathcal{C}(\mathfrak{X}, \mathfrak{W}(Z))\text{ abbreviate }\mathcal{C}(\mathfrak{X}, \mathfrak{W}(Z)), \mathcal{C}(\mathfrak{X}, \mathfrak{W}(Z)), \text{ and } \mathcal{C}(\mathfrak{X}, \mathfrak{W}(Z)),\text{ respectively, where }\mathfrak{W}(\mathfrak{Z}) = (Z, \to_\mathfrak{Z}).\]
\[ \mathfrak{X} = (\mathbb{N}^\omega, \rightarrow_\infty) \text{ and } \mathfrak{3} = (\mathbb{N}, \tau_\mathbb{N}). \] Note that for any topological space \((X, \tau)\) we have defined the compact open topology \(\tau_{(X,\tau),3}\) by considering the compact sets of the sequentialization \((X, \text{seq}(\tau))\) rather than those of \((X, \tau)\). If \((X, \tau)\) is a Hausdorff space with admissible representation then by Proposition 3.3.2(3) the compact sets of \((X, \text{seq}(\tau))\) are exactly the compact sets of \((X, \tau)\). Hence at least in this case our definition coincides with the usual definition of the compact open topology in [Wil70, Eng89, Smy92]. Moreover, in this case we have
\[
\tau_{(X,\tau),3}^{\text{co}} = \tau_{(X,\tau),3}^{\text{coco}} = \tau_{(X,\text{seq}(\tau)),3}^{\text{co}} = \tau_{(X,\text{seq}(\tau)),3}^{\text{coco}} = \tau_{(X,\text{seq}(\tau)),3}^{\text{seq}} = \tau_{(X,\text{seq}(\tau)),3}^{\text{coco}} = \tau_{(X,\text{seq}(\tau)),3}^{\text{seq}} = \tau_{(X,\text{seq}(\tau)),3}^{\text{co}}.
\]

as compactness, sequential compactness, and countable compactness coincide by Propositions 3.3.1(5) and 3.3.2(3) for Hausdorff spaces with admissible representations.

We show that the four topologies\(^7\) induce the convergence relation \(\Rightarrow_{\mathfrak{X},3}\) of continuous convergence on \(C(\mathfrak{X}, \mathfrak{3})\). In the case that \(\mathfrak{X}\) and \(\mathfrak{3}\) are sequential topological spaces, this property of the sequentially compact open topology and of the simple topology belongs to the folklore of sequential spaces (cf. [Hyl79]).

**Lemma 4.2.2 (Topologies inducing \(\Rightarrow_{\mathfrak{X},3}\))**

Let \(\mathfrak{X} = (X, \rightarrow_\mathfrak{X})\) be a weak limit space, and let \(\mathfrak{3} = (Z, \tau_3)\) be a topological space. Then \(\Rightarrow_{\mathfrak{X},3}^{\text{coco}} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{co}} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{sco}} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{seq}} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{simple}}.\)

**Proof:**

We show \(\Rightarrow_{\mathfrak{X},3} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{coco}} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{co}} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{sco}} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{seq}} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{simple}}.\)

\(\Rightarrow_{\mathfrak{X},3} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{coco}}:\) Let \((f_n)_{n=1}^{\infty}\) be a convergent sequence of \(C(\mathfrak{X}, \mathfrak{3})\). Let \(K \neq \emptyset\) be a countably compact set of \(\mathfrak{X}\), and let \(O \in \tau_3\) be an open set with \(f_\infty[K] \subseteq O\). As \(S_{\mathfrak{X},3}^{\text{coco}}\) is a subbase of \(\tau_{\mathfrak{X},3}^{\text{coco}}\), it suffices to show that \((f_n)_{n=1}^{\infty}\) is eventually in \(F(K, O)\). Let \(m \in \mathbb{N}\). Since the transpose \(f^* : \mathbb{N} \times \mathfrak{X} \to Y\) of \((f_n)_{n=1}^{\infty}\) is \((\rightarrow_\mathfrak{X}, \rightarrow, \rightarrow)\)-continuous and \(\{m, \ldots, \infty\}\) is a sequentially compact subset of \((\mathbb{N}, \rightarrow_\mathfrak{X})\), the set
\[
U_m := \{x \in X \mid (\forall n \in \{m, \ldots, \infty\}) f_n(x) \in O\}
\]
is sequentially open in \(\mathfrak{X}\) by Lemma 2.2.9. For every \(x \in K\) there is some \(m_x \in \mathbb{N}\) with \((\forall n \geq m_x) f_n(x) \in O\), because \((f_n(x))_{n=1}^{\infty}\) converges to \(f_\infty(x)\). Thus \(\{U_m \mid m \in \mathbb{N}\}\) is a countable sequentially open cover of \(K\). Since \(K\) is countably compact in \((\mathfrak{X}, \text{seq}(\mathfrak{X}))\), there are \(k \geq 1\) and \(m_1, \ldots, m_k \in K\) with \(K \subseteq U_{m_1} \cup \ldots \cup U_{m_k}\). For \(m_0 := \max\{m_1, \ldots, m_k\}\) we have \(K \subseteq U_{m_1} \cup \ldots \cup U_{m_k} \subseteq U_{m_0}\). This implies \(f_n \in F(K, O)\) for all \(n \geq m_0\). Hence \((f_n)_{n=1}^{\infty}\) is a convergent sequence of \((C(\mathfrak{X}, 3), \tau_{\mathfrak{X},3}^{\text{coco}})\).

\(\Rightarrow_{\mathfrak{X},3}^{\text{co}} \subseteq \Rightarrow_{\mathfrak{X},3}^{\text{coco}}:\) This follows from the fact that \(\tau_{\mathfrak{X},3}^{\text{coco}}\) is finer (larger) than or equal to \(\tau_{\mathfrak{X},3}^{\text{co}},\) because compact sets are countably compact implying \(S_{\mathfrak{X},3}^{\text{coco}} \supseteq S_{\mathfrak{X},3}^{\text{co}}\).

\(^7\)and of course in the case that \(\tau\) is a sequential topology.

\(^8\)If \((X, \tau)\) is a sequential space, then also the Isbell topology on \(C((X, \tau), 3)\) can be shown to induce \(\Rightarrow_{(X,\tau),3}\) in a similar way as the compact open topology. A subbase of the Isbell topology is given by the family of the sets \(N(H, O) := \{f \in C((X, \tau), 3) \mid f^{-1}[O] \in H\}\) for \(H \subseteq \tau\) a Scott–open set and \(O\) an open set of \(3\) (cf. [Smy92]). A set \(H \subseteq \tau\) is called Scott–open iff \(U \subseteq H\) and \(U \subseteq V \subseteq \tau\) imply \(V \in H\) and for all \(\alpha \subseteq \tau\) with \(\bigcup \alpha \in H\) there is some finite subset \(\beta \subseteq \alpha\) with \(\bigcup \beta \in H\).
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→ \tau_{x3} \subseteq \tau_{x3}^{\text{simple}}: For any convergent sequence \((x_n)_{n \leq \infty}\) of \(X\) the set \(K := \{x_n \mid n \in \mathbb{N}\}\) is compact in \(T(X)\), because for every sequentially open cover \(O\) of \(K\) the sequence \((x_n)_{n}\) is eventually in each of those sets \(O \in \mathcal{O}\) that contain \(x_{\infty}\). Thus we have \(\mathcal{S}_{x3}^{\text{co}} \supseteq \mathcal{S}_{x3}^{\text{simple}}\) and hence \(\tau_{x3}^{\text{co}} \supseteq \tau_{x3}^{\text{simple}}\). Therefore convergence w.r.t. the compact topology \(\tau_{x3}^{\text{co}}\) implies convergence w.r.t. the simple topology \(\tau_{x3}^{\text{simple}}\).

→ \tau_{x3}^{\text{simple}} \subseteq \tau_{x3}^{\text{co}}: Let \((f_n)_{n \leq \infty}\) be a convergent sequence of the space \((C(X, 3), \tau_{x3}^{\text{simple}})\).

Let \((x_n)_{n \leq \infty}\) be a convergent sequence of \(X\), and let \((m_n)_{n}\) converge to \(\infty\) in \((\mathbb{N}, -\infty)\). Let \(O \in \tau_3\) be an open set containing \(f_\infty(x_\infty)\). By sequential continuity of \(f_\infty\), there is some \(n_1 \in \mathbb{N}\) with \(f_\infty(x_n) \in O\) for all \(n \geq n_1\). Since \((f_{m_n})_{n}\) converges to \(f_\infty\) in \((C(X, 3), \tau_{x3}^{\text{simple}})\) and \(F(\{x_n \mid n_1 \leq n \leq \infty\}, O)\) is an open neighbourhood of \(f_\infty\) in this space, there is some \(n_2 \in \mathbb{N}\) with \(f_{m_n} \in F(\{x_n \mid n_1 \leq n \leq \infty\}, O)\) for all \(n \geq n_2\). We obtain \(f_{m_n}(x_n) \in O\) for all \(n \geq \max\{n_1, n_2\}\). Hence \((f_{m_n}(x_n))_{n}\) converges to \(f_\infty(x_\infty)\) in \(3\). We conclude \((f_n)_{n} \Rightarrow \tau_{x3}^{\text{co}}\) by (4.3), because \(\mathcal{W}(3)\) is a limit space.

\(\Rightarrow \tau_{x3} \subseteq \tau_{x3}^{\text{co}}: \) Let \((f_n)_{n \leq \infty}\) be a convergent sequence of \(C(X, 3)\). Let \(K\) be a sequentially compact set of \(X\) and let \(O \in \tau_3\) be an open set with \(f_\infty[K]\subseteq O\). We show that \((f_n)_{n}\) is eventually in \(F(K, O)\).

Suppose for contradiction \(f_n \notin F(K, O)\) for infinitely many \(n \in \mathbb{N}\). Then there is a strictly increasing function \(\varphi : \mathbb{N} \rightarrow \mathbb{N}\) and a sequence \((x_n)_{n}\) in \(K\) with \(f_{\varphi(n)}(x_n) \notin O\).

By sequential compactness of \(K\), there is a strictly increasing function \(\psi : \mathbb{N} \rightarrow \mathbb{N}\) such that \((x_{\psi(n)})_{n}\) converges to some \(x_\infty \in K\). As \(C(X, 3)\) satisfies Axiom (L5) by Proposition 4.2.1(1), \((f_{\varphi(n)}(x_{\psi(n)}))_{n}\) converges continuously to \(f_\infty\). Hence \((f_{\varphi(n)}(x_{\psi(n)}))_{n}\) converges to \(f_\infty(x_\infty)\). Thus \((f_{\varphi(n)}(x_{\psi(n)}))_{n}\) is eventually in \(O\), because \(O\) contains \(f_\infty(x_\infty)\). This contradicts \((\forall n \in \mathbb{N}\) \(f_{\varphi(n)}(x_{\psi(n)}) \notin O)\). Therefore \((f_n)_{n}\) converges to \(f_\infty\) with respect to the sequentially compact open topology \(\tau_{x3}^{\text{co}}\) on \(C(X, 3)\).

→ \tau_{x3}^{\text{co}} \subseteq \tau_{x3}^{\text{simple}}: As for every convergent sequence \((x_n)_{n \leq \infty}\) the set \(K := \{x_n \mid n \in \mathbb{N}\}\) is sequentially compact (cf. Subsection 2.2.5), we have \(\mathcal{S}_{x3}^{\text{co}} \supseteq \mathcal{S}_{x3}^{\text{simple}}\) and hence \(\tau_{x3}^{\text{co}} \supseteq \tau_{x3}^{\text{simple}}\). Therefore convergence w.r.t. the sequentially compact open topology \(\tau_{x3}^{\text{co}}\) implies convergence w.r.t. the simple topology \(\tau_{x3}^{\text{simple}}\).

4.2.2 Cartesian Closure of WeakLim

Now we prove that the Cartesian category WeakLim of weak limit spaces and of total sequentially continuous functions is Cartesian closed.

Let \(C\) be a Cartesian category (cf. Subsection 4.1.4), and let \(X\) and \(Y\) be objects of \(C\).

An exponent of \(X, Y\) in \(C\) is an object \(Y^X\) and a morphism \(\text{eval} : Y^X \times C X \rightarrow Y\) such that for all objects \(Z\) and all morphisms \(h : Z \times C X \rightarrow Y\) there is exactly one morphism \(\hat{h} : Z \rightarrow Y^X\) with \(h = \text{eval} \circ (\hat{h} \times \text{id}_X)\), where the morphism \(\hat{h} \times \text{id}_X : Z \times C X \rightarrow Y^X \times C X\) is defined by \(\hat{h} \times \text{id}_X := (\hat{h} \circ p_Z, X, q_Z, X)Y^X, X\). A category \(D\) is called Cartesian closed
Chapter 4. Constructions

if D is Cartesian and for all objects X, Y in D there exists an exponent \( Y^X \) in D, cf. [AL91].

**Proposition 4.2.3 (WeakLim is Cartesian closed)**

Let \( \mathcal{X} = (X, \rightarrow_X) \), \( \mathcal{Y} = (Y, \rightarrow_Y) \), and \( \mathcal{Z} = (Z, \rightarrow_Z) \) be weak limit spaces.

1. The evaluation function \( \text{eval} : C(\mathcal{X}, \mathcal{Y}) \times X \to Y \) defined by \( \text{eval}(f, x) := f(x) \) is \((\exists x, y, \rightarrow_x, \rightarrow_y)\)-continuous.

2. If \( h : Z \times X \to Y \) is sequentially continuous, then the function \( \Lambda(h) : Z \to C(\mathcal{X}, \mathcal{Y}) \)
   defined by \( (\Lambda(h)(z))(x) := h(z, x) \) is continuous with respect to \( \rightarrow_Z \) and \( \Rightarrow_{\mathcal{X}, \mathcal{Y}} \).

3. WeakLim is Cartesian closed.

**Proof:**

1. Let \((f_n)_{n \leq \infty}\) and \((x_n)_{n \leq \infty}\) be convergent sequences of the spaces \( C(\mathcal{X}, \mathcal{Y}) \) and \( \mathcal{X} \), respectively. Since \((n)\) converges to \( \infty \) and the transpose of \((f_n)_{n \leq \infty}\) is continuous, the sequence \( (\text{eval}(f_n, x_n))_n = (f_n(x_n))_n \) converges to \( f_\infty(x_\infty) \) in \( \mathcal{Y} \). Hence \( \text{eval} \) is continuous with respect to \( \Rightarrow_{\mathcal{X}, \mathcal{Y}} \) and \( \rightarrow_x \) and \( \rightarrow_y \).

2. Let \((z_n)_{n \leq \infty}\) be a convergent sequence of \( \mathcal{Z} \). Let \((m_n)_{n \leq \infty}\) and \((x_n)_{n \leq \infty}\) be convergent sequences of \( (\mathbb{R}, \rightarrow_\mathbb{R}) \) and \( \mathcal{X} \). Then \((z_{m_n})_n \) converges to \( z_{m_\infty} \), because \( \mathcal{Z} \) satisfies Axiom \((L5)\). Hence \((\Lambda(h)(z_{m_n})(x_n))_n = (h(z_{m_n}, x_n))_n \) converges to \( h(z_{m_\infty}, x_\infty) = \Lambda(h)(z_{m_\infty})(x_\infty) \) by continuity of \( h \). This implies that the function \((m, x) \mapsto \Lambda(h)(z_m)(x) \) is \((\rightarrow_\mathbb{R}, \rightarrow_x, \rightarrow_y)\)-continuous and that \((\Lambda(h)(z_n))_n \) converges continuously to \( \Lambda(h)(z_\infty) \). Thus \( \Lambda(h) \) is sequentially continuous.

3. By Subsections 2.2.3 and 4.1.4, \( (\mathcal{X} \otimes \mathcal{Y}, pr_1, pr_2) \) is a categorical product of \( \mathcal{X} \) and \( \mathcal{Y} \) in WeakLim. Statements (1) and (2) imply that \( C(\mathcal{X}, \mathcal{Y}) \) and \( \text{eval} \) form an exponent of \( \mathcal{X} \) and \( \mathcal{Y} \) in WeakLim.

Proposition 4.2.1 and 4.2.3 imply the well–known result that also the categories Lim and Seq are Cartesian closed (cf. [Hyl79]). By the following lemma, being a consequence of the fact that \( L \) and \( T \) preserve finite products, the subcategories Seq and Lim are even exponential ideals of WeakLim. A subcategory \( D \) of a Cartesian closed category \( \mathcal{C} \) is called an exponential ideal iff the exponent \( Y^X \) formed in \( \mathcal{C} \) represents an object of \( D \), whenever \( Y \) represents an object of \( D \).

**Lemma 4.2.4**

Let \( \mathcal{X} = (X, \rightarrow_X) \) be a weak limit space.

1. For every topological space \( \mathcal{Y} = (Y, \tau_\mathcal{Y}) \), the function space \( C(T(\mathcal{X}), \mathcal{Y}) \) is equal to \( C(\mathcal{X}, \mathcal{Y}) \).

2. For every limit space \( \mathcal{Y} = (Y, \rightarrow_\mathcal{Y}) \), the function space \( C(L(\mathcal{X}), \mathcal{Y}) \) is equal to \( C(\mathcal{X}, \mathcal{Y}) \).
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Proof:

(1) It suffices to show that a function \( f : \mathbb{N} \times X \to Y \) is \((\pi, \tau_X, \eta)\)-continuous if and only if it is \((\pi, \tau_{\mathcal{T}(X)}, \eta)\)-continuous.

\("\Longleftarrow\): This follows from \( \pi \subseteq \tau_{\mathcal{T}(X)} \) (cf. Lemma 2.2.7).

\("\implies\): If \( f \) is \((\pi, \tau_X, \eta)\)-continuous, then \( f \) is \((\pi, \tau_{\mathcal{T}(X)}, \tau, \eta)\)-continuous by Theorem 2.2.10(2). As \( \pi \) and \( \tau \) are induced by topologies, we have \( \tau_{\mathcal{T}(\pi, \tau)} = \tau \) and \( \tau = \tau \) by Lemma 2.2.7. Thus \( f \) is \((\pi, \tau_{\mathcal{T}(X)}, \eta)\)-continuous.

This equivalence implies both \( C(\mathcal{T}(X), \eta) = C(X, \eta) \) and \( \Rightarrow_{\mathcal{T}(X), \eta} = \Rightarrow_X, \eta \).

(2) Similar to (1) by Theorem 2.2.10(1) and Lemma 2.2.8.

4.2.3 The Representation \( \eta \)

For constructing an admissible multirepresentation of \( \mathfrak{C}(X, \eta) \), we use an “effective” representation \( \eta : \omega \to \mathcal{F}^{\omega} \) of all continuous functions \( g : \omega \to \omega \) having a \( G_{\delta} \)-domain. Effectivity of \( \eta \) means here that \( \eta \) satisfies the extension property, the utm–Theorem, the computable smn–Theorem, and the continuous smn–Theorem, i.e.:

(1) for every continuous function \( g : \omega \to \omega \) there is some \( p \in \omega \) such that \( \eta(p) \) extends \( g \) (extension property);

(2) the universal function \( u_\eta : \omega \times \omega \to \omega \) defined by \( u_\eta(p, q) := \eta(p)(q) \) is computable (utm–Theorem) and thus continuous (topological utm–Theorem);

(3) for every computable function \( g : \omega \times \omega \to \omega \) there is some computable function \( s : \omega \to \omega \) with \( \eta(s(p))(q) = g(p, q) \) for all \( p, q \in \omega \) (computable smn–Theorem);

(4) for every continuous function \( g : \omega \times \omega \to \omega \) there is some continuous function \( s : \omega \to \omega \) with \( \eta(s(p))(q) = g(p, q) \) for all \( p, q \in \text{dom}(g) \) (continuous smn–Theorem).

From Property (3) one can easily conclude that a function \( g : \omega \to \omega \) is computable if and only if there is a computable \( \omega \)-word \( p \) with \( g = \eta(p) \). In the following, we will often write \( \eta_p \) instead of \( \eta(p) \). These effectivity properties guarantee that \( \omega \) and the application function \( (\alpha | \beta) := \eta_\alpha(\beta) \) form a partial combinatory algebra (cf. [Bau00, Bau01]).

A representation \( \eta \) satisfying these properties can be found in [Wei85, Wei87, Bau00, Bau01] or constructed from the one for the Cantor space \( \Sigma^\omega \) in [Wei90], for which we here write \( \eta^{(\Sigma^\omega)} \), by defining

\[
\eta_p(q) = \eta(p)(q) := \vartheta_{\Sigma^\omega} \left( \eta^{(\Sigma^\omega)}(q) \right),
\]
where \( q_{\Sigma^*} : \subseteq \mathbb{N}^\omega \to \Sigma^* \) and \( \delta_{\Sigma^*} : \subseteq \Sigma^* \to \mathbb{N}^\omega \) are the representations defined in Subsection 2.1.7. We sketch the construction in [Bau01]: At first one defines a function \( * : \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N} \) for all \( p, q \in \mathbb{N}^\omega \) and \( m \in \mathbb{N} \) by
\[
p * q = m :\iff (\exists i \in \mathbb{N}) (p(\nu_m^{-1}(q^{<i}))) = m + 1 \land (\forall j < i) p(\nu_m^{-1}(q^{<j})) = 0).
\]
Here \( \nu_{\eta^*} : \mathbb{N} \to \mathbb{N}^\omega \) is the canonical bijection from Subsection 1.1.3. The partial function \( \eta_p \in F^{\omega\omega} \) is then defined by
\[
dom(\eta_p) := \{ q \in \mathbb{N}^\omega \mid (\forall n \in \mathbb{N}) p * (n::q) \neq \text{div} \} \quad \text{and} \quad \eta_p(q) := p * (n::q)
\]
for \( p \in \mathbb{N}^\omega \), \( q \in \dom(\eta_p) \) and \( n \in \mathbb{N} \), where \( n::q \) denotes the \( \omega \)-word that has \( n \) as its first symbol followed by \( q \). Properties (2) and (3) imply that all these definitions yield computably equivalent representations of \( F^{\omega\omega} \). Hence the exact choice of \( \eta \) is of minor importance.

### 4.2.4 Admissible Multirepresentations of \( \mathcal{C}(\mathcal{X}, \mathcal{Y}) \)

Now we will construct an admissible multirepresentation of \( \mathcal{C}(\mathcal{X}, \mathcal{Y}) \) starting from admissible ones of given weak limit spaces \( \mathcal{X} = (X, \to_{\chi}) \) and \( \mathcal{Y} = (Y, \to_{\gamma}) \).

Let \( \delta : \subseteq \mathbb{N}^\omega \to X \) and \( \gamma : \subseteq \mathbb{N}^\omega \to Y \) be multirepresentation of sets \( X \) and \( Y \), respectively. By \( \mathcal{C}(\delta, \gamma) \) we denote the set of all total functions \( f : X \to Y \) which are relatively continuous w.r.t. \( \delta \) and \( \gamma \), and by \( \mathcal{C}_{cp}(\delta, \gamma) \) the set of all total \( (\delta, \gamma) \)-computable functions between \( X \) and \( Y \). Proposition 2.4.12 implies that these functions are continuous functions between the generated weak limit spaces \( (X, \to_{\chi}) \) and \( (Y, \to_{\gamma}) \), hence \( \mathcal{C}(\delta, \gamma) \) is a subset of \( \mathcal{C}((X, \to_{\chi}), (Y, \to_{\gamma})) \).

Analogously to [Wei00, Def. 3.3.13], we define the multirepresentation \( [\delta \to \gamma] : \subseteq \mathbb{N}^\omega \to \mathcal{C}(\delta, \gamma) \) by
\[
[\delta \to \gamma][p] \ni f :\iff \eta_p \text{ realizes } f \text{ with respect to } \delta \text{ and } \gamma
\]
for all \( p \in \mathbb{N}^\omega \) and \( f \in \mathcal{C}(\delta, \gamma) \). The extension property of \( \eta \) guarantees the surjectivity of \([\delta \to \gamma]\). One easily verifies that \([\delta \to \gamma]\) is single–valued if and only if \( \gamma \) is single–valued.

If \( \gamma \) is \( \to_{\gamma} \)-admissible, then \( \mathcal{C}(\delta, \gamma) \) is a subset of \( \mathcal{C}((X, \to_{\gamma}), (Y, \to_{\gamma})) \) by Lemma 2.4.26 implying that \([\delta \to \gamma]\) is a multirepresentation of \( \mathcal{C}((X, \to_{\gamma}), (Y, \to_{\gamma})) \). By the next proposition, in this case \([\delta \to \gamma]\) is even an admissible multirepresentation of the function space \( \mathcal{C}((X, \to_{\gamma}), (Y, \to_{\gamma})) \).

We investigate which conditions guarantee that \([\delta \to \gamma]\) is an admissible multirepresentation of the function space \( \mathcal{C}(\mathcal{X}, \mathcal{Y}) \).

**Proposition 4.2.5 (Admissibility of \([\delta \to \gamma]\))**

1. Let \( \delta : \subseteq \mathbb{N}^\omega \to X \) be a \textbf{WeakLim–quotient} multirepresentation of a weak limit space \( \mathcal{X} = (X, \to_{\chi}) \), and let \( \gamma : \subseteq \mathbb{N}^\omega \to Y \) be an admissible multirepresentation of a weak limit space \( \mathcal{Y} = (Y, \to_{\gamma}) \).

   Then \([\delta \to \gamma]\) is an admissible multirepresentation of \( \mathcal{C}(\mathcal{X}, \mathcal{Y}) \).

2. Let \( \delta : \subseteq \mathbb{N}^\omega \to X \) be a \textbf{Lim–quotient} multirepresentation of a limit space \( \mathcal{X} = (X, \to_{\chi}) \), and let \( \gamma : \subseteq \mathbb{N}^\omega \to Y \) be an admissible multirepresentation of a limit space \( \mathcal{Y} = (Y, \to_{\gamma}) \).

   Then \([\delta \to \gamma]\) is an admissible multirepresentation of \( \mathcal{C}(\mathcal{X}, \mathcal{Y}) \).
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(3) Let $\delta : \subseteq 2^{\mathbb{N}^\omega} \Rightarrow X$ be a WeakLim–quotient or a Lim–quotient multirepresentation of a (weak) limit space $X = (X, -\gamma)$, and let $\gamma : \subseteq 2^{\mathbb{N}^\omega} \Rightarrow Z$ be an admissible multirepresentation of a topological space $Z = (Z, \tau_Z)$.

Then $[\delta \rightarrow \gamma]$ is an admissible multirepresentation of $C(X, Z)$. Moreover, $[\delta \rightarrow \gamma]$ is admissible w.r.t. the compact open topology and w.r.t. the sequentially compact open topology on $C(X, Z)$.

(4) Let $\delta : \subseteq 2^{\mathbb{N}^\omega} \Rightarrow X$ be a Top–quotient multirepresentation of a topological space $X = (X, \tau_X)$, and let $\gamma : \subseteq 2^{\mathbb{N}^\omega} \Rightarrow Z$ be an admissible multirepresentation of a topological space $Z = (Z, \tau_Z)$.

Then $[\delta \rightarrow \gamma]$ is an admissible multirepresentation of $C(X, Z)$. Moreover, $[\delta \rightarrow \gamma]$ is admissible w.r.t. the compact open topology and w.r.t. the sequentially compact open topology on $C(X, Z)$.

Proof:

(1) Continuity: Let $(p_n)_{n \leq \infty}$ be a convergent sequence of $(\mathbb{N}^\omega, -\gamma_{\mathbb{N}^\omega})$ and let $(f_n)_{n \leq \infty}$ be a sequence of sequentially continuous functions with $f_n \in [\delta \rightarrow \gamma][p_n]$ for every $n \in \mathbb{N}$. Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $X$, and let $(m_n)_{n \leq \infty}$ converge to $\infty$ in $(\mathbb{N}, -\gamma)$. Since $\delta$ is a WeakLim–quotient multirepresentation, there is a convergent sequence $(\eta_n)_{n \leq \infty}$ of $(\mathbb{N}^\omega, -\gamma_{\mathbb{N}^\omega})$ with $(\forall n \in \mathbb{N}) x_n \in \delta[\eta_n]$. For every $n \in \mathbb{N}$, we have $q_n \in \text{dom}(\eta_{p_{m_n}})$ and $\gamma[\eta_{p_{m_n}}(q_n)] \ni f_{m_n}(x_{n})$, because $\eta_{p_{m_n}}$ realizes $f_{m_n}$ w.r.t. $\delta$ and $\gamma$. By the topological utm–theorem, $(\eta_{p_{m_n}}(q_n))_{n}$ converges to $\eta_{p_{\infty}}(q_{\infty})$, thus $(f_{m_n}(x_{n}))_{n}$ converges to $f_{\infty}(x_{\infty})$ by sequential continuity of $\gamma$. We conclude that $(f_n)_{n}$ converges continuously to $f_{\infty}$ by (4.2).

Universality: Let $\phi : \subseteq 2^{\mathbb{N}^\omega} \Rightarrow C(X, Y)$ be continuous w.r.t. $-\gamma_{\mathbb{N}^\omega}$ and $\Rightarrow$. We define the correspondence $\psi : \subseteq 2^{\mathbb{N}^\omega} \times 2^{\mathbb{N}^\omega} \Rightarrow Y$ by

$$\psi[(p, q)] := \{ f(x) | f \in \phi[p] \text{ and } x \in \delta[q] \}$$

for all $p, q \in \mathbb{N}^\omega$. In order to prove sequential continuity of $\psi$, let $(\{(p_n, q_n)\}_{n \leq \infty})$ be a convergent sequence in $\text{dom}(\psi)$ and $(y_n)_{n \leq \infty} \in \prod_{n \in \mathbb{N}} \psi[(p_n, q_n)]$. For every $n \in \mathbb{N}$, there are $f_n \in \phi[p_n]$ and $x_n \in \delta[q_n]$ with $f_n(x_n) = y_n$. By sequential continuity of $\phi$ and $\delta$, we have $(f_n)_{n} \Rightarrow f_{\infty}$ and $(x_n)_{n} \rightarrow \infty$. Thus, by definition of $\Rightarrow$, $(y_n)_{n}$ converges to $y_{\infty}$ showing the sequential continuity of $\psi$.

By Lemma 2.4.19, there is a continuous function $g : \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega \Rightarrow \mathbb{N}^\omega$ with $\psi[(p, q)] \subseteq \gamma[g(p, q)]$ for all $(p, q) \in \text{dom}(\psi)$. By the continuous smn–Theorem, there is a continuous function $s : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ with $\eta_{s(p)}(q) = g(p, q)$ for all $(p, q) \in \text{dom}(g)$.

Let $p \in \text{dom}(\phi)$ and $f \in \phi[p]$. For every $q \in \text{dom}(\delta)$ and $x \in \delta[q]$, we have

$$f(x) \in \psi[(p, q)] \subseteq \gamma[g(p, q)] = \gamma[\eta_{s(p)}(q)].$$

Thus $\eta_{s(p)} = \eta(s(p))$ realizes $f$ w.r.t. $\delta$ and $\gamma$, hence $f \in [\delta \rightarrow \gamma][s(p)]$. Therefore the continuous function $s$ translates $\phi$ to $[\delta \rightarrow \gamma]$.

(2) As $\delta$ is a Lim–quotient multirepresentation, we have $\sim_{\gamma} = -\chi$. Thus $C(X, -\chi) = (X, -\chi_{\chi}) = X$ by Lemma 2.4.11. Hence $C(X, Y)$ is equal to $C((X, -\chi), Y)$ by Lemma 4.2.4. By (1), $[\delta \rightarrow \gamma]$ is an admissible multirepresentation of the space $C((X, -\chi), Y)$ and hence of $C(X, Y)$. 

(3) From (1) or (2) and the fact that $\langle Z, \rightarrow_{\gamma} \rangle$ is a limit space it follows that $[\delta \rightarrow \gamma]$ is an admissible multirepresentation of $C(\mathcal{X}, \mathcal{Z})$. By Lemma 4.2.2, $[\delta \rightarrow \gamma]$ is admissible w.r.t. the compact open topology and w.r.t. the sequentially compact open topology on $C(\mathcal{X}, \mathcal{Z})$.

(4) As $\text{seq}(\rightarrow_\eta)$ is equal to the final topology $\tau_\delta$ by Lemma 2.4.11, we have $T(\mathcal{X}, \rightarrow_\eta) = \mathcal{X}$ and thus $C(\mathcal{X}, \mathcal{Z}) = C(\langle X, \rightarrow_\eta \rangle, \mathcal{Z})$ by Lemma 4.2.4. By (1), $[\delta \rightarrow \gamma]$ is an admissible multirepresentation of $C(\langle X, \rightarrow_\eta \rangle, \mathcal{Z})$ and hence of $C(\mathcal{X}, \mathcal{Z})$. By Lemma 4.2.2, $[\delta \rightarrow \gamma]$ is admissible w.r.t. the compact open topology and w.r.t. the sequentially compact open topology on $C(\mathcal{X}, \mathcal{Z})$.

Propositions 4.2.1, 4.2.3, 4.2.5, and Corollary 4.1.10 imply:

**Theorem 4.2.6**

The categories $\text{AdmWeakLim}$, $\text{AdmLim}$, and $\text{AdmSeq}$ are Cartesian closed.

Moreover, $\text{AdmLim}$ and $\text{AdmSeq}$ are exponential ideals of $\text{AdmWeakLim}$ by Lemma 4.2.4. Since we have $\text{AdmSeq} = \text{PQ}$ and $\text{AdmLim} = \text{PQL}$ by Theorem 3.4.8, we already know from [MS02] that $\text{AdmSeq}$ is an exponential ideal of $\text{AdmLim}$.

We present another proof the fact that $\text{AdmWeakLim}$ is Cartesian closed by showing how to construct countable limit bases for the product and the exponent in $\text{WeakLim}$ of two weak limit spaces with countable limit bases (cf. Subsection 3.1.1). Remember that a weak limit space has an admissible multirepresentation if and only if it has a countable limit base (cf. Proposition 3.1.7). Similarly, the Cartesian closure of $\text{AdmSeq}$ can be shown by means of countable pseudobases.

**Proposition 4.2.7 (Limit bases for products and exponents)**

Let $\mathcal{X} = \langle X, \rightarrow_\chi \rangle$ and $\mathcal{Y} = \langle Y, \rightarrow_\eta \rangle$ be weak limit spaces. Let $A \subseteq 2^X$ and $B \subseteq 2^Y$ be countable limit bases of $\mathcal{X}$ and $\mathcal{Y}$, respectively.

1. The family $\mathcal{D} := \{A \times B \mid A \in A, B \in B\}$ is a countable limit base of $\mathcal{X} \times \mathcal{Y}$.

2. The family $\mathcal{E} := \{F(A_1 \cap \cdots \cap A_k, B) \mid k \in \mathbb{N}, \{A_1, \ldots, A_k\} \subseteq A, B \in B\}$ is a countable limit base of $C(\mathcal{X}, \mathcal{Y})$.

3. If $A$ and $B$ are countable pseudobases of sequential spaces $\mathcal{X}' = \langle X, \tau_X \rangle$ and $\mathcal{Y}' = \langle Y, \tau_Y \rangle$, respectively, then $\mathcal{D}$ as defined in (1) is a countable pseudobase of $\langle X \times Y, \text{seq}(\tau_X \otimes \tau_Y) \rangle$ and $\mathcal{E}$ as defined in (2) is a countable pseudobase of $\langle C(\mathcal{X}', \mathcal{Y}'), \text{seq}(\tau_{X'} \otimes \tau_{Y'}) \rangle$.

**Proof:**

1. Let $\langle x_n \rangle_{n \leq \infty}$ be a convergent sequence of $\mathcal{X} \otimes \mathcal{Y}$, and let $\langle z_n \rangle_n$ be a sequence in $X \times Y$ that does not converge to $x_\infty$ in $\mathcal{X} \otimes \mathcal{Y}$. Let $pr_1 : X \times Y \to X$, $pr_2 : X \times Y \to Y$ denote the projection functions. W.l.o.g. we can assume that $\langle pr_1(z_n) \rangle_n$ does not converge to $pr_1(x_\infty)$ in $\mathcal{X}$. Since $A$ and $B$ are limit bases, there are sets $A \in A$ and $B \in B$ with

$$\forall n \in \mathbb{N} \left( pr_1(x_n) \in A \land pr_2(x_n) \in B \right) \quad \text{and} \quad \exists n \in \mathbb{N} \left( pr_1(z_n) \notin A \right).$$
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It follows $(\forall^\infty n \in \mathbb{N}) x_n \in A \times B$ and $(\exists^\infty n \in \mathbb{N}) z_n \notin A \times B$. We conclude that $D$ is a countable limit base of $X \otimes \mathcal{Y}$.

(2) Let $(f_n)_{n \leq \infty}$ be a convergent sequence of $C(X, \mathcal{Y})$, and let $(g_n)_{n \leq \infty}$ be a non convergent sequence of $C(X, \mathcal{Y})$ with $g_\infty = f_\infty$. By Implication (4.2), there is a convergent sequence $((m_n, x_n))_{n \leq \infty}$ of $(\mathbb{N}, -\mathfrak{p}) \otimes X$ with $m_\infty = \infty$ such that $(z_n)_n := (g_m(x_n))_n$ does not converge to $f_\infty(x_\infty)$. Moreover, the transpose $f^t$ is a continuous function from $(\mathbb{N}, -\mathfrak{p}) \otimes X$ to $\mathcal{Y}$. Since $\{ \{l\}, \{l, \ldots, \infty\} \mid l \in \mathbb{N}\}$ is a limit base of $(\mathbb{N}, -\mathfrak{p})$, we conclude from (1) that

$$D := \{ \{l\} \times A, \{l, \ldots, \infty\} \times A \mid l \in \mathbb{N}, A \in A \}$$

is a countable limit base of $(\mathbb{N}, -\mathfrak{p}) \otimes X$. As $((m, x_n))_n$ converges to $(\infty, x_\infty)$ and $f^t$ is sequentially continuous, by Lemma 3.1.5 there are some $k, n_1 \in \mathbb{N}$, some sets $D_0, \ldots, D_k \in D$ and a set $B \in B$ with $(n, x_n) \in D_0 \cap \ldots \cap D_k$ for all $n \in \{n_1, \ldots, \infty\}$, $f^t[D_0 \cap \ldots \cap D_k] \subseteq B$ and $(\exists^\infty n \in \mathbb{N}) z_n \notin B$. For every $i \in \{0, \ldots, k\}$ there is some $l_i \in \mathbb{N}$ and some $A_i \in A$ with $D_i = \{l_i, \ldots, \infty\} \times A_i$. For all $n \geq \max\{l_0, \ldots, l_k\}$ (including $n = \infty$) it follows $f_n \in F(A_0 \cap \ldots \cap A_k, B)$. Since $\forall n \geq n_1) x_n \in A_0 \cap \ldots \cap A_k$, we have $g_{m_n}[A_0 \cap \ldots \cap A_k] \notin B$ and thus $g_n \notin F(A_0 \cap \ldots \cap A_k, B)$ for infinitely many $n \in \mathbb{N}$. As $(m_n)_n$ converges to $\infty$ and $g_\infty = f_\infty \in F(A_0 \cap \ldots \cap A_k, B)$, there are also infinitely many $n$ with $g_n \notin F(A_0 \cap \ldots \cap A_k, B)$. Thus $F(A_0 \cap \ldots \cap A_k, B)$ “separates” the sequences $(f_n)_n \leq \infty$ and $(g_n)_n$.

We conclude that $E$ is a limit base of $C(X, \mathcal{Y})$. Note that in the case that every function sequence converges continuously to $f_\infty$ there is some $y_0 \in Y$ which is the limit of every sequence in $\mathcal{Y}$, implying $Y \in B$ and thus $C(X, \mathcal{Y}) \in E$.

(3) This follows from (1), (2), Lemmas 4.1.5, 4.2.2, 2.2.7(3), and 3.1.10.

✓

For the case of single–valued representations, it is proved in [Wei00] that $[\delta \rightarrow \gamma]$ guarantees relative computability of the evaluation function; moreover, the operator $(\delta, \gamma) \mapsto [\delta \rightarrow \gamma]$ preserves computable equivalence of representations. We extend these results to multirepresentations.

**Lemma 4.2.8 (Effectivity properties of $[\delta \rightarrow \gamma]$)**

Let $\delta, \delta' \subseteq \mathbb{N}^\omega \Rightarrow X$, $\gamma, \gamma' \subseteq \mathbb{N}^\omega \Rightarrow Y$, and $\zeta \subseteq \mathbb{N}^\omega \Rightarrow Z$ be multirepresentations.

(1) A total function $f : X \rightarrow Y$ is $(\delta, \gamma)$–computable if and only if $f$ has a computable $[\delta \rightarrow \gamma]$–name.

(2) The evaluation function $\text{eval} : C(\delta, \gamma) \times X \rightarrow Y$ defined by $\text{eval}(f, x) := f(x)$ is computable w.r.t. $[\delta \rightarrow \gamma]$, $\delta$, and $\gamma$.

(3) A function $f : Z \times X \rightarrow Y$ is $(\zeta, \delta, \gamma)$–computable if and only if the function $\Lambda(f) : Z \rightarrow C(\delta, \gamma)$ defined by $\Lambda(f)(z)(x) := f(z, x)$ is computable w.r.t. $\zeta$ and $[\delta \rightarrow \gamma]$.
(4) If $\delta \geq_{cp} \delta'$ and $\gamma \leq_{cp} \gamma'$, then $[\delta \to \gamma] \leq_{cp} [\delta' \to \gamma']$.

Proof:

(1) This follows from the fact that a function $g : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ is computable if and only if there is a computable $\omega$-word $s \in \mathbb{N}^\omega$ with $\eta_s = g$ (cf. [Wei85, Lemma 2.9]).

(2) Let $u_\eta : \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ be the computable universal function of $\eta$ (cf. Subsection 4.2.3). For $p, s \in \mathbb{N}^\omega$, $f \in [\delta \to \gamma][s]$, and $x \in \delta[p]$ we have

$$\text{eval}(f, x) = f(x) \in \gamma[\eta_s(p)] = \gamma[u_\eta(s, p)].$$

Thus $u_\eta$ realizes eval w.r.t. $[\delta \to \gamma]$, $\delta$, and $\gamma$ computably.

(3) $\implies$ Let $g : \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ be a computable function realizing $f$ w.r.t. $\zeta$, $\delta$, and $\gamma$. By the computable smn–Theorem, there is some computable function $s : \mathbb{N}^\omega \to \mathbb{N}^\omega$ with $(\forall p, q \in \mathbb{N}^\omega) \eta_s(p)(q) = g(p, q)$. Let $p, q \in \mathbb{N}^\omega$, $z \in \zeta[p]$ and $x \in \delta[q]$. Then we have

$$\Lambda(f)(z)(x) = f(z, x) \in \gamma[g(p, q)] = \gamma[\eta_s(p)(q)],$$

hence $\eta_s(p)$ realizes $\Lambda(f)(z)$ w.r.t. $\delta$ and $\gamma$. Thus $\Lambda(f)(z) \in [\delta \to \gamma][s(p)]$. Therefore $\Lambda(f)$ is computable w.r.t. $\zeta$ and $[\delta \to \gamma]$.

$\Leftarrow$ Let $s : \mathbb{N}^\omega \to \mathbb{N}^\omega$ be a computable function realizing $\Lambda(f)$ w.r.t. $\zeta$ and $[\delta \to \gamma]$. By the utm–Theorem for $\eta$, the function $g : \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ defined by $g(p, q) = \eta_s(p)(q)$ is computable. Let $p, q \in \mathbb{N}^\omega$, $z \in \zeta[p]$, and $x \in \delta[q]$. Since $\Lambda(f)(z) \in [\delta \to \gamma][s(p)]$ which means that $\eta_s(p)$ realizes the function $\Lambda(f)(z)$ w.r.t. $\delta$ and $\gamma$, we have

$$f(z, x) = \text{eval}(\Lambda(f)(z), x) \in \gamma[\eta_s(p)(q)] = \gamma[g(p, q)]$$

by (2). Hence $g$ realizes $f$ w.r.t. $\zeta$, $\delta$, and $\gamma$.

(4) Let $g_\delta, g_\gamma : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ be computable functions translating $\delta'$ to $\delta$ and $\gamma$ to $\gamma'$, respectively. By the utm–Theorem and the computable smn–Theorem for $\eta$, there is a computable function $l : \mathbb{N}^\omega \to \mathbb{N}^\omega$ with

$$(\forall p, s \in \mathbb{N}^\omega) \eta_l(s)(p) = g_{\gamma'}(\eta_\delta(g_\delta(p))).$$

Let $s \in \mathbb{N}^\omega$ and $f \in [\delta \to \gamma][s]$. For $p \in \mathbb{N}^\omega$ and $x \in \delta'[p]$ we have $x \in \delta[g_\delta(p)]$ and hence

$$f(x) \in \gamma[\eta_s(g_\delta(p))] = \gamma'[\eta_{\gamma'}(\eta_\delta(g_\delta(p)))] = \gamma'[\eta_l(s)(p)].$$

This implies $f \in [\delta' \to \gamma'][l(s)]$. Thus $l$ translates $[\delta \to \gamma]$ to $[\delta' \to \gamma']$ computably. √
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4.2.5 Partial Functions

Let \( X = (X, \to) \) and \( Y = (Y, \to) \) be weak limit spaces, and let \( \delta : \subseteq \mathbb{N}^\omega \Rightarrow X \) and \( \gamma : \subseteq \mathbb{N}^\omega \Rightarrow Y \) be multirepresentations. We denote by \( \mathcal{P}(X, Y) \) the set of all partial sequentially continuous functions between \( X \) and \( Y \), by \( \mathcal{P}(\delta, \gamma) \) the set of all partial functions \( f : \subseteq X \to Y \) which are relatively continuous w.r.t. \( \delta \) and \( \gamma \), and by \( \mathcal{P}_{\text{cp}}(\delta, \gamma) \) the set of all partial functions with are computable w.r.t. \( \delta \) and \( \gamma \).

Similar to \( [\delta \to \gamma] \), we define the multirepresentation \( [\delta \to \gamma] : \subseteq \mathbb{N}^\omega \Rightarrow \mathcal{P}(\delta, \gamma) \) by

\[
[\delta \to \gamma][p] \ni f \iff \eta_p \text{ realizes } f \text{ with respect to } \delta \text{ and } \gamma
\]

for all \( p \in \mathbb{N}^\omega \) and \( f \in \mathcal{P}(\delta, \gamma) \). The extension property of \( \eta \) guarantees the surjectivity of \( [\delta \to \gamma] \). The following effectivity properties of the operator \([\cdot \to \cdot]\) can be proved in a similar way as Lemma 4.2.8.

**Lemma 4.2.9 (Effectivity properties of \([\delta \to \gamma]\))**

Let \( \delta, \delta' : \subseteq \mathbb{N}^\omega \Rightarrow X \) and \( \gamma, \gamma' : \subseteq \mathbb{N}^\omega \Rightarrow Y \) be multirepresentations.

1. A partial function \( f : \subseteq X \to Y \) is \((\delta, \gamma)\)-computable if and only if \( f \) has a computable \([\delta \to \gamma]\)-name.

2. The evaluation function \( \text{eval} : \subseteq \mathcal{P}(\delta, \gamma) \times X \to Y \) defined by \( \text{eval}(f, x) := f(x) \) for \( f \in \mathcal{P}(\delta, \gamma) \) and \( x \in \text{dom}(f) \) is computable w.r.t. \([\delta \to \gamma] \), \( \delta \), and \( \gamma \).

3. If \( \delta \geq_{\text{cp}} \delta' \) and \( \gamma \leq_{\text{cp}} \gamma' \), then \([\delta \to \gamma] \leq_{\text{cp}} [\delta' \to \gamma'] \).

Now we assume additionally that \( \delta \) is a WeakLim–quotient multirepresentation of \( X \) and that \( \gamma \) is an admissible one of \( Y \). Then \( \mathcal{P}(\delta, \gamma) \) is equal to \( \mathcal{P}(X, Y) \) by Lemma 2.4.26. We will now prove that \([\delta \to \gamma]\) is admissible w.r.t. the convergence relation \( \Rightarrow_p \) of partial continuous convergence. It is defined by saying that a sequence \( (f_n)_n \) of functions in \( \mathcal{P}(X, Y) \) converges continuously to a function \( f_\infty \in \mathcal{P}(X, Y) \) iff the transpose

\[
f^\dagger : \subseteq \mathbb{N} \times X \to Y, \quad (n, x) \mapsto f_n(x)
\]

is \((-\to, -\to)\)-continuous. Similar to Proposition 4.2.1(1), one can verify that the space \( \mathcal{P}(X, Y) := (\mathcal{P}(X, Y), \Rightarrow_p) \) is a weak limit space.

We show admissibility of \([\delta \to \gamma]\) by reduction to Proposition 4.2.5. Similar to [MS02], we introduce \( \mathcal{Y} \) as the space that has \( Y \cup \{ \uparrow \} \) as its underlying set (we assume \( \uparrow \not\in Y \)) and as its convergence relation the one defined by

\[
(y_n)_n \mathcal{Y} y_\infty \iff y_\infty = \uparrow \text{ or } (y'_n)_n \mathcal{Y} y_\infty
\]

(4.4)

where \( y'_n \) is given by

\[
y'_n := \begin{cases} y_\infty & \text{if } y_n = \uparrow \\ y_n & \text{otherwise.} \end{cases}
\]

Since for every element \( y \in Y \) the constant sequences \((y)_\mu \) and \((\uparrow)_\mu \) both converge to \( \uparrow \) and to \( y \), \( \mathcal{Y} \) is not a \( T_0 \)-space (unless \( Y = \emptyset \)). For most topological spaces \( \mathcal{Z} \), the convergence relation of \( \mathcal{Z} \) is not induced by any topology. The space \( \mathcal{Y} \) enjoys the following properties.
Lemma 4.2.10 (Properties of $\tilde{\mathcal{Y}}$)

Let $\mathcal{X} = (X, -\chi)$ and $\mathcal{Y} = (Y, -\varphi)$ be weak limit spaces.

1. The space $\tilde{\mathcal{Y}} = (Y \cup \{\uparrow\}, -\varphi)$ is a weak limit space.

2. If $\mathcal{Y}$ is a limit space, then $\tilde{\mathcal{Y}}$ is a limit space.

3. The operator $\text{tot}: \mathcal{P}(\mathcal{X}, \mathcal{Y}) \to \mathcal{C}(\mathcal{X}, \tilde{\mathcal{Y}})$ defined by

$$\text{tot}(f)(x) := \begin{cases} f(x) & \text{if } x \in \text{dom}(f) \\ \uparrow & \text{otherwise} \end{cases}$$

is a homomorphism between the spaces $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{C}(\mathcal{X}, \tilde{\mathcal{Y}})$.

Proof:

1. We omit the straightforward proof.

2. Since $\tilde{\mathcal{Y}}$ is a weak limit space by (1), Axioms (L1) and (L2) are fulfilled. It remains to show that $\tilde{\mathcal{Y}}$ satisfies Axiom (L3). Let $(y_n)_{n \leq \infty}$ be a non–convergent sequence of $\tilde{\mathcal{Y}}$. Then $y_\infty \in Y$ and there is a strictly increasing function $\chi: \mathbb{N} \to \mathbb{N}$ with $(\{y_{\chi(n)} | n \in \mathbb{N}\} \subseteq Y$ and $(y_{\chi(n)})_n \not\to y_\infty).$ Since $\mathcal{Y}$ satisfies Axiom (L3), there is a strictly increasing function $\varphi: \mathbb{N} \to \mathbb{N}$ such that no subsequence of $(y_{\chi(\varphi(n))})_n$ converges to $y_\infty$ in $\mathcal{Y}$. Since $\mathcal{Y}$ is a subspace of $\tilde{\mathcal{Y}}$, in $\tilde{\mathcal{Y}}$ no subsequence of $(y_{\chi(\varphi(n))})_n$ converges to $y_\infty$. Hence $\tilde{\mathcal{Y}}$ satisfies Axiom (L3).

3. At first we prove that, for every weak limit space $\mathcal{Z} = (Z, -\zeta)$, a function $g : \subseteq Z \to Y$ is $(\neg \zeta, \neg \chi)$–continuous if and only if the function $\text{tot}(g) : Z \to Y \cup \{\uparrow\}$ with $\text{tot}(g)(z) := \begin{cases} g(z) & \text{if } z \in \text{dom}(g) \\ \uparrow & \text{otherwise} \end{cases}$ is $(\neg \zeta, \neg \chi)$–continuous.

"$\Rightarrow$" : Let $(z_n)_{n \leq \infty}$ be a convergent sequence of $\mathcal{Z}$. If $z_\infty \notin \text{dom}(g)$ or $z_n \notin \text{dom}(g)$ for almost all $n \in \mathbb{N}$, then $(\text{tot}(g)(z_n))_n$ converges to $\text{tot}(g)(z_\infty)$, because in $\tilde{\mathcal{Y}}$ every sequence converges to $\uparrow$ and every sequence which is eventually equal to $\uparrow$ converges to every element.

Otherwise $z_\infty \in \text{dom}(g)$ holds and there is a strictly increasing function $\varphi: \mathbb{N} \to \mathbb{N}$ with $\text{range}(\varphi) = \{n \in \mathbb{N} | z_n \in \text{dom}(g)\}$. By sequential continuity of $g$, $(g(z_{\varphi(n)}))_n$ converges to $g(z_\infty)$ in $\mathcal{Y}$. Since $\text{tot}(g)(z_n)$ is equal to $\uparrow$ for all $n \notin \text{range}(\varphi)$ and $\mathcal{Y}$ satisfies Axiom (L5), this implies $(\text{tot}(g)(z_n))_n \to \varphi \to \text{tot}(g)(z_\infty)$. Hence $\text{tot}(g)$ is $(\neg \zeta, \neg \varphi)$–continuous.

"$\Leftarrow$" : Let $(z_n)_{n \leq \infty}$ be a convergent sequence of $\mathcal{Z}$ contained in $\text{dom}(g)$. By sequential continuity of $\text{tot}(g)$, $(g(z_n))_{n \leq \infty}$ is a convergent sequence of $\mathcal{Y}$ and thus of the subspace $\tilde{\mathcal{Y}}$. Hence $g$ is $(\neg \zeta, \neg \varphi)$–continuous. Hence, $\text{tot}$ actually maps $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ to $\mathcal{C}(\mathcal{X}, \tilde{\mathcal{Y}})$. By applying the above equivalence to the product space $(\mathbb{N}, -\pi) \otimes \mathcal{X}$, we obtain for every sequence $(f_n)_{n \leq \infty}$ in $\mathcal{P}(\mathcal{X}, \mathcal{Y})$:

$$(f_n)_n \Rightarrow_{\mathcal{P}} f_\infty \iff f^t \text{ is } (\neg \pi, \neg \chi, \neg \varphi) \text{–continuous}$$

$$\iff \text{tot}(f^t) \text{ is } (\neg \pi, \neg \chi, \neg \varphi) \text{–continuous}$$

$$\iff (\text{tot}(f_n))_n \Rightarrow_{\mathcal{X}, \tilde{\mathcal{Y}}} \text{tot}(f_\infty).$$
Thus \( \text{tot} \) is a homomorphism between \( \mathfrak{P}(X, \mathcal{Y}) \) and \( \mathfrak{E}(X, \mathcal{Y}) \).

For constructing an admissible multirepresentation of \( \mathcal{Y} \), we use the function \( g_{\text{til}} : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega \) defined by having

\[
G(g_{\text{til}}) := \left\{ (0^{n_0}(q(0) + 1)0^{n_1}(q(1) + 1)0^{n_2}(q(2) + 1) \ldots, q) \right\}
\]
\[
q \in \mathbb{N}^\omega, \ n_0, n_1, n_2, \ldots \in \mathbb{N}
\]

as its graph. Obviously, \( g_{\text{til}} \) is computable.

Let \( \gamma : \subseteq \mathbb{N}^\omega \Rightarrow Y \) be a multirepresentation of \( Y \). We define \( \tilde{\gamma} : \mathbb{N}^\omega \Rightarrow Y \cup \{\uparrow\} \) by

\[
\tilde{\gamma}[q] := \{\uparrow\} \cup \gamma^{g_{\text{til}}}(q)
\]

for all \( q \in \mathbb{N}^\omega \) and show that \( \tilde{\gamma} \) inherits from \( \gamma \) the property of being admissible. Thus \( \mathcal{Y} \in \text{AdmWeakLim} \) implies \( \mathcal{Y} \in \text{AdmWeakLim} \).

**Lemma 4.2.11 (Admissibility of \( \tilde{\gamma} \))**

Let \( \gamma : \subseteq \mathbb{N}^\omega \Rightarrow Y \) be an admissible multirepresentation of a weak limit space \( Y = (Y, \rightarrow_Y) \). Then \( \tilde{\gamma} \) is an admissible multirepresentation of the weak limit space \( \mathcal{Y} \).

**Proof:**

At first we prove that there is a computable function \( g_{\text{rev}} : \mathbb{N}^\omega \to \mathbb{N}^\omega \) satisfying

\[
g_{\text{til}}(\eta_{\text{rev}}(s))(q) = \eta_{s}(q) \quad \text{and} \quad \text{dom}(\eta_{\text{rev}}(s)) = \mathbb{N}^\omega
\]

for all \( s, q \in \mathbb{N}^\omega \).

Let \( h : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^* \) be a computable monotone word function approximating the computable universal function of \( \eta \). We define \( \tilde{h} : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^* \) inductively by

\[
\tilde{h}(\varepsilon, \varepsilon) := \varepsilon, \quad \tilde{h}(uw, v) := \tilde{h}(u, vw) := \tilde{h}(u, v)
\]

and

\[
\tilde{h}(ua, vb) := \left\{
\begin{array}{ll}
\tilde{h}(u, v)(c_1 + 1)(c_2 + 1) \ldots (c_l + 1) & \text{if } h(ua, vb) \neq h(u, v) \text{ and } \\
 & \text{l}, c_1, \ldots, c_l \in \mathbb{N} \text{ satisfy } h(u, v)c_1 \ldots c_l = h(ua, vb) \\
\tilde{h}(u, v)0 & \text{if } h(ua, vb) = h(u, v)
\end{array}
\right.
\]

for all \( u, v, w \in \mathbb{N}^* \) and \( a, b \in \mathbb{N} \) with \( \lg(u) = \lg(v) \) and \( w \neq \varepsilon \). Obviously, \( \tilde{h} \) is monotone and computable, hence \( \tilde{h}^\omega \) is computable. Since we have \( \lg(h(u, v)) \geq \min \{ \lg(u), \lg(v) \} \), \( \tilde{h}^\omega \) is total. Moreover, \( \tilde{h}^\omega \) satisfies

\[
(\forall s, p \in \mathbb{N}^\omega) g_{\text{til}}(\tilde{h}^\omega(s, p)) = \eta_{s}(p)
\]

because for \( p \in \text{dom}(\eta_{s}) \) there are \( n_0, n_1, \ldots \in \mathbb{N} \) with

\[
\tilde{h}^\omega(s, p) = 0^{n_0}(\eta_{s}(p)(0) + 1)0^{n_1}(\eta_{s}(p)(1) + 1)0^{n_2}(\eta_{s}(p)(2) + 1) \ldots
\]
and for \( p \in \mathbb{N}^\omega \setminus \text{dom}(\eta) \) the set \( \{ \lg(h(s^{<i}, p^{<i})) \mid i \in \mathbb{N} \} \) is finite implying \( \tilde{h}^\omega(s, p) \in \mathbb{N}^*0^\omega \). The computable smn–Theorem yields a computable function \( g_{\text{rev}} : \mathbb{N}^\omega \to \mathbb{N}^\omega \) with

\[
(\forall s, p \in \mathbb{N}^\omega) \eta_{\text{rev}}(s)(p) = \tilde{h}^\omega(s, p).
\]

This function has the desired properties.

Now we are able to prove admissibility of \( \tilde{\gamma} \).

Continuity of \( \tilde{\gamma} \): Let \( (q_n)_{n \leq \infty} \) be a convergent sequence of \( (\mathbb{N}^\omega, \rightarrow^{\omega}) \). Let \( (y_n)_{n \leq \infty} \in \prod_{n \leq \infty} \tilde{\gamma}[q_n] \). Since every sequence converges to \( \uparrow \) in \( \tilde{\mathcal{Y}} \), the only interesting case is \( y_\infty \neq \uparrow \). Let \( \xi : \mathbb{N} \to \mathbb{N} \) be defined by

\[
\xi_n := \begin{cases} \infty & \text{if } y_n = \uparrow \\ n & \text{otherwise.} \end{cases}
\]

Since \( (q_n)_{n \leq \infty} \) is a convergent sequence and since \( \gamma \) and \( g_{\text{til}} \) are sequentially continuous, \( (\xi_n)_{n \leq \infty} \) is a convergent sequence of \( \mathcal{Y} \). By definition of \( \rightarrow \), this means that \( (y_n)_{n \leq \infty} \) converges to \( y_\infty \) in \( \mathcal{Y} \). Hence \( \tilde{\gamma} \) is sequentially continuous.

Universality of \( \tilde{\gamma} \): Let \( \phi : \subseteq \mathbb{N}^\omega \Rightarrow Y \cup \{ \uparrow \} \) be \( (\rightarrow^{\omega}, \rightarrow_{\tilde{\gamma}}) \)-continuous. Then the restriction \( \phi' := \phi|^Y \) of \( \phi \) to \( Y \) is \( (\rightarrow^{\omega}, \rightarrow_{\tilde{\gamma}}) \)-continuous. Thus there is a continuous function \( g : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega \) translating \( \phi' \) to the admissible multirepresentation \( \gamma \). By the extension property of \( \eta \), there is some \( s \in \mathbb{N}^\omega \) with \( (\forall q \in \text{dom}(g)) \eta_{\text{til}}(q) = g(q) \).

We define the continuous and total function \( g' : \mathbb{N}^\omega \to \mathbb{N}^\omega \) by \( g' := \eta_{\text{rev}}(s) \). By Condition (4.6), we obtain

\[
\phi[q] \subseteq \{ \uparrow \} \cup \phi'[q] \subseteq \{ \uparrow \} \cup \gamma[g(q)] = \{ \uparrow \} \cup \gamma[\eta_{\text{til}}(q)] = \{ \uparrow \} \cup \gamma[\tilde{\gamma}[g'(q)] = \tilde{\gamma}[g'(q)]
\]

for all \( q \in \text{dom}(\phi) \) with \( \phi[q] \cap Y \neq \emptyset \). Since \( g' \) is total, \( \phi[q] \subseteq \tilde{\gamma}[g'(q)] \) is satisfied as well by all \( q \in \text{dom}(\phi) \) with \( \phi[q] = \{ \uparrow \} \). Thus \( g' \) translates \( \phi \) to \( \tilde{\gamma} \).

\( \checkmark \)

From Lemma 4.2.11 we obtain the following admissibility result about \( [\cdot \rightarrow \cdot \cdot \cdot \rightarrow \cdot \cdot \cdot \cdot] \).

**Proposition 4.2.12 (Admissibility of \([\delta \rightarrow \gamma]\))**

Let \( \delta : \subseteq \mathbb{N}^\omega \Rightarrow X \) be a WeakLim–quotient multirepresentation of a weak limit space \( X = (X, \rightarrow_{\tilde{\chi}}) \), and let \( \gamma : \subseteq \mathbb{N}^\omega \Rightarrow Y \) be an admissible multirepresentation of a weak limit space \( \mathcal{Y} = (Y, \rightarrow_{\tilde{\eta}}) \).

1. \( [\delta \rightarrow \gamma] \) is computably equivalent to \( \text{tot}^{-1} \circ [\delta \rightarrow \tilde{\gamma}] \).

2. \( [\delta \rightarrow \gamma] \) is an admissible multirepresentation of \( \mathcal{Y}(X, \mathcal{Y}) \).
Proof:

(1) \( \text{tot}^{-1} \circ [\delta \rightarrow \gamma] \leq_{cp} [\delta \rightarrow \gamma] \):

By the computable smn–Theorem, there is a computable function \( h : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) with \((\forall s, p \in \mathbb{N}^\omega) \eta_{h(s)}(p) = g_{\text{til}}(\eta_s(p))\), because \( g_{\text{til}} \) and the universal function of \( \eta \) are computable. Let \( s \in \mathbb{N}^\omega \) and \( f \in \text{tot}^{-1} \circ [\delta \rightarrow \gamma][s] \). For every \( p \in \mathbb{N}^\omega \) and \( x \in \delta[p] \cap \text{dom}(f) \) we have

\[
f(x) = \text{tot}(f)(x) \in \tilde{\gamma}[\eta_s(p)] \setminus \{1\} = \gamma[g_{\text{til}}(\eta_s(p))] = \gamma[\eta_{h(s)}(p)].
\]

This proves \( f \in [\delta \rightarrow \gamma][h(s)] \) and \( \text{tot}^{-1} \circ [\delta \rightarrow \gamma] \leq_{cp} [\delta \rightarrow \gamma] \).

(2) By Proposition 4.2.5 and Lemma 4.2.11, \([\delta \rightarrow \gamma]\) is an admissible multirepresentation of \( \mathcal{C}(X, \#) \). Lemma 4.2.10 and Proposition 4.1.6 imply that \( \text{tot}^{-1} \circ [\delta \rightarrow \gamma] \) is an admissible multirepresentation of \( \mathcal{P}(X, \#) \). From Proposition 2.4.21 it follows that \([\delta \rightarrow \gamma]\) is an admissible multirepresentation of \( \mathcal{P}(X, \#) \).

\[\checkmark\]

4.26 \textbf{AdmWeakLim is locally Cartesian closed}

A category \( \mathcal{C} \) is called \textit{locally Cartesian closed} iff every slice category \( \mathcal{C}/A \) is Cartesian closed. For an object \( A \) in \( \mathcal{C} \) the slice category \( \mathcal{C}/A \) has as objects all morphism in \( \mathcal{C} \) with codomain \( A \). The morphisms \( i : f \rightarrow g \) between two objects \( f : X \rightarrow A \) and \( g : Y \rightarrow A \) in the slice category \( \mathcal{C}/A \) is a morphism \( i : X \rightarrow Y \) in the original category \( \mathcal{C} \) with \( f = g \circ i \). Composition of two morphisms in \( \mathcal{C}/A \) is defined as the corresponding composition in \( \mathcal{C} \) (cf. [AL91]).

From Theorem 3.4.8 we know \( \text{AdmLim} = \text{PQL} \) and \( \text{AdmSeq} = \text{PQ} \). Thus we obtain from [MS02] that \( \text{AdmLim} \) is locally Cartesian closed, whereas \( \text{AdmSeq} \) does not. We show that \( \text{AdmWeakLim} \) is locally Cartesian closed as well.

\textbf{Proposition 4.2.13}

The categories \textbf{WeakLim} and \textbf{AdmWeakLim} are locally Cartesian closed.

\textbf{Proof}:

We sketch the proof for \( \text{AdmWeakLim} \). Let \( \mathfrak{A} = (A, \rightarrow) \) be a weak limit space with an admissible multirepresentation. In order to show that the slice category \( \mathcal{S} := \text{AdmWeakLim}/\mathfrak{A} \) is Cartesian closed, let \( X = (X, \rightarrow) \) and \( \mathcal{Y} = (Y, \rightarrow) \) be weak limit spaces with admissible multirepresentations, and let \( f : X \rightarrow A \) and \( g : Y \rightarrow A \) be sequentially continuous functions.
Products:
We define $X \otimes_{(f,g)} \mathcal{Y}$ to be the subspace of $X \otimes \mathcal{Y}$ with underlying set
$$X \times_{(f,g)} Y := \{(x,y) \mid f(x) = g(y)\}.$$ 

By Proposition 4.1.7 and Subsection 4.1.5, $X \otimes_{(f,g)} \mathcal{Y}$ is an object of $\text{AdmWeakLim}$. Moreover, the functions $f \times_S g : X \times_{(f,g)} Y \to A, p_{f,g} : X \times_{(f,g)} Y \to X$ and $q_{f,g} : X \times_{(f,g)} Y \to Y$ defined by $(f \times_S g)(x,y) := f(x) = g(y), p_{f,g}(x,y) := x$ and $q_{f,g}(x,y) := y$ for $(x,y) \in X \times_{(f,g)} Y$ are sequentially continuous. Hence $f \times_S g$ is an object of $S$. Since $f \times_S g = f \circ p_{f,g}$ and $f \times_S g = g \circ q_{f,g}$ hold, the functions $p_{f,g}$ and $q_{f,g}$ are morphisms in $S$ from $f \times_S g$ to, respectively, $f$ and $g$. One easily verifies that the triple $(f \times_S g, p_{f,g}, q_{f,g})$ forms a categorical product of the pair $(f,g)$ in $S$. Hence $S = \text{AdmWeakLim}/\mathcal{A}$ is a Cartesian category.

Exponents:
W.l.o.g. we can assume $\uparrow \notin Y$. By Lemma 4.2.10, $\mathcal{Y}$ is a weak limit space. We define $E = (E, \rightarrow_e)$ to be the subspace of $\mathcal{A} \otimes \mathcal{C}(X, \mathcal{Y})$ with underlying set
$$E := \{(a,t) \in A \times \mathcal{C}(X, \mathcal{Y}) \mid t[f^{-1}\{a\}] \subseteq g^{-1}\{a\}, \ t[X \setminus f^{-1}\{a\}] \subseteq \{\uparrow\}\}.$$ 

By Propositions 4.2.5, 4.1.7, Lemma 4.2.11, and Subsection 4.1.5, the weak limit space $E$ has an admissible multirepresentation. We define the function $g^l : E \to A$ by $g^l(a,t) := a$ for all $(a,t) \in E$. Since $g^l$ is sequentially continuous, $g^l$ is an object of $S$. Moreover, we define $ev : E \times (g^l,f) X \to Y$ by $ev((a,t),x) := t(x)$ for all $((a,t),x) \in E \times (g^l,f) X$. Since $ev$ is sequentially continuous by Proposition 4.2.3(1) and fulfills $g^l \times_S f = g \circ ev$, $ev$ is a morphism in $S$ from $g^l \times_S f$ to $g$.

Now let $S = (Z, \rightarrow_S)$ be a weak limit space with an admissible multirepresentation $\zeta : \mathbb{N}^c \to Z$, and let $h : Z \to A$ be sequentially continuous. Moreover, let $k : h \times_S f \to g$ be a morphism from $h \times_S f$ to $g$ in $S$, i.e., $k$ is a sequentially continuous function from $S \otimes_{(h,f)} X$ to $\mathcal{Y}$ with $g(k(z,x)) = h(z) = f(x)$ for all $(z,x) \in Z \times (h,f) X$. By definition of $\mathcal{A}$, the function $k' : Z \times X \to Y \cup \{\uparrow\}$ defined by
$$k'(z,x) := \begin{cases} k(z,x) & \text{if } (z,x) \in Z \times (h,f) X \\ \uparrow & \text{otherwise} \end{cases}$$

for $z \in Z$ and $x \in X$ is $(\rightarrow_S, \rightarrow_X, \rightarrow_{\mathcal{Y}})$-continuous. By Proposition 4.2.3(2), there is a sequentially continuous function $s : Z \to \mathcal{C}(X, \mathcal{Y})$ with $s(z)(x) = k'(z,x)$ for all $z \in Z$ and $x \in X$. We define $l : Z \to \mathcal{C}(X, \mathcal{Y})$ continuously by $l(z) := (h(z), s(z))$. Then $l(z) \in E$ and $g^l(l(z)) = h(z)$ for all $z \in Z$, i.e., $l$ is a morphism from $h$ to $g^l$ in $S$. Moreover, we have $ev(l(z),x) = s(z)(x) = k(z,x)$ for all $(z,x) \in Z \times (h,f) X$, hence $k = ev \circ (l \times id_X)$. Let $l' : h \to g^l$ be a further morphism in $S$ satisfying $k = ev \circ (l' \times id_X)$. Let $z \in Z$. Then $pr_1(l'(z)) = g^l(l'(z)) = h(z)$. For every $x \in f^{-1}\{(h(z))\}$ we have $pr_2(l'(z))(x) = ev(l'(z),x) = k(z,x)$ and for every $x \in X \setminus f^{-1}\{(h(z))\}$ we have $pr_2(l'(z))(x) = \uparrow$. Hence $pr_2(l'(z)) = s(z)$ and $l' = l$. We conclude that $l$ is the unique transpose of $k$ in $S$. Thus $g^l$ and $ev$ form an exponent of $f$ and $g$ in $S$.

Thus the slice category $S = \text{AdmWeakLim}/\mathcal{A}$ is Cartesian closed. Hence $\text{AdmWeakLim}$ (and also $\text{WeakLim}$) is locally Cartesian closed.
4.3 Effectivization

Admissibility of multirepresentations is defined in order to guarantee relative continuity of all sequentially continuous functions. From Proposition 3.1.7 and Lemma 3.1.4 it follows that for arbitrary multirepresentations $\delta : \subseteq \mathbb{N}^\omega \Rightarrow X$ and $\gamma : \subseteq \mathbb{N}^\omega \Rightarrow Y$ there exist admissible multirepresentations $\delta'$ and $\gamma'$ of the weak limit spaces $(X, \rightarrow_\gamma)$ and $(Y, \rightarrow_\gamma)$ generated by $\delta$ and $\gamma$. By Theorem 2.4.25 and Proposition 2.4.12, admissibility implies that every $(\delta, \gamma)$–continuous function is also $(\delta', \gamma')$–continuous, i.e. $C(\delta, \gamma) \subseteq C(\delta', \gamma')$. However, the construction of $\delta'$ and $\gamma'$, which is done via limit bases, does not guarantee that also $(\delta, \gamma)$–computability implies $(\delta', \gamma')$–computability. The question arises whether it is possible to construct admissible multirepresentations $\delta^*$ and $\gamma^*$ which admit $(\delta^*, \gamma^*)$–computability of every $(\delta, \gamma)$–computable function.

In this section we demonstrate that this is indeed possible. We define three operators (denoted by $\circ^w$, $\circ^c$, and $\circ^e$) which transform a multirepresentation $\zeta$ into admissible ones of the three quotient spaces generated by $\zeta$ (cf. Subsections 4.3.1 and 4.3.2). We show that these operators have nice computability properties (cf. Subsection 4.3.3). In particular, they preserve relative computability of total multivariate functions (cf. Proposition 4.3.12). This generalizes P. Hertling’s result that every function on the reals which is computable w.r.t. the (non topologically admissible) decimal representation is also computable w.r.t. to the admissible signed–digit representation (cf. [Her99, Theorem 3.7]). Moreover, two of these operators maintain also relative computability of partial functions. Hence functions which are computable w.r.t. arbitrary multirepresentations are also computable w.r.t. admissible ones. The operators $\circ^w$, $\circ^c$, and $\circ^e$ can be used to prove the if–parts of the Characterization Theorems 3.2.1, 3.2.3, and 3.2.4 (cf. Proposition 4.5.1).

In Subsection 4.3.5 we introduce the “effective” categories $\text{EffWeakLim}$, $\text{EffLim}$, and $\text{EffSeq}$. Their objects are pairs $(X, \delta)$, where $X$ is a space and $\delta$ is an effectively admissible multirepresentation of $X$, and their morphisms are the relatively computable total functions. Effectively admissible multirepresentations are defined in Subsection 4.3.4. Several examples of effectively admissible multirepresentations and of objects of the aforementioned categories are presented in Subsection 4.3.6. We end this section with an effective version of Kisynski’s Theorem about the relationship between limit spaces and topological spaces (cf. Subsection 4.3.7).

4.3.1 Universal Codomains

For defining the operators $\circ^w$, $\circ^c$, and $\circ^e$, we need the notion of a universal codomain. Let $\mathfrak{X} = (X, \rightarrow_\delta)$ and $\mathfrak{U} = (U, \rightarrow_\gamma)$ be weak limit spaces. Since the evaluation function $\text{eval} : C(\mathfrak{X}, \mathfrak{U}) \times X \rightarrow U$ is sequentially continuous by Proposition 4.2.3, for every $x \in X$ the function $e_{X,\mathfrak{U}}(x) : C(\mathfrak{X}, \mathfrak{U}) \rightarrow U$ defined by

$$e_{X,\mathfrak{U}}(x)(h) := h(x)$$

is continuous w.r.t. $\Rightarrow_{X,\mathfrak{U}}$ and $\mathfrak{U}$; moreover, the transpose $e_{X,\mathfrak{U}} : X \rightarrow C(\mathfrak{C}(\mathfrak{X}, \mathfrak{U}), \mathfrak{U})$ is a continuous function between $\mathfrak{X}$ and $\mathfrak{C}(\mathfrak{C}(\mathfrak{X}, \mathfrak{U}), \mathfrak{U})$ by Proposition 4.2.3, i.e., the only–if–part in Equivalence (4.7) is always satisfied.

9For topological spaces $\mathfrak{X}$ we will write $e_{\mathfrak{X},\mathfrak{U}}$ instead of $e_{\mathfrak{W}(\mathfrak{X}),\mathfrak{U}}$. 


\[ 4.3 \text{ Effectivization} \]
We are now looking for spaces \( \mathcal{U} \) guaranteeing that \( e_{X,\mathcal{U}} \) is even a homomorphism between \( X \) and \( C(\mathcal{E}(X, \mathcal{U}), \mathcal{U}) \), i.e., the following equivalence holds (cf. Subsection 4.1.3):

\[
(x_n)_n \to_X x_\infty \iff (e_{X,\mathcal{U}}(x_n))_n \Rightarrow e_{X,\mathcal{U}}(x_\infty)
\]  

(4.7)

If \( e_{X,\mathcal{U}} \) is injective and Condition (4.7) holds, then \( e_{X,\mathcal{U}} \) is an embedding of \( X \) into \( C(\mathcal{E}(X, \mathcal{U}), \mathcal{U}) \). This resembles the fact that a normed vector space \( V \) over the complex numbers \( \mathbb{C} \) embeds into the dual of its dual, i.e., into the function space \( L(L(V, \mathbb{C}), \mathbb{C}) \), where \( L(V, \mathbb{C}) \) denotes the vector space of linear functions from \( V \) to \( \mathbb{C} \). The embedding is defined in the same way as \( e_{X,\mathcal{U}} \) (cf. [HS71, Lemma 7.6]).

Let \( \mathcal{W} \) be a class of weak limit spaces. We say that a weak limit space \( \mathcal{U} = (U, \to_\mathcal{U}) \) is an universal codomain for \( \mathcal{W} \) iff \( \mathcal{U} \) is an element of \( \mathcal{W} \) and for every space \( X \) in \( \mathcal{W} \) the transpose \( e_{X,\mathcal{U}} \) satisfies Equivalence (4.7), i.e., \( e_{X,\mathcal{U}} \) is a homomorphism between \( X \) and \( C(\mathcal{E}(X, \mathcal{U}), \mathcal{U}) \).

The Sierpiński space \( \mathcal{S}_i = (Si, \tau_{\mathcal{S}_i}) \) from Example 2.3.7, more precisely the Sierpiński limit space \( \mathcal{W}(\mathcal{S}_i) = (Si, \to_{\mathcal{S}_i}) \), can be easily verified (cf. Lemma 4.3.1) to be an universal codomain for the topological spaces (viewed as weak limit spaces).

The space \( \mathcal{L}_i := \mathcal{S}_i \), being a limit space by Lemma 4.2.10, turns out to be an universal codomain for the class of limit spaces (cf. Lemma 4.3.1). Its convergence relation \( \to_{\mathcal{S}_i} \) on the underlying set \( Li := \{\bot, \top, \omega \} \) is given by

\[
(b_n)_n \to_{\mathcal{S}_i} b_\infty \iff (b_\infty \neq \top \text{ or } (\forall n \in \mathbb{N}) b_n \neq \bot)
\]

(cf. Equivalence (4.4)). We define the multirepresentation \( \varrho_{\mathcal{S}_i} : \mathbb{N}^\omega \rightrightarrows Li \) as an extension of the admissible representation \( \varrho_{\mathcal{S}_i} \) (cf. Example 2.3.7) of \( \mathcal{S}_i \) by

\[
\varrho_{\mathcal{S}_i}[p] := \{\varrho_{\mathcal{S}_i}(p)\} \cup \{\top\}.
\]

The multirepresentation \( \varrho_{\mathcal{S}_i} \) is admissible, because it is the restriction of the admissible multirepresentation \( \varrho_{\mathcal{S}_e} \) (see Equation (4.8)) to the codomain \( Li \).

As an universal codomain for the class of weak limit spaces with admissible multirepresentation we use the space \( \mathcal{W}e \) having\(^{10}\) \( \mathcal{W}e := \{\bot, \top, \omega \} \cup 2^\mathbb{N} \) as its underlying set and as its convergence relation the one defined by

\[
(b_n)_n \to_{\mathcal{W}e} b_\infty \iff (b_\infty \neq \top \text{ or } (\exists j \in \mathbb{N})(\forall n \in \mathbb{N}) (j \in b_n \lor b_n \in \{\top, \omega \})).
\]

Note that \( \mathcal{W}(\mathcal{S}_i) \) and \( \mathcal{L}_i \) are subspaces of \( \mathcal{W}e \). We equip \( \mathcal{W}e \) with the multirepresentation \( \varrho_{\mathcal{W}e} : \mathbb{N}^\omega \rightrightarrows \mathcal{W}e \) defined by

\[
\varrho_{\mathcal{W}e}[p] := \begin{cases} 
\{\top, \omega\} \cup \{M \subseteq \mathbb{N} \mid (p(k) - 1) \in M\} & \text{if } k := \min\{i \mid p(i) \neq 0\} \text{ exists} \\
\{\bot, \top\} \cup 2^\mathbb{N} & \text{otherwise.}
\end{cases}
\]

(4.8)

Its admissibility is shown within the proof of the following proposition.

\(^{10}\)By \( 2^\mathbb{N} \) we denote the power set \( \{M \mid M \subseteq \mathbb{N}\} \) of \( \mathbb{N} \).
4.3 Effectivization

Lemma 4.3.1

(1) \( W(\mathfrak{S}i) \) is an universal codomain for the class of topological weak limit spaces.

(2) \( \mathfrak{L}i \) is an universal codomain for the class of limit spaces.

(3) \( \mathcal{W}e \) is an universal codomain for the class of weak limit spaces with admissible multirepresentation.

Proof:

(1) Let \( X = (X, \tau_X) \) be a topological space. The only–if–part of Equivalence (4.7) (for \( \Omega := \mathfrak{S}i \)) follows from the continuity of\(^{11} \) \( e_{X,\mathfrak{S}i} \). Let \( (x_n)_{n \leq \infty} \) be a sequence such that \( (e_{X,\mathfrak{S}i}(x_n))_{n \leq \infty} \) is a convergent sequence of \( \mathcal{C}(\mathfrak{C}(X, \mathfrak{S}i), \mathfrak{S}i) \). Let \( O \in \tau_X \) be an open neighbourhood of \( x_\infty \). Then the characteristic function \( cf_O : X \to \{\bot, \top\} \) of \( O \) defined by

\[
 cf_O(y) := \begin{cases} 
 \top & \text{if } y \in O \\
 \bot & \text{otherwise}
\end{cases}
\]

is sequentially continuous and hence in \( \mathcal{C}(X, \mathfrak{S}i) \), because every sequence converging to an element \( y_\infty \in cf_O^{-1}[\top] \) is eventually in the open set \( O \). Since \( (e_{X,\mathfrak{S}i}(x_n))_{n \leq \infty} \) and \( (cf_O)_{n \leq \infty} \) are convergent sequences, \( (e_{X,\mathfrak{S}i}(x_n)(cf_O))_{n \leq \infty} \) converges to \( e_{X,\mathfrak{S}i}(x_\infty)(cf_O) = \top \) in \( \mathfrak{S}i \), i.e., \( (\forall n \in \mathbb{N}) cf_O(x_n) = \top \). Thus \( (x_n)_{n \leq \infty} \) is eventually in \( O \). We conclude that \( (x_n)_{n \leq \infty} \) converges to \( x_\infty \) in \( X \). Hence \( e_{X,\mathfrak{S}i} \) reflects convergent sequences between \( X \) and \( \mathcal{C}(\mathfrak{C}(X, \mathfrak{S}i), \mathfrak{S}i) \).

(2) Let \( X = (X, \neg X) \) be a limit space. Let \( (x_n)_{n \leq \infty} \) be a non–convergent sequence of \( X \). By Axiom (L3), there is some strictly increasing function \( \varphi : \mathbb{N} \to \mathbb{N} \) such that no subsequence of \( (x_{\varphi(n)})_{n \leq \infty} \) converges to \( x_\infty \). For \( n \in \mathbb{N} \) and \( y \in X \), we define the functions \( h_n, h_\infty : X \to \{\bot, \top\} \) by

\[
 h_n(y) := \begin{cases} 
 \bot & \text{if } y = x_{\varphi(n)} \\
 \top & \text{otherwise}
\end{cases} \quad \text{and} \quad h_\infty(y) := \begin{cases} 
 \top & \text{if } y = x_\infty \\
 \bot & \text{otherwise}.
\end{cases}
\]

Since the ranges of these functions do not contain both \( \bot \) and \( \top \), they are sequentially continuous.

Let \( (y_n)_{n \leq \infty} \) be a convergent sequence of \( X \). Suppose for contradiction that \( (h_n(y_n))_n \) does not converge to \( h_\infty(y_\infty) \). Then \( h_\infty(y_\infty) = \top \) and there is a strictly increasing \( \psi : \mathbb{N} \to \mathbb{N} \) with \( h_\psi(n)(y_\psi(n)) = \bot \) for all \( n \in \mathbb{N} \). It follows \( y_\infty = x_\infty \) and \( (y_\psi(n))_n = (x_{\varphi(n)})_n \), i.e., \( (y_\psi(n))_n \) is a subsequence of \( (x_{\varphi(n)})_n \) converging to \( x_\infty \), a contradiction.

Therefore \( (h_n)_n \) converges continuously to \( h_\infty \) by Equivalence (4.3). Since we have \( e_{X,\mathfrak{L}i}(x_{\varphi(n)})(h_n) = \bot \) for all \( n \in \mathbb{N} \) and \( e_{X,\mathfrak{L}i}(x_\infty)(h_\infty) = \top \), \( (e_{X,\mathfrak{L}i}(x_{\varphi(n)}))_n \) and thus \( (e_{X,\mathfrak{L}i}(x_n))_n \) do not converge to \( e_{X,\mathfrak{L}i}(x_\infty) \) in \( \mathfrak{L}i \).

Since \( e_{X,\mathfrak{L}i} \) is sequentially continuous, this implies that \( e_{X,\mathfrak{L}i} \) is a homomorphism between \( X \) and \( \mathcal{C}(\mathfrak{C}(X, \mathfrak{L}i), \mathfrak{L}i) \).

\(^{11}\)By convention, we write \( e_{X,\mathfrak{S}i} \) for \( e_{W(X),W(\mathfrak{S}i)} \) and \( \mathcal{C}(\mathfrak{C}(X, \mathfrak{S}i), \mathfrak{S}i) \) for \( \mathcal{C}(\mathfrak{C}(W(X), W(\mathfrak{S}i)), W(\mathfrak{S}i)) \).
Chapter 4. Constructions

(3) One easily verifies that \( W \) satisfies the axioms of weak limit spaces.

We show at first that \( \varphi \) is an admissible multirepresentation of \( W \). For \( j \in \mathbb{N} \) let \( W_j \) denote the set \( \{1, \top\} \cup \{M \subseteq \mathbb{N} \mid j \in M\} \).

Continuity: Let \((p_n)_{n \leq \infty}\) be a convergent sequence of \((\mathbb{N}^\omega, \rightarrow_{\omega})\) and \((b_n)_{n \leq \infty} \in \prod_{n \in \mathbb{N}} \varphi_{\mathbb{M}}[p_n] \). The only interesting case is \( b_\infty = \top \), because in \( W \) every sequence converges to every element \( \notin \top \). Then \( k := \min \{ i \in \mathbb{N} \mid p_\infty(i) \neq 0 \} \) exists. It follows \( 0^k p_\infty(k) \subseteq p_n \) for almost all \( n \). Thus \((b_n)_n \) is eventually in \( W_{p_\infty(k)-1} \). This implies that \((b_n)_n \) converges to \( \top \) in \( W \).

Universality: Let \( \phi :\subseteq \mathbb{N}^\omega \Rightarrow W \) be sequentially continuous. For \( q \in \mathbb{N}^\omega \) we define \( g : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) by

\[
g(q) := \begin{cases} 0^{n_q}(j_q + 1) : 0^\omega & \text{if } n_q < \infty \\ 0^\omega & \text{otherwise,} \end{cases}
\]

where

\[
n_q := \min \{ \infty \cup \{ n \in \mathbb{N} \mid (\exists j \in \mathbb{N}) \phi[q^{\ll_n} \mathbb{N}^\omega] \subseteq W_j \} \},
\]

\[
j_q := \begin{cases} \min \{ j \in \mathbb{N} \mid \phi[q^{\ll_n} \mathbb{N}^\omega] \subseteq W_j \} & \text{if } n_q < \infty \\ \infty & \text{otherwise.} \end{cases}
\]

Then \( g \) is continuous, because \( q \mapsto n_q \) and \( q \mapsto j_q \) are continuous functions between \( (\mathbb{N}^\omega, \rightarrow_{\omega}) \) and \( (\mathbb{N}, \rightarrow) \).

We show that \( g \) translates \( \phi \) to \( \varphi \). Let \( p \in \text{dom}(\phi) \).

Case \( n_p = \infty \): Then \( g(p) = 0^\omega \). Suppose for contradiction \( \top \in \phi[p] \). As \( n_p = \infty \), for every \( m \in \mathbb{N} \) there is some \( q_m \in p^{\ll_m} \mathbb{N}^\omega \) and some element \( b_m \in \phi[q_m] \setminus W_{m - \lfloor \sqrt{m} \rfloor^2} \).

Since \((q_m)_m\) converges to \( p \), \((b_m)_m\) converges to \( \top \) by sequential continuity of \( \phi \). Thus there are some \( j, m_0 \) with \( b_m \in W_j \) for all \( m \geq m_0 \). But we have chosen \( b(m_0 + j)^{2+i} \) from \( W_{j} \) \( \setminus W_{j} \), a contradiction.

Therefore we have \( \phi[p] \subseteq \{ \bot, \top \} \cup 2^\mathbb{N} \) and thus \( \phi[p] \subseteq \varphi_{\mathbb{M}}[g(p)] \).

Case \( n_p \neq \infty \): Then we have \( j_p \neq \infty \) and \( g(p) = 0^{\tau_p}(j_p + 1) : 0^\omega \). Therefore \( \phi[p] \subseteq \phi[p^{\ll_{n_p}}] \subseteq W_{j_p} = \varphi_{\mathbb{M}}[g(p)] \).

Hence \( g \) translates \( \phi \) continuously to \( \varphi \) showing the universal property of \( \varphi \).

Let \( X = (X, \rightarrow_\mathbb{N}) \) be a weak limit space with an admissible multirepresentation. Let \((x_n)_{n \leq \infty}\) be a non–convergent sequence of \( X \). By Proposition 3.1.7 \( X \) has a countable limit base \( \{ \beta_0, \beta_1, \beta_2, \ldots \} \subseteq 2^X \) (cf. Subsection 3.1.1). This means that for every sequence \((y_n)_n\) that converges to \( x_\infty \) in \( X \) there is some \( i \in \mathbb{N} \) with \( x_\infty \in \beta_i, (\forall n \in \mathbb{N}) y_n \in \beta_i, \) and \( (\exists n \in \mathbb{N}) x_n \notin \beta_i \). Hence the set

\[
I := \{ i \in \mathbb{N} \mid x_\infty \in \beta_i \text{ and } (\exists n \in \mathbb{N}) x_n \notin \beta_i \}
\]

is not empty. We define \( h : X \rightarrow W \) by

\[
h(y) := \begin{cases} \top & \text{if } y = x_\infty \\ \{ i \in I \mid y \in \beta_i \} & \text{otherwise} \end{cases}
\]
and show that $h$ is sequentially continuous. Let $(y_n)_{n \leq \infty}$ be a convergent sequence of $X$. The only crucial case is $y_\infty = x_\infty$. By the definition of a limit base, there is some $i \in I$ such that $y_n$ is contained in $\beta_i$ for almost all $n \in \mathbb{N}$. Thus $(h(y_n))_n$ is eventually in $\{\uparrow, \top\} \cup \{M \subseteq \mathbb{N} \mid i \in M\}$ and hence converges to $h(y_\infty)$. This proves that $h$ is sequentially continuous.

For every $j \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ with

$$e_{X,\mathfrak{M}e}(x_n)(h) = h(x_n) \notin \{\uparrow, \top\} \cup \{M \subseteq \mathbb{N} \mid j \in M\}.$$ 

Thus $(e_{X,\mathfrak{M}e}(x_n)(h))_n$ does not converge to $e_{X,\mathfrak{M}e}(x_\infty)(h) = \top$. We conclude that $(e_{X,\mathfrak{M}e}(x_n))_{n \leq \infty}$ is a non-convergent sequence of $\mathcal{C}(\mathcal{C}(X, \mathfrak{W}e), \mathfrak{W}e)$ and, by continuity of $e_{X,\mathfrak{M}e}$, that $e_{X,\mathfrak{M}e}$ is a homomorphism between $X$ and $\mathcal{C}(\mathcal{C}(X, \mathfrak{W}e), \mathfrak{W}e)$.

In contrast to $\mathfrak{Li}$, the space $\mathfrak{W}e$ is a universal codomain of the class of weak limit spaces with admissible multirepresentation satisfying the following stronger property for all weak limit spaces $X = (X, \rightarrow_X)$ and all sequences $(x_n)_{n \leq \infty}$ in $X$:

$$(x_n)_n \rightarrow_X x_\infty \iff (\forall h \in \mathcal{C}(X, \mathfrak{W}e)) (h(x_n))_n \rightarrow_{\mathfrak{W}e} h(x_\infty).$$

By definition of convergence in topological spaces, the analogue is satisfied by the universal codomain $\mathcal{W}(\mathfrak{Si})$ of the sequential topological spaces. A limit space $\mathfrak{Li} = (Le, \rightarrow_{\mathfrak{Li}})$ satisfying

$$(x_n)_n \rightarrow_{\mathfrak{Li}} x_\infty \iff (\forall h \in \mathcal{C}(X, \mathfrak{Li})) (h(x_n))_n \rightarrow_{\mathfrak{Li}} h(x_\infty)$$

(4.9)

for all limit spaces $X$ can be constructed as follows: its underlying set is $Le := \{\perp, \top\} \cup \mathbb{N}$ and its convergence relation $\rightarrow_{\mathfrak{Li}}$ is given by

$$(b_n)_n \rightarrow_{\mathfrak{Li}} b_\infty :\iff b_\infty \neq \top \text{ or } (\exists j \in \mathbb{N})(\forall n \in \mathbb{N}) (b_n \leq j \lor b_n \in \{\uparrow, \top\}).$$

An admissible multirepresentation $\varrho_{\mathfrak{Li}} : \mathbb{N}^\omega \rightrightarrows Le$ of $\mathfrak{Li}$ can be defined by

$$\varrho_{\mathfrak{Li}}[p] := \begin{cases} \{\top\} \cup \{b \in \mathbb{N} \mid b \leq p(k_p) - 1\} & \text{if } k_p := \min\{i \mid p(i) \neq 0\} \text{ exists} \\ \{\perp, \top\} \cup \mathbb{N} & \text{otherwise.} \end{cases}$$

for all $p \in \mathbb{N}^\omega$. We omit the details.

### 4.3.2 The Multirepresentations $\delta^\omega$, $\delta^e$, and $\delta^\omega e$

Now we are ready to define the operators $\sigma^\omega$, $\sigma^e$, and $\sigma^\omega e$. Let $\delta : \subseteq \mathbb{N}^\omega \rightrightarrows X$ be a multirepresentation of a set $X$, and let $\varrho_\mathfrak{U} : \subseteq \mathbb{N}^\omega \rightrightarrows U$ be an admissible multirepresentation of a weak limit space $\mathfrak{U} = (U, \rightarrow_\mathfrak{U})$. Proposition 4.2.5 guarantees that $[[\delta \rightarrow \varrho_\mathfrak{U} \rightarrow \varrho_\mathfrak{U}]]$ is an admissible multirepresentation of the space $\mathcal{C}(\mathcal{C}((X, \rightarrow_\mathfrak{U}), \mathfrak{U}), \mathfrak{U})$. If $\varrho_{(X, \rightarrow_\mathfrak{U})}$ is a homomorphism between $(X, \rightarrow_\mathfrak{U})$ and $\mathcal{C}(\mathcal{C}((X, \rightarrow_\mathfrak{U}), \mathfrak{U}), \mathfrak{U})$, then the correspondence $\delta^{\rightarrow_\mathfrak{U}}$ defined by

$$\delta^{\rightarrow_\mathfrak{U}} := (\varrho_{(X, \rightarrow_\mathfrak{U})})^{-1} \circ [[\delta \rightarrow \varrho_\mathfrak{U} \rightarrow \varrho_\mathfrak{U}]]$$

is an admissible multirepresentation of $(X, \rightarrow_\mathfrak{U})$ by Proposition 4.1.6.
We obtain from Propositions 4.3.2, 2.4.18 and Lemma 2.4.11 the following corollary. (cf. [Sch02b]) and show that they are admissible multirepresentations of the three quotient spaces generated by $\delta$ (cf. Subsection 2.4.5).

**Proposition 4.3.2 (Admissibility of $\delta^m$, $\delta^c$, and $\delta^e$)**

Let $\delta : \subseteq N^\omega \Rightarrow X$ be a multirepresentation of a set $X$.

1. $\delta^m$ is an admissible multirepresentation of the weak limit space $(X, \to^-)$.  
2. $\delta^c$ is an admissible multirepresentation of the limit space $(X, \sim^-)$.  
3. $\delta^e$ is an admissible multirepresentation of the sequential topological space $(X, \tau^\delta)$.  

**Proof:**

1. Let $\mathfrak{X} := (X, \to^-)$. From Proposition 4.2.5 we know that $[[\delta \rightarrow \varphi_{\mathfrak{m}}] \rightarrow \varphi_{\mathfrak{m}}]$ is an admissible multirepresentation of the space $\mathcal{C}(\mathcal{C}(\mathfrak{X}, \mathfrak{M}), \mathfrak{M})$. By Lemma 4.3.1, $e_{\mathfrak{X}, \mathfrak{m}}$ is a homomorphism between the spaces $\mathfrak{X}$ and $\mathcal{C}(\mathcal{C}(\mathfrak{X}, \mathfrak{M}), \mathfrak{M})$. Thus $\delta^m$ is an admissible multirepresentation of the weak limit space $\mathfrak{X} = (X, \to^-)$ by Proposition 4.1.6.

2. Similar to (3); here by using the equality $e_{(X, \sim^-), \mathfrak{c}} = e_{\mathfrak{c}(X, \sim^-), \mathfrak{c}} = e_{(X, \sim^-), \mathfrak{c}}$.

3. By Lemma 4.2.4, the function space $\mathcal{C}(T(X, \sim^-), \mathfrak{c})$ is equal to $\mathcal{C}((X, \sim^-), \mathfrak{c})$. This implies $e_{(X, \sim^-), \mathfrak{c}} = e_{(X, \sim^-), \mathfrak{c}}$, thus $\delta^e = (e_{T(X, \sim^-), \mathfrak{c}})^{-1} \circ [\delta \rightarrow \varphi_{\mathfrak{e}}] \rightarrow \varphi_{\mathfrak{e}}]$. By Lemma 4.3.1, $e_{T(X, \sim^-), \mathfrak{c}}$ is a homomorphism between the spaces $T(X, \sim^-)$ and $\mathcal{C}(\mathcal{C}(T(X, \sim^-), \mathfrak{c}), \mathfrak{c})$. The latter has $[[\delta \rightarrow \varphi_{\mathfrak{e}}] \rightarrow \varphi_{\mathfrak{e}}]$ as an admissible multirepresentation by Proposition 4.2.5. Thus Proposition 4.1.6 implies that $\delta^e$ is an admissible multirepresentation of $T(X, \sim^-)$ and thus of $(X, \tau^\delta)$ by Lemma 2.4.11.

We obtain from Propositions 4.3.2, 2.4.18 and Lemma 2.4.11 the following corollary.

**Corollary 4.3.3 (Characterization of admissibly representable spaces)**

1. A weak limit space $\mathfrak{X} = (X, \to^-)$ has an admissible multirepresentation if and only if $\mathfrak{X}$ has a WeakLim–quotient multirepresentation.

2. A limit space $\mathfrak{X} = (X, \sim^-)$ has an admissible multirepresentation if and only if $\mathfrak{X}$ has a Lim–quotient multirepresentation.

3. A sequential space $\mathfrak{X} = (X, \tau^\delta)$ has an admissible multirepresentation if and only if $\mathfrak{X}$ has a Top–quotient multirepresentation.
4.3 Effectivization

4.3.3 Computability w.r.t. $\delta^m$, $\delta^c$, and $\delta^e$

We will now prove that the operators $\sigma^m$, $\sigma^c$, and $\sigma^e$ preserve relative computability of total functions. Relative computability of partial functions is only maintained by the operators $\sigma^m$ and $\sigma^c$.

Unary Functions

Let $\mathcal{U} = (U, \rightarrow_\mathcal{U})$ be a weak limit space and $\varphi_\mathcal{U} : \mathbb{N}^\omega \rightharpoonup U$ be an admissible multirepresentation of $\mathcal{U}$. For weak limit spaces $\mathcal{X} = (X, \rightarrow_\mathcal{X})$, $\mathcal{Y} = (Y, \rightarrow_\mathcal{Y})$ and for sequentially continuous functions $f \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$, $h \in \mathcal{C}(\mathcal{Y}, \mathcal{U})$ we define the function $\mathcal{H}_\mathcal{U}(f) : \mathcal{C}(\mathcal{Y}, \mathcal{U}) \to \mathcal{C}(\mathcal{X}, \mathcal{U})$ by

$$\mathcal{H}_\mathcal{U}(f)(h) := h \circ f$$

(4.11)

Since, for any convergent sequence $(h_n)_{n \leq \infty}$ of $\mathcal{C}(\mathcal{Y}, \mathcal{U})$, the function $(n, x) \mapsto h_n(f(x))$ is $(\rightarrow_\mathcal{X}, \rightarrow_\mathcal{Y}, \rightarrow_\mathcal{U})$-continuous, $(\mathcal{H}_\mathcal{U}(f)(h_n))_{n \leq \infty}$ is a convergent sequence of $\mathcal{C}(\mathcal{X}, \mathcal{U})$. Thus $\mathcal{H}_\mathcal{U}$ maps $f$ to a continuous function from $\mathcal{C}(\mathcal{Y}, \mathcal{U})$ to $\mathcal{C}(\mathcal{X}, \mathcal{U})$. Hence the operator $\mathcal{H}_\mathcal{U}$ can also be applied to $\mathcal{H}_\mathcal{U}(f)$ yielding a continuous function from $\mathcal{C}(\mathcal{X}, \mathcal{U}, \mathcal{U})$ to $\mathcal{C}(\mathcal{C}(\mathcal{Y}, \mathcal{U}), \mathcal{U})$.

We prove the following properties of the operator $\mathcal{H}_\mathcal{U}$.

**Lemma 4.3.4 (Properties of the operator $\mathcal{H}_\mathcal{U}$)**

Let $\varphi_\mathcal{U} : \mathbb{N}^\omega \rightharpoonup U$ be an admissible multirepresentation of a weak limit space $\mathcal{U} = (U, \rightarrow_\mathcal{U})$.

1. Let $\mathcal{X} = (X, \rightarrow_\mathcal{X})$ and $\mathcal{Y} = (Y, \rightarrow_\mathcal{Y})$ be weak limit spaces. Then

   $$(\mathcal{H}_\mathcal{U} \circ \mathcal{H}_\mathcal{U}(f))(e_{\mathcal{X}, \mathcal{U}}(x)) = e_{\mathcal{Y}, \mathcal{U}}(f(x))$$

   holds for every continuous function $f \in \mathcal{C}(\mathcal{Y}, \mathcal{U})$ and every $x \in X$.

2. Let $\delta \subseteq \mathbb{N}^\omega \rightharpoonup X$ and $\gamma \subseteq \mathbb{N}^\omega \rightharpoonup Y$ be multirepresentations. There is a computable function $\varphi_\mathcal{Y} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ realizing the function

   $$\mathcal{H}_\mathcal{U} : \mathcal{C}(\delta, \gamma) \to \mathcal{C}(\mathcal{C}((Y, \rightarrow_\mathcal{Y}, \mathcal{U}), \mathcal{C}((X, \rightarrow_\mathcal{X}, \mathcal{U}))), f \mapsto \mathcal{H}_\mathcal{U}(f))$$

   with respect to $[\delta \rightarrow \gamma]$ and $[\gamma \rightarrow \varphi_\mathcal{Y}]$ independently of $\mathcal{U}$, $\delta$, and $\gamma$.

**Proof:**

1. For all $G \in \mathcal{C}(\mathcal{C}(\mathcal{X}, \mathcal{U}), \mathcal{U})$ and $h \in \mathcal{C}(\mathcal{Y}, \mathcal{U})$, we have

   $$\mathcal{H}_\mathcal{U}(\mathcal{H}_\mathcal{U}(h))(h) = (G \circ \mathcal{H}_\mathcal{U}(h))(h) = G(\mathcal{H}_\mathcal{U}(h)(h)) = G(h \circ f).$$

   (4.12)

   Since $e_{\mathcal{X}, \mathcal{U}}(x) \in \mathcal{C}(\mathcal{C}(\mathcal{X}, \mathcal{U}), \mathcal{U})$, this implies for every $x \in X$ and $h \in \mathcal{C}(\mathcal{Y}, \mathcal{U})$

   $$\mathcal{H}_\mathcal{U}(\mathcal{H}_\mathcal{U}(h))(e_{\mathcal{X}, \mathcal{U}}(x))(h) = e_{\mathcal{X}, \mathcal{U}}(x)(h \circ f) = h(f(x)) = e_{\mathcal{Y}, \mathcal{U}}(f(x))(h).$$

   Therefore, $(\mathcal{H}_\mathcal{U}(\mathcal{H}_\mathcal{U}(h)))(e_{\mathcal{X}, \mathcal{U}}(x)) = e_{\mathcal{Y}, \mathcal{U}}(f(x))$. 

(2) By admissibility of $\varrho_\mathcal{U}$ and by Propositions 4.2.5 and 2.4.18, $[\gamma \to \varrho_\mathcal{U} \to [\delta \to \varrho_\mathcal{U}]]$ is a multirepresentation of the set $C(\mathcal{E}(Y, \rightarrow), \mathcal{U}), \mathcal{E}((X, \rightarrow), \mathcal{U})$.

By the utm–Theorem and the computable smn–Theorem for $\eta$, there is a computable function $g_\mathcal{H} : \omega^\omega \to \omega^\omega$ satisfying

$$\forall q, r \in \omega^\omega \exists \eta_\mathcal{H}(\eta_\mathcal{H}(q))(s) = \eta_\mathcal{H}(\eta_\mathcal{H}(r))$$.

Let $s, q, r \in \omega^\omega$, let $f \in [\delta \to \gamma][s], h \in [\gamma \to \varrho_\mathcal{U}][q]$, and $x \in [\delta][r]$. Then we have $f(x) \in \gamma[\eta_\mathcal{H}(r)]$, hence

$$h(f(x)) \in \varrho_\mathcal{U}[\eta_\mathcal{H}(\eta_\mathcal{H}(q))(r)] = \varrho_\mathcal{U}[\eta_\mathcal{H}(\eta_\mathcal{H}(q))(r)]$$.

This implies

$$\mathcal{H}_\mathcal{U}(f)(h) = h \circ f \in [\delta \to \varrho_\mathcal{U}][\eta_\mathcal{H}(\eta_\mathcal{H}(q))].$$

Therefore,

$$\mathcal{H}_\mathcal{U}(f) \in [[\gamma \to \varrho_\mathcal{U} \to [\delta \to \varrho_\mathcal{U}])[g_\mathcal{H}(s)],$$

which means that $g_\mathcal{H}$ realizes $\mathcal{H}_\mathcal{U}$ w.r.t. $[\delta \to \gamma]$ and $[[\gamma \to \varrho_\mathcal{U} \to [\delta \to \varrho_\mathcal{U}]].$

This lemma implies the following effectivity properties of the operator $\circ^\omega \mathcal{U}$.

**Proposition 4.3.5 (Effectivity properties of $\circ^\omega \mathcal{U}$)**

Let $\delta : \subseteq \omega^\omega \Rightarrow X$ and $\gamma : \subseteq \omega^\omega \Rightarrow Y$ be multirepresentations. Let $\varrho_{\mathcal{U}} : \subseteq \omega^\omega \Rightarrow \mathcal{U}$ be an admissible multirepresentation of a weak limit space $\mathcal{U} = (U, \rightarrow)$. Then:

1. $[\delta \to \gamma] \leq_{\mathcal{U}} [\delta \to \gamma] \mathcal{U}$.
2. Every total $(\delta, \gamma)$–continuous function $f : X \to Y$ is $(\delta \to \gamma, \mathcal{U})$–continuous.
3. Every total $(\delta, \gamma)$–computable function $f : X \to Y$ is $(\delta \to \gamma, \mathcal{U})$–computable.

**Proof:**

1. Let $s \in \omega^\omega$ and $f \in [\delta \to \gamma][s]$. A first application of Lemma 4.3.4(2) yields

$$\mathcal{H}_\mathcal{U}(f) \in [[\gamma \to \varrho_\mathcal{U} \to [\delta \to \varrho_\mathcal{U}])[g_\mathcal{H}(s)],$$

and a second

$$\mathcal{H}_\mathcal{U}^2(f) \in [[\delta \to \varrho_\mathcal{U} \to \varrho_\mathcal{U} \to [\gamma \to \varrho_\mathcal{U} \to \varrho_\mathcal{U}][g_\mathcal{H}(g_\mathcal{H}(s))].$$

Thus $\eta_\mathcal{H}(\eta_\mathcal{H}(s))$ realizes $\mathcal{H}_\mathcal{U}^2(f)$ w.r.t. $[\delta \to \varrho_\mathcal{U} \to \varrho_\mathcal{U}$ and $[[\gamma \to \varrho_\mathcal{U} \to \varrho_\mathcal{U}$.

Let $p \in \omega^\omega$ and $x \in \delta \to \mathcal{U}[p]$. Then we have $e_{(X, \rightarrow), \mathcal{U}}(x) \in [[\delta \to \varrho_\mathcal{U} \to \varrho_\mathcal{U}][p]$. By Lemma 4.3.4(1), we obtain

$$e_{(Y, \rightarrow), \mathcal{U}}(f)(x) = \mathcal{H}_\mathcal{U}^2(f)(e_{(X, \rightarrow), \mathcal{U}}(x)) \in [[\gamma \to \varrho_\mathcal{U} \to \varrho_\mathcal{U}][\eta_\mathcal{H}(\mathcal{H}(s))](p)].$$

This implies $f(x) \in \gamma \to \mathcal{U}[\eta_\mathcal{H}(\mathcal{H}(s))](p)$ and $f \in [\delta \to \gamma] \mathcal{U}$. Hence $g_\mathcal{H} \circ g_\mathcal{H}$ translates $[\delta \to \gamma]$ computably to $[\delta \to \gamma \mathcal{U}]$. 

\[ \square \]
Let \( \delta \subseteq N^\omega \Rightarrow X \) and \( \gamma \subseteq N^\omega \Rightarrow Y \) be multirepresentations. Let \( \varrho_U : \subseteq N^\omega \Rightarrow U \) be an admissible multirepresentation of a weak limit space \( \Lambda = (U, \neg U) \).

1. If \( \delta \leq_{cp} \gamma \), then \( \delta \leq_{cp} \gamma \).
2. \( \delta \leq_{cp} \delta \equiv_{cp} (\delta \leq_{cp}) \).
3. \( \varrho_U \equiv_{cp} \varrho_U \).
4. \( \delta \leq_{cp} \delta^e \leq_{cp} \delta \).
5. \( \delta^e \equiv_{cp} (\delta^e)^e \equiv_{cp} (\delta^e)^e \).

Proof:
(1) If \( \delta \leq_{cp} \gamma \), then \( \delta \leq_{cp} \gamma \).
(2) \( \delta \leq_{cp} \delta \equiv_{cp} (\delta \leq_{cp}) \).
(3) Since the identical function \( id_{\Lambda} \) is computable, there is a computable \( \omega \)-word \( I \in N^\omega \) with \( \eta_I = id_{\Lambda} \) by Lemma 4.2.8. By the utm-Theorem, the function \( g : \subseteq N^\omega \Rightarrow N^\omega \) defined by \( g(p) := \eta_I(I) \) is computable.
(4) \( \delta \leq_{cp} \delta^e \leq_{cp} \delta \).
(5) \( \delta^e \equiv_{cp} (\delta^e)^e \equiv_{cp} (\delta^e)^e \).

Hence \( g \) translates \( (\varrho_U)^{-1} \) computably to \( \varrho_U \). From the first part of the proof of (2) which does not base on (3), we know \( \varrho_U \leq_{cp} \varrho_U^{-1} \). Therefore \( \varrho_U \equiv_{cp} \varrho_U^{-1} \).

(2) By the utm-Theorem and the computable smn-Theorem there is a computable function \( g_1 : N^\omega \Rightarrow N^\omega \) with \( (\forall p, q \in N^\omega) \eta_{g_1(p)}(q) = g_1(p) \). Let \( p \in N^\omega \) and \( x \in g_1(p) \). For \( q \in N^\omega \) and \( h \in [\delta \leq_{cp} \varrho_U][q] \) we have

\[ e_{(X, \gamma), \Lambda}(x)(h) = h(x) \in \varrho_U[\eta_I(p)] = \varrho_U[g_1(p)] \]

This implies \( e_{(X, \gamma), \Lambda}(x) \in [\delta \leq_{cp} \varrho_U][g_1(p)] \) and thus \( x \in \delta^{-1} \). Hence \( g_1 \) translates \( \delta \) computably to \( \delta \), i.e. \( \delta \leq_{cp} \delta \) and \( \delta \leq_{cp} (\delta^{-1})^{-1} \).
Chapter 4. Constructions

Proposition 4.3.5, Statements (1),(3) and Lemma 4.2.8 imply

\[ [\delta \to \varrho_u] \leq_{cp} [\delta \to \varrho_u] \leq_{cp} [\delta \to \varrho_u] \leq_{cp} [\delta \to \varrho_u] \]

and therefore the existence of a computable function \( g_2 : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega \) with

\[ (\forall q \in \text{dom}(\delta \to \varrho_u)) [\hat{\delta} - \varrho_u] \subseteq [\delta - \varrho_u][g_2(q)] . \]

By the utm–Theorem and the computable smn–Theorem, there is a computable function \( g_3 : \mathbb{N}^\omega \to \mathbb{N}^\omega \) with \((\forall p, q \in \mathbb{N}^\omega) \eta_{g_3(p)}(q) = \eta_p(g_2(q))\). Let \( p \in \mathbb{N}^\omega \) and \( x \in (\delta - \varrho_u)[p] \). For \( q \in \mathbb{N}^\omega \) and \( h \in [\delta \to \varrho_u][q] \), we have, as \( h \in [\delta \to \varrho_u][g_2(q)] \) \( \subseteq \mathcal{C}((X, \to_{\delta - \varrho_u}), \varrho_u) \) and \( e_{(X, \to_{\delta - \varrho_u}), \varrho_u}(x) \in [[\delta \to \varrho_u] \to \varrho_u][p], \)

\[ e_{(X, \to_{\delta - \varrho_u}), \varrho_u}(x)(h) = h(x) = e_{(X, \to_{\delta - \varrho_u}), \varrho_u}(x(h) \in \varrho_u[\eta_p(g_2(q))] = \varrho_u[\eta_p(g_2(q))] . \]

This implies \( e_{(X, \to_{\delta - \varrho_u}), \varrho_u}(x) \in [[\delta \to \varrho_u] \to \varrho_u][g_3(p)] \) and \( x \in \delta - \varrho_u[g_3(p)] \). Thus \( g_3 \) translates \((\delta - \varrho_u)[p] \) computably to \( \delta - \varrho_u \). Hence \((\delta - \varrho_u)[p] \equiv_{cp} \delta - \varrho_u \).

(4) We prove that \( \delta^c \) is a restriction of \( \delta^e \). Let \( p, q \in \mathbb{N}^\omega \), \( x \in \delta^c[p] \) and \( h \in [\delta \to \varrho_{\Theta_i}] \) of the continuous function \( h|_{\Theta_i} : X \to \{ \bot, \top, \bot \} \). We conclude

\[ e_{(X, \to_{\delta^c}), \Theta_i}(x)(h) = h(x) = h|_{\Theta_i}(x) \]

\[ = e_{(X, \to_{\delta^c}), \Theta_i}(x)(h) \in \varrho_{\Theta_i}[\eta_p(q)] \cap \{ \bot, \top, \bot \} = \varrho_{\Theta_i}[\eta_p(q)] . \]

This implies \( e_{(X, \to_{\delta^c}), \Theta_i}(x) \in [[\delta \to \varrho_{\Theta_i}] \to \varrho_{\Theta_i}] \), hence \( x \in \delta^c[p] \) and \( \delta^c \leq_{cp} \delta^e \). In a similar way, \( \delta^m \leq_{cp} \delta^c \) can be shown. From Statement (2) we know \( \delta \leq_{cp} \delta^m \).

(5) By (4), \( \zeta \leq_{cp} \zeta^m \leq_{cp} \zeta^c \) holds for every multirepresentation \( \zeta \). This implies together with (1) and (3)

\[ \delta^c \leq_{cp} (\delta^m)^c \leq_{cp} (\delta^c)^c \leq_{cp} \delta^c \quad \text{and} \quad \delta^c \leq_{cp} (\delta^e)^m \leq_{cp} (\delta^e)^c \leq_{cp} \delta^c . \]

We conclude \( \delta^c \equiv_{cp} (\delta^m)^c \equiv_{cp} (\delta^e)^m \equiv_{cp} (\delta^c)^c \). In a similar way, the other statements can be proved.

\( \checkmark \)

Let \( \delta : \subseteq \mathbb{N}^\omega \to X \) be a multirepresentation. By Equivalence (4.9), \( \delta \to_{\Delta^*} \) is an admissible multirepresentation of the limit space \( (X, \to_{\delta}) \) generated by \( \delta \). Hence \( \delta \to_{\Delta^*} \) is continuously equivalent to \( \delta^c \). By showing \( (\varrho_{\Delta^*})^c \leq_{cp} \varrho_{\Delta^*} \), one can prove that \( \delta \to_{\Delta^*} \) and \( \delta^c \) are even computably equivalent, thus \( \delta \leq_{cp} \delta^m \leq_{cp} \delta^c \equiv_{cp} \delta \to_{\Delta^*} \leq_{cp} \delta^c \).

We obtain the following corollary from Proposition 4.3.5 and Lemma 4.3.6.
Corollary 4.3.7
Let \( \delta : \subseteq \mathbb{N}^\omega \to X \) and \( \gamma : \subseteq \mathbb{N}^\omega \to Y \) be multirepresentations. Then:

1. \[
\begin{align*}
\delta^a \to \gamma &< \delta^c \to \gamma &< \delta^m \to \gamma &< \delta \to \gamma \\
\wedge &\wedge &\wedge &\wedge
\end{align*}
\]

2. \[
\begin{align*}
\delta^a \to \gamma^m &< \delta^c \to \gamma^m &< \delta^m \to \gamma^m &\equiv \delta \to \gamma^m \\
\wedge &\wedge &\wedge &\wedge
\end{align*}
\]

where \( \delta^a \to \gamma^d \) means \( \delta^a \to \gamma^d \equiv \delta^c \to \gamma^d \) and that there are multirepresentations \( \delta \) and \( \gamma \) with \( \delta^c \to \gamma^d \not\subseteq \delta^a \to \gamma^b \).

Proof:
The reducibilities from left to right and the vertical ones follow from Lemma 4.3.6(4) and 4.2.8(4). The reducibilities from right to left follow from \( \delta^c \to \gamma \) (cf. Proposition 4.3.5) and from Lemma 4.3.6(5) and Lemma 4.2.8(4). The positive results of (2) follow from the positive ones of (1) by Lemma 4.2.8(1). The negative statements of (1) and (2) can be obtained from the following counterexamples, Lemma 2.4.4 and the fact that \( \delta^a \to \gamma^b \) is a multirepresentation of \( \mathcal{C}(\delta^a, \gamma^b) \).

\[
\begin{align*}
\mathcal{C}_\text{cp}(\delta^a, \gamma) &\subseteq \mathcal{C}_\text{cp}(\delta^c, \gamma) &\subseteq \mathcal{C}_\text{cp}(\delta^m, \gamma) &\subseteq \mathcal{C}_\text{cp}(\delta, \gamma) \\
\mathcal{C}_\text{cp}(\delta^a, \gamma^m) &\subseteq \mathcal{C}_\text{cp}(\delta^c, \gamma^m) &\subseteq \mathcal{C}_\text{cp}(\delta^m, \gamma^m) &\equiv \mathcal{C}_\text{cp}(\delta, \gamma^m) \\
\mathcal{C}_\text{cp}(\delta^a, \gamma^c) &\subseteq \mathcal{C}_\text{cp}(\delta^c, \gamma^c) &\equiv \mathcal{C}_\text{cp}(\delta^m, \gamma^c) &\equiv \mathcal{C}_\text{cp}(\delta, \gamma^c) \\
\mathcal{C}_\text{cp}(\delta^a, \gamma^e) &\equiv \mathcal{C}_\text{cp}(\delta^c, \gamma^e) &\equiv \mathcal{C}_\text{cp}(\delta^m, \gamma^e) &\equiv \mathcal{C}_\text{cp}(\delta, \gamma^e)
\end{align*}
\]

where \( \mathcal{C}_\text{cp}(\delta^a, \gamma^b) \subseteq \mathcal{C}_\text{cp}(\delta^c, \gamma^d) \) means \( \mathcal{C}_\text{cp}(\delta^a, \gamma^b) \subseteq \mathcal{C}_\text{cp}(\delta^c, \gamma^d) \) and that there are multirepresentations \( \delta \) and \( \gamma \) with \( \mathcal{C}(\delta^a, \gamma^b) \not\supseteq \mathcal{C}(\delta^c, \gamma^d) \).
Now we show that the operators \( \partial F \) and \( \partial \) preserve the computability of partial functions, whereas \( \partial^e \) does not. This is due to the fact that the weak limit spaces \( 2^\mathcal{R} \) and \( 2^\mathfrak{I} \), in contrast to \( \mathfrak{S}^\mathfrak{I} \), contain the value \( \top \) playing the role of an “undefined” element (cf. Lemma 4.2.10). For the proof, we use the following lemma about multirepresentations of subspaces (cf. Subsection 4.1.5).
4.3 Effectivization

Lemma 4.3.8 (Comparison of subspace multirepresentations)
Let $\delta \subseteq \mathbb{N}^\omega \Rightarrow X$ be a multirepresentation of a set $X$, and let $M$ be a subset of $X$. Then $(\delta^{|M|})^c \leq_{cp} (\delta^c)^{|M|}$, $(\delta^{|M|})^c \equiv_{cp} (\delta^c)^{|M|}$, and $(\delta^{|M|})^m \equiv_{cp} (\delta^m)^{|M|}$.

Proof:
Let $\mathfrak{U} = (U, \rightarrow_u) \in \{W(\mathfrak{G}_i), \mathfrak{L}_i, \mathfrak{M}_i\}$.

$(\delta^{|M|})^{-\mathfrak{U}} \leq_{cp} (\delta^{-\mathfrak{U}})^{|M|}$.

Let $p \in \mathbb{N}^\omega$ and $x \in (\delta^{|M|})^{-\mathfrak{U}}[p]$. Let $q \in \mathbb{N}^\omega$ and $h \in [\delta \rightarrow \mathfrak{U}][q]$. Then we have $h^{|M|} \in [\delta^{|M|} \rightarrow q_u][q]$ and hence

\[ e_\mathfrak{U}(x, q, h)(x) = h(x) = h^{|M|}(x) = e(M, \rightarrow, h^{|M}|, \mathfrak{U})(x) \in q_u[q] \]

because $e(M, \rightarrow, h^{|M}|, \mathfrak{U})(x) \in [[\delta^{|M|} \rightarrow q_u] \cap q_u][p]$. This implies

\[ e_\mathfrak{U}(x, q, h)(x) \in [[\delta \rightarrow q_u] \cap q_u][p] \]

Thus $x \in (\delta^{-\mathfrak{U}})^{|M|} \cap M = (\delta^{-\mathfrak{U}})^{|M|}[p]$ proving $(\delta^{|M|})^{-\mathfrak{U}} \leq_{cp} (\delta^{-\mathfrak{U}})^{|M|}$.

$(\delta^c)^{|M|} \leq_{cp} (\delta^{|M|})^c$, $(\delta^m)^{|M|} \leq_{cp} (\delta^{|M|})^m$.

Let $\mathfrak{U} \in \{\mathfrak{U}_i, \mathfrak{W}_i\}$. Let $\sigma : \mathbb{N}^* \times \mathbb{N}^* \rightarrow \mathbb{N}^*$ be a computable monotone word function approximating the universal function of $\eta$. We define $g : \mathbb{N}^\omega \times \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ by

\[ g(q, r) := \begin{cases} 0^\omega & \text{if } (\forall i \in \mathbb{N}) \sigma(q^{<i}, r^{<i}) \in \{0\}^\omega \\ 0^c a : 0^c & \text{if } a, b \in \mathbb{N} \text{ and } c > 0 \text{ satisfy} \\ & a = \min \{ i \mid \sigma(q^{<i}, r^{<i}) \notin \{0\}^\omega \} \text{ and} \\ & 0^c a \subseteq \sigma(q^{<a}, r^{<a}) \end{cases} \]

Obviously, $g$ is computable and total. By the computable smn–Theorem for $\eta$, there is a computable function $g_{\text{tot}} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ with

\[ (\forall q, r \in \mathbb{N}^\omega) \eta_{g_{\text{tot}}(q)}(r) = g(q, r). \] (4.13)

These functions are constructed in such a way that $\eta_{g_{\text{tot}}(q)}$ is total for all $q \in \mathbb{N}^\omega$ and

\[ \varrho_{\mathfrak{M}_i}[\eta_{g_{\text{tot}}(q)}(r)] = \varrho_{\mathfrak{M}_i}[\eta_q(r)] \quad \text{and} \quad \varrho_{\mathfrak{L}_i}[\eta_{g_{\text{tot}}(q)}(r)] = \varrho_{\mathfrak{L}_i}[\eta_q(r)] \] (4.14)

holds for all $r \in \text{dom}(\eta_q)$.

By the utm–Theorem and the computable smn–Theorem there is computable function $g_{\text{sub}} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ with

\[ (\forall q, p \in \mathbb{N}^\omega) \eta_{g_{\text{sub}}(p)}(q) = \eta_p(g_{\text{tot}}(q)). \]

Let $p \in \mathbb{N}^\omega$ and $x \in (\delta^{-\mathfrak{U}})^{|M|}[p]$. Let $q \in \mathbb{N}^\omega$ and $h \in [\delta^{|M|} \rightarrow q_u][q]$. We define $h' : X \rightarrow U$ by

\[ h'(z) := \begin{cases} h(z) & \text{if } z \in M \\ \uparrow & \text{otherwise} \end{cases} \]

for all $z \in X$. Then we have $h' \in [\delta \rightarrow q_u][g_{\text{tot}}(q)]$, because all $z \in M$ and all $r \in (\delta^{-1})^{|M|}$ satisfy

\[ h'(z) = h(z) \in q_u[\eta_q(r)] = q_u[\eta_{g_{\text{tot}}(q)}(r)] \]
and \(\varrho_\mu[\eta_{\mathrm{gTot}}(q)(r')]\) contains \(\uparrow\) for every \(r' \in \mathbb{N}^\omega\), as \(\eta_{\mathrm{gTot}}(q)\) is total. Therefore \(h' \in C((X, \to), \Omega)\). Hence

\[
e(\delta, \gamma) \ne \delta \ne \gamma.
\]

This implies \(e(\delta, \gamma) \ne \delta \ne \gamma\). Furthermore, \((\delta, \gamma)\)-computability of partial functions does not necessarily imply \((\delta, \gamma)\)-continuity.

Example 4.3.9 shows that the operator \(\circ\) satisfies neither \((\delta^e)^{|M|} \leq_{\mathrm{cp}} (\delta^M)^e\) nor \((\delta^e)^{|M|} \leq_1 (\delta^M)^e\). Furthermore, \((\delta, \gamma)\)-computability of partial functions does not necessarily imply \((\delta, \gamma)\)-continuity.

Example 4.3.9

We consider the non topological limit space \(\mathfrak{z} = (Z, \to)\) and its admissible representation \(\zeta\) from Example 2.3.16. Let \(\varrho_n : \mathbb{N}^\omega \to \mathbb{N}\) be the admissible representation of \((\mathbb{N}, \to)\) from Example 2.3.6 with \(\varrho_n(p) = p(0)\). We define the partial function \(f : \subseteq Z \to \mathbb{N}\) by

\[
f(z) := \begin{cases} 
0 & \text{if } z = (1, 0) \\
1 & \text{if } z \in \{3\} \times \mathbb{N} \\
\text{div} & \text{otherwise}.
\end{cases}
\]

Then \(f\) is not a sequentially continuous function between the topological space \(T(\mathfrak{z})\) and \((\mathbb{N}, \to)\), because \(((3, b))_b\) converges to \((1, 0)\) in \(T(\mathfrak{z})\) (cf. Example 2.3.16) and the image sequence \((f(3, b))_b\) does not converge to \(f(1, 0)\) in \((\mathbb{N}, \to)\). By Proposition 4.3.2 and Lemmas 2.2.7, 2.4.11, \(\zeta^e\) and \(\varrho_n^e\) are admissible representations of the spaces \(T(\mathfrak{z})\) and \((\mathbb{N}, \gamma\mathbb{N})\), respectively. Thus \(f\) is not relatively continuous w.r.t. \(\zeta^e\) and \(\varrho_n^e\) by Theorem 2.3.18.

Suppose for contradiction \((\zeta^e)^{\mathrm{dom}(f)} \leq_1 (\zeta_{\mathrm{dom}(f)}^e)^e\). The total restriction \(f_{\mathrm{dom}(f)} : \mathrm{dom}(f) \to Y\) to the domain of \(f\) is \((\zeta_{\mathrm{dom}(f)}^e, \varrho_n^e)\)-computable, because the identity \(\mathrm{id}_{\mathbb{N}^\omega}\) is a computable realizer. Thus \(f_{\mathrm{dom}(f)}\) is \(((\zeta_{\mathrm{dom}(f)}^e)^e, \varrho_n^e)\)-computable by Proposition 4.3.5. By Lemma 2.4.4, \((\zeta^e_{\mathrm{dom}(f)})^e\) would imply that \(f_{\mathrm{dom}(f)}\) is \(((\zeta^e)^{\mathrm{dom}(f)}, \varrho_n^e)\)-continuous and hence that \(f\) is \((\zeta^e, \varrho_n^e)\)-continuous. This contradicts the above observation.

From Lemma 4.3.8 we obtain the following generalization of Corollary 4.3.7(2) to partial functions. Note that \(\mathcal{P}_{\mathrm{cp}}(\delta^e, \gamma^e)\) need not be equal to \(\mathcal{P}_{\mathrm{cp}}(\delta, \gamma)\) and \((\delta, \gamma)\)-computable partial functions are not necessarily \((\delta^e, \gamma^e)\)-computable, whereas we have \(\mathcal{C}_{\mathrm{cp}}(\delta, \gamma) \subseteq \mathcal{C}_{\mathrm{cp}}(\delta^e, \gamma^e) = \mathcal{C}_{\mathrm{cp}}(\delta^e, \gamma^e)\).
Corollary 4.3.10
Let $\delta \subseteq \mathbb{N}^w \Rightarrow X$ and $\gamma \subseteq \mathbb{N}^w \Rightarrow Y$ be multirepresentations. Then we have
\[
P_{cp}(\delta^e, \gamma) \subseteq P_{cp}(\delta^c, \gamma) \subseteq P_{cp}(\delta^m, \gamma) \subseteq P_{cp}(\delta, \gamma)
\]
and by Lemma 4.2.9(3) this implies the subset relations from right to left.
\[
P_{cp}(\delta^e, \gamma^m) \subseteq P_{cp}(\delta^e, \gamma) \subseteq P_{cp}(\delta^m, \gamma^m) \subseteq P_{cp}(\delta, \gamma^m)
\]
\[
P_{cp}(\delta^e, \gamma^e) \subseteq P_{cp}(\delta^c, \gamma^e) = P_{cp}(\delta^m, \gamma^e) = P_{cp}(\delta, \gamma^e)
\]
\[
P_{cp}(\delta^e, \gamma^e) \subseteq P_{cp}(\delta^m, \gamma^e) = P_{cp}(\delta^e, \gamma^e)
\]
where $P_{cp}(\delta^e, \gamma^b) \subseteq P_{cp}(\delta^e, \gamma^d)$ means $P_{cp}(\delta^e, \gamma^b) \subseteq P_{cp}(\delta^e, \gamma^d)$ and that there are multirepresentations $\delta$ and $\gamma$ with $P(\delta^e, \gamma^b) /\notin P_{cp}(\delta^e, \gamma^d)$.

Proof:
The subset relations from left to right and the vertical ones follow from Lemmas 4.3.6(4) and 4.2.9(3). From Proposition 4.3.5(3) and Lemma 4.3.8 we know
\[
C_{cp}(\delta^M, \gamma) \subseteq C_{cp}((\delta^M)^a, \gamma^a) = C_{cp}(\delta^a)^M, \gamma^a
\]
for every $M \subseteq X$ and $a \in \{\mathbb{w}, \varepsilon\}$. Since a partial function $f : X \rightarrow Y$ is $(\delta, \gamma)$-computable if and only if its restriction $f|_{dom(f)} : dom(f) \rightarrow Y$ is $(\delta|_{dom(f)}, \gamma)$-computable, we obtain $P_{cp}(\delta, \gamma) \subseteq P_{cp}(\delta^m, \gamma^m)$ and $P_{cp}(\delta, \gamma) \subseteq P_{cp}(\delta^e, \gamma^e)$. By Lemma 4.3.6(4),(5) and by Lemma 4.2.9(3) this implies the subset relations from right to left.

The representation $\zeta$ and the function $f$ from Example 4.3.9 fulfill $f \in P_{cp}(\zeta, \eta_N) \setminus P(\zeta^e, \eta_N^c) \subseteq P_{cp}(\zeta^e, \eta_N^c) \setminus P(\zeta^e, \eta_N)$ showing $P(\zeta^e, \eta_N^c) \not\subseteq P_{cp}(\zeta^e, \eta_N^c)$. As $\eta_N^c \subseteq \eta_N$ can be shown easily, we even have $P(\zeta^e, \eta_N^c) \not\subseteq P_{cp}(\zeta^e, \eta_N)$. Examples for the other $\not\subseteq$-statements are given in the proof of Corollary 4.3.7.

\[
\checkmark
\]

Multivariate Functions

Now we prove that the operators $\circ^m$, $\circ^e$, and $\circ^a$ preserve computability of multivariate functions as well.

Let $U = (U, \rightarrow_U)$ be a weak limit space and $\varrho_U : \mathbb{N}^w \Rightarrow U$ be an admissible multirepresentation of $U$. For weak limit spaces $X = (X, \rightarrow_X)$, $Y = (Y, \rightarrow_Y)$, $Z = (Z, \rightarrow_Z)$, for sequentially continuous functions $f \in C(X \otimes Y, Z)$, $h \in C(Z, U)$, and for $y \in Y$, $x \in X$ we define the function $H_{2,U}(f) : C(Z, U) \times Y \rightarrow C(X, U)$ by
\[
H_{2,U}(f)(h, y)(x) := h(f(x, y))
\]
Since for all convergent sequences $(h_n)_{n \leq \infty}$ of $C(Z, U)$ and $(y_n)_{n \leq \infty}$ of $Y$ the function $(n, x) \mapsto h_n(f(x, y_n))$ is $(\rightarrow_X, \rightarrow_Y)$-continuous, $(H_{2,U}(f)(h_n, y_n))_{n \leq \infty}$ is a convergent sequence of $C(X, U)$. Consequently, $H_{2,U}$ maps $f$ to a continuous function between the spaces $C(Z, U)$, $Y$, and $C(X, U)$. Hence the operator $H_{2,U}$ can be also applied to $H_{2,U}(f)$ yielding a continuous function between $C(C(X, U), U)$, $Y$, and $C(C(Z, U), U)$.

The operator $H_{2,U}$ enjoys similar properties as $H_U$. 
Proposition 4.3.11 (Properties of $\mathcal{H}_{2,\mathfrak{U}}$)
Let $\delta \subseteq \mathbb{N}^\omega \Rightarrow X$, $\gamma \subseteq \mathbb{N}^\omega \Rightarrow Y$, and $\zeta \subseteq \mathbb{N}^\omega \Rightarrow Z$ be multirepresentations, and let $\mathfrak{X} = (X, \rightarrow_\gamma)$, $\mathfrak{Y} = (Y, \rightarrow_\gamma)$, and $\mathfrak{Z} = (Z, \rightarrow_\gamma)$ be the weak limit spaces generated by these multirepresentations. Let $\eta_{\mathfrak{U}} : \subseteq \mathbb{N}^\omega \Rightarrow U$ be an admissible multirepresentation of a weak limit space $\mathfrak{U} = (U, \rightarrow_\mathfrak{U})$.

1. For all $f \in C(\mathfrak{X} \otimes \mathfrak{Y}, \mathfrak{Z})$, $x \in X$ and $y \in Y$, we have
   \[
   (\mathcal{H}_{2,\mathfrak{U}} \circ \mathcal{H}_{2,\mathfrak{U}}(f))(e_{\mathfrak{X},\mathfrak{U}}(x), y) = e_{3,\mathfrak{U}}(f(x, y)).
   \]

2. The function $\mathcal{H}_{2,\mathfrak{U}} : C(\delta \bowtie \gamma, \zeta) \rightarrow C(C(\mathfrak{X}, \mathfrak{U}) \otimes \mathfrak{Y}, C(\mathfrak{X}, \mathfrak{U}))$ is computable w.r.t. $[\delta \bowtie \gamma \rightarrow \zeta]$ and $[[\zeta \rightarrow \eta_{\mathfrak{U}}] \bowtie \gamma \rightarrow [\delta \rightarrow \eta_{\mathfrak{U}}]]$.

3. It holds
   \[
   [\delta \bowtie \gamma \rightarrow \zeta] \leq \text{cp} \left\{ \begin{array}{c}
   [\delta^{\mathfrak{U}} \bowtie \gamma \rightarrow \zeta^{\mathfrak{U}}] \\
   [\delta \bowtie \gamma \rightarrow \zeta^{\mathfrak{U}}]
   \end{array} \right\} \leq \text{cp} [\delta^{\mathfrak{U}} \bowtie \gamma \rightarrow \zeta^{\mathfrak{U}}].
   \]

4. Every $(\delta, \gamma, \zeta)$–computable function $f : X \times Y \rightarrow Z$ is $(\delta^{\mathfrak{U}}, \gamma, \zeta^{\mathfrak{U}})$–computable, $(\delta, \gamma^{\mathfrak{U}}, \zeta^{\mathfrak{U}})$–computable, and $(\delta^{\mathfrak{U}}, \gamma^{\mathfrak{U}}, \zeta^{\mathfrak{U}})$–computable.

Proof:

1. Let $h \in C(\mathfrak{X}, \mathfrak{U})$. Since $e_{\mathfrak{X},\mathfrak{U}}(x) \in C(C(\mathfrak{X}, \mathfrak{U}), \mathfrak{U})$, we have
   \[
   \mathcal{H}_{2,\mathfrak{U}}(f)(e_{\mathfrak{X},\mathfrak{U}}(x), y)(h) = e_{\mathfrak{X},\mathfrak{U}}(x)(\mathcal{H}_{2,\mathfrak{U}}(f)(h, y))
   = e_{\mathfrak{X},\mathfrak{U}}(x)(a \mapsto h(f(a, y))) = h(f(x, y)) = e_{3,\mathfrak{U}}(f(x, y))(h).
   \]
   Thus $\mathcal{H}_{2,\mathfrak{U}}(f)(e_{\mathfrak{X},\mathfrak{U}}(x), y) = e_{3,\mathfrak{U}}(f(x, y))$.

2. At first, we verify that $[[\zeta \rightarrow \eta_{\mathfrak{U}}] \bowtie \gamma \rightarrow [\delta \rightarrow \eta_{\mathfrak{U}}]]$ is actually a multirepresentation of the set $C(C(\mathfrak{X}, \mathfrak{U}) \otimes \mathfrak{Y}, C(\mathfrak{X}, \mathfrak{U}))$. By admissibility of $\eta_{\mathfrak{U}}$ and Propositions 4.2.5 and 2.4.18, $[\delta \rightarrow \eta_{\mathfrak{U}}]$ is an admissible multirepresentation of $C(\mathfrak{X}, \mathfrak{U})$ and $[\zeta \rightarrow \eta_{\mathfrak{U}}]$ is a WeakLim–quotient multirepresentation of $C(\mathfrak{X}, \mathfrak{U})$. Since the $\bowtie$–operator can be easily shown to preserve the property of being a WeakLim–quotient multirepresentation, Proposition 4.2.5 implies that $[[\zeta \rightarrow \eta_{\mathfrak{U}}] \bowtie \gamma \rightarrow [\delta \rightarrow \eta_{\mathfrak{U}}]]$ is an admissible multirepresentation of $C(C(\mathfrak{X}, \mathfrak{U}) \otimes \mathfrak{Y}, C(\mathfrak{X}, \mathfrak{U}))$.

By the utm–Theorem and the computable smn–Theorem for $\eta$, there is a computable function $g_{\mathfrak{H}_2} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ satisfying
\[
(\forall q, r, s, t \in \mathbb{N}^\omega) \eta_{g_{\mathfrak{H}_2}(t)}((q, s))((r, s)) = \eta_{q}(\eta_{r}(\langle q, s \rangle))
\]
Let $q, r, s, t \in \mathbb{N}^\omega$, $f \in [\delta \bowtie \gamma \rightarrow \zeta][t]$, $h \in [\zeta \rightarrow \eta_{\mathfrak{U}}][q]$, $y \in \gamma[s]$, and $x \in \delta[r]$. Then we have $f(x, y) \in \zeta[\eta_{r}(r, s)]$ and hence
\[
h(f(x, y)) \in \eta_{\mathfrak{U}}[\eta_{q}(\langle r, s \rangle)] = \eta_{\mathfrak{U}}[\eta_{g_{\mathfrak{H}_2}(t)}(q, s)(r)].
\]
Consequently,
\[
\mathcal{H}_{2,\mathfrak{U}}(f)(h, y) \in [\delta \rightarrow \eta_{\mathfrak{U}}][\eta_{g_{\mathfrak{H}_2}(t)}(q, s)].
\]
4.3 Effectivization

Since \((h, y) \in ([\zeta \rightarrow \varrho_\Omega] \boxtimes \gamma)((q, s))\), this implies

\[\mathcal{H}_{2,1}\langle f \rangle \in [[[\zeta \rightarrow \varrho_\Omega] \boxtimes \gamma \rightarrow [\delta \rightarrow \varrho_\Omega]]\mathcal{g}_{H_2}(t)].\]

Thus \(g_{H_2}\) realizes \(\mathcal{H}_{2,1}\) w.r.t. \([\delta \boxtimes \gamma \rightarrow \zeta]\) and \([[\zeta \rightarrow \varrho_\Omega] \boxtimes \gamma \rightarrow [\delta \rightarrow \varrho_\Omega]]\).

(3) Let \(t \in \mathbb{N}^\omega\) and \(f \in [\delta \boxtimes \gamma \rightarrow \zeta][t]\). A first application of (2) yields

\[\mathcal{H}_{2,1}\langle f \rangle \in [[[\zeta \rightarrow \varrho_\Omega] \boxtimes \gamma \rightarrow [\delta \rightarrow \varrho_\Omega]]\mathcal{g}_{H_2}(t)]\]

and a second

\[\mathcal{H}_{2,1}\langle f \rangle \in [[[\delta \rightarrow \varrho_\Omega] \rightarrow \varrho_\Omega] \boxtimes \gamma \rightarrow [[\zeta \rightarrow \varrho_\Omega] \rightarrow \varrho_\Omega]]\mathcal{g}_{H_2}(g_{H_2}(t))].\]

Thus \(\eta_{H_2}(\mathcal{g}_{H_2}(t))\) realizes \(\mathcal{H}_{2,1}\langle f \rangle\) w.r.t. \([\delta \rightarrow \varrho_\Omega] \rightarrow \varrho_\Omega\) and \([[\zeta \rightarrow \varrho_\Omega] \rightarrow \varrho_\Omega]]\).

Let \(p, q \in \mathbb{N}^\omega\), \(x \in \delta^{\rightarrow\Omega}[p]\) and \(y \in \gamma[q]\). Then \(e_{(X, \gamma)}(t) \in [\delta \rightarrow \varrho_\Omega] \rightarrow \varrho_\Omega]][p]\), hence

\[e_{(Z, \gamma)}(f(x, y)) = \mathcal{H}_{2,1}\langle f \rangle(e_{(X, \gamma)}(x, y)) \in [[[\delta \rightarrow \varrho_\Omega] \rightarrow \varrho_\Omega][\eta_{H_2}(\mathcal{g}_{H_2}(t)))(p, q)]\]

by (1). This implies \(f(x, y) \in \zeta^{\rightarrow\Omega}[\eta_{H_2}(\mathcal{g}_{H_2}(t)))(p, q)]\) and therefore

\[f \in [\delta^{\rightarrow\Omega} \boxtimes \gamma \rightarrow \zeta^{\rightarrow\Omega}]\mathcal{g}_{H_2} \circ \mathcal{g}_{H_2}(t),\]

showing \([\delta \boxtimes \gamma \rightarrow \zeta] \leq \mathcal{g}_{H_2} [\delta^{\rightarrow\Omega} \boxtimes \gamma \rightarrow \zeta^{\rightarrow\Omega}]\).

In a similar way, \([\delta \boxtimes \gamma \rightarrow \zeta] \leq \mathcal{g}_{H_2} [\delta^{\rightarrow\Omega} \gamma \rightarrow \zeta^{\rightarrow\Omega}]\) can be proved. From these two results we can deduce

\[\left[\begin{array}{c}
[\delta^{\rightarrow\Omega} \boxtimes \gamma \rightarrow \zeta^{\rightarrow\Omega}] \\
[\delta \boxtimes \gamma^{\rightarrow\Omega} \rightarrow \zeta^{\rightarrow\Omega}]
\end{array}\right] \leq \mathcal{g}_{H_2} \left[\begin{array}{c}
[\delta^{\rightarrow\Omega} \boxtimes \gamma \rightarrow \zeta^{\rightarrow\Omega}] \\
[\delta^{\rightarrow\Omega} \gamma \rightarrow \zeta^{\rightarrow\Omega}]
\end{array}\right]
\]

by Lemmas 4.3.6(2) and 4.2.8(4).

(4) Since by Lemmas 4.2.8 and 4.1.9 the function \(f\) is \((\delta, \gamma, \zeta)-\text{computable}\) if and only if \(f\) has a computable \([\delta \boxtimes \gamma \rightarrow \zeta]-\text{name}\), (4) follows from (3).

\(\checkmark\)

From Proposition 4.3.11 we obtain the following effectivity properties. Statements (3) and (5) can be viewed as effective versions of Theorem 2.2.10.

**Proposition 4.3.12 (Computability of multivariate functions)**

For \(i \in \{1, \ldots, k + 1\}\), let \(\delta_i \subseteq \mathbb{N}^\omega \Rightarrow X_i\) be a multirepresentation.

(1) \((\delta_1^{\rightarrow\Omega} \boxtimes \cdots \boxtimes \delta_k^{\rightarrow\Omega}) \equiv \mathcal{g}_{\delta_1 \boxtimes \cdots \boxtimes \delta_k}^{\rightarrow\Omega}.

(2) Every total \((\delta_1, \ldots, \delta_{k+1})-\text{continuous function}\) \(f\) is \((\delta_1^{\rightarrow\Omega}, \ldots, \delta_{k+1}^{\rightarrow\Omega})-\text{continuous}.

(3) Every total \((\delta_1, \ldots, \delta_{k+1})-\text{computable function}\) \(f\) is \((\delta_1^{\rightarrow\Omega}, \ldots, \delta_{k+1}^{\rightarrow\Omega})-\text{computable}.

\(\checkmark\)
(4) Every \((\delta_1, \ldots, \delta_{k+1})\)-continuous function is \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-continuous, every \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-continuous function is \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-continuous, and every total \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-continuous function is \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-continuous.

(5) Every \((\delta_1, \ldots, \delta_{k+1})\)-computable function is \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-computable, every \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-computable function is \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-computable, and every total \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-computable function is \((\delta_1^\omega, \ldots, \delta_{k+1}^\omega)\)-computable.

Proof:

(1) \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)^{-u} \leq \text{cp} \left( \delta_1^{-u} \boxtimes \ldots \boxtimes \delta_k^{-u} \right)\): For every \(i \in \{1, \ldots, k\}\), the projection \(p_{k,i} : X_1 \times \ldots \times X_k \rightarrow X_i\) is computable with respect to \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)\) and \(\delta_i\) via the computable function \(\pi_{k,i} : N^\omega \rightarrow N^\omega\) (cf. Subsection 2.1.7). By Proposition 4.3.5 there is a computable function \(g_i : \subseteq N^\omega \rightarrow N^\omega\) realizing \(p_{k,i}\) w.r.t. \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)^{-u}\) and \(\delta_i^{-u}\). We define \(G : \subseteq N^\omega \rightarrow N^\omega\) computably by \(G(p) := \langle g_1(p), \ldots, g_k(p) \rangle\). For \(p \in N^\omega\) and \((x_1, \ldots, x_k) \in (\delta_1 \boxtimes \ldots \boxtimes \delta_k)^{-u}[p]\) we have

\[
(x_1, \ldots, x_k) \in \delta_i^{-u}[g_1(p)] \times \ldots \times \delta_k^{-u}[g_k(p)] = (\delta_i^{-u} \boxtimes \ldots \boxtimes \delta_k^{-u})[G(p)],
\]

i.e., \(G\) translates \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)^{-u}\) computably to \((\delta_1^{-u} \boxtimes \ldots \boxtimes \delta_k^{-u})\).

(2) \((\delta_1^{-u} \boxtimes \ldots \boxtimes \delta_k^{-u}) \leq \text{cp} \left( \delta_1 \boxtimes \ldots \boxtimes \delta_k \right)^{-u}\): We prove this statement by induction.

\(k = 2^n\): The identity \(id_{X_1 \times X_2}\) is computable w.r.t. \(\delta_1, \delta_2,\) and \(\delta_1 \boxtimes \delta_2\) via the computable function \((p, q) \mapsto \langle p, q \rangle\). By Proposition 4.3.11(4) there is a computable function \(g_1 : \subseteq N^\omega \times N^\omega \rightarrow N^\omega\) realizing \(id_{X_1 \times X_2}\) w.r.t. \(\delta_1^{-u}, \delta_2^{-u},\) and \((\delta_1 \boxtimes \delta_2)^{-u}\). It follows that the computable function \(g_2 : \subseteq N^\omega \rightarrow N^\omega\) defined by \(g_2(p, q) := g_1(p, q)\) translates \((\delta_1^{-u} \boxtimes \delta_2^{-u})\) computably to \((\delta_1 \boxtimes \delta_2)^{-u}\).

\(k = k + 1\): The induction hypothesis and Lemma 4.1.9 yield

\[(\delta_1^{-u} \boxtimes \ldots \boxtimes \delta_k^{-u}) \leq \text{cp} \left( \delta_1 \boxtimes \ldots \boxtimes \delta_k \right)^{-u} \delta_{k+1}^{-u} \leq \text{cp} \left( \delta_1 \boxtimes \ldots \boxtimes \delta_k \right)^{-u} \delta_{k+1}^{-u}.
\]

By applying case \(k = 2^n\) to \(\gamma_1 := (\delta_1 \boxtimes \ldots \boxtimes \delta_k)\) and \(\gamma_2 := (\delta_{k+1})\) we obtain

\[(\delta_1 \boxtimes \ldots \boxtimes \delta_k)^{-u} \boxtimes \delta_{k+1}^{-u} \leq \text{cp} \left( (\delta_1 \boxtimes \ldots \boxtimes \delta_k) \boxtimes \delta_{k+1} \right)^{-u} \delta_{k+1}^{-u}.
\]

Hence the identity function on \((X_1 \times \ldots \times X_k) \times X_{k+1}\) is computable w.r.t. \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)^{-u} \boxtimes \delta_{k+1}^{-u}\) and \((\delta_1 \boxtimes \ldots \boxtimes \delta_k) \boxtimes \delta_{k+1}^{-u}\). Let \(h : X_1 \times \ldots \times X_{k+1} \rightarrow (X_1 \times \ldots \times X_k) \times X_{k+1}\) defined by \(h(x_1, \ldots, x_{k+1}) := ((x_1, \ldots, x_k), x_{k+1})\). One easily verifies that \(h\) is computable with respect to \((\delta_1 \boxtimes \ldots \boxtimes \delta_k) \boxtimes \delta_{k+1}^{-u}\) and \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)^{-u} \boxtimes \delta_{k+1}^{-u}\) and that the inverse \(h^{-1}\) is computable w.r.t. \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)^{-u} \boxtimes \delta_{k+1}^{-u}\) and \((\delta_1 \boxtimes \ldots \boxtimes \delta_k) \boxtimes \delta_{k+1}^{-u}\). Hence \(h^{-1}\) is computable w.r.t. \((\delta_1 \boxtimes \ldots \boxtimes \delta_k) \boxtimes \delta_{k+1}\) and \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)^{-u} \boxtimes \delta_{k+1}\) by Proposition 4.3.5. This shows that the identity function on \((X_1 \times \ldots \times X_{k+1})\) is computable w.r.t. \((\delta_1^{-u} \boxtimes \ldots \boxtimes \delta_{k+1}^{-u})\) and \((\delta_1 \boxtimes \ldots \boxtimes \delta_{k+1})^{-u}\) which means \((\delta_1^{-u} \boxtimes \ldots \boxtimes \delta_{k+1}^{-u}) \leq \text{cp} \left( \delta_1 \boxtimes \ldots \boxtimes \delta_{k+1} \right)^{-u} \delta_{k+1}^{-u}.

(2) Similar to (3).

(3) Let \(f : X_1 \times \ldots \times X_k \rightarrow X_{k+1}\) be \((\delta_1, \ldots, \delta_{k+1})\)-computable. By Lemma 4.1.9, \(f\) is computable w.r.t. \((\delta_1 \boxtimes \ldots \boxtimes \delta_k)\) and \(\delta_{k+1}\). Hence \(f\) is computable w.r.t.
(δ₁ × ... × δₖ)−u and δ−uₖ₊₁ by Proposition 4.3.5. By (1) and Lemma 2.4.4, f is computable w.r.t. (δ¹−u₁ × ... × δ−uₖ) and δ−uₖ₊₁ which again by Lemma 4.1.9 implies that f is (δ¹−u₁, ..., δ−uₖ₊₁)-computable.

(4) Let \( f : X_1 × \ldots × X_k \to X_{k+1} \) be \( (δ_1, \ldots, δ_{k+1}) \)-continuous. By Proposition 2.4.12, f is a continuous function between the weak limit spaces generated by the multirepresentations \( δ_1, \ldots, δ_{k+1} \). Since \( δ_{1w}, \ldots, δ_{kw} \) are admissible multirepresentations of these spaces by Proposition 4.3.2, f is \( (δ_{1w}, \ldots, δ_{kw}) \)-continuous by Theorem 4.2.45.

Let \( f : X_1 × \ldots × X_k \to X_{k+1} \) be \( (δ_{1w}, \ldots, δ_{kw}) \)-continuous. Then f is sequentially continuous w.r.t. \( \sim_{\delta₁}^{\delta_{1w}}, \sim_{\delta₂}^{\delta_{2w}}, \ldots, \sim_{\deltaₖ}^{\delta_{kw}} \) and \( \sim_{\delta₁}^{\delta_{1w}} \) by Proposition 2.4.12. For every \( i \in \{1, \ldots, k+1\} \), we have \( δₖ \equiv_t (δ_{kw})^c \) by Lemma 4.3.6, hence \( δₖ \) is an admissible multirepresentation of \( (X_{i}, \sim_{\delta_{kw}}) \) by Propositions 4.3.2 and 2.4.21. Theorem 4.2.45 implies that f is \( (δ₁, \ldots, δ_{k+1}) \)-continuous.

Let \( f : X_1 × \ldots × X_k \to X_{k+1} \) be \( (δ₁, \ldots, δ_{k+1}) \)-continuous. By Proposition 2.4.12, f is a sequentially continuous function between the generated topological spaces \( (X_1, τ_{δ₁}), \ldots, (X_{k+1}, τ_{δ_{k+1}}) \). For every \( i \in \{1, \ldots, k+1\} \), we have \( δ_{kw} \equiv_t (δ_{kw})^c \) by Lemma 4.3.6, hence \( δ_{kw} \) is an admissible multirepresentation of \( (X_{i}, τ_{δ_{kw}}) \) by Propositions 4.3.2 and 2.4.21. Theorem 4.2.45 implies that f is \( (δ₁, \ldots, δ_{k+1}) \)-continuous.

(5) Let \( f : X_1 × \ldots × X_k \to X_{k+1} \) be \( (δ_{1w}, \ldots, δ_{kw}) \)-computable. Then f is computable with respect to \( (δ_{1w} × \ldots × δ_{kw}) \) and \( δ_{kw} \) by Proposition 4.3.10, hence \( f ∈ P_{cp}((δ₁ × \ldots × δₖ, δ_{kw})_{k+1}) \) by (1) and Lemma 2.4.4. Corollary 4.3.10 implies f is \( (δ₁, \ldots, δ_{kw}) \)-computable by Lemma 4.1.9. Again by (1) and Lemma 2.4.4, f is computable with respect to \( (δ₁ × \ldots × δₖ) \) and \( δ_{kw} \), hence f is \( (δ₁, \ldots, δ_{kw}) \)-computable by Lemma 4.1.9.

The first statement follows analogously, the third one by using Corollary 4.3.7 instead of Corollary 4.3.10.

From Proposition 4.3.21 it follows that every partial \( (δ_{kw} × \ldots × δ₁) \)-computable function is \( (δ₁ × \ldots × δ_{kw}) \)-computable, whenever \( δ₁, \ldots, δₖ \) are representations that admit relative computability of the inequality test (cf. Subsection 4.3.7).

A further consequence of Proposition 4.3.11 are the following reducibility properties of the operator \( [\to] \) (see also Corollary 4.3.7).

**Proposition 4.3.13**

Let \( δ : \subseteq N^ω \Rightarrow X \) and \( γ : \subseteq N^ω \Rightarrow Y \) be multirepresentations. Let \( g_{\Omega} : \subseteq N^ω \Rightarrow U \) be an admissible multirepresentation of a weak limit space \( Ω = (U, \to_u) \). Then we have

\[
[δ^{−u} \to γ] < [δ \to γ] < [δ^{−u} \to γ^{−u}] \equiv_{cp} [δ \to γ^{−u}]
\]

\[\land \quad \land \quad \equiv_{cp} \equiv_{cp} \equiv_{cp} \equiv_{cp}\]

\[
[δ^{−u} \to γ^{−u}]^{−u} < [δ \to γ]^{−u} < [δ^{−u} \to γ^{−u}]^{−u} \equiv_{cp} [δ \to γ^{−u}]^{−u}
\]

\(\equiv\)
where $[\delta^a \to \gamma^b]^c < [\delta^d \to \gamma^e]^f$ means $[\delta^a \to \gamma^b]^c \leq_{cp} [\delta^d \to \gamma^e]^f$ and that there are multi-
representations $\delta, \gamma, \rho$ with $[\delta^d \to \gamma^e]^f \not\leq_t [\delta^a \to \gamma^b]^c$.

Proof:

(i) At first, we show $[\delta \to \gamma]^{-\mathfrak{U}} \leq_{cp} [\delta^a \to \gamma^b]^c$ The evaluation function $eval : C(\delta, \gamma) \times X \to Y$ is computably realized w.r.t. $[\delta \to \gamma]$, $\delta$, and $\gamma$ by the universal
data function of $\eta$. Proposition 4.3.11 implies that there is a computable function $g_1 : \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ realizing $eval$ w.r.t. $[\delta \to \gamma]^{-\mathfrak{U}}, [\delta^a \to \gamma^b]^c$, and $\gamma^{-\mathfrak{U}}$. The computable smn-Theorem yields a computable function $g_2 : \mathbb{N}^\omega \to \mathbb{N}^\omega$ with $(\gamma p, t \in \mathbb{N}^\omega) \eta_{g_2(t)}(p) = g_1(t, p)$. For all $p, t \in \mathbb{N}^\omega, f \in [\delta \to \gamma]^{-\mathfrak{U}}[t]$, and $x \in [\delta \to \gamma]^{-\mathfrak{U}}[p]$ we have

$$f(x) = eval(f, x) \in [\delta \to \gamma]^{-\mathfrak{U}}[g_1(t, p)] \in [\delta \to \gamma]^{-\mathfrak{U}}[\eta_{g_2(t)}(p)].$$

Thus $g_2(t)$ is a $[\delta \to \gamma]^{-\mathfrak{U}}$-name of $f$. Hence $[\delta \to \gamma]^{-\mathfrak{U}} \leq_{cp} [\delta^a \to \gamma^b]^c$.

From this result we obtain by Lemmas 4.3.6(2) and 4.2.8(4)

$$[\delta \to \gamma]^{-\mathfrak{U}} \leq_{cp} [\delta^a \to \gamma^b]^c \equiv_{cp} [\delta^a \to \gamma^b]^c.$$

The other computable reducibilities follow from the above ones, Lemma 4.3.6, Lemma 4.2.8, and the transitivity of $\leq_{cp}$.

(ii) Let $\gamma : \subseteq \mathbb{N}^\omega \to Y$ be some representation which is not an admissible one of its
generated weak limit space $\mathcal{Q} := (Y, \rightarrow)$ (cf. Example 2.3.10). Then $C(\gamma^m, \gamma) \not\subseteq C(\gamma, \gamma)$ and $C(\gamma^m, \gamma) \not\subseteq C((\gamma^m)^m, \gamma^m)$, because $C(\gamma^m, \gamma)$ does not contain the identity function $id_Y$ on $Y$. This shows

$$[\delta^m \to \gamma^m] \not\leq_t \left\{ [\delta_1 \to \gamma] \land [\delta_2 \to \gamma] \right\} \not\leq_t \left\{ [\delta^m_2 \to \gamma^m] \right\}$$

for $\delta_1 := \gamma$ and $\delta_2 := \gamma^m$.

Let $g_1 : \mathbb{N}^\omega \to \{1\}$ be the admissible representation of the terminal space $1 = \mathcal{W}({\{1\}, \emptyset, \{1\}})$ defined by $g_1(p) := 1$ for all $p \in \mathbb{N}^\omega$.

Assume that there is a continuous function $g : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ translating $[g_1 \to \gamma]^{\mathfrak{W}}$ to $[g_1 \to \gamma]$. By the utm-Theorem and the smn-Theorem there are continuous functions $s_1, s_2, s_3 : \mathbb{N}^\omega \to \mathbb{N}^\omega$ with

$$\eta_{s_1(r)}(q) = r, \quad \eta_{s_2(q)}(r) = \eta_{q}(s_1(r)), \quad \eta_{s_3(p)}(q) = \eta_{p}(s_2(q))$$

for all $p, q, r \in \mathbb{N}^\omega$. Let $y \in Y$ and $h \in C(C(1, \mathcal{Q}), \mathcal{W}e)$. We denote by $\dot{y} : 1 \to Y$ and $\dot{h} : Y \to \mathcal{W}e$ We the functions defined by $\dot{y}(1) := y$ and $\dot{h}(b) := h(b)$ for all $b \in Y$.

Clearly, $\dot{y} \in C(1, \mathcal{Q})$. For $q, r \in \mathbb{N}^\omega$ with $y \in \gamma[r]$ and $\gamma \in [\dot{g}_1 \to \gamma] \to \mathcal{W}e[\eta](\dot{h})(q)$, one easily verifies

$$\dot{y} \in [\dot{g}_1 \to \gamma] \land \dot{h} \in [\gamma \to \mathcal{W}e][s_2(q)],$$

hence $\dot{h} \in C(\mathcal{Q}, \mathcal{W}e)$ by Proposition 2.4.12. For $p, q \in \mathbb{N}^\omega$ with $y \in \gamma^m[p]$ and $h \in [\dot{g}_1 \to \gamma] \to \mathcal{W}e[\eta](\dot{h})(q)$, we obtain

$$e_{C(1, \mathcal{Q}), \mathcal{W}e}(\dot{y})(h) = h(\dot{y}) = \dot{h}(y) = e_{\mathcal{W}e}(\dot{y})(\dot{h}) \in \mathcal{W}e[\eta_{s_2}(q)] = \mathcal{W}e[\eta_{s_3}(p)](q).$$
4.3 Effectivization

This implies \( e_{c(1,q),2}^2 \in (\varrho_1 \to \gamma) \to \varrho_{2m} \to \varrho_{2n} \) and

\[
\dot{y} \in [\varrho_1 \to \gamma]^{\omega} \subseteq [\varrho_1 \to \gamma][g(s_3(p))].
\]

Therefore, \( y \in \gamma[\eta_{\varrho(s_3(p))}(0^\omega)] \). Hence \( p \mapsto \eta_{\varrho(s_3(p))}(0^\omega) \) translates \( \gamma^{\omega} \) continuously to \( \gamma \). From Propositions 4.3.2 and 2.4.21 it follows that \( \gamma \) is an admissible multirepresentation of \( (Y, \rightarrow, \gamma) \), a contradiction.

We obtain \( [\varrho_1 \to \gamma]^{\omega} \subseteq [\varrho_1 \to \gamma] \) as well as \( [\varrho_1^{\omega} \to \gamma]^{\omega} \subseteq [\varrho_1^{\omega} \to \gamma] \), because \( \varrho_1 \equiv \varrho_1^{\omega} \) holds by admissibility of \( \varrho_1 \).

\( \checkmark \)

4.3.4 Effectively Admissible Representations

By Propositions 4.3.2, 2.4.18, and 2.4.21, a multirepresentation \( \delta : \subseteq \mathbb{N}^\omega \Rightarrow X \) is admissible if and only if \( \delta \) is continuously equivalent to \( \delta^{\omega} \). Moreover, \( \delta \) is an admissible multirepresentation of a topological space (of a limit space) if and only if \( \delta \) is continuously equivalent to \( \delta^e \) (to \( \delta^c \)). Hence it is reasonable to define \( \delta \) to be effectively admissible iff \( \delta \) is computably equivalent to one of the multirepresentations \( \delta^{\omega}, \delta^c, \delta^e \).

We say that \( \delta \) effectively topologically admissible iff \( \delta \equiv_{cp} \delta^e \) holds.

We prove the following closure properties of effectively admissible multirepresentations.

Proposition 4.3.14 (Effectively admissible multirepresentations)

Let \( U \in \{ \mathfrak{m}, \mathfrak{z}, \mathfrak{s} \} \). For \( i \in \{1, \ldots, k\} \), let \( \delta_i : \subseteq \mathbb{N}^\omega \Rightarrow X_i \) and \( \gamma : \subseteq \mathbb{N}^\omega \Rightarrow Y \) be multirepresentations with \( \delta_i \equiv_{cp} \delta_i^{\omega} \) and \( \gamma \equiv_{cp} \gamma^{\omega} \).

1. \( (\delta_1 \otimes \ldots \otimes \delta_k) \equiv_{cp} (\delta_1 \otimes \ldots \otimes \delta_k)^{\omega} \).
2. \( (\delta_1 \land \ldots \land \delta_k) \equiv_{cp} (\delta_1 \land \ldots \land \delta_k)^{\omega} \).
3. \( [\delta_1 \to \gamma] \equiv_{cp} [\delta_1 \to \gamma]^{\omega} \).
4. \( (\bigsqcup_{i \in \mathbb{N}} \gamma) \equiv_{cp} (\bigsqcup_{i \in \mathbb{N}} \gamma)^{\omega} \).
5. For every function \( f : X \to Y, \zeta := f^{-1} \circ \gamma \) satisfies \( \zeta \equiv_{cp} \zeta^{\omega} \).
6. For \( M \subseteq Y, \gamma^M \equiv_{cp} (\gamma^M)^{\omega} \).

Proof:

1. By Proposition 4.3.12 and Lemma 4.1.9, we have

\[
(\delta_1 \otimes \ldots \otimes \delta_k)^{\omega} \equiv_{cp} \delta_1^{\omega} \otimes \ldots \otimes \delta_k^{\omega} \equiv_{cp} \delta_1 \otimes \ldots \otimes \delta_k.
\]

2. We define the function \( f : X_1 \cap \ldots \cap X_k \to X_1 \times \ldots \times X_k \) by \( f(z) := (z, \ldots, z) \) for all \( z \in X_1 \cap \ldots \cap X_k \). By the definition of product and conjunction of multirepresentations (cf. Subsection 4.1.4), we have \( (\delta_1 \land \ldots \land \delta_k) = f^{-1} \circ (\delta_1 \otimes \ldots \otimes \delta_k) \).

By (1) and (5) we obtain \( (\delta_1 \land \ldots \land \delta_k) \equiv_{cp} (\delta_1 \land \ldots \land \delta_k)^{\omega} \).
(3) By Proposition 4.3.13 and Lemmas 4.2.8(4), 4.3.6(2) we have

\[ [\delta_1 \to \gamma]^{-\|} \leq \text{cp} [\delta_1 \to \gamma] \equiv \text{cp} [\delta_1 \to \gamma] \leq \text{cp} [\delta_1 \to \gamma]^{-\|}, \]

hence \([\delta \to \gamma] \equiv \text{cp} [\delta_1 \to \gamma]^{-\|}.

(4) We may identify \(\prod_{i \in \mathbb{N}} Y\) with \(C((\mathbb{N}, \to), (Y, \to))\). Hence \([\eta_\mathbb{N} \to \gamma]\) can be viewed as a multirepresentation of \(\prod_{i \in \mathbb{N}} Y\). By Subsections 4.2.3 and 2.1.7, the functions \(g_1, g_2 : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega\) defined by

\[ g_1(p) := \langle \eta_p(0^\omega), \eta_p(1^\omega), \eta_p(2^\omega), \ldots \rangle \quad \text{and} \quad \eta_{g_2(p)}(q) := \pi_{\infty, q(0)}(p) \]

for all \(p, q \in \mathbb{N}^\omega\) are computable. Clearly, \(g_1\) translates \([\eta_\mathbb{N} \to \gamma]\) to \((\mathbb{N}_{i \in \mathbb{N}} \gamma)\) and \(g_2\) translates \((\mathbb{N}_{i \in \mathbb{N}} \gamma)\) to \([\eta_\mathbb{N} \to \gamma]\), hence \((\mathbb{N}_{i \in \mathbb{N}} \gamma) \equiv \text{cp} [\eta_\mathbb{N} \to \gamma]\) (cf. [Wei00, Ex. 3.3.14]). By Statement (3) and Lemma 4.3.6(1), it follows

\[ (\mathbb{N}_{i \in \mathbb{N}} \gamma) \equiv \text{cp} [\eta_\mathbb{N} \to \gamma] \equiv \text{cp} [\eta_\mathbb{N} \to \gamma]^{-\|} \equiv \text{cp} (\mathbb{N}_{i \in \mathbb{N}} \gamma)^{-\|}. \]

(5) Let \(g : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega\) be a computable function translating \(\psi^{-\|}\) to \(\gamma\). We prove that \(g\) also translates \(\psi^{-\|}\) to \(\gamma\). Let \(p \in \mathbb{N}^\omega\) and \(x \in \psi^{-\|}[p]\). Let \(q \in \mathbb{N}^\omega\) and \(h \in [\gamma \to \eta_\mathbb{N}\{q]\]. For every \(r \in \mathbb{N}^\omega\) and \(z \in \psi[r]\) we have \(f(z) \in \gamma[r]\) and thus \(h(f(z)) \in \eta_\mathbb{N}[\eta_q(r)]\), implying \(h \circ f \in [\psi \to \eta_\mathbb{N}[q]\) and \(h \circ f \in C((x, \to, \psi), \Omega)\). It follows

\[ \epsilon_{\psi, \Omega}(f(x))(h) = h \circ f(x) = \epsilon_{(x, \psi), \Omega}(x)(h \circ f) \in \eta_\mathbb{N}[\eta_q(q)]. \]

Therefore \(\epsilon_{\psi, \Omega}(f(x)) \in [\gamma \to \eta_\mathbb{N}\{q]\] and \(f(x) \in \gamma^{-\|}[p]\). This implies \(f(x) \in \gamma[g(p)]\) and \(x \in \psi[g(p)]\). We conclude that \(g\) also translates \(\psi^{-\|}\) computably to \(\psi\), thus \(\psi^{-\|} \leq \text{cp} \psi\). From Lemma 4.3.6 we know \(\psi \leq \text{cp} \psi^{-\|}\), hence \(\psi \equiv \text{cp} \psi^{-\|}\).

(6) This follows from (5), because the injection \(\iota : M \to Y\) of \(M\) into \(Y\) satisfies \(\gamma|^M = \iota^{-1} \circ \gamma\).

\(\square\)

**4.3.5 The Categories \(\text{EffWeakLim}, \text{EffLim}, \text{and EffSeq}\)**

We define \(\text{EffWeakLim} (\text{EffLim}, \text{EffSeq})\) to be the category which has pairs \((X, \delta)\) as objects, where \(X\) is a weak limit space (a limit space, a sequential topological space) and \(\delta : \subseteq \mathbb{N}^\omega \Rightarrow X\) is an admissible multirepresentation of \(X\) with \(\delta \equiv_{\text{cp}} \delta^\Omega (\delta \equiv_{\text{cp}} \delta^\Pi, \delta \equiv_{\text{cp}} \delta^\Sigma)\). The morphisms between two objects \((X, \delta)\) and \((Y, \gamma)\) are the total \((\delta, \gamma)\)-computable functions between the underlying sets of \(X\) and \(Y\). From Lemma 4.3.6(3) we obtain \((\mathbb{W}, \varrho_{\mathbb{W}}) \in \text{EffWeakLim}, (\mathbb{L}, \varrho_{\mathbb{L}}) \in \text{EffLim}, \) and \((\mathbb{S}, \varrho_{\mathbb{S}}) \in \text{EffSeq}\).

Let \((X, \delta)\) and \((Y, \gamma)\) be objects of \(\text{EffWeakLim}\). Proposition 4.3.14 states \(\delta \otimes \gamma \equiv_{\text{cp}} (\delta \otimes \gamma)^\Omega\) and \([\delta \to \gamma] \equiv_{\text{cp}} [\delta \to \gamma]^\Omega\). Moreover, \(\delta \otimes \gamma\) and \([\delta \to \gamma]\) are admissible multirepresentations of, respectively, \(X \otimes Y\) and \(C(X, Y)\) by Propositions 4.1.7 and 4.2.5. Hence \((X \otimes Y, \delta \otimes \gamma)\) and \((C(X, Y), [\delta \to \gamma])\) are objects of \(\text{EffWeakLim}\). Lemmas 4.1.9 imply
that \((X \otimes \mathcal{G}, \delta \boxtimes \gamma)\) together with the projections \(pr_1, pr_2\) (which are computable w.r.t. \(\delta \boxtimes \gamma, \delta,\) and \(\gamma\)) form a categorical product of \(X\) and \(\mathcal{G}\) in \(\text{EffWeakLim}\). From Lemma 4.2.8 it follows that \((C(X, \mathcal{G}), [\delta \rightarrow \gamma])\) and \(\text{eval}\) form an exponent of \((X, \delta)\) and \((\mathcal{G}, \gamma)\) in \(\text{EffWeakLim}\). Similarly, the categories \(\text{EffLim}\) and \(\text{EffSeq}\) can be shown to provide categorical products and exponents. We obtain:

**Theorem 4.3.15**

*The categories \(\text{EffWeakLim}, \text{EffLim}, \text{and EffSeq} *are Cartesian closed.*

We show that \(\text{EffWeakLim}, \text{EffLim}, \text{and EffSeq} *have equalizers for all parallel pairs of morphisms (cf. Subsection 4.1.6). Let \(X = ((X, \rightarrow_{\chi}), \delta), \mathcal{G} = ((Y, \rightarrow_{\eta}), \gamma), \) and \(\mathcal{Z} = ((Z, \rightarrow_{\zeta}), \zeta)\) be objects of \(\text{EffSeq}\), and let \(f, g : X \rightarrow Y\) be \((\delta, \gamma)\)-computable. An equalizer of \((f, g)\) can be constructed as follows: we equip the subset \(M := \{x \in X \mid f(x) = g(x)\}\) with the “sequentialized” subspace topology \(\tau_M := \text{seq}(\tau_{X|M})\). By Subsection 4.1.5, Corollary 2.4.31, and Proposition 4.3.2, \(\delta^{[\mathcal{M}]}\) *is an admissible multi-representations of \((M, \tau_M)\). Proposition 4.3.14 implies that \(((M, \tau_M), \delta^{[\mathcal{M}]})\) *is an object of \(\text{EffSeq}\). Clearly, the injection \(\iota : M \rightarrow X, x \mapsto x\) is \((\delta^{[\mathcal{M}]}, \delta)\)-computable. Moreover, for every \((\zeta, \delta)\)-computable function \(h : Z \rightarrow X\) with \(f \circ h = g \circ h\), the function \(h^{[\mathcal{M}] : Z \rightarrow M}\) is \((\zeta, \delta^{[\mathcal{M}]}\)-computable. Hence \(((M, \tau_M), \delta^{[\mathcal{M}]})\) *together with the injection \(\iota\) forms an equalizer of \((f, g)\) in \(\text{EffSeq}.

Similarly, one can prove that also \(\text{EffWeakLim}\) and \(\text{EffLim}\) *have equalizers for every pair of parallel morphisms. Since these categories are Cartesian, we conclude from [AL91, Theorem 6.3.1] that \(\text{EffWeakLim}, \text{EffLim}, \text{and EffSeq} *have all finite limits."

We define functors \(T_{\text{eff}} : \text{EffWeakLim} \rightarrow \text{EffSeq}\) and \(W_{\text{eff}} : \text{EffSeq} \rightarrow \text{EffWeakLim}\) as follows. Let \((\mathfrak{A}, \zeta)\) and \((\mathfrak{G}, \gamma)\) be objects of \(\text{EffWeakLim}\) and \(f\) be a morphism between them. Let \((\mathfrak{A}, \zeta)\) be an object and \(g\) be a morphism of \(\text{EffSeq}\). Then we define \(T_{\text{eff}}(\mathfrak{A}, \delta) := ((X, \text{seq}(X)), \delta^{[\mathcal{A}]})\), \(T_{\text{eff}}(f) := f, W_{\text{eff}}(\mathfrak{A}, \zeta) := (W(\mathfrak{A}), \zeta)\) and \(W_{\text{eff}}(g) := g.\)

Lemma 2.4.11, 4.3.6 and Propositions 4.3.2, 4.3.5 imply that \((T(X), \delta^{[\mathcal{X}]})\) is actually an object of \(\text{EffSeq}\) and that \(f\) is a morphism between \(T_{\text{eff}}(\mathfrak{A}, \delta)\) and \(T_{\text{eff}}(\mathfrak{G}, \gamma).\) From Lemma 4.3.6 it follows \(\zeta \equiv_{\text{cp}} \zeta^{[\mathcal{A}]},\) \(\equiv_{\text{cp}} \zeta^{[\mathcal{G}]},\) i.e., \((W(\mathfrak{A}), \zeta)\) is an object of \(\text{EffWeakLim}\). Propositions 4.3.12 and 4.3.13 imply that \(T_{\text{eff}}\) preserves finite products and exponents. From Lemma 4.3.6(4) and [AL91, Theorem 5.3.2] one can conclude that \(T_{\text{eff}}\) *is a left–adjoint to \(W_{\text{eff}}\) for the object \((\mathfrak{A}, \delta)\) the unit morphism \(\eta_{(\mathfrak{A}, \delta)} : (\mathfrak{A}, \delta) \rightarrow W_{\text{eff}}(T_{\text{eff}}(\mathfrak{A}, \delta))\) *of the adjunction is simply the \((\delta, \delta^{[\mathcal{A}]})\)-computable identical function on the underlying set of \(\mathfrak{A}.\) Thus \(\text{EffSeq}\) is a full reflective subcategory of \(\text{EffWeakLim}.

Since the image of \(W_{\text{eff}}\) is contained in \(\text{EffLim},\) also the restriction of \(T_{\text{eff}}\) to \(\text{EffLim}\) is a left–adjoint to \(W_{\text{eff}}.\) Hence \(\text{EffSeq}\) is a full reflective subcategory of \(\text{EffLim}\). In a similar way, one can show that \(\text{EffLim}\) is a full reflective subcategory of \(\text{EffWeakLim}\). The corresponding reflection functor \(L_{\text{eff}} : \text{EffWeakLim} \rightarrow \text{EffLim}\) maps any object \((\mathfrak{A}, \delta)\) to \((L(\mathfrak{A}), \delta^{[\mathcal{A}]})\).

**4.3.6 Examples of Effectively Admissible Multirepresentations**

We now give some useful examples for effectively admissible multirepresentations and for objects of the categories \(\text{EffWeakLim}, \text{EffLim}, \text{and EffSeq}.

In [Wei00], the notion of a *computable topological space* is introduced. A triple \(X = (X, \mathcal{B}, \beta)\) is called a *computable topological space* iff \(X\) is a non–empty set, \(\mathcal{B} \cup \{X\}\)
is a countable subbase of a topology on \( X \) and \( \beta : \subseteq \mathbb{N} \to \mathcal{B} \) is a numbering\(^{12}\) of \( \mathcal{B} \) such that

1. \( \{ B \in \mathcal{B} \mid x \in B \} = \{ B \in \mathcal{B} \mid y \in B \} \) implies \( x = y \) for every \( x, y \in X \);
2. \( \{ (u, v) \in \text{dom}(\beta) \times \text{dom}(\beta) \mid \beta(u) = \beta(v) \} \) is recursively enumerable.

Let \( \mathcal{X} = (X, \mathcal{B}, \beta) \) be a computable topological space. We denote by \( \tau_\mathcal{X} \) the topology generated by the subbase \( \mathcal{B} \cup \{ X \} \) and by \( T(\mathcal{X}) \) the countably based and thus sequential topological space \( (X, \tau_\mathcal{X}) \). Property (1) implies that \( \tau_\mathcal{X} \) has the \( T_0 \)-property. We equip the computable topological space \( \mathcal{X} \) with a standard representation \( \rho_\mathcal{X} : \subseteq \mathbb{N}^\omega \to X \). It is defined in a similar way as in [Wei00] by

\[
\rho_\mathcal{X}(p) = x :\iff (\text{En}(p) \subseteq \text{dom}(\beta) \text{ and } \{ \beta(i) \mid i \in \text{En}(p) \} = \{ B \in \mathcal{B} \mid x \in B \})
\]

for all \( p \in \mathbb{N}^\omega \) and \( x \in X \), where \( \text{En}(p) := \{ p(j) - 1 \mid j \in \mathbb{N} \text{, } p(j) > 0 \} \). From [Wei00] we know that \( \rho_\mathcal{X} \) is an admissible representation of the topological space \( (X, \tau_\mathcal{X}) \), i.e. \( \rho_\mathcal{X} \equiv_{\mathcal{E}} (\rho_\mathcal{X})^\mathcal{E} \), hence the final topology of \( \rho_\mathcal{X} \) is equal to the sequential topology \( \tau_\mathcal{X} \). We prove that \( \rho_\mathcal{X} \) is even computably equivalent to the representation \( (\rho_\mathcal{X})^\mathcal{E} \).

**Proposition 4.3.16** (\( \rho_\mathcal{X} \) is effectively admissible)

Let \( \mathcal{X} = (X, \mathcal{B}, \beta) \) be a computable topological space. Then its standard representation \( \rho_\mathcal{X} \) satisfies \( \rho_\mathcal{X} \equiv_{\mathcal{CP}} \rho_\mathcal{X}^\mathcal{E} \equiv_{\mathcal{CP}} \rho_\mathcal{X}^\mathcal{S} \equiv_{\mathcal{CP}} \rho_\mathcal{X}^\mathcal{W} \), thus \( ((X, \tau_\mathcal{X}), \rho_\mathcal{X}) \in \text{EffSeq}, ((X, \neg_\mathcal{X}), \rho_\mathcal{X}) \in \text{EffLim}, \) and \( ((X, \neg_\mathcal{X}), \rho_\mathcal{X}) \in \text{EffWeakLim} \).

**Proof:**

We show \( (\rho_\mathcal{X})^\mathcal{E} \subseteq_{\mathcal{CP}} \rho_\mathcal{X} \). There are computable functions \( l_1, l_2 : \mathbb{N} \to \mathbb{N} \) with

\[
\{ (l_1(j), l_2(j)) \mid j \in \mathbb{N} \} = \{ (u, v) \in \text{dom}(\beta) \times \text{dom}(\beta) \mid \beta(u) = \beta(v) \}.
\]

By the smm-Theorem, there is a computable function \( g_1 : \mathbb{N}^\omega \to \mathbb{N}^\omega \) with

\[
\eta_{g_1(t)}(i, j) = \begin{cases} 1 & \text{if } r(i) - 1 = l_1(j) \text{ and } l_2(j) = t(0) \\ 0 & \text{otherwise} \end{cases}
\]

for all \( r, t \in \mathbb{N}^\omega \) and \( i, j \in \mathbb{N} \). If \( t(0) \in \text{dom}(\beta) \), then \([\rho_\mathcal{X} \to \rho_{\mathcal{E}}]^* (g_1(t))\) is equal to the characteristic function \( \text{cf}_{\beta(t(0))} : X \to \text{SI} \) of \( \beta(t(0)) \) defined by

\[
\text{cf}_{\beta(t(0))}(x) := \begin{cases} \top & \text{if } x \in \beta(t(0)) \\ \bot & \text{otherwise,} \end{cases}
\]

because every \( r \in \text{dom}(\rho_\mathcal{X}) \) satisfies

\[
\text{cf}_{\beta(t(0))}(x) = \top \iff \rho_\mathcal{X}(r) \in \beta(t(0))
\]

\[
\iff (\exists k \in \text{En}(r))(\beta(k) = \beta(t(0)))
\]

\[
\iff (\exists i, j \in \mathbb{N})(r(i) - 1 = l_1(j) \text{ and } l_2(j) = t(0))
\]

\[
\iff \eta_{g_1(t)}(r) \neq 0^2
\]

\[
\iff \rho_{\mathcal{E}}(\eta_{g_1(t)}(r)) = \top.
\]

\(^{12}\)In the original definition in [Wei00], a notation \( \beta : \subseteq \Sigma^* \to \mathcal{B} \) rather than a numbering of \( \mathcal{B} \) is used.
Since the function $g_2 : \subseteq N^\omega \times N^\omega \rightarrow N^\omega$ defined by $g_2(p, t) := \eta_p(g_1(t))$ is computable, there is a computable word function $\sigma : N^* \times N^* \rightarrow N^*$ approximating $g_2$. We define $\text{gcts} : \subseteq N^\omega \rightarrow N^\omega$ by

$$\text{gcts}(p)((i, j)) := \begin{cases} l_1(i) + 1 & \text{if } \sigma(p^{<j}, l_1(i)0^j) \notin \{0\}^* \\ 0 & \text{otherwise} \end{cases}$$

for all $p \in N^\omega$ and $i, j \in N$. Clearly, $\text{gcts}$ is computable. We show that $\text{gcts}$ translates $\rho^e_X$ to $\rho_X$. Let $p \in \text{dom}(\rho^e_X)$ and $x := \rho^e_X(p)$. Obviously, $\text{En}(p) \subseteq \text{dom}(\beta)$. For every $k \in \text{dom}(\beta)$, we have

$$x \in \beta(k) \iff c_{\beta(k)}(x) = \top$$

$$\iff e_{(X, r_X), \xi_i}(x)(c_{\beta(k)}) = \top$$

$$\iff \eta_p(g_{1+}(k::0^\omega)) \neq 0^\omega$$

$$\iff (\exists i, j \in N) (\sigma(p^{<j}, l_1(i)0^j) \notin \{0\}^* \text{ and } k = l_1(i))$$

$$\iff (\exists i, j \in N) \text{gcts}(p)((i, j)) = l_1(i) + 1 = k + 1$$

$$\iff k \in \text{En}(\text{gcts}(p)),$$

because $c_{\beta(k)} \in C((X, r_X), \xi_i)$ and $[\rho_X \rightarrow \rho_{\xi_i} \rightarrow \rho_{\xi_i}](p) = e_{(X, r_X), \xi_i}(x)$. Thus $\text{En}(\text{gcts}(p)) = \{k \in \text{dom}(\beta) \mid x \in \beta(k)\}$. We conclude $\rho_X(\text{gcts}(p)) = x = \rho^e_X(p)$. Hence $\rho^e_X \equiv_{\text{cp}} \rho_X$. By Lemmas 4.3.6(4) and 2.4.4 we obtain $\rho_X \equiv_{\text{cp}} \rho^e_X \equiv_{\text{cp}} \rho^e_X \equiv_{\text{cp}} \rho^e_X$.

With the help of Proposition 4.3.16, one can prove that the admissible representation $\varrho_{\mathbb{R}}$ of the Euclidean space $(\mathbb{R}, \tau_{\mathbb{R}})$ from Example 2.3.8 satisfies $\varrho_{\mathbb{R}} = \rho^e_{\mathbb{R}}$.

**Example 4.3.17 (Computable Euclidean space)**

Similar to [Wei00], we define the computable topological space $\mathfrak{R} := (\mathbb{R}, \text{Cb}, \nu_{\text{Cb}})$ by

$$\text{Cb} := \{(a - r; a + r) \mid a, r \in \mathbb{Q}, r > 0\},$$

$$\nu_{\text{Cb}}((i, j, k, l)) := \begin{cases} \left(\frac{p(i)}{j} - \frac{k_i}{l_i}, \frac{p(i)}{j} + \frac{k_i}{l_i}\right) & \text{div} \\ \text{otherwise} \end{cases} \iff j, k, l > 0 \quad (i, j, k, l \in \mathbb{N}).$$

Clearly, the triple $(\mathbb{R}, \text{Cb}, \nu_{\text{Cb}})$ fulfils the properties of a computable topological space. Similar to [Wei00, Theorem 7.2.1], one can show that the signed digit representation $\rho_{\mathbb{R}}$ from Example 2.3.8 is computably equivalent to the standard representation $\rho_{(\mathbb{R}, \text{Cb}, \nu_{\text{Cb}})}$. Proposition 4.3.16 and Lemma 4.3.6 imply $\rho_{\mathbb{R}} \equiv_{\text{cp}} \rho^e_{\mathbb{R}} \equiv_{\text{cp}} \rho^e_{\mathbb{R}} \equiv_{\text{cp}} \rho^e_{\mathbb{R}}$. Thus $((\mathbb{R}, \tau_{\mathbb{R}}), \varrho_{\mathbb{R}}) \in \text{EffSeq}$, $((\mathbb{R}, \tau_{\mathbb{R}}), \varrho_{\mathbb{R}}) \in \text{EffLim}$, and $((\mathbb{R}, \tau_{\mathbb{R}}), \varrho_{\mathbb{R}}) \in \text{EffWeakLim}$. ☐

Proposition 4.3.16 can be generalized from the Sierpiński–space $\mathfrak{S}_i$ to $\mathfrak{L}_i$ and $\mathfrak{M}_e$.

**Lemma 4.3.18**

Let $\mathfrak{U} = (U, \tau_U) \in \{\mathfrak{M}_e, \mathfrak{L}_i, \mathfrak{W}(\mathfrak{S}_i)\}$. Let $X$ be a set, and let $(h_i)_i$ be a sequence of total functions from $X$ to $U$. Define the multirepresentation $\delta_{(h_i)_i} : \subseteq N^\omega \Rightarrow X$ by

$$x \in \delta_{(h_i)_i}[p] : \iff (\forall i \in N) h_i(x) \in \varrho_{\mathfrak{U}}[\tau_{\mathfrak{U}, i}(p)]$$

for $p \in N^\omega$ and $x \in X$ (cf. Proposition 4.1.1).

Then $\delta_{(h_i)_i} \equiv_{\text{cp}} (\delta_{(h_i)_i})^{-\mathfrak{U}}$. ☐
Proof:
By Lemma 4.3.6(2), it suffices to show \((\delta(h_{i_i}))^{-\Delta} \subseteq \delta(h_{i_i})\). By the computability of \((t,r) \rightarrow \pi_{\omega,t(0)}(r)\) (cf. Subsection 2.1.7) and by the computable smm-Theorem, there is a computable function \(g_1 : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega\) with \((\forall r,t \in \mathbb{N}^\omega) \eta_{\delta_1(t)}(r) = \pi_{\omega,t(0)}(r)\). Then \(h_k \in [\delta(h_{i_i}) \rightarrow \varphi_U][g_1(k::0^\omega)]\) for all \(k \in \mathbb{N}\), because \(\pi_{\omega,k}\) realizes \(h_k\) w.r.t. \(\delta(h_{i_i})\) and \(\varphi_U\). We define \(g_2 : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega\) by
\[
g_2(p) := \langle \eta_p(g_1(00^\omega)), \eta_p(g_1(10^\omega)), \eta_p(g_1(20^\omega)), \ldots \rangle
\]
for all \(p \in \mathbb{N}^\omega\). Clearly, \(g_2\) is computable. Let \(p \in \mathbb{N}^\omega\), \(x \in (\delta(h_{i_i}))^{-\Delta}[p]\), and \(k \in \mathbb{N}\). Since \(h_k \in \mathcal{C}((X, \rightarrow \delta(h_{i_i}), \Delta))\), we have
\[
h_k(x) = e_i(X, \rightarrow \delta(h_{i_i}), \Delta)(x)(h_k) \in \varphi_U[\eta_p(g_1(k::0^\omega))] = \varphi_U[\pi_{\omega,k}(g_2(p))].
\]
This implies \(x \in \delta(h_{i_i})[g_2(p)]\) and \((\delta(h_{i_i}))^{-\Delta} \subseteq \delta(h_{i_i})\).
\[
\square
\]

With the help of the construction in Lemma 4.3.18, we can verify that the decimal representation \(\rho_{10}\) from Example 2.1.2 is an effectively admissible representation of its generated weak limit space \((\mathbb{R}, \rightarrow \rho_{10})\), which is not equal to the Euclidean limit space \((\mathbb{R}, \rightarrow \tau_{\mathbb{R}})\), because \(\rho_{10}\) is not admissible w.r.t. the Euclidean topology \(\tau_{\mathbb{R}}\) (cf. Example 2.2.5).

Example 4.3.19 (Effective admissibility of the decimal representation)
For all \(m,n \in \mathbb{N}\) we define the functions \(h_{2(m,n)} : h_{2(m,n)+1} : \mathbb{R} \rightarrow \mathbb{N}^\omega\) by
\[
h_{2(m,n)}(x) := \begin{cases} \top & \text{if } x = \frac{\nu_2(m)}{10^{m+n}} \\ \{0\} & \text{if } x \in (\frac{\nu_2(m)-1}{10^{m+n}}; \frac{\nu_2(m)}{10^{m+n}}) \\ \{5\} & \text{if } x \in (\frac{\nu_2(m)}{10^{m+n}}; \frac{\nu_2(m)+1}{10^{m+n}}) \\ \bot & \text{otherwise} \end{cases}
\]
and
\[
h_{2(m,n)+1}(x) := \begin{cases} \top & \text{if } x \in (\frac{\nu_2(m)}{10^{m+n}}; \frac{\nu_2(m)+1}{10^{m+n}}) \\ \{0\} & \text{if } x = \frac{\nu_2(m)+1}{10^{m+n}} \\ \{5\} & \text{if } x = \frac{\nu_2(m)+1}{10^{m+n}} \\ \bot & \text{otherwise} \end{cases}
\]
for all \(x \in \mathbb{R}\). Then the representation \(\delta(h_{i_i}) : \mathbb{N}^\omega \rightarrow X\) defined by
\[
\delta(h_{i_i})(p) := x : \iff \langle \forall i \in \mathbb{N}\rangle h_i(x) \in \varphi_M[p]
\]
turns out to be computably equivalent to \(\rho_{10}\). We give a sketch of the proof.

\(\delta(h_{i_i}) \subseteq \rho_{10}\): Let \(p \in \text{dom}(\delta(h_{i_i}))\) and \(x := \delta(h_{i_i})(p)\). For every \(n \in \mathbb{N}\) there is at least one \(m \in \mathbb{N}\) such that \(h_{2(m,n)}(x)\) or \(h_{2(m,n)+1}(x)\) is equal to \(\top\), thus \(\pi_{\omega,2(m,n)}(p) \neq 0^\omega\) or \(\pi_{\omega,2(m,n)+1}(p) \neq 0^\omega\). Since for all \(i \in \mathbb{N}\) and \(a \geq 1\) the sets \(h_{2(m,n)}^{-1}[\varphi_M[a0^\omega]]\) and \(h_{2(m,n)+1}^{-1}[\varphi_M[a0^\omega]]\) are subsets of \([\frac{\nu_2(m)-1}{10^{m+n}}; \frac{\nu_2(m)}{10^{m+n}}]\) or of \([\frac{\nu_2(m)}{10^{m+n}}; \frac{\nu_2(m)+1}{10^{m+n}}]\), one...
can compute from some finite prefix of $p$ some integer $z \in \mathbb{Z}$ satisfying $x \in [\frac{z}{10^n}; \frac{z+1}{10^n}]$. From $z$ the $n$–th digit to the right of the decimal point of a $\rho_{10}$–name of $x$ can be deduced. We conclude $\delta_{(h_i)} \leq_{cp} \rho_{10}$.

$\rho_{10} \leq_{cp} \delta_{(h_i)}$: Let $p \in \text{dom}(\rho_{10})$ and $x := \rho_{10}(p)$. Let $m, n \in \mathbb{N}$. From the prefix of $p$ up to the $(n+1)$–st digit to the right of the decimal point, one can compute some integer $z \in \mathbb{Z}$ satisfying $x \in [\frac{z}{10^n}; \frac{z+1}{10^n}]$. Then

\[
q_{2(m,n)} := \begin{cases} 
10^\omega & \text{if } \frac{z+1}{10^{n+1}} = \nu_2(m) / 10^n \\
60^\omega & \text{if } \frac{z}{10^{n+1}} = \nu_2(m) \\
0^\omega & \text{otherwise}
\end{cases}
\]

is a $\varrho_{m\kappa}$–name of $h_{2(m,n)}(x)$ and

\[
q_{2(m,n)+1} := \begin{cases} 
10^\omega & \text{if } \left[\frac{z}{10^{n+1}}; \frac{z+1}{10^{n+1}}\right] \subseteq \left[\frac{\nu_2(m)}{10^n}; \frac{\nu_2(m)+0.5}{10^n}\right] \\
60^\omega & \text{if } \left[\frac{z}{10^{n+1}}; \frac{z+1}{10^{n+1}}\right] \subseteq \left[\frac{\nu_2(m)+0.5}{10^n}; \frac{\nu_2(m)+1}{10^n}\right] \\
0^\omega & \text{otherwise}
\end{cases}
\]

is a $\varrho_{m\kappa}$–name of $h_{2(m,n)+1}(x)$. Hence $\delta_{(h_i)}, \langle q_0, q_1, \ldots \rangle = x$. As the function $p \mapsto \langle q_0, q_1, \ldots \rangle$ can be shown to be computable, it follows $\rho_{10} \leq_{cp} \delta_{(h_i)}$.

From Propositions 4.1.1, 2.4.21 and Lemma 2.4.11(4) we conclude that $\rho_{10}$ is an admissible representation of the weak limit space $(\mathbb{R}, \to_{\rho_{10}})$. Furthermore $\rho_{10}$ is computably equivalent to $(\rho_{10} \circ \varrho_{\Sigma_{dec}})^\omega$ by Lemma 4.3.18.

4.3.7 An Effective Version of Kisyński’s Theorem

A theorem by J. Kisyński (cf. [Kis60]) states that every limit space $\mathfrak{X} = (X, \to_{\mathfrak{X}})$ which satisfies Axiom (L0)\footnote{i.e., every sequence has at most one limit, cf. Subsection 2.2.2} is topological, i.e., $\to_{\mathfrak{X}}$ is induced by its associated topology $\text{seq}(\to_{\mathfrak{X}})$ (cf. Subsection 2.2.6). We will now prove an effective version of this theorem.

For this purpose we need an effectivization of Axiom (L0). The idea is provided by the following lemma about relative continuity of the inequality test $\not\equiv_X$ on a set $X$. The inequality test $\not\equiv_X: X \times X \to \{\top, \bot\}$ is defined by

$$
\not\equiv_X(x, y) := \begin{cases} 
\top & \text{if } x \neq y \\
\bot & \text{otherwise}
\end{cases}
$$

for all $x, y \in X$.

**Lemma 4.3.20 (Axiom (L0) versus relative continuity of $\not\equiv_X$)**

Let $\delta : \subseteq \mathbb{N}^\omega \rightarrow X$ be a multipresentation. Then the weak limit space $(X, \to_{\delta})$ generated by $\delta$ satisfies Axiom (L0) if and only if the inequality test $\not\equiv_X$ on $X$ is $(\delta, \delta, \varrho_{\Sigma})$–continuous.
Proof: We define \( g : \mathbb{N}^\omega \times \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) by
\[
g(p, q)(n) := \begin{cases} 
1 & \text{if } \delta[p^{<n}\mathbb{N}^\omega] \cap \delta[q^{<n}\mathbb{N}^\omega] = \emptyset \\
0 & \text{otherwise}
\end{cases}
\]
for all \( p, q \in \mathbb{N}^\omega \). Since \( g(p, q)(n) \) depends only on finite prefixes of \( p \) and \( q \), namely on \( p^{<n} \) and \( q^{<n} \), \( g \) is continuous.

We prove that \( g \) realizes \( \not\equiv_X \) w.r.t. \( \delta \) and \( \varrho_{\mathfrak{S}i} \). Let \( p, q, x \in \mathbb{N}^\omega \), \( x \in \delta[p] \), and \( y \in \delta[q] \). If \( g(p, q) \neq 0^\omega \), then there is some \( n \in \mathbb{N} \) with \( \delta[p^{<n}\mathbb{N}^\omega] \cap \delta[q^{<n}\mathbb{N}^\omega] = \emptyset \). This implies \( x \neq y \). Otherwise, for every \( n \in \mathbb{N} \) there are \( p_n \in p^{<n}\mathbb{N}^\omega \), \( q_n \in q^{<n}\mathbb{N}^\omega \) and \( z_n \in \delta[p_n] \cap \delta[q_n] \). As \( (p_n)_n \) converges to \( p \) and \( (q_n)_n \) converges to \( q \), \( (z_n)_n \) converges in \( (X, \rightarrow) \) both to \( x \) and to \( y \). Hence \( x = y \), because in \( (X, \rightarrow) \) every sequence has at most one limit.

We conclude that \( \not\equiv_X \) is \((\delta, \delta, \varrho_{\mathfrak{S}i})\)-continuous.

Proof: Let \( (z_n)_n \) be a converging sequence in \( (X, \rightarrow) \), and let \( x, y \in X \) be limits of \( (z_n)_n \). By Proposition 2.4.27, the function \( \not\equiv_X \) is \((\rightarrow, \rightarrow, \rightarrow)\)-continuous, because \( \varrho_{\mathfrak{S}i} \) is an admissible representation of \( \mathfrak{S}i \). Since the only limit of the sequence \( \not\equiv_X(z_n, z_n)_n = (\perp)_n \) in \( \mathfrak{S}i \) is \( \perp \), we have \( \not\equiv_X(x, y) = \perp \), hence \( x = y \). Therefore \( (X, \rightarrow) \) satisfies Axiom (L0).

Of course, if a multirepresentation \( \delta \subseteq \mathbb{N}^\omega \rightarrow X \) admits relative continuity of the inequality test \( \not\equiv_X \), then \( \delta \) (as well as \( \delta^\omega \), \( \delta^\omega \), \( \delta^\omega \) by Proposition 4.3.12 and Lemma 4.3.20) is single–valued.

Now we can formulate our effective version of Kisyński’s Theorem.

Proposition 4.3.21 (Effective Theorem by Kisyński)
Let \( \delta \subseteq \mathbb{N}^\omega \rightarrow X \) be a representation of \( X \) such that the inequality test \( \not\equiv_X : X \times X \rightarrow \mathfrak{S}i \) on \( X \) is \((\delta, \delta, \varrho_{\mathfrak{S}i})\)-computable. Then \( \delta^\omega \equiv_{cp} \delta^\omega \).

Proof: From Lemma 4.3.6 we know \( \delta^\omega \leq_{cp} \delta^\omega \).

Let \( X := (X, \rightarrow) \) be the weak limit space generated by \( \delta \). Proposition 4.3.11 and Lemmas 4.3.6, 2.4.4 imply that \( \not\equiv_X \) is \((\delta^\omega, \delta, \varrho_{\mathfrak{S}i})\)-computable, i.e., there is a computable function \( g_1 : \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) with \( \varrho_{\mathfrak{S}i}(g_1(p, r)) = \not\equiv_X(\delta^\omega(p), \delta(r)) \) for all \( p \in \text{dom}(\delta^\omega) \) and \( r \in \text{dom}(\delta) \). Let \( g_{\text{tot}} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) be the computable function from Equation (4.13) on page 127. By the utm–Theorem and the smn–Theorem, there are computable functions \( g_2 : \mathbb{N}^\omega \times \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) and \( g_{\mathfrak{K}is} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) with
\[
\eta_{g_2(p, q)}(r) = \langle g_1(p, r), \eta_\delta(r) \rangle \quad \text{and} \quad \eta_{g_{\mathfrak{K}is}(p)}(q) = \eta_{g_{\text{tot}}(p)}(g_2(p, q))
\]
for all \( p, q, r \in \mathbb{N}^\omega \).

We show that \( g_{\mathfrak{K}is} \) translates \( \delta^\omega \) to \( \delta^\omega \). Let \( p, q, x := \delta^\omega(p) \) and \( h \in [\delta \rightarrow \varrho_{\mathfrak{S}i}][q] \). If \( h(x) = \top \), then \( \varepsilon_{x, \mathfrak{S}i}(x)(h) = \top \in \varrho_{\mathfrak{S}i}[\eta_{g_{\mathfrak{K}is}(p)}(q)] \), because \( \eta_{g_{\text{tot}}(p)} \) and \( g_2 \) are total functions and every element of \( \mathbb{N}^\omega \) is a \( \varrho_{\mathfrak{S}i} \)-name of \( \top \).

If \( h(x) \neq \top \), then we can define the function \( \hat{h} : X \rightarrow \mathfrak{S}i \) by
\[
\hat{h}(y) := \begin{cases} 
 h(x) & \text{if } x = y \\
 \top & \text{otherwise.}
\end{cases}
\]
4.4 Multirepresentations of Hyperspaces

for all \( y \in X \). For all \( r \in \text{dom}(\delta) \) we have \( \hat{h}(\delta(r)) = \varrho_{\mathcal{Ei}}(\eta_{g_{2}(p,q)}(r)) \), because

\[
\eta_{g_{2}(p,q)}(r) = 0^\omega \iff g_{1}(p,r) = \eta_{p}(r) = 0^\omega \iff (x = \delta(r) \land h(x) = \bot).
\]

This implies \( \hat{h} = [\delta \to \varrho_{\mathcal{Ei}}](g_{2}(p,q)) \), hence \( \hat{h} \in C(X, \mathcal{Si}) \). By Equation (4.14) on page 127, we obtain

\[
e_{X,\mathcal{L}i}(x)(\hat{h}) = h(x) = \hat{h}(x) = e_{X,\mathcal{Si}}(x)(\hat{h}) = \varrho_{\mathcal{Si}}(\eta_{p}(g_{2}(p,q))) \\
\in \varrho_{\mathcal{L}}[\eta_{p}(g_{2}(p,q))] = \varrho_{\mathcal{L}}[\eta_{g_{\text{tot}}(p)}(g_{2}(p,q))] = \varrho_{\mathcal{L}}[\eta_{g_{\text{Kis}}(p)}(q)].
\]

This implies \( e_{X,\mathcal{L}i}(x) \in \left( [\delta \to \varrho_{\mathcal{Ei}}] \right) [g_{\text{Kis}}(p)] \), thus \( x \in \delta^\omega [g_{\text{Kis}}(p)] \). Therefore \( g_{\text{Kis}} \) translates \( \delta^\omega \) computably to \( \delta^\omega \).

As a consequence of Proposition 4.3.21 we obtain that for any object \((X, \delta)\) in \( \text{EffLim} \) such that the inequality test is \((\delta, \delta, \varrho_{\mathcal{Ei}})\)-computable the pair \((T(X), \delta)\) is an object of \( \text{EffSeq} \).

4.4 Multirepresentations of Hyperspaces

In this section we construct some admissible (multi–) representations of classes of sub–sets of admissibly represented spaces. These constructions generalize some representations in [BrWei99, Wei00, BrPr01] for the closed subsets, for the open subsets, and for the compact subsets of the Euclidean space.

Subsection 4.4.1 is devoted to the admissible Sierpiński representations of the sequentially open subsets and of the sequentially closed subsets of weak limit spaces. In Subsection 4.4.2 we define at first an admissible representation of the closed sets of sequential spaces yielding “positive” information and construct then a representation that is admissible w.r.t. to the Fell topology on closed sets. In Subsection 4.4.3 we define various admissible multirepresentations of the sequentially compact subsets of limit spaces and of the compact subsets of sequential spaces. One of them is admissible w.r.t. the Vietoris topology on compact sets. In Subsection 4.4.4 we present an example of a naturally defined weak limit space which is not a limit space. Its underlying set is the family of the regularly closed subsets of real numbers and its convergence relation is induced by a useful representation of regularly closed sets.

4.4.1 The Sierpiński Representations of Sequentially Open and Sequentially Closed Subsets

In this subsection we construct admissible representations of the family of all sequentially open subsets and the family of all sequentially closed subsets of an admissibly represented weak limit space. We show that in the case of the Euclidean space the corresponding representation of all closed sets is computably equivalent to the representations \( \psi_{\geq} \) considered in [Wei00]. In Proposition 4.4.2 we give an effectivization of topological continuity for functions between sequential spaces.
Let $\mathfrak{X} = (X, \rightarrow)$ be a weak limit space. It is easy to see that the sequentially open subsets of $\mathfrak{X}$ are in one–to–one correspondence with the sequentially continuous functions from $\mathfrak{X}$ to the Sierpiński space $\mathfrak{S}_i$ (cf. Example 2.3.7), because every subset $M \subseteq X$ satisfies

$$M \in \text{seq}(\mathfrak{X}) \iff cf_M \in \mathcal{C}(\mathfrak{X}, \mathfrak{S}_i).$$

Here $cf_M$ denotes the characteristic function $cf_M : X \to \mathfrak{S}_i$ of $M$ defined by

$$cf_M(x) := \begin{cases} \top & \text{if } x \in M \\ \bot & \text{otherwise.} \end{cases}$$

Correspondingly, a subset $A$ is sequentially closed in $\mathfrak{X}$ if and only if $cf_{X \setminus A}$ is sequentially continuous. These equivalences give rise to two straightforward representations of $\text{seq}(\mathfrak{X})$ and $\text{sclo}(\mathfrak{X})$ obtained from a multirepresentation of $\mathfrak{X}$ inducing $\text{seq}(\mathfrak{X})$ as its final topology.

Let $\delta : \subseteq N^\omega \Rightarrow X$ be a multirepresentation such that the final topology $\tau_\delta$ of $\delta$ is equal to the topology $\text{seq}(\mathfrak{X})$ of sequentially open sets of $\mathfrak{X}$ (cf. Subsections 2.2.5 and 2.4.5). By using the admissible representation $q_{\mathfrak{S}_i}$ of $\mathfrak{S}_i$ (cf. Example 2.3.7), we define the Sierpiński representations $\delta^\ominus : \subseteq N^\omega \to \text{seq}(\mathfrak{X})$ and $\delta^\oplus : \subseteq N^\omega \to \text{sclo}(\mathfrak{X})$ by

$$\delta^\ominus(q) = O :\iff [\delta \to q_{\mathfrak{S}_i}](q) = cf_O,$$

$$\delta^\oplus(q) = A :\iff [\delta \to q_{\mathfrak{S}_i}](q) = cf_{X \setminus A}$$

for all $q \in N^\omega$, $O \in \text{seq}(\mathfrak{X})$, and $A \in \text{sclo}(\mathfrak{X})$. Since $[\delta \to q_{\mathfrak{S}_i}]$ is a representation of $\mathcal{C}(T(\mathfrak{X}), \mathfrak{S}_i) = \mathcal{C}(\mathfrak{X}, \mathfrak{S}_i)$ by Proposition 4.2.5 and Lemma 4.2.4, $\delta^\ominus$ and $\delta^\oplus$ are actually representations. Moreover, $\delta^\ominus$ and $\delta^\oplus$ are single–valued, even if $\delta$ is not single–valued, because $q_{\mathfrak{S}_i}$ and thus $[\delta \to q_{\mathfrak{S}_i}]$ are single–valued and $M \mapsto cf_M$ is injective.

For every $q \in \text{dom}(\delta^\ominus) = \text{dom}(\delta^\oplus)$, $r \in N^\omega$, and $x \in \delta[r]$ we have

$$x \in \delta^\ominus(q) \iff x \notin \delta^\oplus(q) \iff \eta_r(r) \neq 0^\omega.$$

Hence we obtain from a finite prefix of any $\delta$–name $r$ of any element $x \in \delta^\ominus(q)$ the information that $x$ is contained in $\delta^\ominus(q)$ and that $x$ is not contained in $\delta^\oplus(q)$. Therefore the representation $\delta^\ominus$ yields “positive” information about open sets, whereas $\delta^\oplus$ supplies “negative” information about closed sets. This fact motivates the above notations $\delta^\ominus$ and $\delta^\oplus$.

We equip the sets $\text{seq}(\mathfrak{X})$ and $\text{sclo}(\mathfrak{X})$ with topologies $\tau^\ominus_\mathfrak{X} \subseteq 2^{\text{seq}(\mathfrak{X})}$ and $\tau^\oplus_\mathfrak{X} \subseteq 2^{\text{sclo}(\mathfrak{X})}$. They are defined by having, respectively, the sets

$$\mathcal{S}^\ominus_\mathfrak{X} := \{ \{ O \in \text{seq}(\mathfrak{X}) \mid K \subseteq O \} \mid K \text{ is compact in } (X, \text{seq}(\mathfrak{X})) \},$$

$$\mathcal{S}^\oplus_\mathfrak{X} := \{ \{ A \in \text{sclo}(\mathfrak{X}) \mid K \cap A = \emptyset \} \mid K \text{ is compact in } (X, \text{seq}(\mathfrak{X})) \}$$

as their subbases. In the case that $\mathfrak{X}$ is a sequential space, $\tau^\ominus_\mathfrak{X}$ is the upper Fell topology on $\text{sclo}(\mathfrak{X})$ which has

$$\{ \{ A \in \text{sclo}(\mathfrak{X}) \mid K \cap A = \emptyset \} \mid K \text{ is compact in } \mathfrak{X} \}$$

as its subbase (cf. [Bee93, BrPr01]).

We prove that the representations $\delta^\ominus$ and $\delta^\oplus$ are admissible w.r.t. the topologies $\tau^\ominus_\mathfrak{X}$ and $\tau^\oplus_\mathfrak{X}$, respectively.
Proposition 4.4.1 (Admissibility of $\delta^\oplus$ and $\delta^\ominus$)

Let $\delta, \subseteq \mathbb{N}^\omega \Rightarrow X$ be a multirepresentation of a weak limit space $X$ with $\tau_\delta = \text{seq}(X)$. Then $\delta^\oplus$ is an admissible representation of the topological space $(\text{seq}(X), \tau_X)$ and $\delta^\ominus$ is an admissible one of $(\text{sclo}(X), \tau_X)$. Moreover, the pairs $14$ $(\text{seq}(X), \tau_X)$, $\delta^\oplus$ and $(\text{sclo}(X), \tau_X)$, $\delta^\ominus$ are objects of the category EffSeq.

Proof:

By Proposition 4.2.5 and Lemma 4.2.4, $[\delta \rightarrow g_{\text{si}}]$ is admissible w.r.t. the compact open topology $\tau_X^\text{co}_{\text{si}}$ on $C(X, \text{si})$. A subbase of $\tau_X^\text{co}_{\text{si}}$ is

$$S := \{ \{ h \in C(X, \text{si}) \mid h[K] \subseteq \{ \top \} \} \mid K \text{ is compact in } X \}.$$ 

For all subsets $F \subseteq C(X, \text{si})$ we have

$$F \in S \iff \{ O \in \text{seq}(X) \mid \text{cf}_O \in F \} \in S_X^\oplus \iff \{ A \in \text{sclo}(X) \mid \text{cf}_{X \setminus A} \in F \} \in S_X^\ominus.$$ 

This implies that $\tau_X^\oplus$ is the coarsest topology on $\text{seq}(X)$ for which $O \mapsto \text{cf}_O$ is topologically continuous, and that $\tau_X^\ominus$ is the coarsest topology on $\text{sclo}(X)$ for which $A \mapsto \text{cf}_{X \setminus A}$ topologically continuous (w.r.t. the topology $\tau_X^\text{co}_{\text{si}}$ on $C(X, \text{si})$). Thus $\delta^\oplus$ is an admissible representation of $(\text{seq}(X), \tau_X^\oplus)$ and $\delta^\ominus$ is an admissible one of $(\text{sclo}(X), \tau_X^\ominus)$ by Proposition 4.1.4.

By Corollary 2.4.31, $\delta^\oplus$ and $\delta^\ominus$ are also admissible representations of the sequential spaces $(\text{seq}(X), \tau_X^\oplus)$ and $(\text{sclo}(X), \tau_X^\ominus)$, respectively. Proposition 4.3.14(3),(5) and Lemma 4.3.6(3) imply $\delta^\oplus \equiv_{cp} (\delta^\ominus)^c$ and $\delta^\ominus \equiv_{cp} (\delta^\oplus)^c$. Therefore the pairs $((\text{seq}(X), \text{seq}(\tau_X^\oplus)), \delta^\oplus)$ and $((\text{sclo}(X), \text{seq}(\tau_X^\ominus)), \delta^\ominus)$ are objects of EffSeq.

The next proposition can be regarded as an effectivization of topological continuity. In the case of total functions it generalizes Proposition 25 in [Spr01] for functions between computable topological spaces.

Proposition 4.4.2 (Effective topological continuity I)

Let $X = ((X, \tau_X), \delta)$ and $Y = ((Y, \tau_Y), \gamma)$ be two objects of EffSeq. Then a total function $f : X \rightarrow Y$ is $(\delta, \gamma)$-computable if and only if the function $F : \tau_Y \rightarrow \tau_X$ defined by $F(O) := f^{-1}[O]$ is $(\gamma^\oplus, \delta^\ominus)$-computable.

Proof:

"$\Rightarrow$" Let $f$ be $(\delta, \gamma)$-computable. Then $[\gamma \rightarrow g_{\text{si}}]$ and $[\delta \rightarrow g_{\text{si}}]$. Let $q \in \text{dom}(\gamma^\oplus)$ and $O = \gamma^\oplus(q)$. Then $[\gamma \rightarrow g_{\text{si}}](q) = \text{cf}_O$. For all $r \in \mathbb{N}^\omega$ and $x \in \delta[r]$ we have

$$\text{cf}_{F(O)}(x) = \begin{cases} \top & \text{if } x \notin f^{-1}[O] \\ \bot & \text{otherwise} \end{cases} = \begin{cases} \top & \text{if } f(x) \in O \\ \bot & \text{otherwise} \end{cases} = \text{cf}_O(f(x)) = (\text{H}_{\text{sclo}}(f)(\text{cf}_O))(x) = [\delta \rightarrow g_{\text{si}}](g(q))(x).$$

Thus $\delta^\ominus(g(q)) = F(O)$. We conclude that $g$ realizes $F$ w.r.t. $\gamma^\ominus$ and $\delta^\oplus$.

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14One can prove that $\text{seq}(\tau_X^\ominus)$ is the Scott-topology on the cpo $(\text{seq}(X), \subseteq)$. Moreover, if $B$ is a countable pseudobase of $(X, \text{seq}(X))$, then the family $\{ (O \in \text{seq}(X) \mid B \subseteq O) \mid B \in B \} \cup \{ \text{seq}(X) \}$ can be shown to be a pseudobase of $(\text{seq}(X), \text{seq}(\tau_X^\ominus))$. 
Let \( g_1 : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega \) be a computable function realizing \( F \) w.r.t. \( \gamma^\oplus \) and \( \delta^\oplus \).

By the utm-Theorem and the computable sum-Theorem, there is a computable function \( g_2 : \mathbb{N}^\omega \to \mathbb{N}^\omega \) with \( (\forall p, q \in \mathbb{N}^\omega) \eta_{g_2(p)}(q) = \eta_{p}(g_1(q)) \).

Let \( p \in \mathbb{N}^\omega \) and \( x \in \delta^\oplus(p) \). Then \( \epsilon_{\mathcal{E}_1}(x) = \lfloor [\delta \to \epsilon_{\mathcal{E}_1}] \to \epsilon_{\mathcal{E}_1}(p) \rfloor \). Moreover, let \( q \in \text{dom}(\lfloor \gamma \to \epsilon_{\mathcal{E}_1} \rfloor) \), \( h := [\gamma \to \epsilon_{\mathcal{E}_1}](q) \), and \( O := \gamma^\oplus(q) = h^{-1}[\top] \). For every \( y \in X \) we have

\[
\begin{align*}
\text{cf}_{f^{-1}[O]}(y) &= \begin{cases} \top & \text{if } y \in F(O) \\ \bot & \text{otherwise} \end{cases} = \begin{cases} \top & \text{if } y \in \delta^\oplus(g_1(q)) \\ \bot & \text{otherwise} \end{cases} \\
&= [\delta \to \epsilon_{\mathcal{E}_1}](g_1(q))(y).
\end{align*}
\]

Thus \( g_1(q) \) is a \([\delta \to \epsilon_{\mathcal{E}_1}]\)-name of \( \text{cf}_{f^{-1}[O]} \). We obtain

\[
\begin{align*}
\epsilon_{\mathcal{E}_1}(f(x))(h) &= h(f(x)) = \begin{cases} \top & \text{if } x \in O \\ \bot & \text{otherwise} \end{cases} \quad = \begin{cases} \top & \text{if } x \in f^{-1}[O] \\ \bot & \text{otherwise} \end{cases} \\
&= \text{cf}_{f^{-1}[O]}(x) = \epsilon_{\mathcal{E}_1}(x)(\text{cf}_{f^{-1}[O]}) = \epsilon_{\mathcal{E}_1}(\eta_p(g_1(q))) = \epsilon_{\mathcal{E}_1}(\eta_{g_2(p)}(q)) .
\end{align*}
\]

This implies \( [\gamma \to \epsilon_{\mathcal{E}_1}] \to \epsilon_{\mathcal{E}_1} \) \( (g_2(p) = \epsilon_{\mathcal{E}_1}(f(x)) \) and \( f(x) \in \gamma^\oplus(g_2(p)) \). We conclude that \( f \) is \( (\delta^\oplus, \gamma^\oplus) \)-computable. Since \( \gamma^\oplus \leq_{cp} \gamma \) (by Definition of EffSeq) and \( \delta \leq_{cp} \delta^\oplus \) (cf. Lemma 4.3.6), \( f \) is computable w.r.t. \( \delta \) and \( \gamma \) by Lemma 2.4.4.

\[ \checkmark \]

Let \( \mathcal{X} = (X, \mathcal{B}, \beta) \) be a computable topological space (cf. Subsection 4.3.6). We define the families \( \mathcal{B}^\oplus \) and \( \mathcal{B}^\ominus \) of subsets of \( X \) by

\[
\begin{align*}
\mathcal{B}^\ominus &:= \{ B_1 \cap \ldots \cap B_k \mid k \geq 1, \{B_1, \ldots, B_k\} \subseteq \mathcal{B} \} \cup \{ X \} , \\
\mathcal{B}^\ominus &:= \{ B_1 \cup \ldots \cup B_k \mid k \geq 1, \{B_1, \ldots, B_k\} \subseteq \mathcal{B} \} \cup \{ \emptyset \} ,
\end{align*}
\]

and the numberings \( \beta^\ominus : \subseteq \mathbb{N} \to \mathcal{B}^\ominus \) and \( \beta^\ominus : \subseteq \mathbb{N} \to \mathcal{B}^\ominus \) by

\[
\begin{align*}
\beta^\ominus(a) &:= \bigcap \{ X, \beta(j) \mid j \in \nu_{\mathcal{F}N}(a) \} \quad \text{if } \nu_{\mathcal{F}N}(a) \subseteq \text{dom}(\beta) \\
&\quad \text{div} \quad \text{otherwise}, \\
\beta^\ominus(a) &:= \bigcup \{0, \beta(j) \mid j \in \nu_{\mathcal{F}N}(a) \} \quad \text{if } \nu_{\mathcal{F}N}(a) \subseteq \text{dom}(\beta) \\
&\quad \text{div} \quad \text{otherwise},
\end{align*}
\]

where \( \nu_{\mathcal{F}N} \) denotes the bijective numbering of the set \( \mathcal{F}N := \{ M \subseteq \mathbb{N} \mid M \text{ finite} \} \) defined by \( \nu_{\mathcal{F}N}^{-1}(M) := \sum_{j \in M} 2^j \) for all \( M \subseteq \mathbb{N} \).

Since the topology \( \tau_X \) of \( \mathcal{X} \) has \( \mathcal{B}^\ominus \) as a countable base, \( (X, \tau_X) \) is a sequential space and every open set is the countable union of certain sets in \( \mathcal{B}^\ominus \). This fact motivates the following representations of the family of \( \tau_X \) of all open sets of \( \mathcal{X} \) and of the family \( \text{clo}(\mathcal{X}) \) of all closed sets of \( \mathcal{X} \). Analogously to [BrPr01], we define the enumeration representations \( \theta^\ominus_X \subseteq \mathbb{N}^\omega \to \tau_X = \text{seq}(\tau_X) \) and \( \psi^\ominus_X \subseteq \mathbb{N}^\omega \to \text{clo}(\mathcal{X}) \) by

\[
\begin{align*}
\theta^\ominus_X(q) &:= \bigcup \{ \beta^\ominus(a) \mid a \in \text{En}(q) \} \quad \text{if } \text{En}(q) \subseteq \text{dom}(\beta^\ominus) \\
&\quad \text{div} \quad \text{otherwise}, \\
\psi^\ominus_X(q) &:= X \setminus \theta^\ominus_X(q)
\end{align*}
\]
for all \( q \in \mathbb{N}^\omega \), where \( En(q) = \{ q(j) - 1 \mid j \in \mathbb{N} \} \cap \mathbb{N} \).

We prove that \( \theta^\eta_X \) and \( \psi^\eta_X \) are computably equivalent to the representations \( (\rho_X)^\oplus \) and \( (\rho_X)^\boxdot \) (respectively) which are obtained from the admissible standard representation \( \rho_X \) of \((X,\tau_X)\) defined in Subsection 4.3.6. Therefore \( \psi^\eta_X \) is admissible w.r.t. the upper Fell topology \( (_\tau^\boxdot(X,\tau_X)) \) by Propositions 4.4.1 and 2.3.5.

**Proposition 4.4.3 (Computable equivalence of \( \theta^\eta_X \) and \( (\rho_X)^\oplus \))**

Let \( X = (X,B,\beta) \) be a computable topological space.
Then \( \theta^\eta_X \equiv_{cp} (\rho_X)^\oplus \) and \( \psi^\eta_X \equiv_{cp} (\rho_X)^\boxdot \).

**Proof:**

There are computable functions \( l_1, l_2 : \mathbb{N} \to \mathbb{N} \) with

\[
\{ (l_1(j), l_2(j)) \mid j \in \mathbb{N} \} = \{ (u, v) \in \text{dom}(\beta) \times \text{dom}(\beta) \mid \beta(u) = \beta(v) \},
\]

because the set on the right hand side is recursively enumerable.

\[
\theta^\eta_X \leq_{cp} (\rho_X)^\oplus \text{ and } \psi^\eta_X \leq_{cp} (\rho_X)^\boxdot:
\]

Obviously, the function \( g_1 : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega \) defined by

\[
g_1(q,r) \langle a, b, c \rangle := \begin{cases} 1 & \text{if } q(a) > 0, \nu_{FN}(q(a) - 1) = \{ l_1(j) \mid j \in \nu_{FN}(b) \} \\ 0 & \text{otherwise} \end{cases}
\]

and \( \{ l_2(j) \mid j \in \nu_{FN}(b) \} \subseteq En(r^{<\omega},0^\omega) \)

for all \( q,r \in \mathbb{N}^\omega \) and \( a,b,c \in \mathbb{N} \) is computable. By the smn–Theorem, there is a computable function \( g_2 : \mathbb{N}^\omega \to \mathbb{N}^\omega \) with \((\forall q,r \in \mathbb{N}^\omega) \eta_{g_2(q)}(r) = g_1(q,r)\).

Let \( q \in \text{dom}(\theta^\eta_X) = \text{dom}(\psi^\eta_X) \) and \( r \in \text{dom}(\rho_X) \). Then we have

\[
\rho_X(r) \in \theta^\eta_X(q) = X \setminus \psi^\eta_X(q)
\]

\[
\iff \exists a \in \mathbb{N} \left( q(a) > 0 \land \rho_X(r) \in \beta^\cap(q(a) - 1) \right)
\]

\[
\iff \exists a \in \mathbb{N} \left( q(a) > 0 \land (\forall u \in \nu_{FN}(q(a) - 1)) \rho_X(r) \in \beta(u) \right)
\]

\[
\iff \exists a \in \mathbb{N} \left( q(a) > 0 \land (\exists u \in \nu_{FN}(q(a) - 1)) \rho_X(r) \in \beta(u) \right)
\]

\[
\iff \exists a,b \in \mathbb{N} \left( q(a) > 0 \land \nu_{FN}(q(a) - 1) \subseteq \{ \nu_{FN}(b) \mid j \in \text{dom}(r^{<\omega},0^\omega) \} \right)
\]

\[
\iff \exists a,b \in \mathbb{N} \left( q(a) > 0 \land (\forall u \in \nu_{FN}(q(a) - 1)) \rho_X(r) \in \beta(u) \right)
\]

\[
\iff \exists a,b,c \in \mathbb{N} \left( g_1(q,r) \langle a, b, c \rangle = 1 \right)
\]

\[
\iff \exists a \in \mathbb{N} \left( g_1(q,r) \langle a, b, c \rangle = 1 \right)
\]

Since \( \text{dom}(\rho_X) \subseteq \text{dom}(\eta_{g_2(q)}) \), we obtain \([\rho_X \rightarrow \eta_{g_2(q)}](g_2(q)) = c_{\theta^\eta_X(q)}\) and hence \((\rho_X)^\oplus(g_2(q)) = \theta^\eta_X(q)\). Therefore \( \theta^\eta_X \leq_{cp} (\rho_X)^\oplus \) and \( \psi^\eta_X \leq_{cp} (\rho_X)^\boxdot \).

\((\rho_X)^\oplus \leq_{cp} \theta^\eta_X \) and \((\rho_X)^\boxdot \leq_{cp} \psi^\eta_X \).

By the utm–Theorem, there is a computable word function \( \sigma : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^* \)

approximating the computable function \( (q, r) \mapsto \eta_{q}(r) \). We define the obviously computable function \( g_3 : \mathbb{N}^\omega \to \mathbb{N}^\omega \) by

\[
g_3(q) \langle a, b, c, d \rangle := \begin{cases} a + 1 & \text{if } \nu_{FN}(a) \subseteq \{ l_1(j) \mid j \in \nu_{FN}(b) \} = En(\nu_{FN}(c);0^\omega) \\ 0 & \text{otherwise} \end{cases}
\]

for all \( q \in \mathbb{N}^\omega \) and \( a,b,c,d \in \mathbb{N} \).

Let \( q \in \text{dom}(\theta^\eta_X) = \text{dom}(\rho_X) \). We show \( \theta^\eta_X(g_3(q)) = \rho_X(q) \).


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"⊆": Let \( a \in En(g_3(q)) \). There are \( b,c,d \in \mathbb{N} \) with \( \nu_{\mathcal{F}(a)} = \{ l_1(j) \mid j \in \nu_{\mathcal{F}(b)} \} \), \( \nu_{\mathcal{F}(a)} = En(\nu_{\mathcal{F}(c)};0^\omega) \) and \( \sigma(q^{<\omega},\nu_{\mathcal{F}(c)}) \notin \{ 0 \}^* \). The first equality implies \( a \in \text{dom}(\beta) \). Let \( x \in \beta(a) \). There is some \( r \in \mathbb{N}^x \) with \( En(r) = \{ i \in \mathbb{N} \mid x \in \beta(i) \} \). Since \( x \in \bigcap \{ \beta(i) \mid i \in \nu_{\mathcal{F}(a)} \} = \bigcap \{ \beta(i) \mid i \in En(\nu_{\mathcal{F}(c)};0^\omega) \} \), the \( \omega \)-word \( s := \nu_{\mathcal{F}(c)};r \) is also a \( \rho_X \)-name of \( x \). As \( \eta_i(s) \neq 0^\omega \), we have \( x \in \rho^n_X(q) \). Hence \( \beta(a) \subseteq \rho^n_X(q) \). This shows "⊆".

"⊇": Let \( x \in \rho^n_X(q) \). Let \( r \) be some \( \rho_X \)-name of \( x \). There are some \( a,b,c,d \in \mathbb{N} \) with \( \sigma(q^{<\omega},r^{<\omega}) \notin \{ 0 \}^* \), \( \nu_{\mathcal{F}(c)}(r) = r^{<\omega} \) and \( En(r^{<\omega};0^\omega) = \{ l_1(j) \mid j \in \nu_{\mathcal{F}(b)} \} = \nu_{\mathcal{F}(a)}(r) \).

Since \( a \in \text{dom}(\beta) \cap En(g_3(q)) \) and \( x \in \beta(a) \), we have \( x \in \theta^n_{\rho_X}(g_3(q)) \).

Therefore, \( \theta^n_{\rho_X}(g_3(q)) = (\rho_X)^\Xi(q) \) and \( \psi^n_{\rho_X}(g_3(q)) = (\rho_X)^\Box(q) \). Hence \( (\rho_X)^\Xi \leq_{cp} \theta^n_{\rho_X} \) and \( (\rho_X)^\Box \leq_{cp} \psi^n_{\rho_X} \).

Proposition 4.4.3 extends the analogue result in [BrPr01] for separable metric spaces. The representation \( \psi_{\Box} \) of the family of all closed sets of real numbers defined in [Wei00] can be shown in a similar way as in [BrPr01] to be computably equivalent to \( (\rho_{\mathbb{R},\mathcal{Cb},\nu_{\mathcal{Cb}}})^\Box \), where \( (\mathbb{R},\mathcal{Cb},\nu_{\mathcal{Cb}}) \) is the computable topological space from Example 4.3.17. Thus the concept of Sierpiński representations of sequentially closed sets generalizes \( \psi_{\Box} \).

4.4.2 Positive Representations of Closed Sets

In this subsection, we introduce an admissible multirepresentation of all subsets of an admissibly represented sequential space \( X \) providing positive information. From this, we construct a representation of the closed sets of \( X \) which is admissible w.r.t. the Fell topology (cf. [Bee93]). We show that it generalizes the representation \( \psi \) from [Wei00] to the Fell topology of \( X \), for the closed sets of the Euclidean space. In Proposition 4.4.6, we give a further effective visualization of topological continuity.

Let \( \mathcal{X} = (X,\tau_X) \) be a sequential space, i.e. \( seq(\tau_X) = \tau_X \). We equip the set \( 2^X := \{ M \mid M \subseteq X \} \) of subsets with the hit topology (or lower topology) \( \tau_X^{hit} \). It is defined by having

\[
\mathcal{S}_X^{hit} := \{ \{ M \subseteq X \mid M \cap U \neq \emptyset \} \mid U \in \tau_X \} \cup \{ 2^X \}
\]

as a subbase. Note that for all sets \( M \subseteq X \) and all open sets \( U \) we have

\[
M \cap U \neq \emptyset \iff \text{Cls}(M) \cap U \neq \emptyset,
\]

where \( \text{Cls}(M) = \bigcap \{ A \subseteq X \mid A \text{ closed in } X, M \subseteq A \} \) denotes the closure of the set \( M \) in \( X \). Thus \( M \) and \( \text{Cls}(M) \) have the same open neighbourhoods in the hyperspace \( (2^X,\tau_X^{hit}) \). Hence \( \tau_X^{hit} \) is not a \( T_0 \)-topology, unless \( X \) is discrete.

The lower Fell topology \( \tau_X^{\Box} \) on the family \( \text{sclo}(\mathcal{X}) \) of all closed subsets of \( \mathcal{X} \) is defined by having

\[
\mathcal{S}_X^{\Box} := \{ \{ A \in \text{sclo}(\mathcal{X}) \mid A \cap U \neq \emptyset \} \mid U \in \tau_X \} \cup \{ \text{sclo}(\mathcal{X}) \}
\]

as a subbase (cf. [Bee93, BrPr01]). Of course, \( \tau_X^{\Box} \) is the subspace topology on \( \text{sclo}(\mathcal{X}) \) inherited from \( \tau_X^{hit} \). Moreover, \( \tau_X^{\Box} \) is a \( T_0 \)-topology, because for all closed sets \( A,B \subseteq X \)
4.4 Multirepresentations of Hyperspaces

with \( A \setminus B \neq \emptyset \), the open set \( H := \{ C \in \text{scl}(X) \mid C \cap (X \setminus B) \neq \emptyset \} \in S^B_X \) contains \( A \), but not \( B \). The Fell topology \( \tau^\text{Fell}_X \) is defined to be the topology that is generated by the subbase \( S^B_X \cup S^A_X \) (cf. [Bee93, BrPr01]), hence it is the conjunction of the lower Fell topology \( \tau^B_X \) and the upper Fell topology \( \tau^A_X \) (cf. Subsections 4.1.4, 4.4.1).

We will now prove that \((2^X, \tau^\text{hit}_X)\) and \((\text{scl}(X), \tau^\text{scl}_X)\) have admissible multirepresentations, whenever there is an admissible multirepresentation of the space \( X \). The idea is to embed the space \((2^X, \tau^\text{hit}_X)\) into the function space \( \mathcal{C}(\mathcal{C}(X, \mathcal{S}_i), \mathcal{S}_i) \). For \( M \subseteq X \) and \( h \in \mathcal{C}(X, \mathcal{S}_i) \), we define the function \( e^+_{X, \mathcal{S}_i}(M) : \mathcal{C}(X, \mathcal{S}_i) \to \mathcal{S}_i \) by

\[
e^+_{X, \mathcal{S}_i}(M)(h) := \begin{cases} \top & \text{if } (\exists x \in M) \ h(x) = \top \\ \bot & \text{otherwise}. \end{cases}
\]

For every \( x \in X \), \( e^+_{X, \mathcal{S}_i}(\{x\}) \) is equal to the function \( e_{X, \mathcal{S}_i}(x) \) from Subsection 4.3.1. Moreover, since \( h^{-1}[\top] \) is open for all \( h \in \mathcal{C}(X, \mathcal{S}_i) \), we have

\[
e^+_{X, \mathcal{S}_i}(M) = e^+_{X, \mathcal{S}_i}(\text{Cls}(M)) \tag{4.18}
\]

by Equivalence (4.17). The function \( e^+_{X, \mathcal{S}_i} \) enjoys the following properties.

**Lemma 4.4.4 (The space \((2^X, \tau^\text{hit}_X)\) embeds in \( \mathcal{C}(\mathcal{C}(X, \mathcal{S}_i), \mathcal{S}_i) \))**

Let \( X = (X, \tau_X) \) be a sequential topological space.

(1) For every \( M \subseteq X \), \( e^+_{X, \mathcal{S}_i}(M) \in \mathcal{C}(\mathcal{C}(X, \mathcal{S}_i), \mathcal{S}_i) \).

(2) For every sequence \( (M_n)_{n \leq \infty} \) of subsets of \( X \),

\[
(M_n)_{n \to \tau^\text{hit}_X} M_\infty \iff (e^+_{X, \mathcal{S}_i}(M_n))_{n \to \mathcal{C}(\mathcal{C}(X, \mathcal{S}_i), \mathcal{S}_i)} e^+_{X, \mathcal{S}_i}(M_\infty).
\]

**Proof:**

(1) Let \( (h_n)_{n \leq \infty} \) be a convergent sequence of \( \mathcal{C}(X, \mathcal{S}_i) \). The only interesting case is \( e^+_{X, \mathcal{S}_i}(M)(h_\infty) = \top \), i.e., there is some \( x \in M \) with \( h_\infty(x) = \top \). Since \( (x)_\mu \) converges to \( x \in X \) and \( (h_n) \) converges continuously to \( h_\infty \), there is some \( n_0 \geq n_0 \) with \( (\forall n \geq n_0) h_n(x) = \top \). This implies \( e^+_{X, \mathcal{S}_i}(M)(h_n) = \top \) for all \( n \geq n_0 \), hence \( (e^+_{X, \mathcal{S}_i}(M)(h_n))_{n \to \mathcal{S}_i} e^+_{X, \mathcal{S}_i}(M)(h_\infty) \). Therefore \( e^+_{X, \mathcal{S}_i}(M) \) is sequentially continuous.

(2) “\( \iff \)” : Let \( (e^+_{X, \mathcal{S}_i}(M_n))_n \) converge to \( e^+_{X, \mathcal{S}_i}(M_\infty) \) in \( \mathcal{C}(\mathcal{C}(X, \mathcal{S}_i), \mathcal{S}_i) \). Let \( H \in \mathcal{S}^\text{hit}_X \) with \( M_\infty \in H \). W.l.o.g. we can assume \( H \neq 0 \), hence there are an open set \( U \in \tau_X \) with \( M_\infty \cap U \neq \emptyset \) and \( H = \{ M \subseteq X \mid M \cap U \neq \emptyset \} \). Since \( (e^+_{X, \mathcal{S}_i}(M_n))(cf_U)(h_\infty) = \top \), there is some \( n_0 \in N \) with \( (\forall n \geq n_0) e^+_{X, \mathcal{S}_i}(M_n)(cf_U) = \top \). This implies \( (\forall n \in N) M_n \in H \). We conclude that \( (M_n)_n \) converges to \( M_\infty \) in \((2^X, \tau^\text{hit}_X)\).

“\( \implies \)” : Let \( (M_n)_n \) converge to \( M_\infty \) in \((2^X, \tau^\text{hit}_X)\). Let \( (h_n)_{n \leq \infty} \) be a convergent sequence of \( \mathcal{C}(X, \mathcal{S}_i) \). The only interesting case is \( e^+_{X, \mathcal{S}_i}(M_\infty)(h_\infty) = \top \), i.e., there exists some \( x \in M_\infty \) with \( h_\infty(x) = \top \). As \( (h_n)_n \) converges continuously to \( h_\infty \), there is some \( n_1 \in N \) with \( (\forall n \geq n_1) h_n(x) = \top \). Since the transpose...
Lemma 4.4.4 motivates the definition of the following multirepresentations of \(2^X\) which can be interpreted as follows: a family \(\delta\) is an admissible multirepresentation of \(X\) if \(\delta\) is \(h\)-continuous, and \(K := \{n \in \mathbb{N} \mid n \geq n_1\}\) is a sequentially compact subset of \((\mathbb{N}, -p)\), the set

\[
U := \{y \in X \mid (\forall n \in K) h_n(y) \in \{\top\}\}
\]

is sequentially open in \(X\) by Lemma 2.2.9 and thus contained in the sequential topology \(\tau_\mathcal{X}\). As the hyper-open \(H := \{M \subseteq X \mid M \cap U \neq \emptyset\}\) in \(\mathcal{S}_\mathcal{X}\) contains \(M_\infty\), there is some \(n_2 \in \mathbb{N}\) with \((\forall n \geq n_2) M_n \in H\). This implies \(e^{+}_{X, E_{\mathcal{S}i}}(M_n)(h_n) = \top\) for all \(n \geq \max\{n_1, n_2\}\), because we have \(U \subseteq h_n^{-1}[\top]\) for all \(n \in K\). Hence \((e^{+}_{X, E_{\mathcal{S}i}}(M_n))(h_n), e^{+}_{X, E_{\mathcal{S}i}}(M_\infty)(h_\infty)\). From Equivalence (4.3) we conclude that \((e^{+}_{X, E_{\mathcal{S}i}}(M_n))_{n \leq \infty}\) is a convergent sequence of \(\mathcal{C}(\mathcal{E}(X, \mathcal{S}i), \mathcal{S}i)\).

Proposition 4.4.5 (Admissibility of \(\delta_{hit}\) and \(\delta_{\Box}\))

Let \(\delta \subseteq \mathbb{N}^\omega \rightrightarrows X\) be a Top-quotient multirepresentation of \(X\). We define \(\delta_{hit} \subseteq \mathbb{N}^\omega \rightrightarrows 2^X\) and \(\delta_\Box \subseteq \mathbb{N}^\omega \rightrightarrows sclo(X)\) by

\[
\delta_{hit}[p] M : \iff [\delta \to g_{E_{\mathcal{S}i}} \to g_{E_{\mathcal{S}i}}](p) = e^{+}_{X, E_{\mathcal{S}i}}(M) \quad \text{and} \quad \\
\delta_\Box(p) A \iff [\delta \to g_{E_{\mathcal{S}i}} \to g_{E_{\mathcal{S}i}}](p) = e^{+}_{X, E_{\mathcal{S}i}}(A)
\]

for all \(p \in \mathbb{N}^\omega\), \(M \subseteq X\) and \(A \in sclo(X)\). The name \(\delta_{hit}\) is motivated by the equivalence

\[
M \in \delta_{hit}[p] \iff \{O \in \text{seq}(X) \mid M \cap O \neq \emptyset\} = (\delta_\Box)^\Box(p)
\]

which can be interpreted as follows: a \(\delta_{hit}\)-name \(p\) of \(M\) yields the family of those sequentially open sets \(O\) that hit \(M\) in the sense that \(O\) intersects \(M\). By Lemma 4.4.4, this family is actually a sequentially open set of the hyperspace \((\text{seq}(X), \tau_{hit})\).

We prove that the multirepresentation \(\delta_{hit}\) is admissible w.r.t. the hit topology \(\tau_{hit}\) and that the representation \(\delta_{\Box}\) is admissible w.r.t. the lower Fell topology on \(sclo(X)\).

\[\Box\]

One can show that the family \(\{(A \in sclo(X) \mid A \cap B \neq \emptyset) \mid B \in \mathcal{B}^\omega\} \cup \{sclo(X)\}\) is a pseudosubbase of \((sclo(X), \tau_{\Box})\), whenever \(\mathcal{B}\) is a countable pseudobase of \(X\).
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Proof:

(1) By Propositions 4.2.5 and 2.3.2, \( \{\delta \rightarrow \varrho_{\mathcal{E}_i} \rightarrow \varrho_{\mathcal{E}_i} \} \) is an admissible representation of \( \mathcal{C}(\mathcal{C}(\mathfrak{X}, \mathcal{G}_i), \mathcal{G}_i) \). Since \( \varepsilon_{\mathfrak{X}, \mathcal{G}_i} : 2^X \rightarrow \mathcal{C}(\mathcal{C}(\mathfrak{X}, \mathcal{G}_i), \mathcal{G}_i) \) is a homomorphism between the limit spaces \( (2^X, \rightarrow_{2^X}) \) and \( \mathcal{C}(\mathcal{C}(\mathfrak{X}, \mathcal{G}_i), \mathcal{G}_i) \), \( \delta_{\text{hit}} \) is an admissible multirepresentation of \( (2^X, \rightarrow_{2^X}) \) by Proposition 4.1.6. Thus \( \delta_{\text{hit}} \) is \( \tau_{\mathfrak{X}} \)–admissible. By Proposition 4.3.14(3) and Lemma 4.3.6(3), we have \( \{\delta \rightarrow \varrho_{\mathcal{E}_i} \rightarrow \varrho_{\mathcal{E}_i} \} \equiv_{\text{cp}} \{[\delta \rightarrow \varrho_{\mathcal{E}_i} \rightarrow \varrho_{\mathcal{E}_i}] \}^\delta \). This implies \( \delta_{\text{hit}} \equiv_{\text{cp}} \{\delta_{\text{hit}}\}^\delta \) by Proposition 4.3.14(5).

(2) As the lower Fell topology \( \tau_{\mathfrak{X}} \) is the subspace topology on \( \text{sclo}(\mathfrak{X}) \) inherited from \( \tau_{\mathfrak{X}} \), \( \delta_{\text{hit}} \) is equal to \( (\delta_{\text{hit}})^{\text{sclo}(\mathfrak{X})} \), the \( \tau_{\mathfrak{X}} \)–admissibility of \( \delta_{\text{hit}} \) follows from (1) and results in Subsection 4.1.5. Since \( \tau_{\mathfrak{X}} \) is a \( T_0 \)–topology, \( \delta_{\text{hit}} \) is single–valued by Lemma 2.4.6 and thus actually a representation. Since \( \delta_{\text{hit}} = (\delta_{\text{hit}})^{\text{sclo}(\mathfrak{X})} \), \( \delta_{\text{hit}} = (\delta_{\text{hit}})^\delta \) follows from (1) by Proposition 4.3.14(6).

(3) Since the Fell topology \( \tau_{\mathfrak{X}} \) is equal to the conjunction of \( \tau_{\mathfrak{X}} \) and \( \delta_{\text{hit}} \), \( \delta_{\text{hit}} \) is admissible w.r.t. \( \tau_{\mathfrak{X}} \) by (2), Proposition 4.1.6 and Proposition 4.1.8. Statements (1), (2) and Proposition 4.3.14(2) imply \( (\delta_{\text{hit}} \land \delta_{\text{hit}}) \equiv_{\text{cp}} (\delta_{\text{hit}} \land \delta_{\text{hit}})^\delta \).

From [Wil70, Eng89, Smy92] we know the following characterization of topological continuity for functions \( f : X \rightarrow Y \) between two topological spaces \( \mathfrak{X} = (X, \tau_{\mathfrak{X}}) \) and \( \mathfrak{Y} = (Y, \tau_{\mathfrak{Y}}) \):

\[
f \text{ is } (\tau_{\mathfrak{X}}, \tau_{\mathfrak{Y}})\text{–continuous } \iff \ (\forall M \subseteq X) f[\text{Cls}(M)] \subseteq \text{Cls}(f[M]) . \quad (4.19)
\]

Since \( f[\text{Cls}(M)] \subseteq \text{Cls}(f[M]) \) implies \( \text{Cls}(f[\text{Cls}(M)]) = \text{Cls}(f[M]) \), we can reformulate this equivalence as follows:

\[
f \text{ is } (\tau_{\mathfrak{X}}, \tau_{\mathfrak{Y}})\text{–continuous } \iff \ (\forall M_1, M_2 \subseteq X) (\text{Cls}(M_1) = \text{Cls}(M_2) \iff \text{Cls}(f[M_1]) = \text{Cls}(f[M_2])) . \quad (4.20)
\]

The following proposition can be viewed as an effectivization of these equivalences.

**Proposition 4.4.6 (Effective topological continuity II)**

Let \( \delta : \mathbb{N}^\omega \rightarrow X \) be a \( \text{Top} \)–quotient multirepresentation of a sequential space \( \mathfrak{X} = (X, \tau_{\mathfrak{X}}) \). Let \( \mathfrak{Y} = ((Y, \tau_{\mathfrak{Y}}), \gamma) \) be an object of \( \text{EffSeq} \). Let \( f : X \rightarrow Y \) be a total function.

(1) The function \( f \) is computable w.r.t. \( \delta \) and \( \gamma \) if and only if the function \( F_1 : \text{sclo}(\mathfrak{X}) \rightarrow \text{sclo}(\mathfrak{Y}) \) defined by \( F_1(A) := \text{Cls}(f[A]) \) is computable w.r.t. \( \delta_{\text{hit}} \) and \( \gamma_{\text{hit}} \) and \( f \) is topologically continuous.

(2) If \( \mathfrak{X} \) is a \( T_1 \)–space, then \( f \) is computable w.r.t. \( \delta \) and \( \gamma \) if and only if the function \( F_1 : \text{sclo}(\mathfrak{X}) \rightarrow \text{sclo}(\mathfrak{Y}) \) defined in (1) is computable w.r.t. \( \delta_{\text{hit}} \) and \( \gamma_{\text{hit}} \).

(3) The function \( f \) is computable w.r.t. \( \delta \) and \( \gamma \) if and only if the function \( F_2 : 2^X \rightarrow 2^Y \) defined by \( F_2(M) := f[M] \) is computable w.r.t. \( \delta_{\text{hit}} \) and \( \gamma_{\text{hit}} \).
Chapter 4. Constructions

Proof:

(1) “⇒”: By Proposition 2.4.34, \( f \) is topologically continuous. Clearly, the identity function \( id_{\mathbb{N}} \) realizes the function \( \text{Cls}: 2^Y \rightarrow \text{sclo}(\mathbb{Q}) \) w.r.t. \( \delta^{\text{hit}} \) and \( \gamma^{\text{hit}} \). Since \( F_1(A) = \text{Cls}(F_2(A)) \) and \( \delta^{\text{hit}}(\text{sclo}(X)) \), the \( (\delta^{\text{hit}}, \gamma^{\text{hit}}) \)-computability of \( F_1 \) follows from (3).

“⇐” Let \( g : \mathbb{N}^w \rightarrow \mathbb{N}^w \) be a computable function realizing \( F_1 \) w.r.t. \( \delta^{\text{hit}} \) and \( \gamma^{\text{hit}} \). Let \( p \in \mathbb{N}^w \) and \( M \in \delta^{\text{hit}}[p] \). Then \( \delta^{\text{hit}}(p) = \text{Cls}(M) \). From Equivalence (4.20) it follows

\[
\text{Cls}(F_2(M)) = \text{Cls}(f[M]) = \text{Cls}(f[\text{Cls}(M)]) = F_1(\text{Cls}(M)) = \gamma^{\text{hit}}(g(p)).
\]

Thus \( F_2(M) \in \gamma^{\text{hit}}[g(p)] \). We conclude that \( g \) realizes \( F_2 \) w.r.t. \( \delta^{\text{hit}} \) and \( \gamma^{\text{hit}} \).

By (3), \( f \) is computable with respect to \( \delta \) and \( \gamma \).

(2) “⇒”: This follows from (1).

“⇐”: Let \( g : \mathbb{N}^w \rightarrow \mathbb{N}^w \) be a computable function realizing \( F_1 \) w.r.t. \( \delta^{\text{hit}} \) and \( \gamma^{\text{hit}} \). Let \( p \in \text{dom}(\delta^{\text{hit}}) \) and \( x := \delta^{\text{hit}}(p) \). Then \( \left[ [\delta \rightarrow g_{\mathbb{N}}] \rightarrow f_{\mathbb{N}} \right](p) = e_{\mathbb{N}, \mathbb{N}}(x) = \delta^{\text{hit}}(x) \) and hence \( \delta^{\text{hit}}(p) = \{ x \} \), because \( \{ x \} \) is closed by the \( T_1 \)-property. Since \( \gamma^{\text{hit}}(g(p)) = F_1(\{ x \}) \), we have by Equivalence (4.18)

\[
\left[ \gamma \rightarrow g_{\mathbb{N}} \rightarrow g_{\mathbb{N}} \right](g(p)) = e_{\mathbb{N}, \mathbb{N}}(F_1(\{ x \})) = e_{\mathbb{N}, \mathbb{N}}(\text{Cls}(f(x))) = e_{\mathbb{N}, \mathbb{N}}(f(x)).
\]

We obtain \( f(x) \in \gamma^{\text{hit}}[g(p)] \). Thus \( g \) realizes \( f \) w.r.t. \( \delta^{\text{hit}} \) and \( \gamma^{\text{hit}} \). Since \( \delta \leq_{cp} \delta^{\text{hit}} \) and \( \gamma^{\text{hit}} \leq_{cp} \gamma \), \( f \) is \( (\delta, \gamma) \)-computable by Lemma 2.4.4.

(3) “⇒”: The function \( \mathcal{H}_{\mathbb{N}} \circ \mathcal{H}_{\mathbb{N}}(f) : \mathcal{C}(\mathbb{X}, \mathbb{N}), \mathbb{N}) \rightarrow \mathcal{C}(\mathbb{Q}, \mathbb{N}), \mathbb{N}) \) is computable with respect to \( \left[ [\delta \rightarrow g_{\mathbb{N}}] \rightarrow g_{\mathbb{N}} \right] \) and \( \left[ [\gamma \rightarrow g_{\mathbb{N}}] \rightarrow g_{\mathbb{N}} \right] \) by Lemmas 4.3.4 and 4.2.8(1). Thus there is a computable function \( g : \mathbb{N}^w \rightarrow \mathbb{N}^w \) realizing \( \mathcal{H}_{\mathbb{N}} \circ \mathcal{H}_{\mathbb{N}}(f) \) w.r.t. these representations. Let \( p \in \mathbb{N}^w \) and \( M \in \delta^{\text{hit}}[p] \). Then \( \left[ [\delta \rightarrow g_{\mathbb{N}}] \rightarrow g_{\mathbb{N}} \right](p) = e_{\mathbb{N}, \mathbb{N}}^+(M) \). For every \( h \in \mathcal{C}(\mathbb{X}, \mathbb{N}) \), we obtain from Equation (4.12) on page 121

\[
e_{\mathbb{N}, \mathbb{N}}^+(f[M])(h) = \begin{cases} \top & \text{if } (\exists y \in f[M]) h(y) = \top \\ \perp & \text{otherwise} \end{cases}
\]

\[
e_{\mathbb{N}, \mathbb{N}}^+(M)(h \circ f) = \mathcal{H}_{\mathbb{N}}(f)(e_{\mathbb{N}, \mathbb{N}}^+(M))(h).
\]

Therefore

\[
e_{\mathbb{N}, \mathbb{N}}^+(F_2(M)) = e_{\mathbb{N}, \mathbb{N}}^+(f[M]) = (\mathcal{H}_{\mathbb{N}} \circ \mathcal{H}_{\mathbb{N}}(f))(e_{\mathbb{N}, \mathbb{N}}^+(M)) = \left[ [\gamma \rightarrow g_{\mathbb{N}}] \rightarrow g_{\mathbb{N}} \right](g(p)).
\]

Thus \( F_2(M) \in \gamma^{\text{hit}}[g(p)] \). We conclude that \( F_2 \) is computable with respect to \( \delta^{\text{hit}} \) and \( \gamma^{\text{hit}} \).
“\(\subseteq\)” \(\Longleftrightarrow\): Let \(g : \mathbb{N}^\omega \to \mathbb{N}^\omega\) be a computable function realizing \(F_2\) w.r.t. \(\delta_{\text{hit}}\) and \(\gamma_{\text{hit}}\). Let \(p \in \mathbb{N}^\omega\) and \(x \in \delta^p[p]\). Then \([\delta \to \varphi_{\mathcal{E}}] \to \varphi_{\mathcal{E}}(p) = \epsilon_{\mathcal{X},\mathcal{E}_i}(x) = e^+_{\mathcal{X},\mathcal{E}_i}(x)\) and hence \(\{x\} \in \delta_{\text{hit}}[p]\). This implies \(\{f(x)\} = F_2(\{x\}) \in \gamma_{\text{hit}}[g(p)]\). We obtain
\[
[\gamma \to \varphi_{\mathcal{E}}] \to \varphi_{\mathcal{E}}(g(p)) = e^+_{\mathcal{Y},\mathcal{E}_i}(\{f(x)\}) = e_{\mathcal{Y},\mathcal{E}_i}(f(x)),
\]
hence \(f(x) \in \gamma^p[g(p)]\). Therefore \(g\) realizes \(f\) w.r.t. \(\delta^p\) and \(\gamma^p\). Since \(\delta \leq_{\text{cp}} \delta^p\) and \(\gamma \equiv_{\text{cp}} \gamma^p\), \(f\) is computable w.r.t. \(\delta\) and \(\gamma\) by Lemma 2.4.4.

\(\checkmark\)

Let \(\mathcal{X} = (X, \mathcal{B}, \beta)\) be a computable topological space (cf. Subsection 4.3.6) such that the set \(\{(u, v) \mid \beta^u(u) = \beta^v(v)\}\) is recursively enumerable. We define \(\mathcal{X}^\oplus\) to be the triple \((\text{sclo}(\mathcal{X}), \mathcal{D}, \alpha)\), where the family \(\mathcal{D}\) and the numbering \(\alpha : \mathbb{N} \to \mathcal{D}\) are given by
\[
\mathcal{D} := \left\{\left\{A \in \text{sclo}(\mathcal{X}) \mid A \cap U \neq \emptyset\right\} \mid U \in \mathcal{B}^\ominus\right\}
\text{ and }\n\alpha(n) := \left\{\begin{array}{ll}
A \in \text{sclo}(\mathcal{X}) \mid A \cap \beta^\ominus(n) \neq \emptyset & \text{if } n \in \text{dom}(\beta^\ominus) \\
\bot & \text{otherwise.}
\end{array}\right\}
\]

Since the set \(\{(u, v) \mid \beta^u(u) = \beta^v(v)\}\) is recursively enumerable, \(\mathcal{X}^\oplus\) is a computable topological space. Clearly, the topology \(\tau_{\mathcal{X}^\oplus}\) of \(\mathcal{X}^\oplus\) is the lower Fell topology \(\tau_{\mathcal{Y}}^\ominus\) on the closed sets of the countably based space \((X, \tau_{\mathcal{X}})\). We show that the standard representation \(\rho_{\mathcal{X}^\oplus}\) of \(\mathcal{X}^\oplus\) is computably equivalent to \((\rho_{\mathcal{X}})^\ominus\).

**Proposition 4.4.7 (Computable equivalence of \(\rho_{\mathcal{X}^\oplus}\) and \((\rho_{\mathcal{X}})^\ominus\))**

Let \(\mathcal{X} = (X, \mathcal{B}, \beta)\) be a computable topological space such that
\[
\left\{(u, v) \in \text{dom}(\beta^\ominus) \times \text{dom}(\beta^\ominus) \mid \beta^u(u) = \beta^v(v)\right\}
\]
is recursively enumerable. Then \(\rho_{\mathcal{X}^\oplus} \equiv_{\text{cp}} (\rho_{\mathcal{X}})^\ominus\).

**Proof:**
Let \(\alpha\) be the subbase numbering of \(\mathcal{X}^\oplus\). There are computable functions \(l_1, l_2 : \mathbb{N} \to \mathbb{N}\) with
\[
\left\{(l_1(j), l_2(j)) \mid j \in \mathbb{N}\right\} = \left\{(u, v) \in \text{dom}(\beta^\ominus) \times \text{dom}(\beta^\ominus) \mid \beta^u(u) = \beta^v(v)\right\},
\]
because the set on the right hand side is recursively enumerable.
\(\rho_{\mathcal{X}^\oplus} \leq_{\text{cp}} (\rho_{\mathcal{X}})^\ominus\).

We define the function \(g_1 : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega\) by
\[
g_1(p, r)((a, b, c)) := \begin{cases} 1 & \text{if } l_1(c) = p(a) - 1 \text{ and } l_2(c) = r(b) - 1 \\ 0 & \text{otherwise} \end{cases}
\]
for all \(p, r \in \mathbb{N}^\omega\) and \(a, b, c \in \mathbb{N}\). Clearly \(g_1\) is computable. By Proposition 4.4.3 there is a computable function \(g_2 : \mathbb{N}^\omega \to \mathbb{N}^\omega\) translating \((\rho_{\mathcal{X}})^\ominus\) to \(\delta_{\text{cp}}^\ominus\). By the computable smm–Theorem, there exists a computable function \(g_3 : \mathbb{N}^\omega \to \mathbb{N}^\omega\) with
\[
(\forall p, q \in \mathbb{N}^\omega) \eta_{g_3(p)}(q) = g_1(p, g_2(q)).
\]
We show that $g_3$ translates $\rho_{X^\omega}$ to $(\rho_X)^\boxdot$. Let $p \in \text{dom}(\rho_{X^\omega})$ and $A := \rho_{X^\omega}(p)$. Let $q \in \text{dom}([\rho_X \to \varrho_{E_1}])$ and $h := [\rho_X \to \varrho_{E_1}](q)$. Then $g_2(q)$ is a $\theta^\omega_X$-name of the open set $h^{-1}[\{\top\}]$. We have $\text{dom}([\rho_X \to \varrho_{E_1}]) \subseteq \text{dom}(g_2) \subseteq \text{dom}(\eta_{g_3(p)})$ and

$$e^+_X(A)(h) = \top \iff A \cap h^{-1}[\{\top\}] \neq \emptyset$$

$$\iff A \cap \bigcup \{\beta^n(n) \mid n \in \text{En}(g_2(q))\} \neq \emptyset$$

$$\iff (\exists n \in \text{En}(g_2(q))) A \cap \beta^n(n) \neq \emptyset$$

$$\iff (\exists m \in \text{En}(p)) (\exists n \in \text{En}(g_2(q))) \beta^n(m) = \beta^n(n)$$

$$\iff (\exists a, b, c \in \mathbb{N}) (p(a) - 1 = l_1(e) \land l_2(e) = g_2(q)(b) - 1)$$

$$\iff \eta_{g_3(p)}(q) \neq \omega^\omega.$$

This implies $\varrho_{E_1}(\eta_{g_3(p)}(q)) = e^+_X(A)(h)$ and $[\rho_X \to \varrho_{E_1}] [\rho_X \to \varrho_{E_1}](g_3(p)) = e^+_X(A)$. Therefore $(\rho_X)^{\boxdot}(g_3(p)) = A$ and $\rho_{X^\omega} \leq_{cp} (\rho_X)^{\boxdot}$.

$(\rho_X)^{\boxdot} \leq_{cp} \rho_{X^\omega}$:

We define the function $g_4 : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ by

$$g_4(t, r)(\langle a, b \rangle) := \begin{cases} 1 & \text{if } l_1(a) = t(0) \text{ and } \nu_{\text{FN}}(l_2(a)) \subseteq \text{En}(r^{<b::0^\omega}) \\ 0 & \text{otherwise.} \end{cases}$$

Since $g_4$ is computable, there is some computable function $g_5 : \mathbb{N}^\omega \to \mathbb{N}^\omega$ with $(\forall t, r \in \mathbb{N}^\omega) \eta_{g_5(t)}(r) = g_4(t, r)$ by the computable smn–Theorem. Moreover, there is a computable monotone word function $\sigma : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ approximating the function $(p, t) \mapsto \eta_p(g_5(t))$, which is computable by the utm–Theorem. We define the total function $g_6 : \mathbb{N}^\omega \to \mathbb{N}^\omega$ by

$$g_6(p)(\langle j, k \rangle) := \begin{cases} k + 1 & \text{if } \sigma(p^{<j}, k0^j) \notin \{0\}^* \\ 0 & \text{otherwise.} \end{cases}$$

for $p \in \mathbb{N}^\omega$ and $j, k \in \mathbb{N}$. Clearly, $g_6$ is computable.

We show that $g_6$ translates $(\rho_X)^{\boxdot}$ to $\rho_{X^\omega}$. Let $p \in \text{dom}((\rho_X)^{\boxdot})$ and $A := (\rho_X)^{\boxdot}(p)$. Let $k \in \text{dom}(\alpha) = \text{dom}(\beta^\tau)$. For all $r \in \text{dom}(\rho_X)$ we have

$$\rho_X(r) \in \beta^\tau(k) \iff (\forall i \in \nu_{\text{FN}}(k)) \rho_X(r) \in \beta(i)$$

$$\iff \{\beta(i) \mid i \in \nu_{\text{FN}}(k)\} \subseteq \{\beta(j) \mid j \in \text{En}(r)\}$$

$$\iff (\exists a \in \mathbb{N})(\beta^n(a) = \beta^n(c) \land \nu_{\text{FN}}(c) \subseteq \text{En}(r))$$

$$\iff (\exists a \in \mathbb{N})(l_1(a) = k \land \nu_{\text{FN}}(l_2(a)) \subseteq \text{En}(r))$$

$$\iff (\exists a, b \in \mathbb{N}) g_4(k^{<0^\omega}, r)(a, b) = 1$$

$$\iff \varrho_{E_1}(\eta_{g_5(k^{<0^\omega})}(r)) = \top.$$

Since $\eta_{g_5(k^{<0^\omega})}$ is total, this shows $[\rho_X \to \varrho_{E_1}](g_5(k^{<0^\omega})) = cf_{\beta^\tau(k)}$. Hence we have

$$A \in \alpha(k) \iff A \cap \beta^\tau(k) \neq \emptyset \iff e^+_X(A)(cf_{\beta^\tau(k)}) = \top$$

$$\iff \eta_p(g_5(k^{<0^\omega})) \neq \omega^\omega \iff (\exists j \in \mathbb{N}) \sigma(p^{<j}, k0^j) \notin \{0\}^*$$

$$\iff k \in \text{En}(g_6(p)).$$

For $k \notin \text{dom}(\alpha)$, we have $k \notin \{l_1(a) \mid a \in \mathbb{N}\}$, hence $[\rho_X \to \varrho_{E_1}](g_5(k^{<0^\omega})) = cf_0$ and $\eta_p(g_5(k^{<0^\omega})) = \omega^\omega$. This implies $\text{En}(g_6(p)) = \{k \in \text{dom}(\alpha) \mid A \in \alpha(k)\}$. Thus $\rho_{X^\omega}(g_6(p)) = A$. We conclude that $g_6$ translates $(\rho_X)^{\boxdot}$ computably to $\rho_{X^\omega}$.
The positive representation $\psi_<$ of the family of all closed sets of real numbers defined in [Wei00] is constructed in a similar way as $\rho_{\mathbb{R}}$, where $\mathcal{R} := (\mathbb{R}, C_b, \nu_{C_b})$ is the computable topological space from Example 4.3.17. Thus $\psi_\equiv \equiv cp \rho_{\mathbb{R}}$ can be verified easily. This implies $\psi_\equiv \equiv cp (\rho_{\mathbb{R}}) \equiv cp (\rho_{\mathbb{R}})\equiv$ by Lemma 4.2.8(4). We conclude that the concept of constructing a “positive” representation of the closed subsets of a sequential space by means of the operator $e^\equiv$ generalizes $\psi_<$.

4.4.3 Multirepresentations of Compact Subsets and of Sequentially Compact Subsets

In this subsection we show that family of compact subsets of an admissibly represented sequential space has an admissible multirepresentation which is admissible w.r.t. the Victoris topology (cf. [Bee93]). It generalizes the representation $\kappa$ from [Wei00] for the compact sets of the Euclidean space. Moreover, we construct an admissible multirepresentation of the sequentially compact sets of an admissibly representable weak limit space. This construction generalizes the representation $\kappa_c$ from [Wei00] for the compact (= sequentially compact) sets of the Euclidean space.

Let $\mathfrak{X} = (\mathcal{X}, \rightarrow_\mathfrak{X})$ be a weak limit space. By $\mathcal{SC}(\mathfrak{X})$ we denote the family of all sequentially compact subsets of $\mathfrak{X}$ and by $\mathcal{K}(\mathfrak{X})$ the family of all compact subsets of the associated topological space $\mathcal{T}(\mathfrak{X}) = (\mathcal{X}, seq(\mathfrak{X}))$ (cf. Subsection 2.2.5). Sequential compactness and countable compactness in $\mathfrak{X}$ can be characterized by means of the following operators $e_{\mathfrak{X}, Li}$ and $e_{\mathfrak{X}, Si}$. For a subset $M \subseteq X$ we define the functions $e^-_{\mathfrak{X}, Si}(M) : \mathcal{C}(\mathfrak{X}, \mathfrak{S}_i) \rightarrow Si$ and $e^-_{\mathfrak{X}, Li}(M) : \mathcal{C}(\mathfrak{X}, \mathfrak{L}_i) \rightarrow Li$ by

$$
e^-_{\mathfrak{X}, Si}(M)(h) := \begin{cases} \top & \text{if } (\exists x \in M) h(x) = \bot \\ \bot & \text{otherwise} \end{cases} \quad (h \in \mathcal{C}(\mathfrak{X}, \mathfrak{S}_i)),
$$

$$
e^-_{\mathfrak{X}, Li}(M)(k) := \begin{cases} \top & \text{if } (\exists x \in M) k(x) = \bot \\ \bot & \text{if } (\forall x \in M) k(x) = \top \\ \uparrow & \text{otherwise} \end{cases} \quad (k \in \mathcal{C}(\mathfrak{X}, \mathfrak{L}_i)).
$$

Clearly, $e^-_{\mathfrak{X}, Si}(\{x\}) = e^+_{\mathfrak{X}, Si}(\{x\}) = e_{\mathfrak{X}, Si}(x)$ and $e^-_{\mathfrak{X}, Li}(\{x\}) = e_{\mathfrak{X}, Li}(x)$ holds for every $x \in \mathfrak{X}$.

**Lemma 4.4.8 (Characterization of countable and sequential compactness)**

Let $\mathfrak{X} = (\mathcal{X}, \rightarrow_\mathfrak{X})$ be a weak limit space, and let $K \subseteq X$.

1. $K$ is sequentially compact in $\mathfrak{X}$ if and only if $e^-_{\mathfrak{X}, Li}(K) \in \mathcal{C}(\mathcal{C}(\mathfrak{X}, \mathfrak{L}_i), \mathfrak{L}_i)$.

2. $K$ is countably compact in $\mathcal{T}(\mathfrak{X})$ if and only if $e^-_{\mathfrak{X}, Si}(K) \in \mathcal{C}(\mathcal{C}(\mathfrak{X}, \mathfrak{S}_i), \mathfrak{S}_i)$.

**Proof:**

1. “$\Rightarrow$”: Let $K$ be sequentially compact. Let $(h_n)_{n \leq \infty}$ be a sequence in $\mathcal{C}(\mathfrak{X}, \mathfrak{L}_i)$ such that $(e^-_{\mathfrak{X}, Li}(K)(h_n))_n$ does not converge to $e^-_{\mathfrak{X}, Li}(K)(h_\infty)$ in $\mathfrak{L}_i$. Then $e^-_{\mathfrak{X}, Li}(K)(h_\infty) = \top$ and there is a strictly increasing function $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ with $e^-_{\mathfrak{X}, Li}(K)(h_{\varphi(n)}) = \bot$. Thus for every $n \in \mathbb{N}$ there is some $x_n \in K$ with $h_{\varphi(n)}(x_n) = \bot$. By sequential compactness of $K$, there exists some $x_\infty \in K$ and a strictly increasing function
ψ : N → N with (x_ψ(n))_n → x_∞. As e_{X,Σ}^−(K)(h_∞) = T, we have h_∞(x_∞) = T. Thus (h_ψ(x_ψ(n))_n does not converge to h_∞(x_∞) which implies that neither (h_ψ(x_ψ(n))_n nor (h_n)_n converges to h_∞ in C(Σ, Li). We conclude that e_{X,Σ}^−(K) is sequentially continuous, i.e. e_{X,Σ}^−(K) ∈ C(C(Σ, Li), Li).

“⇐=”: Let e_{X,Σ}^−(K) be a continuous function between C(Σ, Li) and Li.

Let (x_n)_n be a sequence in K. For n ∈ N and y ∈ X, we define the functions h_n, h_∞ : X → Li by

\[ h_n(y) := \begin{cases} \top & \text{if } y = x_n \\ \bot & \text{otherwise} \end{cases} \quad \text{and} \quad h_∞(y) := \begin{cases} \top & \text{if } y \in K \\ \bot & \text{otherwise} \end{cases} \]

As these functions have ranges that do not simultaneously contain ⊥ and T, they are in C(Σ, Li). Since h_∞[K] ⊆ {T} and (∀n ∈ N) h_n(x_n) = ⊥, we have on the one hand e_{X,Σ}^−(K)(h_∞) = T and on the other hand e_{X,Σ}^−(K)(h_n) = ⊥ for every n ∈ N. Thus (e_{X,Σ}^−(K)(h_n))_n does not converge to e_{X,Σ}^−(K)(h_∞). From continuity of e_{X,Σ}^−(K) it follows that (h_n)_n does not converge continuously to h_∞. By Equivalence (4.3) on page 97, there is a convergent sequence (y_n)_n≤∞ of X with (h_n(y_n))_n → e_{X,Σ}^−(h_∞(y_∞)). Thus h_∞(y_∞) = T and there exists a strictly increasing function ϕ : N → N with h_ϕ(n)(y_ϕ(n)) = ⊥ for all n ∈ N. It follows that (x_ϕ(n))_n is a subsequence of (x_n)_n converging to y_∞ ∈ K. Therefore K is sequentially compact.

(2) “⇒”:

Let K be countably compact in T(Σ).

Let (h_n)_n≤∞ be a convergent sequence of C(Σ, Li). The only interesting case is e_{X,Σ}^−(K)(h_∞) = T, i.e. h_∞[K] ⊆ {T}. By Lemma 4.2.2, ⇒ X,Σ is induced by the countably compact open topology τ_{X,Σ}^CoCo on C(Σ, Li). Since F(K, {T}) ∈ τ_{X,Σ}^CoCo there is some n_0 ∈ N with, for all n ≥ n_0, h_n ∈ F(K, {T}) and hence e_{X,Σ}^−(K)(h_n) = T. Therefore (e_{X,Σ}^−(K)(h_n))_n → e_{X,Σ}^−(h_∞). We conclude that e_{X,Σ}^−(K) is a continuous function from C(Σ, Li) to Li.

“⇐”: Let e_{X,Σ}^−(K) be a continuous function between C(Σ, Li) and Li.

Let (O_n)_n∈N be a sequence of sequentially open sets with K ⊆ \bigcup_{n∈N} O_n. For n ≤ ∞ and x ∈ X, we define the continuous function h_n : X → Li by

\[ h_n(x) := \begin{cases} \top & \text{if } x ∈ \bigcup_{i<n} O_i \\ \bot & \text{otherwise} \end{cases} \]

and prove that (h_n)_n≤∞ is a convergent sequence of C(Σ, Li). Let (x_n)_n≤∞ be a convergent sequence of X. The only interesting case is h_∞(x_∞) = T, in which there is some n_1 ∈ N with x_∞ ∈ O_{n_1}. Since O_{n_1} is sequentially open, there is some n_2 ∈ N with (∀n ≥ n_2) x_n ∈ O_{n_1}. Hence h_n(x_n) = T for all n ≥ max{n_1, n_2}. We conclude by Equivalence (4.3) on page 97 that (h_n)_n converges continuously to h_∞.

Since e_{X,Σ}^−(K)(h_∞) = T, there is some m ∈ N with (∀n ≥ m) e_{X,Σ}^−(K)(h_n) = T by sequential continuity of e_{X,Σ}^−(K). Hence K ⊆ O_0 ∪ ... ∪ O_m. We conclude that K is countably compact.

✓
4.4 Multirepresentations of Hyperspaces

Let $\mathcal{X} = (X, \tau_\mathcal{X})$ be a sequential space, and let $\delta : \subseteq \mathbb{N}^\omega \rightrightarrows X$ be a Top-quotient multirepresentation of $\mathcal{X}$. By Proposition 3.3.1(5), compactness and countable compactness coincide in $\mathcal{X}$, because $\mathcal{X}$ has an admissible multirepresentation by Proposition 4.3.2. Lemma 4.4.8 motivates the following multirepresentation $\delta^{miss}$ of the family $\mathcal{K}(\mathcal{X})$ of all compact sets of $\mathcal{X}$.

We define $\delta^{miss} : \subseteq \mathbb{N}^\omega \rightrightarrows \mathcal{K}(\mathcal{X})$ by

$$
\delta^{miss}[p] := \left\{ K \in \mathcal{K}(\mathcal{X}) \mid \epsilon_{\mathcal{X}, \mathcal{E}_\mathcal{E}}(K) = [\mathcal{K}(\mathcal{X}, \tau_\mathcal{X}) \rightarrow \epsilon_{\mathcal{E}_\mathcal{E}}(p)] \right\}.
$$

Proposition 4.2.5 and Lemma 4.4.8 imply that $\delta^{miss}$ is actually a sequentially open set of the hyperspace $\mathcal{K}(\mathcal{X})$. Moreover, a $\delta^{miss}$-name $p$ of $K$ yields the family of those closed sets $A$ that “miss” $K$ in the sense that $A$ does not intersect $K$. By Lemma 4.4.8, this family is actually a sequentially open set of the hyperspace $(\text{sclo}(\mathcal{X}), \tau_\mathcal{X})$. Moreover, a $\delta^{miss}$-name $p$ of $K$ can also be regarded as a name of the family of all open sets that contain $K$.

On $\mathcal{K}(\mathcal{X})$ we consider two topologies, the miss topology $\tau_{\mathcal{X}}^{\text{miss}}$ and the Vietoris topology $\tau_{\mathcal{X}}^{\text{Viet}}$ (cf. [Smy92, Bee93, BrPr01]). The miss topology $\tau_{\mathcal{X}}^{\text{miss}}$ has the family

$$
\mathcal{S}_{\mathcal{X}}^{\text{miss}} := \left\{ \left( \{ K \in \mathcal{K}(\mathcal{X}) \mid K \cap A = \emptyset \} \mid A \text{ is closed in } \mathcal{X} \right) \right\}
$$

has its subbase; the Vietoris topology $\tau_{\mathcal{X}}^{\text{Viet}}$ is defined to be the conjunction $\tau_{\mathcal{X}}^{\text{hit}} \wedge \tau_{\mathcal{X}}^{\text{miss}}$ of the topologies $\tau_{\mathcal{X}}^{\text{hit}}$ and $\tau_{\mathcal{X}}^{\text{miss}}$ (cf. Subsections 4.4.2 and 4.1.4). The name $\tau_{\mathcal{X}}^{\text{miss}}$ is motivated by the fact that $\tau_{\mathcal{X}}^{\text{miss}}$ provides about a given compact set $K$ the information which closed sets $A$ “miss” $K$ in the sense that $K \cap A$ is empty.

We prove that $\delta^{miss}$ is admissible w.r.t. the miss topology and that the multirepresentation $\delta^K := \delta^{hit} \wedge \delta^{miss}$ is admissible w.r.t. the Vietoris topology.

**Proposition 4.4.9 (Admissibility of $\delta^{miss}$ and $\delta^K$)**

Let $\delta : \subseteq \mathbb{N}^\omega \rightrightarrows X$ be a Top-quotient multirepresentation of a sequential topological space $\mathcal{X} = (X, \tau_\mathcal{X})$.

1. $\delta^{miss}$ is admissible with respect to the miss topology $\tau_{\mathcal{X}}^{\text{miss}}$. Moreover, the pair $((\mathcal{K}(\mathcal{X}), \text{seq}(\tau_{\mathcal{X}}^{\text{miss}})), \delta^{miss})$ is an object of EffSeq.

2. $\delta^K$ is admissible with respect to the Vietoris topology $\tau_{\mathcal{X}}^{\text{Viet}}$. Moreover, the pair $((\mathcal{K}(\mathcal{X}), \text{seq}(\tau_{\mathcal{X}}^{\text{Viet}})), \delta^K)$ is an object of EffSeq.

3. $\delta^{miss}$ is single-valued if and only if $\mathcal{X}$ is a $T_1$-space.

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16One can show that the family $\{ \{ K \in \mathcal{K}(\mathcal{X}) \mid K \subseteq B \} \mid B \in \mathcal{B} \}$ is a pseudosubbase of $(\mathcal{K}(\mathcal{X}), \tau_{\mathcal{X}}^{\text{miss}})$, whenever $\mathcal{B}$ is a countable $\mathcal{M}$-pseudobase of $\mathcal{X}$. 
Proof:

(1) We show at first that $e_{\mathcal{X},\mathcal{E}_i}$ is a homomorphism between $(\mathcal{K}(\mathcal{X}),\rightarrow_{\mathcal{X}})$ and the function space $\mathcal{C}(\mathcal{C}(\mathcal{X},\mathcal{E}_i),\mathcal{E}_i)$, i.e.,

$$(K_n)_{n \rightarrow \infty} \iff (e_{\mathcal{X},\mathcal{E}_i}(K_n))_{n \rightarrow \mathcal{C}(\mathcal{X},\mathcal{E}_i),\mathcal{E}_i} e_{\mathcal{X},\mathcal{E}_i}(K_\infty)$$

holds for all sequences $(K_n)_{n \leq \infty}$ in $\mathcal{K}(\mathcal{X})$.

$\implies$ Let $(K_n)_{n \leq \infty}$ converge to $K_\infty$ in the topological space $(\mathcal{K}(\mathcal{X}),\tau^\mathcal{X}\mathcal{E}_i)$. Let $(h_n)_{n \leq \infty}$ be a convergent sequence of $\mathcal{C}(\mathcal{X},\mathcal{E}_i)$. The only interesting case is $e_{\mathcal{X},\mathcal{E}_i}(K_n)(h_\infty) = \top$, i.e., $h_\infty[K_\infty] \subseteq \{\top\}$. Since the convergence relation on $\mathcal{C}(\mathcal{X},\mathcal{E}_i)$ is induced by the compact open topology $\tau^\mathcal{X}_ep$ (cf. Lemma 4.2.2) and $h_\infty \in \mathcal{F}(K_\infty,\{\top\})$, there exists some $n_1 \in \mathbb{N}$ with $(\forall n \geq n_1) h_n[K_\infty] \subseteq \{\top\}$. As the transpose $h^T : \mathbb{N} \times X \rightarrow Si$ of $(h_n)_{n \leq \infty}$ (cf. Equation (4.1)) is $(-\infty, -\infty, -\infty)$-continuous and $\{n_1, \ldots, \infty\}$ is a sequentially compact subset of $ð\mathbb{N}, -\infty\)$, the set

$$U := \{x \in X \mid (\forall n \in \{n_1, \ldots, \infty\}) h_n(x) \in \{\top\}\}$$

is sequentially open in $\mathcal{X}$ by Lemma 2.2.9. Hence $X \setminus U$ is closed in the sequential space $\mathcal{X}$. As $\{K \in \mathcal{K}(\mathcal{X}) \mid K \subseteq U\}$ is an open set of $(\mathcal{K}(\mathcal{X}),\tau^\mathcal{X}\mathcal{E}_i)$ containing $K_\infty$, there is some $n_2 \in \mathbb{N}$ with $K_n \subseteq U$ for all $n \geq n_2$. This implies $h_n[K_n] \subseteq \{\top\}$ and $e_{\mathcal{X},\mathcal{E}_i}(K_n)(h_n) = \top$ for all $n \geq \max\{n_1, n_2\}$. We conclude by Equivalence (4.3) that $(e_{\mathcal{X},\mathcal{E}_i}(K_n))_{n}$ converges continuously to $e_{\mathcal{X},\mathcal{E}_i}(K_\infty)$.

$\impliedby$ Let $(e_{\mathcal{X},\mathcal{E}_i}(K_n))_{n}$ converge to $e_{\mathcal{X},\mathcal{E}_i}(K_\infty)$ in $\mathcal{C}(\mathcal{X},\mathcal{E}_i),\mathcal{E}_i)$. Let $H \in \mathcal{S}_X^\mathcal{E}_i$ with $K_\infty \subseteq H$. Then there exists an open set $U \in \tau_H$ with $H = \{C \in \mathcal{K}(\mathcal{X}) \mid C \subseteq U\}$. Since $cf_U \in \mathcal{C}(\mathcal{X},\mathcal{E}_i)$ and $e_{\mathcal{X},\mathcal{E}_i}(K_n)(cf_U) = \top$, we have, for almost all $n \in \mathbb{N}$, $e_{\mathcal{X},\mathcal{E}_i}(K_n)(cf_U) = \top$ and thus $K_n \in H$. As $\mathcal{S}_X^\mathcal{E}_i$ is a subbase of $\tau^\mathcal{X}_{ep}$, $(K_n)_{n \leq \infty}$ converges to $K_\infty$ in $(\mathcal{K}(\mathcal{X}),\tau^\mathcal{X}_{ep})$.

Since $[\delta \rightarrow \varrho_{\mathcal{E}_i} \rightarrow \varrho_{\mathcal{E}_i}]$ is an admissible multirepresentation of $\mathcal{C}(\mathcal{C}(\mathcal{X},\mathcal{E}_i),\mathcal{E}_i)$ by Proposition 4.2.5, $\delta^\mathcal{E}_i$ is an admissible multirepresentation of $(\mathcal{K}(\mathcal{X}),\tau^\mathcal{X}_{ep})$ by Proposition 4.1.6. The computable equivalence $\delta^\mathcal{X} \equiv \mathcal{E}_i \delta^\mathcal{X}$ follows from Propositions 4.3.13 and 4.3.14(5).

(2) The admissibility of $\delta^\mathcal{X}$ w.r.t. the Vietoris topology follows from (1), Proposition 4.4.5, and Proposition 4.4.18. Proposition 4.3.14(2) implies $\delta^\mathcal{X} \equiv \mathcal{E}_i (\delta^\mathcal{X})^\mathcal{E}_i$.

(3) $\iff$ Let $X$ be a $T_1$-space.

Let $K_1$ and $K_2$ be two compacts subsets of $X$ and $x \in K_1 \setminus K_2$. As $X$ is a $T_1$-space, $\{x\}$ is a closed set and $H := \{K \in \mathcal{K}(\mathcal{X}) \mid K \cap \{x\} = \emptyset\}$ is an open set of the hyperspace $(\mathcal{K}(\mathcal{X}),\tau^\mathcal{X}_{ep})$. Since $H$ contains $K_2$, but not $K_1$, $\tau^\mathcal{X}_{ep}$ has the $T_0$-property. By Lemma 2.4.6, the continuous correspondence $\delta^\mathcal{X}$ is single-valued.

$\implies$ Let $\delta^\mathcal{X}$ be single-valued.

Let $x \in X$. By Proposition 2.3.13, $\tau^\mathcal{X}_{ep}$ is a $T_0$-topology. Thus for every $y \in X \setminus \{x\}$ there is a closed set $A_y$ with $\{x, y\} \cap A_y \neq \emptyset$ and $\emptyset \cap A_y = \emptyset$, because $\{x, y\}$ and $\emptyset$ are compact and $\mathcal{S}_X^\mathcal{E}_i$ is a subbase of $\tau^\mathcal{X}_{ep}$. It follows $\{x\} = \bigcap \{A_y \mid y \in X \setminus \{x\}\}$. Thus $\{x\}$ is closed. Therefore $X$ is a $T_1$-space.
The first statement of the next proposition can be regarded as an effectivization of the well-known lemma stating that the continuous image of a compact set is compact.

**Proposition 4.4.10**
Let \( \delta \subseteq \mathbb{N}^\omega \rightarrow X \) be a Top-quotient multirepresentation of a sequential topological space \( X = (X, \tau_X) \), and let \( \mathcal{Y} = ((Y, \tau_Y), \gamma) \) be an object of EffSeq. Let \( f : X \rightarrow Y \) be a function, and let \( F : \mathcal{K}(X) \rightarrow \mathcal{K}(\mathcal{Y}) \) be defined by \( F(K) := f[K] \). Then the following statements are equivalent:

(a) \( f \) is \((\delta, \gamma)\)-computable.
(b) \( F \) is \((\delta^{\text{miss}}, \gamma^{\text{miss}})\)-computable.
(c) \( F \) is \((\delta^K, \gamma^K)\)-computable.

**Proof:**

\( (a) \implies (b), (c) \): The function \( \mathcal{H}_{\mathcal{E}_1} \circ \mathcal{H}_{\mathcal{E}_1}(f) : C(C(X, \mathcal{E}_1), \mathcal{E}_1) \rightarrow C(C(\mathcal{Y}, \mathcal{E}_1), \mathcal{E}_1) \) is computable w.r.t. \( [[\delta \rightarrow \mathcal{E}_1] \rightarrow \mathcal{E}_1] \) and \( [[\gamma \rightarrow \mathcal{E}_1] \rightarrow \mathcal{E}_1] \) by Lemmas 4.3.4 and 4.2.8(1). Let \( g_1 : \subseteq \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) be a computable function realizing \( \mathcal{H}_{\mathcal{E}_1} \circ \mathcal{H}_{\mathcal{E}_1}(f) \) w.r.t. \( \delta^{\text{miss}} \) and \( \gamma^{\text{miss}} \). Let \( p \in \mathbb{N}^\omega \) and \( K \in \delta^{\text{miss}}[p] \). For every \( h \in C(X, \mathcal{E}_1) \), we obtain from Equation (4.12) on page 121

\[
e_{\mathcal{E}_1}(f[K])(h) = \begin{cases} \top & \text{if } (\forall y \in f[K]) h(y) = \top \\ \bot & \text{otherwise} \end{cases}
\]

Thus \( e_{\mathcal{E}_1}(f[K]) = H^2(\mathcal{E}_1)(e_{\mathcal{E}_1}(K)) \). Since \( [[\delta \rightarrow \mathcal{E}_1] \rightarrow \mathcal{E}_1](p) = e_{\mathcal{E}_1}(K) \), we obtain

\[
e_{\mathcal{E}_1}(F(K)) = H^2(\mathcal{E}_1)(e_{\mathcal{E}_1}(K)) = H^2(\mathcal{E}_1)(e_{\mathcal{E}_1}(K))(g_1(p)) = H^2(\mathcal{E}_1)(e_{\mathcal{E}_1}(K))(g_1(p)).
\]

Hence \( F(K) \in \gamma^{\text{miss}}[g_1(p)] \). We conclude that \( g_1 \) realizes \( F \) w.r.t. \( \delta^{\text{miss}} \) and \( \gamma^{\text{miss}} \).

By Proposition 4.4.6, there is a computable function \( g_2 : \subseteq \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) realizing \( F \) w.r.t. \( \delta^{\text{hit}} \) and \( \gamma^{\text{hit}} \). Obviously, the computable function \( g_2 : \subseteq \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) defined by \( g_2(p, q) := (g_2(p), g_1(q)) \) realizes \( F \) w.r.t. \( \delta^K = \delta^{\text{hit}} \land \delta^{\text{miss}} \) and \( \gamma^K = \gamma^{\text{hit}} \land \gamma^{\text{miss}} \).

\( (b) \implies (a) \): Let \( g : \subseteq \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) be a computable function realizing \( F \) w.r.t. \( \delta^{\text{miss}} \) and \( \gamma^{\text{miss}} \). Let \( p \in \mathbb{N}^\omega \) and \( x \in \delta^\mathcal{E}[p] \). Equivalence (4.21) implies \( x \in \delta^{\text{miss}}[p] \), hence \( \{f(x)\} = F(\{x\}) \in \gamma^{\text{miss}}[g(p)] \). Again by Equivalence (4.21) it follows \( f(x) \in \gamma^\mathcal{E}[g(p)] \). Hence \( f \) is computable w.r.t. \( \delta^\mathcal{E} \) and \( \gamma^\mathcal{E} \). Since \( \delta \leq \delta^{\text{cp}} \delta^\mathcal{E} \) and \( \gamma \equiv \delta^{\text{cp}} \gamma^\mathcal{E}, f \) is computable w.r.t. \( \delta \) and \( \gamma \) by Lemma 2.4.4.

\( (c) \implies (a) \): Let \( g : \subseteq \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) be a computable function realizing \( F \) w.r.t. \( \delta^K \) and \( \gamma^K \). Let \( p \in \mathbb{N}^\omega \) and \( x \in \delta^\mathcal{E}[p] \). Equivalence (4.21) implies \( x \in \delta^K[p, p] \), hence \( \{f(x)\} = F(\{x\}) \in \gamma^K[g(p, p)] \) and \( \{f(x)\} \in \gamma^{\text{miss}}[\pi_{2, 2}(g(p, p))] \), where \( \langle \cdot, \cdot \rangle : \mathbb{N}^\omega \times \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) and \( \pi_{2, 2} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega \) are the computable functions from Subsection
2.1.7. From Equivalence (4.21) it follows $f(x) \in \gamma^\delta[\pi_{2,2}(g(p,p))]$. Therefore $f$ is computable w.r.t. $\delta^\delta$ and $\gamma^\delta$. Since $\delta \leq_{cp} \delta^\delta$ and $\gamma \equiv_{cp} \gamma^\delta$, $f$ is computable w.r.t. $\delta$ and $\gamma$ by Lemma 2.4.4.

Let $\mathcal{X} = (X, \rightarrow_\mathcal{X})$ be a limit space, and let $\delta : \subseteq \mathbb{N}^\omega \Rightarrow X$ be a $\text{Lim}$–quotient multirepresentation of $\mathcal{X}$. Using Lemma 4.4.8, we define the multirepresentation $\delta^{\text{SC}} : \subseteq \mathbb{N}^\omega \Rightarrow \text{SC}(\mathcal{X})$ of the family of all sequentially compact sets of $\mathcal{X}$ by

$$\delta^{\text{SC}}[p] := \{ K \in \text{SC}(\mathcal{X}) \mid e_{\mathcal{X},\delta}(K) \in [[\delta \rightarrow \varnothing \delta] \rightarrow \varnothing \delta][p] \}.$$  

Lemma 4.4.8 and Proposition 4.2.5 imply that $\delta^{\text{SC}}$ is actually surjective. We equip the family $\text{SC}(\mathcal{X})$ with a convergence relation denoted by $\rightarrow_{\text{sc}(\mathcal{X})}$. For a sequence $(K_n)_{n \leq \infty}$ of sequentially compact sets we define $\rightarrow_{\text{sc}(\mathcal{X})}$ by

$$(K_n)_{n \rightarrow_{\text{sc}(\mathcal{X})} K}$$

$$\iff (\forall \varphi : \mathbb{N} \rightarrow \mathbb{N} \text{ strictly increasing})(\varphi(x)_n \in \prod_{n \in \mathbb{N}} K_{\varphi(n)})$$

$$(\exists \psi : \mathbb{N} \rightarrow \mathbb{N} \text{ strictly increasing})(\exists x_\infty \in K_\infty)(x_{\psi(n)} \rightarrow_\mathcal{X} x_\infty).$$

One easily verifies that $(\text{SC}(\mathcal{X}), \rightarrow_{\text{sc}(\mathcal{X})})$ is a limit space. We prove that $\delta^{\text{SC}}$ is admissible w.r.t. this convergence relation $\rightarrow_{\text{sc}(\mathcal{X})}$. Moreover, we show an effectivization of the lemma that the continuous image of a sequentially compact set is sequentially compact.

**Proposition 4.4.11 (Admissibility of $\delta^{\text{SC}}$)**

Let $\mathcal{X} = (X, \rightarrow_\mathcal{X})$ be a limit space, and let $\delta : \subseteq \mathbb{N}^\omega \Rightarrow X$ be a $\text{Lim}$–quotient multirepresentation of $\mathcal{X}$. Let $\mathcal{Y} = ((Y, \rightarrow_\mathcal{Y}), \gamma)$ be an object of $\text{EffLim}$.

1. $\delta^{\text{SC}}$ is an admissible multirepresentation of the limit space $(\text{SC}(\mathcal{X}), \rightarrow_{\text{sc}(\mathcal{X})})$. Moreover, $((\text{SC}(\mathcal{X}), \rightarrow_{\text{sc}(\mathcal{X})}), \delta^{\text{SC}})$ is an object of $\text{EffLim}$.

2. A function $f : X \rightarrow Y$ is $(\delta, \gamma)$–computable if and only if the function $F : \text{SC}(\mathcal{X}) \rightarrow \text{SC}(\mathcal{Y})$ defined by $F(K) := f[K]$ is $(\delta^{\text{SC}}, \gamma^{\text{SC}})$–computable.

**Proof:**

1. We show at first that $e_{\mathcal{X},\delta}$ is a homomorphism between $(\text{SC}(\mathcal{X}), \rightarrow_{\text{sc}(\mathcal{X})})$ and the function space $\mathcal{C}(\mathcal{C}(\mathcal{X}, \mathcal{L}), \mathcal{L})$, i.e.,

$$(K_n)_{n \rightarrow_{\text{sc}(\mathcal{X})} K} \iff (e_{\mathcal{X},\delta}(K)_n)_{n \in \mathcal{C}(\mathcal{X}, \mathcal{L}), \mathcal{L}} e_{\mathcal{X},\delta}(K_\infty)$$

holds for all sequences $(K_n)_{n \leq \infty}$ in $\text{SC}(\mathcal{X})$. The proof is quite similar to the one of Lemma 4.4.8(1). Since $\mathcal{C}(\mathcal{C}(\mathcal{X}, \mathcal{L}), \mathcal{L})$ is a limit space by Proposition 4.2.1, this equivalence implies that $(\text{SC}(\mathcal{X}), \rightarrow_{\text{sc}(\mathcal{X})})$ is a limit space.

Let $\{ e_{\mathcal{X},\delta}(K_n) \}_{n \leq \infty}$ be a convergent sequence of $\mathcal{C}(\mathcal{C}(\mathcal{X}, \mathcal{L}), \mathcal{L})$. Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function, and let $(x_n)_n$ be a sequence with $(\forall n \in \mathbb{N}) x_n \in K_{\varphi(n)}$. For $n \in \mathbb{N}$ and $y \in X$, we define the functions $h_n, h_\infty : X \rightarrow \mathcal{L}$ by

$$h_n(y) := \begin{cases} \perp & \text{if } y = x_n \\ \uparrow & \text{otherwise} \end{cases}$$

and

$$h_\infty(y) := \begin{cases} \top & \text{if } y \in K_\infty \\ \uparrow & \text{otherwise}. \end{cases}$$
Since these functions have ranges that do not simultaneously contain \( \perp \) and \( \top \), they are in \( C(\mathcal{X}, \mathcal{L}) \). Since \( h_\infty [K_\infty] \subseteq \{ \top \} \) and \( (\forall n \in \mathbb{N}) h_n(x_n) = \perp \), we have on the one hand \( e_{\mathcal{X}, \mathcal{L}}(K_\infty)(h_\infty) = \top \) and on the other hand \( e_{\mathcal{X}, \mathcal{L}}(K_{\psi(n)})(h_n) = \perp \) for every \( n \in \mathbb{N} \). Thus \( (e_{\mathcal{X}, \mathcal{L}}(K_{\psi(n)})(h_n))_n \) does not converge to \( e_{\mathcal{X}, \mathcal{L}}(K_\infty)(h_\infty) \).

Since \( (e_{\mathcal{X}, \mathcal{L}}(K_{\psi(n)}))_n \) converges continuously to \( e_{\mathcal{X}, \mathcal{L}}(K_\infty) \), \( (h_n)_n \) cannot converge to \( h_\infty \) in \( C(\mathcal{X}, \mathcal{L}) \) by Equivalence (4.3). Therefore, again by Equivalence (4.3), there is a convergent sequence \((y_n)_{n<\infty}\) of \( \mathcal{X} \) with \( (h_n(y_n))_n \not\rightarrow h_\infty(y_\infty) \).

Thus \( h_\infty(y_\infty) = \top \) and there exists a strictly increasing function \( \psi : \mathbb{N} \rightarrow \mathbb{N} \) with \( h_\psi(n)(y_\psi(n)) = \perp \) for all \( n \in \mathbb{N} \). It follows that \( (x_\psi(n))_n \), being equal to \( (y_\psi(n))_n \), is a sequence that converges to the element \( y_\infty \in K_\infty \). We conclude that \( (K_n)_n \) converges to \( K_\infty \) in \((SC(\mathcal{X}), \delta_{SC(\mathcal{X}))})\).

"⇐⇒": Let \( (K_n)_{n<\infty} \) be a convergent sequence of \((SC, \delta_{SC(\mathcal{X}))})\).

Suppose for contradiction that \( (e_{\mathcal{X}, \mathcal{L}}(K_n))_n \) does not converge in \( C(\mathcal{C}(\mathcal{X}, \mathcal{L}), \mathcal{L}) \) to \( e_{\mathcal{X}, \mathcal{L}}(K_\infty) \). By Equivalence (4.3), there is convergent sequence \((h_n)_{n<\infty} \) of \( C(\mathcal{X}, \mathcal{L}) \) such that \( e_{\mathcal{X}, \mathcal{L}}(K_\infty)(h_\infty) = \top \) and \( e_{\mathcal{X}, \mathcal{L}}(K_n)(h_n) = \perp \) for infinitely many \( n \in \mathbb{N} \). Hence there are a strictly increasing function \( \varphi : \mathbb{N} \rightarrow \mathbb{N} \) and a sequence \((x_n)_n \) with \( x_n \in K_{\varphi(n)} \) and \( h_{\varphi(n)}(x_n) = \perp \) for all \( n \in \mathbb{N} \). By definition of \( \delta_{SC(\mathcal{X}))} \), there are a strictly increasing function \( \psi : \mathbb{N} \rightarrow \mathbb{N} \) and some \( x_\infty \in K_\infty \) with \( (x_\psi(n))_n \not\rightarrow x_\infty \). Since \( (h_{\varphi(n)}(x_\psi(n)))_n \) converges continuously to \( h_\infty \) and \((h_{\varphi(n)}(x_\psi(n)))_n \) is equal to the constant sequence \((\perp)_n \), we have \( h_\infty(x_\infty) \neq \top \). This contradicts \( h_\infty[K_\infty] \subseteq \{ \top \} \).

Since \( [[\delta \rightarrow \varrho_\mathcal{L}] \rightarrow \varrho_\mathcal{L}] \) is an admissible multirepresentation of \( C(\mathcal{C}(\mathcal{X}, \mathcal{L}), \mathcal{L}) \) by Proposition 4.2.5, \( \delta^{SC} \) is an admissible multirepresentation of \((SC(\mathcal{X}), \delta_{SC(\mathcal{X}))}) \) by Proposition 4.1.6. Propositions 4.3.13 and 4.3.14(5) imply \( \delta^{SC} \equiv_{cp} (\delta^{SC})^\varepsilon \).

(2) This can be proved in a similar way as (a)\(\equiv\)(b) in Proposition 4.4.10.

Now we effectivize the lemma stating that every sequentially compact subset of a topological space is countably compact (cf. [Wil70, Ex. 17G]).

**Lemma 4.4.12 (Seq. compactness effectively implies count. compactness)**

Let \( \delta : \subseteq \mathbb{N}^w \Rightarrow X \) be an admissible multirepresentation of a sequential topological space \( \mathcal{X} = (X, \tau_\mathcal{X}) \). Then \( \delta^{SC} \leq_{cp} \delta^\text{miss} \).

**Proof:**

By Proposition 2.4.18, \( \delta \) is a \( \text{Lim} \)–quotient multirepresentation of \( (X, \rightarrow_\mathcal{X}) \) and \( \text{Top} \)–quotient multirepresentation of \( (X, \tau_\mathcal{X}) \).

Let \( p \in \mathbb{N}^w \) and \( K \in \delta^{SC}[p] \), i.e. \( e_{\mathcal{X}, \mathcal{L}}(K) \in [[\delta \rightarrow \varrho_\mathcal{L}] \rightarrow \varrho_\mathcal{L}][p] \). For \( q \in \text{dom}([[\delta \rightarrow \varrho_\mathcal{L}]) \) and \( h := [[\delta \rightarrow \varrho_\mathcal{L}](q) \), we have \( h|_L \in [[\delta \rightarrow \varrho_\mathcal{L}][q] \) and thus

\[
e_{\mathcal{X}, \mathcal{L}}(K)(h) = \begin{cases} \top & \text{if } (\forall x \in K) h(x) = \top \\ \perp & \text{otherwise} \end{cases} = e_{\mathcal{X}, \mathcal{L}}(K)(h|_L) \in \varrho_\mathcal{L} \eta_p(q) \cap SI = \{ \varrho_\mathcal{L}(\eta_p(q)) \}.
\]

Hence \( [[\delta \rightarrow \varrho_\mathcal{L}] \rightarrow \varrho_\mathcal{L}](p) \) = \( e_{\mathcal{X}, \mathcal{L}}(K) \). We conclude \( K \in \delta^{\text{miss}}[p] \). Thus the identity function on \( \mathbb{N}^w \) translates \( \delta^{SC} \) computably to \( \delta^\text{miss} \).
Let $\mathfrak{X} = (X, B, \beta)$ be a computable topological space (cf. Subsection 4.3.6) such that $\tau_\mathfrak{X}$ is a $T_1$--topology and the set\footnote{We write $\beta^{\cap \cup}$ for the numbering $\left(\beta^\cap \right)^\cup$ of $\left(\beta^\cap \right)^\cup$, cf. Equation (4.16).} $\{(u, v) \mid \beta^{\cap \cup}(u) = \beta^{\cap \cup}(v)\}$ is recursively enumerable. We define $\mathfrak{R}(\mathfrak{X})$ to be the triple $(\mathcal{K}(\mathfrak{X}), D, \alpha)$, where the family $D$ and the numbering $\alpha : \subseteq \mathbb{N} \to D$ are given by

\[
D := \left\{ \{K \in \mathcal{K}(\mathfrak{X}) \mid K \subseteq V \} \mid V \in (B^\cap)^\cup \right\} \quad \text{and} \\
\alpha(n) := \left\{ \begin{array}{ll}
\{K \in \mathcal{K}(\mathfrak{X}) \mid K \subseteq \beta^{\cap \cup}(n)\} & \text{if } n \in \text{dom}(\beta^{\cap \cup}) \\
\text{div} & \text{otherwise}.
\end{array} \right.
\]

Compactness of every $K \in \mathcal{K}(\mathfrak{X})$ implies for every $U \in \tau_\mathfrak{X}$:

\[
\{ K \in \mathcal{K}(\mathfrak{X}) \mid K \subseteq U \} = \bigcup \{ \alpha(n) \mid n \in \text{dom}(\alpha), \beta^{\cap \cup}(n) \subseteq U \}.
\]

Hence $D$ is a subbase of $\tau_\mathfrak{X}^\text{miss}$. Propositions 4.4.9 and 2.3.13 imply that $\tau_\mathfrak{X}^\text{miss}$ is a $T_0$--topology. Since $\alpha(u) = \alpha(v) \iff \beta^{\cap \cup}(u) = \beta^{\cap \cup}(v)$ holds, $\mathfrak{R}(\mathfrak{X})$ is a computable topological space.

We show that the standard representation $\rho_\mathfrak{R}(\mathfrak{X})$ of $\mathfrak{R}(\mathfrak{X})$ is computably equivalent to $(\rho_\mathfrak{X})^\text{miss}$ and to $(\rho_\mathfrak{X})^\text{SC}$. This effectivizes the theorem that sequential compactness and compactness coincide for subsets of countably based spaces (cf. [Wil70, Ex. 17]).

**Proposition 4.4.13 (Computable equivalence of $\rho_\mathfrak{R}(\mathfrak{X})$, $(\rho_\mathfrak{X})^\text{miss}$, and $(\rho_\mathfrak{X})^\text{SC}$)**

Let $\mathfrak{X} = (X, B, \beta)$ be a computable topological space such that the set

\[
\{(u, v) \in \text{dom}(\beta^{\cap \cup}) \times \text{dom}(\beta^{\cap \cup}) \mid \beta^{\cap \cup}(u) = \beta^{\cap \cup}(v)\}
\]

is recursively enumerable and $\tau_\mathfrak{X}$ is a $T_1$--topology. Then $\rho_\mathfrak{R}(\mathfrak{X}) \equiv_{cp} (\rho_\mathfrak{X})^\text{miss} \equiv_{cp} (\rho_\mathfrak{X})^\text{SC}$.

**Proof:**

Let $\alpha$ be the subbase numbering of $\mathfrak{R}(\mathfrak{X})$. There are computable functions $l_1, l_2, l_3, l_4 : \mathbb{N} \to \mathbb{N}$ with

\[
\{(l_1(j), l_2(j)) \mid j \in \mathbb{N}\} = \{(u, v) \in \text{dom}(\beta^\cap) \times \text{dom}(\beta^\cap) \mid \beta^\cap(u) = \beta^\cap(v)\}
\]

and

\[
\{(l_3(j), l_4(j)) \mid j \in \mathbb{N}\} = \{(u, v) \in \text{dom}(\beta^{\cap \cup}) \times \text{dom}(\beta^{\cap \cup}) \mid \beta^{\cap \cup}(u) = \beta^{\cap \cup}(v)\},
\]

because $\{(u, v) \mid \beta^{\cap \cup}(u) = \beta^{\cap \cup}(v)\}$ and thus $\{(u, v) \mid \beta^\cap(u) = \beta^\cap(v)\}$ are recursively enumerable.

$(\rho_\mathfrak{X})^\text{miss} \leq_{cp} \rho_\mathfrak{R}(\mathfrak{X})$:

We define the function $g_1 : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ by

\[
g_1(t, r)(\langle a, b \rangle) := \left\{ \begin{array}{ll}
1 & \text{if } l_1(a) \in \nu_{\mathcal{N}}(t(0)) \text{ and } \nu_{\mathcal{N}}(l_2(a)) \subseteq \text{En}(r^{<b}; 0^\omega) \\
0 & \text{otherwise}
\end{array} \right.
\]

for all $t, r \in \mathbb{N}^\omega$ and $a, b \in \mathbb{N}$. Since $g_1$ is computable, there is computable function $g_2 : \mathbb{N}^\omega \to \mathbb{N}^\omega$ with $(\forall t, r \in \mathbb{N}^\omega)(\forall g_2(t)(r) = g_1(t, r)$ by the computable smn--Theorem.
4.4 Multirepresentations of Hyperspaces

Moreover, there is a computable monotone word function $\sigma_1 : \mathbb{N}^* \times \mathbb{N}^* \rightarrow \mathbb{N}^*$ approximating the computable function $(p, t) \mapsto \eta_p(g_2(t))$. We define the total function $g_3 : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ by

$$g_3(p)(a, b) := \begin{cases} l_3(a) + 1 & \text{if } \sigma_1(p^<b, l_3(a)0^b) \notin \{0\}^* \\ 0 & \text{otherwise.} \end{cases}$$

for $p \in \mathbb{N}^\omega$ and $a, b \in \mathbb{N}$. Clearly, $g_3$ is computable.

We prove that $g_3$ translates $(\rho_\chi)^{\text{miss}}$ to $\rho_{\mathcal{R}(\chi)}$. Let $p \in \text{dom}(\rho_\chi^{\text{miss}})$ and $K := \rho_\chi^{\text{miss}}(p)$.

Let $k \in \text{dom}(\alpha)$. For all $r \in \text{dom}(\rho_\chi)$ we have

$$\rho_\chi(r) \in \beta^{\gamma\cup}(k) \iff \exists i \in \nu_F(n(k)) \rho_\chi(r) \in \beta^\gamma(i) \iff \exists i \in \nu_F(n(k))(\exists j \in \mathbb{N})(\beta^\gamma(i) \land \nu_F(n(j) \subseteq \text{En}(r)) \iff \exists a \in \mathbb{N}(l_1(a) \in \nu_F(n(k) \land \nu_F(n(l_2(a)) \subseteq \text{En}(r)) \iff \exists a, b \in \mathbb{N}(g_1(k;0^\omega, r)(a, b) = 1 \iff \varrho_{\mathcal{E}n}(\eta_{g_2(k;0^\omega)}(r)) = 1.$$ 

Since $\eta_{g_2(k;0^\omega)}$ is total, it follows $[\rho_X \rightarrow \varrho_{\mathcal{E}n}](g_2(k;0^\omega)) = cf_{\beta^{\gamma\cup}(k)}$. Hence we have

$$K \in \alpha(k) \iff K \subseteq \beta^{\gamma\cup}(k) \iff e_{\mathcal{E}n}(K)(cf_{\beta^{\gamma\cup}(k)}) = 1 \iff \eta_p(g_2(k;0^\omega)) \neq 0^\omega \iff (\exists b \in \mathbb{N}) \sigma_1(p^<b, k0^b) \notin \{0\}^* \iff (\exists a, b \in \mathbb{N})(l_3(a) = k \land \sigma_1(p^<b, l_3(a)0^b) \notin \{0\}^*) \iff k \in \text{En}(g_3(p)) .$$

Since $\text{En}(g_3(p)) \subseteq \{l_3(a) \mid a \in \mathbb{N} \} = \text{dom}(\beta^{\gamma\cup}) = \text{dom}(\alpha)$, this implies $\text{En}(g_3(p)) = \{k \in \text{dom}(\alpha) \mid K \in \alpha(k)\}$. Therefore $\rho_{\mathcal{R}(\chi)}(g_3(p)) = K$. We conclude that $g_3$ translates $(\rho_\chi)^{\text{miss}}$ to $\rho_{\mathcal{R}(\chi)}$.

$\rho_{\mathcal{R}(\chi)} \leq_{cp} (\rho_\chi)^{\text{SC}}$.

There is a computable monotone word function $\sigma_2 : \mathbb{N}^* \times \mathbb{N}^* \rightarrow \mathbb{N}^*$ approximating the computable function $(q, r) \mapsto \eta_q(r)$. We define the functions $g_4 : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ and $g_5 : \mathbb{N}^\omega \times \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ by

$$g_4(q)(a, b, c) := \begin{cases} l_1(a) + 1 & \text{if } \nu_F(n(l_1(a)) = \text{En}(\nu_{\mathcal{E}n}(b);0^\omega) \land \sigma_2(q^<c, \nu_{\mathcal{E}n}(b)) \notin \{0\}^* \\ 0 & \text{otherwise.} \end{cases}$$

and

$$g_5(p, r)(a, b, c) := \begin{cases} 1 & \text{if } p(a) - 1 = l_3(b) \land \nu_F(n(l_4(b)) \subseteq \text{En}(r^<c;0^\omega) \\ 0 & \text{otherwise.} \end{cases}$$

for all $p, q, r \in \mathbb{N}^\omega$ and $a, b, c \in \mathbb{N}$. Clearly, $g_4$ and $g_5$ are computable. By the smn-Theorem, there is a computable function $g_6 : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$ with $(\forall p, q \in \mathbb{N}) \eta_{g_6(p)}(q) = g_5(p, g_4(q)).$

We prove that $g_6$ translates $\rho_{\mathcal{R}(\chi)}$ to $(\rho_\chi)^{\text{SC}}$. Let $p \in \text{dom}(\rho_{\mathcal{R}(\chi)})$ and $K := \rho_{\mathcal{R}(\chi)}(p)$. Let $q \in \mathbb{N}^\omega$ and $h \in [[\rho_X \rightarrow \varrho_{\mathcal{E}n} \rightarrow \varrho_{\mathcal{E}n}][q]$. Since $\text{En}(g_4(q)) \subseteq \text{dom}(\beta^\gamma)$, we can
define the open set $O_q$ by $O_q := \bigcup \{ \beta^\top(j) \mid j \in \text{En}(g_4(q)) \} = \theta_X^\top(g_4(q))$. We show $h^{-1}[\{\top\}] \subseteq O_q \subseteq X \setminus h^{-1}[\{\bot\}]$.

"$h^{-1}[\top] \subseteq O_q$":

Let $x \in h^{-1}[\top]$ and let $r$ be some $\rho_X$-name of $x$. As $\eta_q(r) \neq 0^\omega$, there are some $a, b, c \in \mathbb{N}$ with $\sigma_2(q^{<\omega}, r^{<\omega}) \notin \{0\}^\omega$, $\nu r(b) = r^{<\omega}$, and $\nu r(l_1(a)) = \text{En}(r^{<\omega}, 0^\omega)$. Since $l_1(a) \in \text{dom}(\beta^\top) \cap \text{En}(g_4(q))$ and $x \in \beta^\top(l_1(a))$, we have $x \in O_q$.

"$O_q \subseteq X \setminus h^{-1}[\bot]$":

Let $x \in O_q$. Then there are some $a, b, c \in \mathbb{N}$, $j \in \text{En}(g_4(q))$ with $x \in \beta^\top(j)$, $l_1(a) = j$, $\nu r(l_1(a)) = \text{En}(\nu r(b), 0^\omega)$ and $\sigma_2(q^{<\omega}, \nu r(b)) \notin \{0\}^\omega$. There is some $r \in \mathbb{N}^\omega$ with $\text{En}(r) = \{ i \in \mathbb{N} \mid x \in \beta(i) \}$. Since $\text{En}(\nu r(b), 0^\omega) \subseteq \{ i \in \mathbb{N} \mid x \in \beta(i) \}$, the $\omega$-word $s := \nu r(b) : r$ is a $\rho_X$-name of $x$. As $\eta_q(s) \neq 0^\omega$, we have $h(x) \neq \bot$. Hence $O_q \subseteq X \setminus h^{-1}[\bot]$.

The compactness of $K$ implies

$$K \subseteq O_q \iff (\exists d \in \mathbb{N}) (\nu r(d) \subseteq \text{En}(g_4(q)) \land K \subseteq \bigcup \{ \beta^\top(i) \mid i \in \nu r(d) \})$$

$$\iff (\exists a, d \in \mathbb{N}) (\nu r(d) \subseteq \text{En}(g_4(q)) \land \beta^\top(p(a) - 1) = \beta^\top(d))$$

$$\iff (\exists a, b \in \mathbb{N}) (\nu r(l_1(b)) \subseteq \text{En}(g_4(q)) \land l_3(b) = p(a) - 1)$$

$$\iff g_l(p, g_4(q)) \neq 0^\omega \iff \eta_{g_6}(q) \neq 0^\omega.$$ We obtain the implications

$$e_{X, \xi}(K)(h) = \top \implies K \subseteq h^{-1}[\top] \implies K \subseteq O_q \implies \top \in \varrho_N[\eta_{g_6}(q)]$$

and

$$e_{X, \xi}(K)(h) = \bot \implies K \cap h^{-1}[\bot] \neq \emptyset \implies K \not\subseteq O_q \implies \bot \in \varrho_N[\eta_{g_6}(q)].$$

Since $\eta_{g_6}(p)$ is total and $(\varrho_N)^{-1}[\{\top\}] = \mathbb{N}^\omega$, this implies $e_{X, \xi}(K)(h) \in \varrho_N[\eta_{g_6}(p)(q)]$ and $e_{X, \xi}(K)(h) \in [\rho_X \rightarrow \varrho_N] \rightarrow \varrho_N[\eta_{g_6}(p)]$. Thus $(\rho_X)^{SC}(g_6(p)) = K$. Hence $g_6$ translates $\rho_X(\xi)$ to $(\rho_X)^{SC}$.

$(\rho_X)^{SC} \leq cp$ $(\rho_X)^{\text{miss}}$. This follows from Proposition 4.3.16 and Lemma 4.4.12.

Let $\kappa$ and $\kappa_c$ be the two representations of the compact subsets of $\mathbb{R}$ introduced in [Wei00]. The representation $\kappa_c$ is defined in a similar way as $\rho_{\mathbb{R}}(\varnothing)$, where $\mathbb{R} = (\mathbb{R}, C_b, \kappa_{CL})$ is the computable topological space from Example 4.3.17. This substantiates $\kappa_c \equiv_{cp} \rho_{\mathbb{R}}(\varnothing)$. Proposition 4.4.13 implies $\kappa_c \equiv_{cp} (\rho_{\mathbb{R}})^{\text{miss}}$. By [Wei00, Lemma 5.2.10], $\kappa$ is computably equivalent to the conjunction of $\psi_\prec$ and $\kappa_c$. Here $\psi_\prec$ denotes the positive representation of the closed sets of $\mathbb{R}$ from the end of Subsection 4.2.4, which we know to be computably equivalent to $(\rho_{\mathbb{R}})^{\text{mod}}$. We obtain

$$\kappa \equiv_{cp} \psi_\prec \land \kappa_c \equiv_{cp} (\rho_{\mathbb{R}})^{\text{mod}} \land (\rho_{\mathbb{R}})^{\text{miss}} \equiv_{cp} (\rho_{\mathbb{R}})^{\text{hit}} \land \kappa_c \equiv_{cp} (\rho_{\mathbb{R}})^{\text{miss}} = (\rho_{\mathbb{R}})^{\kappa_\mathbb{R}}.$$ Therefore our concept of constructing multirepresentations of the compact subsets of a sequential space by means of the operators $\sigma^\kappa$ and $c^{\text{miss}}$ generalizes the representations $\kappa$ and $\kappa_c$ consistently.
4.4.4 Representations for Regularly Closed Sets

The WeakLim–quotient \((\mathbb{R}, \rightarrow_{\tau_{10}})\) generated by the decimal representation is an example for an admissibly representable weak limit space which is not a limit space (cf. Examples 2.2.5, 4.3.19). However, since real addition is not computable w.r.t. \(\rho_{10}\), this space seems to be a rather useless example of weak limit space failing Axiom (L3). In this subsection we present naturally defined admissible representations of “regularly closed sets” such that the weak limit spaces generated by these representations are not necessarily limit spaces. This demonstrates that the weak limit spaces, just as limit spaces and topological spaces, form a useful mathematical concept.

Let \(3 = (Z, \tau_3)\) be a topological space. A subset \(A\) of \(3\) is called regularly closed iff \(A\) is the closure of its interior, i.e. \(A = \text{Cls}(\text{Int}(A))\) (cf. [Wil70]). Non–empty proper regularly closed sets are called solids (cf. [ZiBr01]). By \(\mathcal{R}(3)\) we denote the family of all regularly closed subsets of \(3\).

Let \(X = (X, \mathcal{B}, \beta)\) be a computable topological space (cf. Subsection 4.3.6). Using the representation \(\theta^\text{en}_X\) of \(\tau_X\) from Subsection 4.4.1, a straightforward representation \(\delta^\text{reg}_X : \mathbb{N}^\omega \rightarrow \mathcal{R}(X)\) can be defined by

\[
\delta^\text{reg}_X(p) = A \iff \theta^\text{en}_X(p) = \text{Int}(A)
\]

for \(p \in \mathbb{N}^\omega\) and \(A \in \mathcal{R}(X)\). By Propositions 4.4.1, 4.4.3, and 4.1.4, \(\delta^\text{reg}_X\) is admissible w.r.t. the topology

\[
\tau^\text{reg}_X := \left\{ \{A \in \mathcal{R}(X) \mid \text{Int}(A) = \mathcal{V}\} \mid \mathcal{V} \in \tau^\text{en}_X \right\}.
\]

The disadvantage of \(\delta^\text{reg}_X\), where \(\mathcal{R} = (\mathbb{R}, \text{Cb}, \nu_{\text{Cb}})\) is the computable topological space from Example 4.3.17, because it is not sequentially continuous w.r.t. the convergence relation induced by \(\tau^\text{reg}_X\): the sequences \((A_n)_n := ((-1:-(\frac{1}{2})^n))_n\) and \((B_n)_n := ([\frac{1}{2}; 1])_n\) converge to, respectively, \(A_\infty := [-1; 0]\) and \(B_\infty := [0; 1]\), but \((A_n \cup B_n)_n\) does not converge to \([-1; 1]\) w.r.t. \(\tau^\text{reg}_X\), as none of the sets \(A_n \cup B_n\) contains the compact set \(\{0\} \subseteq \text{Int}([-1; 1])\).

For constructing a representation of \(\mathcal{R}(X)\) admitting computability of the union operator, we use the fact that all open sets \(U, V\) satisfy \(\text{Cls}(U \cup V) = \text{Cls}(U) \cup \text{Cls}(V)\) and \(\text{Cls}(U) \in \mathcal{R}(X)\) (cf. [Wil70]). We define the representations \(\psi^\text{reg}_X\) and \(\xi_X\) of \(\mathcal{R}(X)\) by

\[
\psi^\text{reg}_X(q) := \text{Cls}(\theta^\text{en}_X(q)) \quad \text{and} \quad \xi_X := \psi^\text{reg}_X \land \psi^\text{en}_X
\]

for all \(q \in \mathbb{N}^\omega\) (cf. [Sch02c]), where \(\theta^\text{en}_X\) and \(\psi^\text{en}_X\) are the representations from Subsection 4.4.1 of, respectively, the open sets and the closed sets of \(X\). The representation \(\xi_X\) is defined in a similar way as the representation \(\xi^\text{en}\) of the regularly closed sets of real numbers in [ZiBr01]. One can easily show that the set union operators on \(\tau_X\) and on \(\text{sclo}(X)\) are, respectively, \((\theta^\text{en}_X, \theta^\text{en}_X, \theta^\text{en}_X)\)–computable and \((\psi^\text{en}_X, \psi^\text{en}_X, \psi^\text{en}_X)\)–computable. It follows that set union on \(\mathcal{R}(X)\) is \((\psi^\text{reg}_X, \psi^\text{reg}_X, \psi^\text{reg}_X)\)–computable and \((\xi_X, \xi_X, \xi_X)\)–computable.

For proving admissibility of \(\psi^\text{reg}_X\), we at first equip \(\mathcal{R}(X)\) with a convergence relation denoted by \(\rightarrow_{\mathcal{R}(X)}\). It is defined by

\[
(A_n)_n \rightarrow_{\mathcal{R}(X)} A_\infty : \iff \forall U \in \tau_X \exists U \neq \emptyset \Rightarrow 
\]

\[
(\exists V \in \tau_X \forall V \neq \emptyset \wedge (\forall^* n \in \mathbb{N}) V \subseteq A_n \cap A_\infty)
\]
By checking Axioms (L1), (L5), (L6), one easily verifies that $(\mathcal{R}_4(\mathfrak{X}), -_{\mathcal{R}_4(\mathfrak{X})})$ is a weak limit space. In general, this space satisfies neither the $T_1$–property nor Axiom (L0), because for all sets $A, B \in \mathcal{R}_4(\mathfrak{X})$ the constant sequence $(A)_n$ converges to $B$ w.r.t. $-_{\mathcal{R}_4(\mathfrak{X})}$ if and only if $A \supseteq B$. On the other hand, one can show that the conjunction of $(\mathcal{R}_4(\mathfrak{X}), -_{\mathcal{R}_4(\mathfrak{X})})$ with the topological space $(\text{sclo}(\mathfrak{X}), \tau^\mathfrak{X})$, where $\tau^\mathfrak{X}$ is the upper Fell topology on $\text{sclo}(\mathfrak{X})$ from Subsection 4.4.1, satisfies Axiom (L0).

We prove that $\psi^\mathfrak{X}$ is an admissible representation of $(\mathcal{R}_4(\mathfrak{X}), -_{\mathcal{R}_4(\mathfrak{X})})$.

**Proposition 4.4.14 (Admissibility of $\psi^\mathfrak{X}$ and $\xi_X$)**

Let $\mathfrak{X} = (X, \mathcal{B}, \beta)$ be a computable topological space. Then $\psi^\mathfrak{X}$ is an admissible representation of the hyperspace $(\mathcal{R}_4(\mathfrak{X}), -_{\mathcal{R}_4(\mathfrak{X})})$. Moreover, $\xi_X$ is an admissible representation of the conjunction $(\mathcal{R}_4(\mathfrak{X}), -_{\mathcal{R}_4(\mathfrak{X})}) \land (\text{sclo}(\mathfrak{X}), -_{\text{sclo}(\mathfrak{X})})$.

**Proof:**

Continuity of $\psi^\mathfrak{X}$: Let $(q_n)_{n \leq \infty}$ be a convergent sequence in $\text{dom}(\psi^\mathfrak{X})$. For every $n \in \mathbb{N}$ let $A_n := \psi^\mathfrak{X}(q_n)$ and $O_n := \theta^\mathfrak{X}_n(q_n)$. Let $U$ be an open set with $U \cap A_{\infty} \neq \emptyset$. Since $A_{\infty}$ is the closure of $O_{\infty}$, there is some $x \in O_{\infty} \cap U$. By definition of $\theta^\mathfrak{X}_n$ there exist $j \in \text{En}(q_{\infty})$ and $m_0 \in x$ with $x \in \beta^\mathfrak{X}(j) \subseteq O_{\infty}$ and $(\forall n \geq m_0) j \in \text{En}(q_n)$. We define $V := \beta^\mathfrak{X}(j)$ and obtain $U \cap V \neq \emptyset$ and $V \subseteq O_n \cap O_{\infty} \subseteq A_n \cap A_{\infty}$ for all $n \geq m_0$. This proves $(A_n)_{n \in \mathbb{N}} \rightarrow_{\mathcal{R}_4(\mathfrak{X})} A_{\infty}$.

Universality of $\psi^\mathfrak{X}$: Let $\phi : \subseteq \omega^\omega \rightarrow \mathcal{R}_4(\mathfrak{X})$ be a continuous function. For $p \in \omega^\omega$ and $i, j \in \mathbb{N}$ we define $g : \omega^\omega \rightarrow \omega^\omega$ by

$$g(p)((i, j)) := \begin{cases} i + 1 & \text{if } i \in \text{dom}(\phi^\mathfrak{X}) \text{ and } (\forall F \in \phi[p^{<j}\omega^\omega]) \beta^\mathfrak{X}(i) \subseteq F \\ 0 & \text{otherwise.} \end{cases}$$

Since $g(p)((i, j))$ depends only on a finite prefix of $p$, $g$ is continuous.

We prove that $g$ translates $\phi$ to $\psi^\mathfrak{X}$. Let $p \in \text{dom}(\phi)$ and $A := \phi(p)$. Since for every $i \in \text{En}(\phi(p))$ we have $\beta^\mathfrak{X}(i) \subseteq A$, the open $O := \theta^\mathfrak{X}_n(g(p))$ is a subset of $A$.

Let $x \in A$. Suppose for contradiction $x \notin \text{Cl}(\phi)$ (Cl). Then there is some open set $U$ with $x \in U$ and $O \cap U = \emptyset$. Thus for every $k \in \text{En}(\phi(p))$ we have $\beta^\mathfrak{X}(k) \cap U = \emptyset$. Hence for every $m \in \mathbb{N}$ there is some $F_m \in \phi[p^{<m}\omega^\omega]$ with

$$\beta^\mathfrak{X}(m - [\sqrt{m}]^2) \cap U \neq \emptyset \implies \beta^\mathfrak{X}(m - [\sqrt{m}]^2) \notin F_m.$$

By continuity of $\phi$, $(F_m)_{m \in \mathbb{N}}$ converges to $A$. Thus there is an open set $S$ and some $m_0 \in \mathbb{N}$ with $U \cap S \neq \emptyset$ and $(\forall m \geq m_0) S \subseteq F_m \cap A$. As $S^\mathfrak{X}$ is a base of $\mathfrak{X}$, there is some $i_0 \in \text{dom}(\phi^\mathfrak{X})$ with $\emptyset \neq S^\mathfrak{X}(i_0) \subseteq U \cap S$. We define $m_1 := (m_0 + i_0)^2 + i_0$. On the one hand we have $\beta^\mathfrak{X}(i_0) \subseteq F_m$, as $m_1 \geq m_0$, but on the other hand $\beta^\mathfrak{X}(i_0) \notin F_m$, because $i_0 = m_1 - [\sqrt{m_1}]^2$ and $\beta^\mathfrak{X}(i_0) \cap U \neq \emptyset$, a contradiction.

We conclude $A = \text{Cl}(\phi)$. Hence $\psi^\mathfrak{X}(g(p)) = A$. Therefore $g$ translates $\phi$ continuously to $\psi^\mathfrak{X}$.

By Proposition 4.1.7, the admissibility of $\xi_X$ follows from the admissibility of $\psi^\mathfrak{X}$ and $\psi^\mathfrak{X}$ (cf. Subsection 4.4.1).

In the next example we show that the hyperspace $((\mathcal{R}_4(\mathfrak{R}), -_{\mathcal{R}_4(\mathfrak{R})})$ as well as its conjunction with $(\text{sclo}(\mathfrak{R}), -_{\text{sclo}(\mathfrak{R})})$, where $\mathfrak{R}$ denotes the computable one-dimensional Euclidean space from Example 4.3.17 and $\tau^\mathfrak{R}$ is the upper Fell topology on the closed subsets of $\mathbb{R}$, are not limit spaces.
Example 4.4.15 ((\(\mathcal{RA}(\mathbb{R})\), \(-\varepsilon^{\text{reg}}\)) and (\(\mathcal{RA}(\mathbb{R}), \rightarrow_{\xi^{\Theta}}\)) are not limit spaces)

We construct a sequence \((A_n)_{n \leq \infty}\) of regularly closed sets of real numbers with the following properties: every subsequence of \((A_n)_{n}\) has a subsequence that converges to \(A_{\infty}\) w.r.t. \(\rightarrow_{\mathcal{RA}(\mathbb{R})}\) and w.r.t. \(\rightarrow_{\xi^{\Theta}}\), but \((A_n)_{n}\) itself does not converge to \(A_{\infty}\) w.r.t. \(\rightarrow_{\mathcal{RA}(\mathbb{R})}\).

At first we define the sequences \((l_n)_{n}\), \((m_n)_{n}\), and \((q_n)_{n}\) by

\[l_n := \lfloor \log_2(n + 1) \rfloor, \quad m_n := n + 1 - 2^l_n \quad \text{and} \quad q_n := \frac{m_n}{2^{l_n}},\]

for \(n \in \mathbb{N}\). Then \(n \mapsto (m_n, l_n)\) is a bijective numbering of the set \(\{(c, k) \in \mathbb{N}^2 \mid c < 2^k\}\).

We define \((A_n)_{n \leq \infty}\) by

\[A_n := [-2; 2] \setminus (q_n - 2^{-l_n}; q_n + 2^{-l_n}) \quad \text{and} \quad A_{\infty} := [-2; 2]\]

for all \(n \in \mathbb{N}\). Clearly, these sets are regularly closed.

Let \(\varphi : \mathbb{N} \rightarrow \mathbb{N}\) be strictly increasing. Since \([0; 1]\) is sequentially compact and contains \((\varphi(n))_{n}\), there is some strictly increasing function \(\chi : \mathbb{N} \rightarrow \mathbb{N}\) such that \(d_{\varphi(n)}\) converges to some \(z \in [0; 1]\). We show that \((A_{\varphi(n)})_{n}\) converges to \(A_{\infty}\). Let \(U\) be an open set intersecting \(A_{\infty}\). Then there is some \(x \in (-2; 2) \cap U\) with \(x \neq z\). Moreover, there exists some \(k \in \mathbb{N}\) with \(V := (x - 2^{-k}; x + 2^{-k}) \subseteq [-2; 2]\) and \(|x - z| > 4 \cdot 2^{-k}\). Finally, there is some \(n_0 \in \mathbb{N}\) with \(|d_{\varphi(n)} - k| > 2^{-k}\) for all \(n \geq n_0\). We obtain \(|d_{\varphi(n)} - x| > 3 \cdot 2^{-k}\), thus \(V \cap (d_{\varphi(n)} - 2^{-l_{\varphi(n)}}; d_{\varphi(n)} + 2^{-l_{\varphi(n)}}) = \emptyset\) and \(V \subseteq A_{\varphi(n)}\) for all \(n \geq n_0\). We conclude that \((A_{\varphi(n)})_{n}\) converges to \(A_{\infty}\) w.r.t. \(\rightarrow_{\mathcal{RA}(\mathbb{R})}\).

Assume that \((A_n)_{n}\) converges to \(A_{\infty}\) w.r.t. \(\rightarrow_{\mathcal{RA}(\mathbb{R})}\). Let \(U := (0; 1)\). Then there are an open set \(V\), \(n_0 \in \mathbb{N}\), and \(x \in \mathbb{R}\) with \(x \in U \cap V \neq \emptyset\) and \(V \subseteq A_n\) for all \(n \geq n_0\). We define \(k := l_{n_0} + 1\). Then there is some \(c < 2^k\) with \(x \in (\frac{c-1}{2^k}; \frac{c+1}{2^k})\). For \(n_1 := c + 2^k - 1\) we have \(l_{n_1} = k > l_{n_0}\) and \(q_{n_1} = \frac{c}{2^k}\), hence \(n_1 \geq n_0\) and \(x \notin A_{n_1}\). This contradicts \(V \subseteq A_{n_1}\).

Therefore \((A_n)_{n}\) does not converge to \(A_{\infty}\) in the hyperspace \((\mathcal{RA}(\mathbb{R}), \rightarrow_{\mathcal{RA}(\mathbb{R})})\).

Now let \(K \subseteq \mathbb{R}\) be a compact set that does not intersect \(A_{\infty}\). Then \(K\) does not intersect \(A_n\) for any \(n \in \mathbb{N}\). Since \(S_{\mathbb{X}}\) is a subbase of \(\tau_{\mathbb{X}}^{\mathbb{E}}\) (cf. Subsection 4.4.1), it follows that \((A_n)_{n}\) as well as all subsequences of \((A_n)_{n}\) converge to \(A_{\infty}\) w.r.t. the upper Fell topology \(\tau_{\mathbb{X}}^{\mathbb{E}}\). Hence \((A_n)_{n \leq \infty}\) also witnesses that the conjunction \((\mathcal{RA}(\mathbb{R}), \rightarrow_{\mathcal{RA}(\mathbb{R})}) \land (\text{sclo}(\mathcal{R}), \rightarrow_{\mathbb{R}})\) is a weak limit space which is not a limit space.

We conclude that both representations \(\psi_{\mathbb{R}}^{\text{reg}}\) and \(\xi_{\mathbb{R}}\) generate weak limit spaces which are not limit spaces.

\(\Box\)

One easily verifies that \((\mathcal{RA}(\mathbb{R}), \rightarrow_{\mathcal{RA}(\mathbb{R})})\) satisfies the “zipping” Axiom (L7). Hence Axioms (L1), (L5), (L6), and (L7) do not imply Axiom (L3).

The representations \(\xi^k\) of \(\mathcal{RA}(\mathbb{R}^k, \tau_{\mathbb{R}^k})\) considered in [ZiBr01] play an important role in linear optimization. They are essentially defined as the representation\(^{18}\) \(\xi_{\mathbb{R}^k}\), where \(\mathbb{R}^k\) is a canonically defined computable topological space having \(\tau_{\mathbb{R}^k}\), the topology of the \(k\)-dimensional Euclidean space, as its topology. For every \(k \geq 1\), it follows from Proposition 4.4.14 that \(\xi^k\) is an admissible representation of the conjunction \((\mathcal{RA}(\mathbb{R}^k), \rightarrow_{\mathcal{RA}(\mathbb{R}^k)}) \land (\text{sclo}(\mathbb{R}^k), \rightarrow_{\mathbb{R}^k})\). Similar to Example 4.4.15, one can prove that this space is not a limit space.

\(^{18}\)Actually, representations which are equivalent to \(\psi_{\mathbb{R}^k}^{\text{reg}}\) and \(\psi_{\mathbb{R}^k}^{\text{ext}}\) (cf. [BrWe99]) are used as the underlying representations of the open and the closed sets.
4.5 Final Constructions

We will now investigate final constructions. Generating a space $\mathcal{Y}$ by a final construction means to equip the underlying $Y$ with the finest (smallest) convergence relation (or the finest topology) such that a given countable family of functions $h_i$ from given spaces $X_i$ onto $Y$ become continuous. In Subsection 4.5.4 we prove that this construction principle preserves admissible representability. Forming quotient spaces (cf. Subsection 4.5.1), disjunction (cf. Subsection 4.5.2), and coproduct (cf. Subsection 4.5.3) are examples of final constructions. In Subsection 4.5.5 we prove that the categories $\text{AdmWeakLim}$, $\text{AdmLim}$, and $\text{AdmSeq}$ of admissibly representable spaces are closed under coequalizers and under countable colimits. Moreover, the effective categories $\text{EffWeakLim}$, $\text{EffLim}$, and $\text{EffSeq}$ are shown to be closed under coequalizers and under finite colimits. Uniform weak limit spaces are considered in Subsection 4.5.6 as a simple and useful way to construct admissibly representable spaces. We characterize the weak limit spaces that have a fiber–compact admissible multirepresentation as a certain subclass of uniform weak limit spaces. In Subsection 4.5.7 inductive limit spaces are studied. As a final example, we consider the space of distributions and show how to construct a reasonable admissible representation of that space by using many of the operators investigated in Chapter 4.

4.5.1 Quotient Spaces

Let $\mathfrak{X} = (X, \rightarrow X)$ be a weak limit space, and let $Q : \subseteq X \rightrightarrows Y$ be a surjective correspondence onto a set $Y$. In Subsection 2.4.5 we have defined three kinds of quotient spaces generated by $Q$, namely the weak limit space $(Y, \rightarrow_{X,Q})$, the limit space $(Y, \Leftarrow_{X,Q})$ and the sequential topological space $(Y, \tau_{\text{seq}(X),Q})$. The convergence relation $\rightarrow_{X,Q}$ is the finest (smallest) one among those on $Y$ such that $Q$ is sequentially continuous, and $\Leftarrow_{X,Q}$ is the finest one among those convergence relations $\rightarrow$ on $Y$ for which $Q$ is $(\rightarrow_{X,Q})$–continuous and $(Y, \rightarrow)$ is a limit space. Moreover, if $Q$ is a total function, then the final topology $\tau_{\text{seq}(X),Q}$ is the finest (largest) topology on $Y$ such that $Q : X \rightarrow Y$ is a topologically continuous function between the sequential topological spaces $(X, \text{seq}(\mathfrak{X}))$ and $(Y, \tau_{\text{seq}(X),Q})$, cf. [Wil70, Eng89, Smy92].

We show by explicit constructions that the quotients $(Y, \rightarrow_{X,Q})$, $(Y, \Leftarrow_{X,Q})$, and, if $Q$ is total, $(Y, \tau_{\text{seq}(X),Q})$ inherit from $\mathfrak{X}$ the property of having an admissible multirepresentation.

Proposition 4.5.1 (Admissible multirepresentations of quotient spaces)

Let $\delta : \subseteq \mathbb{N}^\omega \rightrightarrows X$ be an admissible multirepresentation of a weak limit space $\mathfrak{X} = (X, \rightarrow X)$. Let $Q : \subseteq X \rightrightarrows Y$ be a surjective correspondence onto a set $Y$.

1. $(Q \circ \delta)^w$ is an admissible multirepresentation of the weak limit space $(Y, \rightarrow_{X,Q})$.

2. $(Q \circ \delta)^c$ is an admissible multirepresentation of the limit space $(Y, \Leftarrow_{X,Q})$.

3. If $Q$ is total, then $(Q \circ \delta)^s$ is an admissible multirepresentation of the sequential topological space $(Y, \tau_{\text{seq}(X),Q})$. 


4.5 Final Constructions

Proof:

(1) Let \((y_n)_{n \leq \infty}\) be a sequence in \(Y\). Then we have

\[
(y_n)_{n \leq \infty} \rightarrow_{x,Q} y_{\infty} \\
\iff \left( \exists (x_n)_{n \leq \infty} \right) \left( (x_n)_{n \rightarrow x} \land (\forall n \in \mathbb{N}) y_n \in Q[x_n] \right) \\
\iff \left( \exists (p_n)_{n \leq \infty} \left( \exists (x_n)_{n \leq \infty} \right) \right) \\
\left( (p_n)_{n \rightarrow_{\text{seq}} p_{\infty}} \land (\forall n \in \mathbb{N}) (y_n \in Q[x_n] \land x_n \in \delta[p_n]) \right) \\
\iff \left( \exists (p_n)_{n \leq \infty} \left( (p_n)_{n \rightarrow_{\text{seq}} p_{\infty}} \land (\forall n \in \mathbb{N}) (y_n \in (Q \circ \delta)[p_n]) \right) \\
\iff (y_n)_{n \rightarrow_Q \rightarrow_{x,Q}} y_{\infty}.
\]

Thus \(Q \circ \delta\) is a \(\text{WeakLim}\)-quotient multirepresentation of \((Y, \rightarrow_{x,Q})\). We conclude by Proposition 4.3.2 that \((Q \circ \delta)\) is an admissible multirepresentation of \((Y, \rightarrow_{x,Q})\).

(2) In (1) we have shown that \(\rightarrow_{Q \circ \delta}\) is equal to \(\rightarrow_{x,Q}\). Lemmas 2.4.10 and 2.4.11 imply \(\sim_{Q \circ \delta} = \sim_{L(Y, \rightarrow_{Q \circ \delta})} = \sim_{L(Y, \rightarrow_{x,Q})} = \sim_{x,Q}\). Therefore \(Q \circ \delta\) is a \(\text{Lim}\)-quotient multirepresentation of \((Y, \sim_{x,Q})\). Proposition 4.3.2 implies that \((Q \circ \delta)^\circ\) is an admissible multirepresentation of \((Y, \sim_{x,Q})\).

(3) By (1) we have \(\rightarrow_{Q \circ \delta} = \rightarrow_{x,Q}\). Lemmas 2.4.10 and 2.4.11 imply \(\tau_{Q \circ \delta} = \text{seq}(-\rightarrow_{Q \circ \delta}) = \text{seq}(-\rightarrow_{x,Q}) = \tau_{\text{seq}(x),Q}\). Hence \(Q \circ \delta\) is a \(\text{Top}\)-quotient multirepresentation of \((Y, \tau_{\text{seq}(x),Q})\). Proposition 4.3.2 implies that \((Q \circ \delta)^\circ\) is an admissible multirepresentation of \((Y, \tau_{\text{seq}(x),Q})\).

\(\checkmark\)

In general, \(Q \circ \delta\) is not an admissible multirepresentation of any of the quotient spaces \((Y, \rightarrow_{x,Q})\), \((Y, \sim_{x,Q})\), \((Y, \tau_{\text{seq}(x),Q})\), as the following example shows.

Example 4.5.2

We consider the naive Cauchy representation \(\rho_{\text{Cn}} : \subseteq \mathbb{N}^\omega \rightarrow \mathbb{R}\) from Example 2.3.11. We equip the set \(X := \text{dom}(\rho_{\text{Cn}})\) with the subspace convergence relation \(\rightarrow_{x}\) inherited from \((\mathbb{N}^\omega, -\rightarrow_{\text{seq}})\) (cf. Subsection 2.2.1) and use \(\delta := \text{id}_{\mathbb{N}^\omega}\) as a clearly admissible representation of \(X := (X, -\rightarrow_{x})\). Moreover we define \(Y := \mathbb{R}\) and \(q := \rho_{\text{Cn}}|X\). By Example 2.3.11, \(\rho_{\text{Cn}} = q \circ \delta\) is not an admissible representation, because \(\rightarrow_{\rho_{\text{Cn}}}\) is the chaotic convergence relation on \(\mathbb{R}\).

By Lemma 2.4.10(6), \(\sim_{x,q}\) and \(\tau_{\text{seq}(x),q}\) are coarser than or equal to \(\rightarrow_{x,q} = \rightarrow_{\rho_{\text{Cn}}}\). Hence \(\sim_{x,q}\) and \(\tau_{\text{seq}(x),q}\) are equal to the chaotic convergence relation on \(\mathbb{R}\), too. Thus \(q \circ \delta\) is neither an admissible representation of \((\mathbb{R}, \sim_{x,q})\) nor of \((\mathbb{R}, \tau_{\text{seq}(x),q})\).

\(\otimes\)

From Propositions 4.5.1 and 2.3.2 we obtain the following corollary.

Corollary 4.5.3 (Admissible representability of quotient spaces)

(1) Every \(\text{WeakLim}\)-quotient of a weak limit space with admissible multirepresentation has an admissible multirepresentation.
(2) Every \( \text{Lim} \)-quotient of a limit space with admissible multirepresentation has an admissible multirepresentation.

(3) Every topological quotient of a sequential topological space with admissible multirepresentation has an admissible multirepresentation.

However, topological quotients of non-sequential topological spaces with admissible representations need not have admissible multirepresentations. This can be deduced from a result by E. Michael in [Mi66] (cf. Lemma 3.4.4).

Let \( \mathfrak{X} = (X, \tau_X) \), \( \mathfrak{Y} = (Y, \tau_Y) \), and \( \mathfrak{Z} = (Z, \tau_Z) \) be sequential topological spaces such that \( \mathfrak{Y} \) is the topological quotient of \( \mathfrak{X} \) generated by a surjective function \( q : X \to Y \).

It is well-known from topology (cf. [Wil70, Eng89, Smy92]) that a function \( f : Y \to Z \) is topologically continuous if and only if \( f \circ q \) is topologically continuous. Similar statements hold for weak limit spaces and limit spaces (cf. Lemma 2.4.10). We prove the following effective version of these equivalences.

**Proposition 4.5.4 (Effectivity properties)**

Let \( \mathfrak{U} \in \{ \mathfrak{ML}, \mathfrak{L}, \mathfrak{S} \} \). Let \( \delta : \subseteq \mathbb{N}^w \rightarrow X \) be a multirepresentation and \( Q : \subseteq X \Rightarrow Y \) be a surjective correspondence between sets \( X \) and \( Y \). Let \( \zeta : \subseteq \mathbb{N}^w \Rightarrow Z \) be an admissible multirepresentation of \( \mathfrak{Z} = (Z, \tau_Z) \) with \( \zeta \equiv_{cp} \zeta^{\leftarrow \mathfrak{U}} \).

Then a function \( f : Y \to Z \) is computable w.r.t. \( (Q \circ \delta)^{\leftarrow \mathfrak{U}} \) and \( \zeta \) if and only if \( f \circ Q \) is computable w.r.t. \( \delta \) and \( \zeta \).

**Proof:**

\[ \Rightarrow \text{Since we have } Q \circ \delta \leq_{cp} (Q \circ \delta)^{\leftarrow \mathfrak{U}} \text{ (cf. Lemma 4.3.6), there is a computable function } g : \subseteq \mathbb{N}^w \rightarrow \mathbb{N}^w \text{ realizing } f \text{ w.r.t. } Q \circ \delta \text{ and } \zeta \text{ by Lemma 2.4.4. One easily verifies that } g \text{ also realizes } f \circ Q \text{ with respect to } \delta \text{ and } \zeta. \]

\[ \Leftarrow \text{Let } g : \subseteq \mathbb{N}^w \rightarrow \mathbb{N}^w \text{ be a computable function realizing } f \circ Q \text{ w.r.t. } \delta \text{ and } \zeta. \text{ One can easily verify that } g : \subseteq \mathbb{N}^w \rightarrow \mathbb{N}^w \text{ also realizes } f \text{ w.r.t. } Q \circ \delta \text{ and } \zeta. \text{ By Proposition 4.3.5, } f \text{ is } ((Q \circ \delta)^{\leftarrow \mathfrak{U}}, \zeta^{\leftarrow \mathfrak{U}})-\text{computable. Since } \zeta^{\leftarrow \mathfrak{U}} \leq_{cp} \zeta, \text{ } f \text{ is computable w.r.t. } (Q \circ \delta)^{\leftarrow \mathfrak{U}} \text{ and } \zeta \text{ by Lemma 2.4.4.} \]

\[ \square \]

### 4.5.2 Disjunction

For \( i \in \mathbb{N} \), let \( \mathfrak{X}_i = (X_i, \rightarrow_{x_i}) \) be a weak limit space. We define the disjunction \( \bigvee_{i \in \mathbb{N}} \mathfrak{X}_i \) of these spaces to be the sequential convergence space whose underlying set is \( \bigcup_{i \in \mathbb{N}} X_i \) and whose convergence relation \( (\forall i \in \mathbb{N} \rightarrow_{x_i}) \) is defined by

\[ (x_n)_n \left( \bigvee_{i \in \mathbb{N}} \rightarrow_{x_i} \right) x_\infty : \iff (\exists j, k \in \mathbb{N})(\forall m \geq k) x_m \in X_j \land (x_{n+k})_n \rightarrow_{x_j} x_\infty). \]

One easily verifies that \( (\forall i \in \mathbb{N} \rightarrow_{x_i}) \) is the finest (smallest) convergence relation on \( \bigcup_{i \in \mathbb{N}} X_i \) among those which satisfy Axiom (L6) and for which the injections \( i_j : X_j \rightarrow \bigcup_{i \in \mathbb{N}} X_i \), \( x \mapsto x \) are sequentially continuous. The finite disjunction \( \mathfrak{X}_1 \vee \ldots \vee \mathfrak{X}_k \) is defined accordingly. The class of weak limit spaces is closed under finite and infinite disjunction, in contrast to the class of limit spaces and the class of sequential spaces.
Lemma 4.5.5 (WeakLim is closed under finite disjunction)

For \( i \in \mathbb{N} \), let \( X_i = (X_i, \rightarrow_{X_i}) \) be a weak limit space. Then \( X_1 \vee \ldots \vee X_k \) and \( \bigvee_{i \in \mathbb{N}} X_i \) are weak limit spaces.

Proof:
Axioms (L1) and (L6) can be verified easily. We show that \( \bigvee_{i \in \mathbb{N}} X_i \) satisfies Axiom (L5).
Let \((y_n)_{n \leq \infty}\) and \((k_n)_{n \leq \infty}\) be convergent sequences of \( \bigvee_{i \in \mathbb{N}} X_i \) and \((\mathbb{N}, \rightarrow)\), respectively. Case \( k_\infty \neq \infty \): Then there is some \( n_0 \in \mathbb{N} \) with \((\forall n \geq n_0) k_n = k_\infty \). Let \( j \) be a number with \( y_{k_\infty} \in X_j \). Since \( X_j \) satisfies Axiom (L1), the constant sequence \((y_{n+n_0})_n\) converges to \( y_{k_\infty} \) in \( X_j \). Hence \((y_{n_0})_n\) converges to \( y_{k_\infty} \) in \( \bigvee_{i \in \mathbb{N}} X_i \).

Case \( k_\infty = \infty \): There are some \( j, n_1, n_2 \in \mathbb{N} \) with \((\forall n \geq n_1) y_n \in X_j \), \((y_n + n_1) \rightarrow_{X_j} y_{k_\infty} \), and \((\forall n \geq n_2) k_n \geq n_1 \). Since \( X_j \) satisfies Axiom (L5), \((y_{k+n_2})_n\) converges to \( y_{k_\infty} \) in \( X_j \). Hence \((y_{n_0})_{n \leq \infty}\) is a convergent sequence of \( \bigvee_{i \in \mathbb{N}} X_i \).

Therefore Axiom (L5) is satisfied.

For \( i \in \mathbb{N} \), let \( \delta_i : \subseteq \mathbb{N}^\omega \Rightarrow X_i \) be a WeakLim–quotient multirepresentation of \( X_i \). We define the finite disjunction \((\delta_1 \vee \ldots \vee \delta_k) \subseteq \mathbb{N}^\omega \Rightarrow \bigcup_{i=1}^k X_i \) and the infinite disjunction \((\bigvee_{i \in \mathbb{N}} \delta_i) : \subseteq \mathbb{N}^\omega \Rightarrow \bigcup_{i \in \mathbb{N}} X_i \) of these multirepresentations by

\[
(\delta_1 \vee \ldots \vee \delta_k)[p] := \begin{cases} 
\delta_{p(0)}[p > 0] & \text{if } p(0) \in \{1, \ldots, k\} \\
\emptyset & \text{otherwise}
\end{cases}
\]

and

\[
(\bigvee_{i \in \mathbb{N}} \delta_i)[p] := \delta_{p(0)}[p > 0]
\]

for all \( p \in \mathbb{N}^\omega \), where \( p > 0 \) denotes the \( \omega \)-word that satisfies \((\forall j \in \mathbb{N}) p > 0 (j) = p(j + 1) \).
One easily verifies that \((\delta_1 \vee \ldots \vee \delta_k) \) and \((\bigvee_{i \in \mathbb{N}} \delta_i) \) are WeakLim–quotient multirepresentations of \( X_1 \vee \ldots \vee X_k \) and \( \bigvee_{i \in \mathbb{N}} X_i \), respectively. By Proposition 4.3.2, \((\delta_1 \vee \ldots \vee \delta_k)^{\infty}\) and \((\bigvee_{i \in \mathbb{N}} \delta_i)^{\infty}\) are admissible multirepresentations of these spaces, proving the following proposition.

Proposition 4.5.6
The class of weak limit spaces with admissible multirepresentations is closed under finite and infinite disjunction.

We show that \((\delta_1 \vee \ldots \vee \delta_k) \) and \((\bigvee_{i \in \mathbb{N}} \delta_i) \) need not be admissible, if all the multirepresentations \( \delta_0, \delta_1, \delta_2, \ldots \) are admissible.

Example 4.5.7
We consider the Sierpiński–space \( \mathfrak{S}i \) from Example 2.3.7 and equip its underlying set \( \mathfrak{S}i = \{\bot, \top\} \) with a second convergence relation \( \rightarrow_{\mathfrak{S}} \) defined by

\[
(b_n)_n \rightarrow_{\mathfrak{S}} b_{\infty} \iff (b_{\infty} = \top \text{ or } (\forall n \in \mathbb{N}) b_n = \bot).
\]

Thus \( \rightarrow_{\mathfrak{S}} \) is defined "dually" to \( \rightarrow_{\mathfrak{S}} \), i.e., it holds on the one hand \((\bot)_{\mu} \rightarrow_{\mathfrak{S}} \top, (\top)_{\mu} \not\rightarrow_{\mathfrak{S}} \bot\) and on the other hand \((\bot)_{\mu} \not\rightarrow_{\mathfrak{S}} \top, (\top)_{\mu} \rightarrow_{\mathfrak{S}} \bot\).
Similar to $\varrho_{\mathcal{S}i}$ which is an admissible representation of $\mathcal{S}i$, the function $\delta_2 : \mathbb{N}^\omega \rightarrow \mathcal{S}i$ defined by

$$\delta_2(p) := \begin{cases} 
\top & \text{if } p = 0^\omega \\
\bot & \text{otherwise}
\end{cases}$$

is an admissible representation of $(\mathcal{S}i, \rightarrow_2)$.

The convergence relation of the disjunction $(\mathcal{S}i, \rightarrow_2) \lor (\mathcal{S}i, \rightarrow_2)$ is the chaotic convergence relation on $X_i$, because every sequence $(b_n)_n$ converges in $(\mathcal{S}i, \rightarrow_2)$ to $\bot$ and in $(\mathcal{S}i, \rightarrow_2)$ to $\top$. Hence $(\mathcal{S}i, \rightarrow_2) \lor (\mathcal{S}i, \rightarrow_2)$ misses the $T_0$-property. On the other hand, $\varrho_{\mathcal{S}i} \lor \delta_2$ is single-valued. Therefore, $\varrho_{\mathcal{S}i} \lor \delta_2$ cannot be an admissible (multi)representation of $(\mathcal{S}i, \rightarrow_2) \lor (\mathcal{S}i, \rightarrow_2)$ by Proposition 2.3.13.

We present a useful condition on sequences of weak limit spaces which guarantees that the disjunction operator on multirepresentations preserve admissibility.

**Lemma 4.5.8 (Condition for admissibility of $(\bigvee_{i \in \mathbb{N}} \delta_i)$)**

For every $l \in \mathbb{N}$, let $x_l = (X_l, \rightarrow_{x_l})$ be a weak limit space with an admissible multirepresentation $\delta_l : \subseteq \mathbb{N}^\omega \Rightarrow X_l$. Assume that

$$(x_n)_n \rightarrow_{x_l} x_\infty \iff (x_n)_n \rightarrow_{x_j} x_\infty$$

holds for all $i, j \in \mathbb{N}$ and all sequences $(x_n)_{n < \infty}$ in $X_i \cap X_j$.

Then $(\delta_1 \lor \ldots \lor \delta_k)$ and $(\bigvee_{i \in \mathbb{N}} \delta_i)$ are admissible multirepresentations of $x_1 \lor \ldots \lor x_k$ and $\bigvee_{i \in \mathbb{N}} x_i$, respectively.

**Proof:**

We only show that $(\bigvee_{i \in \mathbb{N}} \delta_i)$ is an admissible multirepresentation of $\bigvee_{i \in \mathbb{N}} x_i$.

**Continuity:** Let $(p_n)_{n < \infty}$ be a convergent sequence in $\text{dom}(\bigvee_{i \in \mathbb{N}} \delta_i)$, and let $(x_n)_{n < \infty}$ be a sequence with $(\forall n \in \mathbb{N}) x_n \in (\bigvee_{i \in \mathbb{N}} \delta_i)[p_n]$. Then there is some $n_0 \in \mathbb{N}$ satisfying $(\forall n \geq n_0) p_n(0) = p_\infty(0)$. By continuity of $\delta_{p_\infty(0)}$, $(x_n + n_0)_n$ converges to $x_\infty$ in $X_{p_\infty(0)}$. Thus $(x_n)_{n < \infty}$ is a convergent sequence of $\bigvee_{i \in \mathbb{N}} x_i$. Hence $(\bigvee_{i \in \mathbb{N}} \delta_i)$ is sequentially continuous.

**Universality:** Let $\phi : \subseteq \mathbb{N}^\omega \Rightarrow \bigcup_{i \in \mathbb{N}} X_i$ be sequentially continuous.

For all $q \in \mathbb{N}^\omega$ we define the numbers $n_q, j_q \in \mathbb{N}$ by

$$n_q := \min \left( \{ \infty \} \cup \{ n \in \mathbb{N} \mid (\exists j \in \mathbb{N}) \phi[q^{< n} \mathbb{N}^\omega] \subseteq X_j \} \right)$$

and

$$j_q := \begin{cases} 
\min \{ j \in \mathbb{N} \mid \phi[q^{< n} \mathbb{N}^\omega] \subseteq X_j \} & \text{if } n_q < \infty \\
\infty & \text{otherwise.}
\end{cases}$$

Let $p \in \text{dom}(\phi)$. Suppose for contradiction $n_p = \infty$. Then for every $m \in \mathbb{N}$ there are some $q_m \in p^{< m} \mathbb{N}^\omega$ and some $y_m \in \phi[q_m]$ with $y_m \notin X_{m-\lfloor \sqrt{m} \rfloor}$. Since $(q_m)_m$ converges to $p$ and $\phi$ is continuous, $(y_m)_m$ converges in $\bigvee_{i \in \mathbb{N}} x_i$ to every element $x \in \phi[p]$. But there is no $j \in \mathbb{N}$ such that $(y_m)_m$ is eventually in $X_j$. This contradicts the definition of $(\bigvee_{i \in \mathbb{N}} \rightarrow_{x_i})$.

Let $j \in \mathbb{N}$. We define $\phi_j : \subseteq \mathbb{N}^\omega \Rightarrow X_j$ by $\phi_j[q] := \phi[q] \cap X_j$. From the continuity of $\phi$,
An important example for the concept of "final constructions" is the coproduct (or coproduct (4.5.3 Coproduct \( \text{Coproduct} \))  has its underlying set and as its convergence relation the one defined by

\[
g(q) := \begin{cases} 
  j_q \cdot g_{j_q}(q) & \text{if } j_q < \infty \\
  \text{div} & \text{otherwise}
\end{cases}
\]

for all \( q \in \mathbb{N}^\omega \). Then \( g \) is continuous, because, for \( q \in \text{dom}(g) \), \( j_q \) only depends on a finite prefix of \( q \), namely on \( q^{<n_0} \). Since we have \( \phi[p^{<n_0}N^\omega] \subseteq X_j \) and \( \phi[p] = \phi_j[p] \subseteq \delta_j \phi_j(p) \) for every \( p \in \text{dom}(\phi) \), \( g \) translates \( \phi \) to \( (\bigvee_{i \in \mathbb{N}} \delta_i) \). Hence \( (\bigvee_{i \in \mathbb{N}} \delta_i) \) has the universal property.

\[ \checkmark \]

4.5.3 Coproduct

An important example for the concept of "final constructions" is the coproduct (or disjoint sum) of weak limit spaces.

For every \( i \in \mathbb{N} \), let \( X_i = (X_i, \rightarrow_{x_i}) \) be a weak limit space. We define the coproduct \( \bigoplus_{i \in \mathbb{N}} X_i \) of the spaces \( X_0, X_1, \ldots \) as the sequential convergence space that has

\[
\bigcup_{i \in \mathbb{N}} X_i := \bigcup_{i \in \mathbb{N}} \{(i) \times X_i \}
\]

has its underlying set and as its convergence relation the one defined by

\[
(y_n)_n \rightarrow_{\bigoplus X_i} y_\infty \iff \\
(\exists n_0 \in \mathbb{N}) \left( (\forall n \geq n_0) pr_1(y_n) = pr_1(y_\infty) \land (pr_2(y_{j+n_0}))_j \rightarrow_{x_{pr_1(y_\infty)}} (pr_2(y_\infty)) \right),
\]

where \( pr_1, pr_2 \) are defined by \( pr_1(i, x) := i \) and \( pr_2(i, x) := x \) for \( i \in \mathbb{N} \) and \( x \in X_i \). The finite coproduct \( X_1 \oplus \ldots \oplus X_k := (\bigcup_{i=1}^k X_i, \rightarrow_{x_1 \oplus \ldots \oplus x_k}) \) is defined accordingly.

Now let \( \tau_i \) be a topology on the set \( X_i \) for every \( i \in \mathbb{N} \). We define \( \bigoplus_{i \in \mathbb{N}} \tau_i \) as the topology on \( \bigcup_{i \in \mathbb{N}} X_i \) that is generated by the subbase

\[
B := \{ \{i\} \times O \mid i \in \mathbb{N} \text{ and } O \in \tau_i \}
\]

(cf. [Eng89]). Since \( B \cup \{\emptyset\} \) is closed under finite intersection, \( B \) is even a base of \( \bigoplus_{i \in \mathbb{N}} \tau_i \).

The topology \( \bigoplus_{i=1}^k \tau_i \) on the set \( \bigcup_{i=1}^k X_i = \bigcup_{i=1}^k \{\{i\} \times X_i\} \) is defined accordingly.

For every \( j \in \mathbb{N} \), we denote by \( \iota_j : X_j \rightarrow \bigcup_{i \in \mathbb{N}} X_i \) the injection of \( X_j \) into \( \bigcup_{i \in \mathbb{N}} X_i \) defined by \( \iota_j(x) := (j, x) \). One easily observes that \( \rightarrow_{\bigoplus \tau_i} \) is the finest (smallest) convergence relation on \( \bigcup_{i \in \mathbb{N}} X_i \) for which the injections \( \iota_0, \iota_1, \ldots \) are sequentially continuous. Similarly, \( \bigoplus_{i \in \mathbb{N}} \tau_i \) is well-known to be the finest (largest) topology on \( \bigcup_{i \in \mathbb{N}} X_i \) for which each injection \( \iota_j \) is topologically continuous.

We show that the weak limit spaces, the limit spaces, and the topological spaces are closed under finite and under countably infinite coproduct.
Lemma 4.5.9 (Properties of coproducts)

For every $i \in \mathbb{N}$, let $X_i = (X_i, \rightarrow_{X_i})$ be a weak limit space and $\tau_i$ be a topology on $X_i$.

1. The coproduct $\bigoplus_{i \in \mathbb{N}} X_i$ is a weak limit space.

2. If $X_0, X_1, \ldots$ are limit spaces, then the coproduct $\bigoplus_{i \in \mathbb{N}} X_i$ is a limit space.

3. If, for every $i \in \mathbb{N}$, $\tau_i$ induces $\rightarrow_{X_i}$, then the topology $\bigoplus_{i \in \mathbb{N}} \tau_i$ induces the convergence relation $\rightarrow_{\bigoplus_{i \in \mathbb{N}} X_i}$.

4. If $\tau_i$ is a sequential topology for every $i \in \mathbb{N}$, then $\bigoplus_{i \in \mathbb{N}} \tau_i$ is sequential as well.

5. Statements (1), (2), (3), (4) hold analogously for the finite coproduct $\bigoplus_{i=1}^k X_i$.

Proof:

(1) For $i \in \mathbb{N}$, we define the sequential convergence space $Y_i = (Y_i, \rightarrow_{Y_i})$ by $Y_i := \{i\} \times X_i$ and

$$ (y_n)_n \rightarrow_{Y_i} y_\infty :\iff (pr_2(y_n))_n \rightarrow_{X_i} pr_2(y_\infty) $$

for all sequences $(y_n)_n \leq \infty$ in $Y_i$. Clearly, $Y_i$ is a weak limit space for every $i \in \mathbb{N}$, and the coproduct $\bigoplus_{i \in \mathbb{N}} Y_i$ is equal to $\bigvee_{i \in \mathbb{N}} Y_i$. Hence $\bigoplus_{i \in \mathbb{N}} X_i$ is a weak limit space by Lemma 4.5.5.

(2) We have to check Axiom (L3). Let $(y_n)_n \leq \infty$ be non–convergent sequence of $\bigoplus_{i \in \mathbb{N}} X_i$. Let $j := pr_1(y_\infty)$.

Case 1: There is some $n_0 \in \mathbb{N}$ with $(\forall n \geq n_0) pr_1(y_n) = j$.

Then $(pr_2(y_{n+n_0}))_n$ does not converge to $pr_2(y_\infty)$ in $X_j$. Since $X_j$ satisfies Axiom (L3), there is some strictly increasing function $\varphi: \mathbb{N} \rightarrow \{k \in \mathbb{N} | k \geq n_0\}$ such that no subsequence of $(pr_2(y_{\varphi(n)}))_n$ converges to $pr_2(y_\infty)$ in $X_j$. It follows that no subsequence of $(y_{\varphi(n)})_n$ converges to $y_\infty$ in $\bigoplus_{i \in \mathbb{N}} X_i$.

Case 2: There are infinitely many $n$ with $pr_1(y_n) \neq j$.

Then there is a strictly increasing function $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ with $pr_1(y_{\varphi(n)}) \neq j$ for all $n \in \mathbb{N}$. Since no subsequence of $(y_{\varphi(n)})_n$ is eventually in $\{j\} \times X_j$, no subsequence of $(y_{\varphi(n)})_n$ converges to $y_\infty$ in $\bigoplus_{i \in \mathbb{N}} X_i$.

Hence $\bigoplus_{i \in \mathbb{N}} X_i$ satisfies Axiom (L3).

(3) $\rightarrow_{\bigoplus_{i \in \mathbb{N}} X_i} \subseteq \rightarrow_{\bigoplus_{i \in \mathbb{N}} \tau_i}$: Let $(y_n)_n \leq \infty$ be a convergent sequence of $\bigoplus_{i \in \mathbb{N}} X_i$. Define $j := pr_1(y_\infty)$. There is some $n_1 \in \mathbb{N}$ with $(\forall n \geq n_1) y_n \in \{j\} \times X_j$. Let $O \in \bigoplus_{i \in \mathbb{N}} \tau_i$ be an open set containing $y_\infty$. Then there is a set $U \in \tau_j$ with $y_\infty \in \{j\} \times U \subseteq O$.

Since $(pr_2(y_{n+n_1}))_n$ converges to $pr_2(y_\infty)$ in $X_j$, there is some $n_2 \geq n_1$ with $(\forall n \geq n_2) pr_2(y_n) \in U$. Hence $(y_n)_n$ is eventually in $O$. We conclude that $(y_n)_n$ converges to $y_\infty$ with respect to the topology $\bigoplus_{i \in \mathbb{N}} \tau_i$.

$\rightarrow_{\bigoplus_{i \in \mathbb{N}} \tau_i} \subseteq \rightarrow_{\bigoplus_{i \in \mathbb{N}} X_i}$: Let $(y_n)_n \leq \infty$ be a convergent sequence of $\bigvee_{i \in \mathbb{N}} X_i, \bigoplus_{i \in \mathbb{N}} \tau_i$.

We define $j := pr_1(y_\infty)$.

Let $U \in \tau_j$ be an open set with $pr_2(y_\infty) \in U$. Since $\{j\} \times U$ is an open set of $\bigvee_{i \in \mathbb{N}} X_i, \bigoplus_{i \in \mathbb{N}} \tau_i$, there exists some $m_U \in \mathbb{N}$ with $y_n \in \{j\} \times U$ for all $n \geq m_U$. We conclude that the sequence $(pr_2(y_{n+m_U}))_n$ converges to $pr_2(y_\infty)$ in $(X_j, \tau_j)$ and thus in $X_j$. Hence $(y_n)_n \leq \infty$ is a convergent sequence of $\bigoplus_{i \in \mathbb{N}} X_i$.
(4) Let \( O \in \text{seq}(\bigoplus_{i \in \mathbb{N}} \tau_i) \). Let \( l \in \mathbb{N} \). We define \( U_l := \{ x \in X_l \mid (l, x) \in O \} \). Let \( x_\infty \in U_l \) and let \((x_n)_n\) be a sequence that converges in \((X_l, \tau_l)\) to \( x_\infty \). From (3) it follows that \(((l, x_n))_{n \leq \infty} \) is a convergent sequence of \((\bigcup_{i \in \mathbb{N}} X_i, \bigoplus_{i \in \mathbb{N}} \tau_i)\). Hence \((x_n)_n\) is eventually in \( U_l \). This proves that \( U_l \) is sequentially open in \((X_l, \tau_l)\) and hence open, because \( \tau_l \) is sequential, implying \( \{l\} \times U_l \in \bigoplus_{i \in \mathbb{N}} \tau_i \). Since \( O = \bigcup_{i \in \mathbb{N}} \{ (i) \times U_i \} \), it follows \( O \in \bigoplus_{i \in \mathbb{N}} \tau_i \). Therefore \( \bigoplus_{i \in \mathbb{N}} \tau_i \) is a sequential topology.

(5) Similar to (1), (2), (3), (4).

Since \( \neg \neg \delta_{X_i} \) is the finest (smallest) convergence relation on \( \bigcup_{i \in \mathbb{N}} X_i \) which satisfies Axiom (L6) and for which the injections \( \iota_0, \iota_1, \ldots \) are sequentially continuous, the following equivalence holds for every weak limit space \( Z = (Z, \neg \neg \omega) \) and every function \( f : \bigcup_{i \in \mathbb{N}} X_i \to Z \):

\[
\text{f is } (\neg \neg \iota_j, \neg \neg \omega)\text{-continuous} \iff f \circ \iota_j \text{ is } (\neg \neg \omega, \neg \neg \omega)\text{-continuous for every } j \in \mathbb{N}.
\]

(4.23)

In particular, the injections themselves are sequentially continuous. Analogous statements hold for the finite coproduct. We conclude by Lemma 4.5.9 that \( \bigoplus_{i \in \mathbb{N}} X_i \) forms a category–theoretical coproduct of \( X_0, X_1, \ldots \) in the categories \( \text{WeakLim} \) and \( \text{Lim} \), and \((\bigcup_{i \in \mathbb{N}} X_i, \bigoplus_{i \in \mathbb{N}} \tau_i)\) forms a category–theoretical coproduct of \((X_0, \tau_0), (X_1, \tau_1), \ldots \) in \( \text{Seq} \). Together with Proposition 4.2.3, this means that \( \text{WeakLim} \), just as it is well–known for \( \text{Lim} \) and \( \text{Seq} \), is bicartesian–closed.

Now let \( \delta_i : \subseteq \mathbb{N}^\omega \Rightarrow X_i \) be a multirepresentation of \( X_i \) for every \( i \in \mathbb{N} \). We define the multirepresentations \((\delta_1 \boxplus \ldots \boxplus \delta_k) : \subseteq \mathbb{N}^\omega \Rightarrow \bigcup_{i=1}^k X_i \) and \((\boxplus_{i \in \mathbb{N}} \delta_i) : \subseteq \mathbb{N}^\omega \Rightarrow \bigcup_{i \in \mathbb{N}} X_i \) by

\[
(\delta_1 \boxplus \ldots \boxplus \delta_k)[p] \ni x := p(0) \in \{1, \ldots, k\} \text{ and } pr_2(x) \in \delta_{p(0)}[pr^0]
\]

and

\[
(\boxplus_{i \in \mathbb{N}} \delta_i)[p] \ni y := pr_2(y) \in \delta_{p(0)}[pr^0]
\]

for all \( p \in \mathbb{N}^\omega \), \( x \in \bigcup_{i=1}^k X_i \), and \( y \in \bigcup_{i \in \mathbb{N}} X_i \), where \( pr^0 \) denotes the \( \omega \)-word satisfying \((\forall j \in \mathbb{N}) (pr^0(j) = p(j + 1))\).

We prove the following admissibility result about the operator \( \boxplus \).

**Proposition 4.5.10 (Admissibility of \((\delta_1 \boxplus \ldots \boxplus \delta_k)\) and \((\boxplus_{i \in \mathbb{N}} \delta_i)\))**

For every \( i \in \mathbb{N} \), let \( \delta_i : \subseteq \mathbb{N}^\omega \Rightarrow X_i \) be an admissible multirepresentation of a weak limit space \( X_i = (X_i, \neg \neg x_i) \). Then:

1. \((\delta_1 \boxplus \ldots \boxplus \delta_k)\) is an admissible multirepresentation of \( X_1 \oplus \ldots \oplus X_k \).
2. \((\boxplus_{i \in \mathbb{N}} \delta_i)\) is an admissible multirepresentation of \( \bigoplus_{i \in \mathbb{N}} X_i \).
We show that the operators $\text{for}$ and $\text{AdmWeakLim}$ are bicartesian–closed categories.

Theorem 4.2.6, Proposition 4.5.10, and Lemma 4.5.9 imply that $\text{AdmSeq}$, $\text{AdmLim}$, and $\text{AdmWeakLim}$ are bicartesian–closed categories.

We effectivize Equivalence (4.23) as follows.

**Lemma 4.5.11 (Effectivity properties of $\boxplus$)**

For $i \in \{1, \ldots, k\}$, let $\delta_i : \subseteq N^\omega \rightrightarrows X_i$ be a multirepresentation of a weak limit space $X_i = (X_i, \rightarrow_{X_i})$. Moreover, let $\zeta : \subseteq N^\omega \rightrightarrows Z$ be a multirepresentation of a weak limit space $Z = (Z, \rightarrow_Z)$.

Then a function $f : \bigcup_{i \in \mathbb{N}} X_i \rightarrow Z$ is computable w.r.t. $(\delta_1 \boxplus \ldots \boxplus \delta_k)$ and $\zeta$ if and only if, for every $j \in \{1, \ldots, k\}$, $f \circ \iota_j$ is computable w.r.t. $\delta_j$ and $\zeta$.

**Proof:**

$\Rightarrow$: Let $j \in \{1, \ldots, k\}$. Obviously, the injection $\iota_j$ is computable w.r.t. $\delta_j$ and $(\delta_1 \boxplus \ldots \boxplus \delta_k)$, because a realizing function is $p \mapsto j::p$. Thus the composition $f \circ \iota_j$ is computable w.r.t. $\delta_j$ and $\zeta$ by Lemma 2.4.4(4).

$\Leftarrow$: For every $j \in \{1, \ldots, k\}$, let $g_j : \subseteq N^\omega \rightarrow N^\omega$ be a computable function realizing $f \circ \iota_j$ w.r.t. $\delta_j$ and $\zeta$. We define $G : \subseteq N^\omega \rightarrow N^\omega$ by

$$G(p) := \begin{cases} g_p(0)(p^{>0}) & \text{if } p(0) \in \{1, \ldots, k\} \\ 0^\omega & \text{otherwise} \end{cases}$$

for all $p \in N^\omega$. Obviously, $G$ is computable and realizes $f$ w.r.t. $(\delta_1 \boxplus \ldots \boxplus \delta_k)$ and $\zeta$.

We show that the operators $\circ^w$, $\circ^c$, and $\circ^s$ “distribute” over the finite and the infinite coproduct operator $\boxplus$.

**Lemma 4.5.12**

For every $i \in \mathbb{N}$, let $\delta_i : \subseteq N^\omega \rightrightarrows X_i$ be an admissible multirepresentation of a weak limit space $X_i = (X_i, \rightarrow_{X_i})$. Let $\Lambda \in \{\mathsf{Wc}, \mathsf{Lc}, \mathsf{Si}\}$.

Then $(\delta_1^{-\Lambda} \boxplus \ldots \boxplus \delta_k^{-\Lambda}) \equiv_{cp} (\delta_1 \boxplus \ldots \boxplus \delta_k)^{-\Lambda}$ and $(\boxplus_{j \in \mathbb{N}} \delta_j^{-\Lambda}) \equiv_{cp} (\boxplus_{j \in \mathbb{N}} \delta_j)^{-\Lambda}$. 

\hspace{1cm} ✓
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Proof:
We only show the second statement. Let $U$ denote the underlying set of $\mathcal{U}$.

“$\leq_{cp}$”: By the utm–Theorem and the computable smn–Theorem, there are computable functions $g_1 : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ and $g_2 : \mathbb{N}^\omega \to \mathbb{N}^\omega$ with

$$(\forall p, q, r \in \mathbb{N}^\omega) \left( \eta_{g_1(p,q)}(r) = \eta_q(p(0):r : r) \right)$$

and

$$\eta_{g_2(p)}(q) = \eta_{p > 0}(g_1(p, q)).$$

We show that $g_2$ translates $(\exists j \in \mathbb{N} \delta_j^{\mathcal{U}})$ computably to $(\exists j \in \mathbb{N} \delta_j^{\mathcal{U}})^{\mathcal{U}}$. Let $p \in \mathbb{N}^\omega$ and $y \in (\exists j \in \mathbb{N} \delta_j^{\mathcal{U}})[p]$. Let $q \in \mathbb{N}^\omega$ and $h \in ([\exists j \in \mathbb{N} \delta_j] \to \eta_{\mathcal{U}})[\eta_q]$. For every $r \in \mathbb{N}^\omega$ and $x \in \delta_{\eta_q}(r)$, we have

$$h \circ \eta_{\mathcal{U}}[\eta_q](p(0):r) = \eta_{g_2(p, q)}(r).$$

This implies $h \circ \eta_{\mathcal{U}}[\eta_q](p(0):r) \in \eta_{g_2(p, q)}[\eta_q]$. It follows

$$e(\Theta_{j \in \mathbb{N}} \mathcal{U})_j(y)(h) = (h \circ \eta_{\mathcal{U}}[\eta_q](p(0):r))(x) = \eta_{g_2(p, q)}(r).$$

Therefore $e(\Theta_{j \in \mathbb{N}} \mathcal{U})_j(y)(h) \in ([\exists j \in \mathbb{N} \delta_j] \to \eta_{\mathcal{U}})[\eta_{\mathcal{U}}][\eta_{g_2(p, q)}]$ and $y \in ([\exists j \in \mathbb{N} \delta_j] \to \eta_{\mathcal{U}})[\eta_{g_2(p, q)}]$. Thus $g_2$ translates $(\exists j \in \mathbb{N} \delta_j^{\mathcal{U}})$ computably to $(\exists j \in \mathbb{N} \delta_j^{\mathcal{U}})^{\mathcal{U}}$.

“$\geq_{cp}$”: By the utm–Theorem there is a computable function $g_3 : \mathbb{N}^\omega \to \mathbb{N}^\omega$ with

$$(\forall r, t \in \mathbb{N}^\omega) \eta_{g_3(t)}(r) = \begin{cases} 1^\omega & \text{if } r(0) = t(0) \\ 0^\omega & \text{otherwise.} \end{cases}$$

Let $\sigma : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ be a computable monotone word function approximating the computable function $(p, t) \mapsto \eta_{\mathcal{U}}[g_3(t)]$ (cf. Subsection 2.1.2). We define the function $g_4 : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ by

$$g_4(p) := \begin{cases} (m_p - \left\lfloor \sqrt{m_p^2} \right\rfloor) : 2^\omega & \text{if } m_p < \infty \\ \text{div} & \text{otherwise,} \end{cases}$$

where

$$m_p := \min \left( \{ \infty \} \cup \{ m \in \mathbb{N} \mid \sigma(p < m, (m - \left\lfloor \sqrt{m^2} \right\rfloor) 2^\omega \notin \{0\}^* \} \right).$$

Clearly, $g_4$ is computable. By the effectivity properties of $\eta$, there are computable functions $g_5 : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ and $g_6 : \mathbb{N}^\omega \to \mathbb{N}^\omega$ with

$$\eta_{g_5(t, q)}(r) = \begin{cases} \eta_q(r > 0) & \text{if } r(0) = t(0) \\ 0^\omega & \text{otherwise,} \end{cases}$$

and

$$\eta_{g_6(p)}(q) = \eta_{\mathcal{U}}[g_5(g_4(p), q)].$$

for all $p, q, r, t \in \mathbb{N}^\omega$. We define $g_7 : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ computably by

$$g_7(p) := g_4(p)(0) : g_6(p)$$

and prove that $g_7$ translates $(\exists j \in \mathbb{N} \delta_j^{\mathcal{U}})$ to $(\exists j \in \mathbb{N} \delta_j^{\mathcal{U}})$. Let $p \in \mathbb{N}^\omega$ and $y \in ([\exists j \in \mathbb{N} \delta_j] \to \eta_{\mathcal{U}})[p]$. At first we show $g_4(p)(0) = \mathrm{pr}_1(y)$. For every $i \in \mathbb{N}$ we define the function $c_i : \bigcup_{j \in \mathbb{N}} X_j \to \mathcal{U}$ by

$$c_i(z) := \begin{cases} \top & \text{if } \mathrm{pr}_1(z) = i \\ \bot & \text{otherwise.} \end{cases}$$
Proposition 4.5.10, Lemmas 4.5.9, 4.5.12, and the obvious implication

From Corollary 4.5.13 and Lemma 4.5.11 we obtain that the categories EffWeakLim, EffLim, and EffSeq have finite coproducts.

Corollary 4.5.13 (EffWeakLim, EffLim, and EffSeq have finite coproducts)

1. For every $i \in \{1, \ldots, k\}$, let $(X_i, \delta_i)$ be an object of EffWeakLim. Then $(X_1 \oplus \ldots \oplus X_k, (\delta_1 \oplus \ldots \oplus \delta_k))$ is an object of EffWeakLim.

2. For every $i \in \{1, \ldots, k\}$, let $(X_i, \delta_i)$ be an object of EffLim. Then $(X_1 \oplus \ldots \oplus X_k, (\delta_1 \oplus \ldots \oplus \delta_k))$ is an object of EffLim.

3. For every $i \in \{1, \ldots, k\}$, let $((X_i, \tau_i), \delta_i)$ be an object of EffSeq. Then $((U_{i=1}^k X_i, \bigoplus_{i=1}^k \tau_i), (\delta_1 \oplus \ldots \oplus \delta_k))$ is an object of EffSeq.

From Corollary 4.5.13 and Lemma 4.5.11 we obtain that the categories EffWeakLim, EffLim, and EffSeq, being Cartesian closed by Theorem 4.3.15, are bicartesian closed.
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4.5.4 Strong Convergence Relations and Strong Topologies

Let $Y$ be a set. For every $i \in \mathbb{N}$, let $X_i = (X_i, \rightarrow_{X_i})$ be a weak limit space, $\tau_i$ be a sequential topology on $X_i$, and $q_i : X_i \rightarrow Y$ be a function such that $Y = \bigcup_{i \in \mathbb{N}} \text{range}(q_i)$. We define the topology $\tau_{(q_i)}$ by

$$
\tau_{(q_i)} := \{ O \subseteq Y \mid (\forall i \in \mathbb{N}) q_i^{-1}[O] \in \tau_i \}.
$$

From [Wil70, Ex. 9H] we know that $\tau_{(q_i)}$ is the finest (largest) topology on $Y$ for which each function $q_i$ is topologically continuous w.r.t. $\tau_i$ on $X_i$. The topology $\tau_{(q_i)}$ is called the strong topology on $Y$ coinduced by the functions $q_i$. Similarly, there exists a finest (smallest) convergence relation on $Y$ which satisfies Axiom (L6) and for which each function $q_i$ is sequentially continuous w.r.t. $\rightarrow_{X_i}$ on $X_i$. This strong convergence relation on $Y$ can be shown to be equal to the convergence relation defined by

$$
(y_n)_n \rightarrow_{(q_i)} y_\infty := (\exists i \in \mathbb{N}) \left( (\exists (x_n)_{n \leq \infty} \in X_i^n) (\exists n_0 \in \mathbb{N}) \right.

\left. ((x_n)_n \rightarrow_{X_i} x_\infty \text{ and } (\forall n \in \{n_0, \ldots, \infty\}) y_n = q_i(x_n) \right).
$$

From Proposition 2.2.4, Lemma 4.5.9 and Equivalence (4.24) it follows that $\mathfrak{Y} := (Y, \rightarrow_{(q_i)})$ is a weak limit space.

In order to show that $\mathfrak{Y}$ inherits from the spaces $X_i$ the property of having an admissible multirepresentation, we consider the coproduct $\bigoplus_{i \in \mathbb{N}} X_i$ and define the function $q : \bigcup_{i \in \mathbb{N}} X_i \rightarrow Y$ by $q(i, x) := q_i(x)$. Since $Y = \bigcup_{i \in \mathbb{N}} \text{range}(q_i)$, for every sequence $(y_n)_n \leq \infty$ in $Y$ we have

$$
(y_n)_n \rightarrow_{(q_i)} y_\infty \iff (\exists i \in \mathbb{N}) \left( (\exists (x_n)_{n \leq \infty} \in X_i^n) (\exists n_0 \in \mathbb{N}) \right.

\left. ((x_n)_n \rightarrow_{X_i} x_\infty \text{ and } (\forall n \in \{n_0, \ldots, \infty\}) y_n = q(i, x_n) \right).
$$

Hence the strong convergence relation $\rightarrow_{(q_i)}$ is equal to the final convergence relation $\rightarrow_{(\bigoplus_{i \in \mathbb{N}} X_i), q}$ induced by $q$ from the coproduct $\bigoplus_{i \in \mathbb{N}} X_i$. Propositions 4.5.1 and 4.5.10 imply that $(q \circ (\bigoplus_{i \in \mathbb{N}} \delta_i))^{\mathfrak{Y}}$ is an admissible multirepresentation of the weak limit space $(Y, \rightarrow_{(q_i)})$, whenever $\delta_i : \subseteq \mathbb{N}^{\omega} \Rightarrow X_i$ is an admissible multirepresentation of $X_i$ for every $i \in \mathbb{N}$.

In a similar way, we show that $(Y, \tau_{(q_i)})$ has an admissible multirepresentation, if for every $i \in \mathbb{N}$ there is an admissible multirepresentation $\gamma_i : \subseteq \mathbb{N}^{\omega} \Rightarrow X_i$ of the space $(X_i, \tau_i)$: The topological coproduct $\mathfrak{Z} := (\bigcup_{i \in \mathbb{N}} X_i, \bigoplus_{i \in \mathbb{N}} \tau_i)$ is a sequential space by Lemma 4.5.9. Since a set $M \subseteq X_i$ is open in $(X_i, \tau_i)$ if and only if $\{i\} \times M$ is open in $\mathfrak{Z}$, every subset $O \subseteq Y$ satisfies

$$
O \in \tau_{\text{seq}(\mathfrak{Z}), q} \iff q^{-1}[O] \in \text{seq}(\mathfrak{Z}) \iff q^{-1}[O] \in \bigoplus_{i \in \mathbb{N}} \tau_i

\iff \bigcup_{i \in \mathbb{N}} \left( \mathfrak{Z} \cap q^{-1}(O) \right) \in \bigoplus_{i \in \mathbb{N}} \tau_i.
$$
4.5.5 Coequalizers and Countable Colimits

A coequalizer of a pair of parallel morphisms \( f, g : A \to X \) in a category \( C \) is an object \( Y \) and a morphism \( q : X \to Y \) in \( C \) such that \( q \circ f = q \circ g \) holds and for all objects \( Z \) and all morphisms \( h : X \to Z \) with \( h \circ f = h \circ g \) there is exactly one morphism \( k : Y \to Z \) satisfying \( k \circ q = h \) (cf. [AL91]). We will now demonstrate that the categories \( \text{WeakLim}, \text{AdmWeakLim}, \text{AdmLim}, \text{AdmSeq} \) as well as \( \text{EffWeakLim}, \text{EffLim}, \text{EffSeq} \) have coequalizers for all pairs of parallel morphisms.

Let \( \mathfrak{A} = (A, \rightarrow_A), \mathfrak{X} = (X, \rightarrow_X), \) and \( \mathfrak{Z} = (Z, \rightarrow_Z) \) be weak limit spaces, and let \( f, g : A \to X \) be sequentially continuous functions. The object of a coequalizer of \( (f, g) \) in \( \text{WeakLim} \) can be constructed as a WeakLim–quotient of \( \mathfrak{X} \). We define the function \( q \) and the set \( Y \subseteq 2^X \) by

\[
q(x) := \{ x' \in X \mid \text{there are } k \in \mathbb{N} \text{ and } a_0, \ldots, a_k \in A \text{ with } (\forall i < k) g(a_i) = f(a_{i+1}) \text{ and } (x, x') = (f(a_0), g(a_k)) \cup (x', x) = (f(a_0), g(a_k)) \}
\]

and \( Y := \text{range}(q) \). Clearly, \( q \), viewed as a total surjection from \( X \) onto \( Y \), is \( (\rightarrow_X, \rightarrow_{q_X}) \)–continuous (cf. Subsection 2.2.4). Moreover, for every continuous function \( h : X \to Z \) between \( \mathfrak{X} \) and \( \mathfrak{Z} \) with \( h \circ f = h \circ g \) there is a function \( k : Y \to Z \) satisfying \( k \circ q = h \), because \( q(x) = q(x') \) implies \( h(x) = h(x') \). Clearly, \( k \) is unique. By Lemma 2.4.10(2), \( k \) is \( (\rightarrow_{q_X}, \rightarrow_Y) \)–continuous. We conclude that the weak limit space \( \mathfrak{Y} := (Y, \rightarrow_{Y,q}) \) and \( q \) form a coequalizer of the pair \( (f, g) \) in the category \( \text{WeakLim} \).

Now let \( \delta : \subseteq \mathbb{N} \to X \) be an admissible multirepresentation of \( \mathfrak{X} \). As \( \gamma := (q \circ \delta)^{\omega} \) is an admissible multirepresentation of \( \mathfrak{Y} \) by Proposition 4.5.1, the coequalizer of a pair of parallel morphisms of \( \text{AdmWeakLim} \) formed in \( \text{WeakLim} \) lies in \( \text{AdmWeakLim} \). Thus the category \( \text{AdmWeakLim} \) has coequalizers. By Lemma 4.3.6 we have \( (q \circ \delta) \leq_{cp} \gamma \equiv_{cp} \gamma^{\omega} \), and by Lemma 2.4.4 the \( (\delta, q \circ \delta) \)–computable function \( q \) is computable w.r.t. \( \delta \) and \( \gamma \). Thus \( (\mathfrak{Y}, \gamma) \) is an object and \( q \) is a morphism of \( \text{EffWeakLim} \). Moreover, if \( \zeta \) is an admissible multirepresentation of \( \mathfrak{Z} \) with \( \zeta \equiv_{cp} \zeta^{\omega} \) such that \( h \) is \( (\delta, \zeta) \)–computable, then \( k \) is \( (\gamma, \zeta) \)–computable by Proposition 4.5.4. Hence \( \text{EffWeakLim} \) has coequalizers for all pairs of parallel morphisms.

Hence the strong topology \( \tau_{\delta(q_i)} \) is equal to the final topology \( \tau_{\delta(q_i)} \) induced by \( q \) from the coproduct \( \delta \). From Propositions 4.5.1, 4.5.10 and Lemma 4.5.9 it follows that \( (q \circ (\mathbb{N}_i \in \mathfrak{Z}_i))^\omega \) is an admissible of the topological space \( (Y, \tau_{\delta(q_i)}) \).

If all the spaces \( \mathfrak{X}_i \) are limit spaces, one can also equip \( \mathfrak{Y} \) with the finest (smallest) convergence relation \( \sim_{\delta(q_i)} \) which satisfies Axioms (L1), (L2), (L3) and for which each function \( q_i \) is sequentially continuous. An admissible multirepresentation of \( (Y, \sim_{\delta(q_i)}) \) is given by \( (q \circ (\mathbb{N}_i \in \mathfrak{Z}_i))^\omega \), if \( \mathfrak{Z}_i := \mathfrak{X}_i \) is an admissible multirepresentation of \( \mathfrak{X}_i \) for every \( i \in \mathbb{N} \).

\[
\iff (\forall i \in \mathbb{N}) \{(i) \times X_i \cap q^{-1}[O] \in \bigoplus_{i \in \mathbb{N}} \tau_i
\iff (\forall i \in \mathbb{N}) \{i \times q_i^{-1}[O] \in \bigoplus_{i \in \mathbb{N}} \tau_i
\iff (\forall i \in \mathbb{N}) q_i^{-1}[O] \in \tau_i
\iff O \in \tau_{\delta(q_i)}
\]
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If \( Y \) and \( X \) are limit spaces, then the \( \lim \)-quotient \( (Y, \sim_{x,q}) \) and the continuous function \( q \) form a coequalizer of \( (f, g) \) in the category \( \lim \). Moreover, if \( \tau_X \) and \( \tau_X \) are sequential topologies on, respectively, \( A \) and \( X \) such that \( f \) and \( g \) are \( (\tau_X, \tau_X) \)-continuous, then the sequential topological space \( (Y, \tau_{\tau_X,q}) \) and the topologically continuous function \( q \) form a coequalizer of \( f, g \) in \( \text{Seq} \). By similar arguments as in the case of weak limit spaces, it follows that the categories \( \text{AdmLim}, \text{AdmSeq}, \text{EffLim}, \) and \( \text{EffSeq} \) have coequalizers for all pairs of parallel morphisms, too.

By Proposition 4.5.10, all countable coproducts exist in the categories \( \text{AdmWeakLim}, \text{AdmLim}, \) and \( \text{AdmSeq} \). Thus these categories, providing coequalizers for all pairs of morphisms, have countable colimits by the dual of [AL91, Theorem 6.3.1]. From Subsection 4.5.3 we know that the categories \( \text{EffWeakLim}, \text{EffLim}, \) and \( \text{EffSeq} \) are bicartesian-closed. Therefore all finite colimits exist in these categories by the dual of [AL91, Corollary 6.3.3].

4.5.6 Uniform Weak Limit Spaces

We define in this subsection the uniform weak limit spaces. They are equipped with quite simple admissible multirepresentations. Moreover, we characterize the class of weak limit spaces that have a fiber–compact admissible multirepresentation as a certain subclass of uniform weak limit spaces.

Let \( (U_{i,j})_{i,j \in \mathbb{N}} \) be a double sequence of sets. We define the weak limit space \( U_{(U_{i,j})} \) by

\[
U_{(U_{i,j})} := \bigwedge_{i \in \mathbb{N}} \bigvee_{j \in \mathbb{N}} (U_{i,j}, -_{\epsilon_{ij}}),
\]

where \(-_{\epsilon_{ij}}\) denotes, for every \( i, j \in \mathbb{N} \), the chaotic convergence relation on the considered set \( U_{i,j} \) declaring every sequence to be convergent. By \(-_{(U_{i,j})}\) we denote the convergence relation of \( U_{(U_{i,j})} \). From Subsections 4.1.4 and 4.5.2 we know that \( U_{(U_{i,j})} \) is actually a weak limit space, because the “chaotic” spaces \( (U_{i,j}, -_{\epsilon_{ij}}) \) are weak limit spaces (cf. Example 2.4.24). We call any sequential convergence space \( \mathcal{Y} \) a uniform weak limit space iff there is double sequence \( (U_{i,j})_{i,j \in \mathbb{N}} \) of subsets of the underlying set of \( \mathcal{Y} \) such that \( \mathcal{Y} = U_{(U_{i,j})} \).

The Euclidean limit space is an example of a uniform weak limit space.

Example 4.5.14 (The Euclidean space as a uniform weak limit space)

For \( m \in \mathbb{N} \), we consider the multirepresentation \( \gamma_m : \subseteq \mathbb{Z}^m \rightsquigarrow \mathbb{R} \) from Example 2.4.2. For \( z \in \mathbb{N} \), let \( U_{m,z} \) denote the interval

\[
U_{m,z} := \left[ (z - 1) \cdot \left(\frac{1}{2}\right)^m; (z + 1) \cdot \left(\frac{1}{2}\right)^m \right].
\]

In Example 2.4.13 we have observed that a sequence \( (x_n)_{n \leq \infty} \) of reals is a convergent sequence of \( (\mathbb{R}, -_{\epsilon_m}) \) if and only if there is an integer \( z \in \mathbb{Z} \) with \( (\forall n \in \mathbb{N}) x_n \in U_{m,z} \). Hence \( (\mathbb{R}, -_{\epsilon_m}) = \bigvee_{z \in \mathbb{Z}} (U_{m,z}, -_{\epsilon_{im}}) \). Moreover, \( (x_n)_{n \leq \infty} \) is a convergent sequence of the Euclidean space if and only if, for every \( m \in \mathbb{N} \), \( (x_n)_{n \leq \infty} \) converges to \( x_{\infty} \) in the space \( (\mathbb{R}, -_{\epsilon_m}) \). Thus we have \( (\mathbb{R}, -_{\epsilon_{im}}) = \bigvee_{m \in \mathbb{N}} (U_{m,z}, -_{\epsilon_{im}}) \).

Actually, every separable metric space, viewed as a limit space, turns out to be a uniform weak limit space.
Lemma 4.5.15
For every separable metric space \((X, d)\), the space \((X, \rightarrow_a)\) is a uniform weak limit space.

Proof:
Let \(\{\alpha_0, \alpha_1, \alpha_2, \ldots\}\) be a countable dense subset of \((X, d)\). For \(i, j \in \mathbb{N}\) we define the set \(U_{i,j}\) by
\[
U_{i,j} := \{ x \in X \mid d(x, \alpha_j) \leq (\frac{1}{2})^i \}.
\]
With standard methods, one can show \(U(U_{i,j}) = (X, \rightarrow_a)\).

Further examples of uniform weak limit spaces are the spaces considered in Examples 2.2.5, 2.3.6, 2.3.15, 2.3.16, and 2.4.14.

Every uniform weak limit space has a simple admissible multirepresentation.

Proposition 4.5.16 (Admissible multirep. of uniform weak limit spaces)
Let \((U_{i,j})_{i,j} \in \mathbb{N}\) be a double sequence of sets, and let \(X := \bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} U_{i,j}\). Let the correspondence \(\delta(U_{i,j}) : \subseteq \mathbb{N}^\omega \Rightarrow X\) be defined by
\[
\delta(U_{i,j})[p] := \bigcap_{i \in \mathbb{N}} U_{i,p(i)}
\]
for all \(p \in \mathbb{N}^\omega\).
Then \(\delta(U_{i,j})\) is an admissible multirepresentation of the uniform weak limit space \(U(U_{i,j})\).

Proof:
For every \(i \in \mathbb{N}\), the multirepresentation \(\zeta_i : \subseteq \mathbb{N}^\omega \Rightarrow \bigcup_{j \in \mathbb{N}} U_{i,j}\) defined by \(\zeta_i[p] := U_{i,p(0)}\) is an admissible multirepresentation of \(\bigvee_{j \in \mathbb{N}} (U_{i,j}, \rightarrow_{\omega})\) by Lemma 4.5.8 and Example 2.4.24. One easily verifies that \(\delta(U_{i,j})\) is computably equivalent to the conjunction \(\bigwedge_{i \in \mathbb{N}} \zeta_i\). Thus \(\delta(U_{i,j})\) is an admissible multirepresentation of \(U(U_{i,j})\) by Propositions 4.1.7 and 2.4.21.

We say that a multirepresentation \(\delta : \subseteq \Sigma^\omega \Rightarrow X\) is fiber–compact iff for every element \(x \in X\) the “fiber” \(\delta^{-1}[x]\) is compact. Moreover, \(\delta : \subseteq \Sigma^\omega \Rightarrow X\) is called proper iff the preimage \(\delta^{-1}[K]\) is compact for every subset \(K \subseteq X\) that is sequentially compact in \((X, \rightarrow_a)\). Fiber–compact admissible multirepresentations and proper admissible multirepresentations play important roles in defining reasonable notions for functions on represented spaces (cf. [Wei01]). We have the following nice characterization of the weak limit spaces that have a fiber–compact admissible multirepresentation.

Proposition 4.5.17 (Spaces with fiber–compact admissible multirep.)
A weak limit space \(X = (X, \rightarrow_a)\) has a fiber–compact admissible multirepresentation if and only if there is a double sequence \((U_{i,j})_{i,j} \in \mathbb{N}\) with \(X = U(U_{i,j})\) such that for every \(x \in X\) and every \(i \in \mathbb{N}\) the set \(J_{x,i} := \{ j \in \mathbb{N} \mid x \in U_{i,j} \}\) is finite.
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Proof:

\(\Rightarrow\): Let \(\delta \subseteq \Sigma^w \Rightarrow X\) be an admissible multirepresentation with compact fibers. At first we show for every \(x \in X\) and every set \(W \subseteq \Sigma^*\) the implication

\[
\delta^{-1}[x] \subseteq W \Sigma^w \implies (\exists k \geq 1)(\exists v_1, \ldots, v_k \in W) \delta^{-1}[x] \subseteq \{v_1, \ldots, v_k\} \Sigma^w \quad (4.25)
\]

Let \(w_0, w_1, \ldots\) be a sequence of words with \(\{w_0, w_1, \ldots\} = W\). Suppose for contradiction \((\forall n \in \mathbb{N})\delta^{-1}[x] \nsubseteq \{w_0, \ldots, w_n\} \Sigma^w\). Then for every \(n \in \mathbb{N}\) there is some \(p_n \in \delta^{-1}[x] \setminus \{w_0, \ldots, w_n\} \Sigma^w\). Since \(\delta^{-1}[x]\) is compact and thus sequentially compact (as \((\Sigma^w, \gamma_{\Sigma^w})\) is countably based), there is some strictly increasing function \(\varphi : \mathbb{N} \to \mathbb{N}\) and some \(p_\infty \in \delta^{-1}[x]\) with \((p_{\varphi(n)})_n \to p_\infty\). Moreover, there is some \(k \in \mathbb{N}\) with \(p_\infty \in w_k \Sigma^w\). Hence \((p_{\varphi(n)})_n\) is eventually in \(w_k \Sigma^w\), a contradiction.

For every \(l \in \mathbb{N}\) and \(w \in \Sigma^l := \{u \in \Sigma^* \mid \text{lg}(u) = l\}\) we define \(U_{l,w} := [\delta[w \Sigma^w]\) and show

\[
(x_n)_n \to_x x_\infty \iff (\forall l \in \mathbb{N})(\exists w \in \Sigma^l)(\forall \in \mathbb{N}) x_n \in U_{l,w}.
\]

\(\Rightarrow\): Let \((x_n)_n \subseteq \mathbb{N}\) be a convergent sequence of \(X\). By Proposition 2.4.18, there is a convergent sequence \((p_n)_n \subseteq \mathbb{N}\) in \(\text{dom}(\delta)\) with \((\forall n \in \mathbb{N}) x_n \in \delta[p_n]\). Let \(l \in \mathbb{N}\). Then there is some \(m_0 \in \mathbb{N}\) with \((\forall n \geq m_0) p_n \in p_\infty \subseteq \Sigma^w\). It follows that \(x_n \in U_{l,p_\infty}\) for all \(n \geq m_0\). Thus \((x_n)_n \subseteq \mathbb{N}\) is a convergent sequence of \(\bigvee_{w \in \Sigma^l}(U_{l,w}, \to_x)\) for all \(l \in \mathbb{N}\).

\(\Leftarrow\): Let \((x_n)_n \subseteq \mathbb{N}\) be a sequence such that \((x_n)_n\) does not converge to \(x_\infty\) in \(X\). By Lemma 2.4.8, for every \(p \in \delta^{-1}[x_\infty]\) there is some \(k_p \in \mathbb{N}\) with \((\exists \in \mathbb{N}) x_n \notin \delta[p \wedge \Sigma^w]\). By Statement (4.25), there are words \(v_1, \ldots, v_m \in \{p \wedge \mid p \in \delta^{-1}[x_\infty]\}\) with \(\delta^{-1}[x] \subseteq \{v_1, \ldots, v_m\} \Sigma^w\). We define \(l := \max\{\text{lg}(v_1), \ldots, \text{lg}(v_m)\}\). Let \(w \in \Sigma^l\) with \(x_n \in U_{l,w}\). Then for some \(i \in \{1, \ldots, k\}\), \(v_i\) is a prefix of \(w\). It follows that \(\delta[w \Sigma^w]\), being a subset of \(\delta[v_i \Sigma^w]\), does not contain \(x_n\) for infinitely many \(n \in \mathbb{N}\). Hence \((x_n)_n \subseteq \mathbb{N}\) is not a convergent sequence of \(\bigvee_{w \in \Sigma^l}(U_{l,w}, \to_x)\).

Therefore \(\mathcal{X} = \bigcup_{U_{l,w}}\). From Statement (4.25) it follows that for every \(x \in X\) the set \(J_x := \{w \in \Sigma^l \mid x \in U_{l,w}\}\) is finite (even if \(X\) is a countably infinite alphabet).

\(\Leftarrow\): By Proposition 4.5.16, the multirepresentation \(\delta \subseteq \mathbb{N}^w \Rightarrow X\) defined by \(\delta[p] := \bigcap_{i \in \mathbb{N}} U_{i,p(i)}\) is an admissible multirepresentation of \(X\).

Let \(x \in X\). We show that \(\delta^{-1}[x]\) is sequentially compact. Let \((p_n)_n\) be a sequence in \(\delta^{-1}[x]\). We construct inductively a decreasing sequence of infinite sets \(M_1 \supseteq M_0 \supseteq M_1 \supseteq \ldots\) and a sequence of natural numbers \(j_0, j_1, \ldots\) with \((\forall i \in \mathbb{N}) \min(M_i) > \min(M_{i-1})\) and \((\forall i \in \mathbb{N})(\forall n \in M_i) p_n(i) = j_i\) as follows:

\(i = 1\): We choose \(M_{i-1} := \mathbb{N}\).

\(i + 1 \rightarrow i\): Since \(M_{i-1}\) is infinite, whereas \(J_x_i\) is finite, and since for every \(m \in M_{i-1}\) we have \(p_m(i) \in J_{x,i}\) by definition of \(\delta\), there is some \(j_i \in J_{x,i}\) with \(p_m(i) = j_i\) for infinitely many \(m \in M_{i-1}\). We choose \(M_i := \{m \in M_{i-1} \mid m > \min(M_{i-1})\} \land p_m(i) = j_i\}.

Using the sequences \((M_i)_i\) and \((j_i)_i\), we define \(p_\infty \in \mathbb{N}^w\) and \(\varphi : \mathbb{N} \to \mathbb{N}\) by \(p_\infty(i) = j_i\) and \(\varphi(i) := \min(M_i)\) for all \(i \in \mathbb{N}\). Then \(\varphi\) is strictly increasing, \(p_\infty \in \delta^{-1}[x]\) and \((p_{\varphi(n)})_n\) converges to \(p_\infty\), because for every \(i \in \mathbb{N}\) and all \(n \geq i\) we have \(p_{\varphi(n)}(i) = j_i\).
Proof: Let \( \varphi(n) \) be a representation if and only if \( X \). Moreover, one can show that a weak limit space \( (\bigcup \tau_i)_{i \in \mathbb{N}} \) of sequentially closed sets such that \( X = \bigcup (U_{i,j}) \) and for every sequentially closed set \( K \subseteq X \) and every \( i \in \mathbb{N} \) the set \( J_{K,i} := \{ j \in \mathbb{N} \mid K \cap U_{i,j} \neq \emptyset \} \) is finite. We omit the proof.

### 4.5.7 Inductive Limit Spaces

Let \((X_i)_i = ((X_i, \tau_{X_i}))_i\) be an “increasing” sequence of topological spaces, i.e., \( X_i \) is a subspace of \( X_{i+1} \) for every \( i \in \mathbb{N} \). We equip \( \bigcup_{i \in \mathbb{N}} X_i \) with the inductive limit topology

\[
\tau_{\text{Lim}, X_i} := \{ O \subseteq \bigcup_{i \in \mathbb{N}} X_i \mid (\forall l \in \mathbb{N}) O \cap X_l \in \tau_{X_l} \}. 
\]

Obviously, \( \tau_{\text{Lim}, X_i} \) is actually a topology on \( \bigcup_{i \in \mathbb{N}} X_i \). The topological space \( \text{Lim} X_i := (\bigcup_{i \in \mathbb{N}} X_i, \tau_{\text{Lim}, X_i}) \) is called the inductive limit of the increasing sequence \( (X_i)_i \). In the case that the spaces \( X_i \) are all in \( \text{AdmSeq} \), \( \text{Lim} X_i \) is the colimit of the spaces \( X_i \) and the injections \( \iota_{i,j} : X_i \to X_{i+j} \) in \( \text{AdmSeq} \) and has therefore an admissible multirepresentation by Subsection 4.5.5. We now construct directly an admissible representation of \( \text{Lim} X_i \) in the case that the spaces \( X_i \) are admissibly representable \( T_1 \)-spaces.

**Lemma 4.5.18 (Inductive limit spaces)**

Let \((X_i)_i = ((X_i, \tau_{X_i}))_i\) be an increasing sequence of topological spaces.

1. If all the spaces \( X_i \) are \( T_1 \)-spaces, then the convergence relation of the inductive limit space \( \text{Lim} X_i \) is equal to the one of the disjunction \( \bigvee_{i \in \mathbb{N}} X_i \).

2. If all the spaces \( X_i \) are sequential, then the inductive limit space \( \text{Lim} X_i \) is a sequential space.

**Proof:**

1. A first we prove that, for every \( l \in \mathbb{N} \), \( X_l \) is a subspace of the inductive limit space \( \text{Lim} X_i \), i.e. \( \tau_{X_l} \) = \( \{ V \cap X_l \mid V \in \tau_{\text{Lim}, X_i} \} \).

   “\( \supseteq \)”: This follows directly from the definition of the topology \( \tau_{\text{Lim}, X_i} \).

   “\( \subseteq \)”: Let \( U \in \tau_{X_l} \). We define inductively a sequence \( (O_i)_i \) of sets with \( U = O_l \), \( O_i \subseteq X_i \), and \( O_j = O_i \cap X_j \) for all \( i \in \mathbb{N} \) and \( j \leq i \).

   \( i \leq l \): Since \( X_i \) is a subspace of \( X_l \), the set \( O_i := U \cap X_i \) is open in \( X_i \). For all \( j \leq i \), we have \( O_j = O_i \cap X_j = U \cap X_i \cap X_j \).

   \( i > l \): Since \( X_{i-l} \) is a subspace of \( X_i \), there is some set \( O_i \cap X_{i-j} \) and \( O_i \cap X_{i-j} \) is open in \( X_i \). For all \( j < i \) we have \( O_j = O_i \cap X_j = O_{i-l} \cap X_{i-j} \cap X_j = O_i \cap X_{i-j} \).

   The set \( O := \bigcup_{i \in \mathbb{N}} O_i \) is open in \( \text{Lim} X_i \), because for every \( k \in \mathbb{N} \) we have

   \[
   O \cap X_k = \left( \bigcup_{j<k} (O_j \cap X_k) \right) \cup (O_k \cap X_k) \cup \left( \bigcup_{j>k} (O_j \cap X_k) \right)
   \]
4.5 Final Constructions

= O_k \cup \bigcup_{i>k} (O_i \cap X_k) = O_k \in \tau_{X_k}.

Since $U = O_l = O \cap X_l$, we obtain $U \in \{ V \cap X_l \mid V \in \tau_{\text{Lim} X_i} \}$.

Now we are ready to prove $\tau_{\text{Lim} X_i} = (\forall i \in \mathbb{N} \rightarrow x_i)$.

“\(\subseteq\)”: Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $\text{Lim} X_i$.

Suppose for contradiction that for all $l \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ with $x_n \notin X_l \cup \{x_{\infty}\}$. Then we can inductively define a strictly increasing function $\varphi : \mathbb{N} \cup \{-1\} \rightarrow \mathbb{N} \cup \{-1\}$ by $\varphi(-1) := -1$ and

$$\varphi(i) := \min\{ n > \varphi(i-1) \mid x_n \notin X_i \cup \{x_{\infty}\} \}.$$

Let $O := X \setminus \{ x_{\varphi(i)} \mid i \in \mathbb{N} \}$. For every $i \in \mathbb{N}$ we have

$$O \cap X_i = X_i \setminus \{ x_{\varphi(0)}, \ldots, x_{\varphi(i)} \},$$

as $(\forall n \geq i) x_{\varphi(n)} \notin X_i$. Hence $O \cap X_i$ is open in $X_i$, because in $T_1$-spaces finite sets are closed. This implies that $O$ is an open set of $\text{Lim} X_i$ containing $x_{\infty}$.

Since $(x_{\varphi(n)})_n$ converges to $x_{\infty}$, $(x_{\varphi(n)})_n$ is eventually in $O$, a contradiction.

Thus there are some $l_0, n_0 \in \mathbb{N}$ with $x_n \in X_{l_0}$ for all $n \in \{n_0, \ldots, \infty\}$. Since $X_{l_0}$ is a subspace of $\text{Lim} X_i$, $(x_{n+n_0})_n$ converges to $x_{\infty}$ in $X_{l_0}$. This implies that $(x_n)_{n \leq \infty}$ is a convergent sequence of $\bigvee_{i \in \mathbb{N}} X_i$.

“\(\supseteq\)”: Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $\bigvee_{i \in \mathbb{N}} X_i$.

Let $O$ be an open set of $\text{Lim} X_i$ with $x_{\infty} \in O$. There are some $j, n_1 \in \mathbb{N}$ with $(\forall n \geq n_1) x_n \in X_j$ and $(x_{n+n_1})_n \rightarrow_{x_j} x_{\infty}$. By definition of $\tau_{\text{Lim} X_i}$, we have $O \cap X_j \in \tau_{X_j}$. Thus there is some $n_2 \geq n_1$ with $(\forall n \geq n_2) x_n \in O \cap X_j \subseteq O$.

Therefore $(x_n)_{n \leq \infty}$ is a convergent sequence of $\text{Lim} X_i$.

(2) Let $O$ be sequentially open in $\text{Lim} X_i$, and let $j \in \mathbb{N}$. Let $(x_n)_{n \leq \infty}$ be a convergent sequence of $X_j$ with $x_{\infty} \in O \cap X_j$. Then $(x_n)_{n \leq \infty}$ is also a convergent sequence of $\text{Lim} X_i$, because for every neighbourhood $U \in \tau_{\text{Lim} X_i}$ of $x_{\infty}$ the set $U \cap X_j$ is a neighbourhood of $x_{\infty}$ in $X_j$. Thus $(x_n)_n$ is eventually in $O$. Hence $O \cap X_j$ is sequentially open in $X_j$ and therefore open in $X_j$, because $X_j$ is sequential. Hence $O$ is open in $\text{Lim} X_i$. This means that $\text{Lim} X_i$ is a sequential topological space.

✓

From Lemmas 4.5.18 and 4.5.8 we obtain the following proposition.

**Proposition 4.5.19 (Admissable representations of inductive limit spaces)**

Let $(X_i)_l = ((X_i, \tau_i))_{l}$ be an increasing sequence of topological $T_1$-spaces. For every $l \in \mathbb{N}$, let $\delta_l : \subseteq \mathbb{N}^l \rightarrow X_l$ be an admissible representation of $X_l$.

Then $(\bigcup_{i \in \mathbb{N}} \delta_l)$ is an admissible representation of the inductive limit $\text{Lim} X_i$.

Nice applications of the concept of inductive limit spaces are the Gevrey classes of infinitely differentiable functions.
Example 4.5.20 (Gevrey class)

For a given real \( \alpha > 0 \), the Gevrey class of order \( \alpha \) is defined to be the set of those infinitely differentiable functions \( f : [-1;1] \to \mathbb{R} \) for which there exists some \( R > 0 \) satisfying

\[
(\forall n \in \mathbb{N}) \| f^{(n)} \|_\infty \leq R^{n+1} \cdot n^{\alpha - n}
\]  

(cf. [LLM01]). Here \( f^{(n)} \) denotes the \( n \)-th derivative of \( f \) and \( \| \cdot \|_\infty \) denotes the maximum norm on the set \( C([-1;1]) \) of continuous functions \( g : [-1;1] \to \mathbb{R} \). The maximum norm is defined by \( \| g \|_\infty := \sup \{ g(x) | x \in [-1;1] \} \). We denote by \( \text{Gev}_\alpha \) the Gevrey class of order \( \alpha \) and, for a given \( R > 0 \), by \( \text{Gev}_{\alpha,R} \) the set of functions \( f \in \text{Gev}_\alpha \) satisfying Estimation (4.26).

Let \( d_\infty \) be the maximum metric on \( C([-1;1]) \) defined by \( d_\infty(g,h) := \| g - h \|_\infty \). The function \( \theta_{\text{Gev}}([-1;1]) := \mathbb{N}^\omega \to C([-1;1]) \) defined by

\[
\theta_{\text{Gev}}([-1;1]) := \left[ \theta_{\mathbb{R}}([-1;1]) \to \theta_{\mathbb{R}} \right]
\]

is an admissible representation of the separable metric space \( (C([-1;1]), d_\infty) \), because \( \theta_{\mathbb{R}} \) is an admissible representation of the Euclidean space (cf. Example 2.3.8), \( \theta_{\text{Gev}}([-1;1]) \) is admissible w.r.t. the compact open topology \( \tau^{co} \) on \( C([-1;1]) \) by Proposition 4.2.5, and the compact open topology \( \tau^{co} \) on \( C([-1;1]) \) is induced by the maximum metric \( d_\infty \) (cf. [Eng89]). By Subsection 4.1.5 it follows that, for every \( l \in \mathbb{N} \), \( \delta_l := \theta_{\text{Gev}}([-1;1])^{(\text{Gev},l)} \) is an admissible representation of the subspace \( X_l := (\text{Gev}_{\alpha,l}, \tau^{co}|_{\text{Gev}_{\alpha,l}}) \). By Proposition 4.5.19, \( \theta_{\text{Gev}} := \bigvee_{l \in \mathbb{N}} \delta_l \) is a \( \tau_{\lim, X_l} \)-admissible representation of \( \text{Gev}_\alpha \).

This representation has nice computability properties. For example, the function \( (k,f) \mapsto f^{(k)} \) is (efficiently) computable w.r.t. \( \theta_{\mathbb{N}} \) and \( \theta_{\text{Gev}} \). This can be deduced from Theorem 5.2.13 in [LLM01]. S. Labhalla, H. Lombardi, E. Moutai proved in [LLM01] some interesting complexity results about operators on the Gevrey classes (like integration and differentiation)

\( \Box \)

The next example is the space of “test functions” equipped with the “natural topology” from which the space of “distributions” is derived (cf. [BN73, ZoWe00]). Distributions play an important role in the field of partial differential equations.

Example 4.5.21 (Test functions and distributions)

A test function on \( \mathbb{R} \) is an infinitely differentiable function \( f : \mathbb{R} \to \mathbb{C} \) such that its support \( \text{supp}(f) := f^{-1}[C \setminus \{0\}] \) lies in some compact set (cf. [BN73]). Distributions are linear functionals from the vector space \( \mathcal{D}(\mathbb{R}) \) of the test functions on \( \mathbb{R} \) to \( \mathbb{C} \).

We denote the set of continuous functions \( f : \mathbb{R} \to \mathbb{C} \) by \( C(\mathbb{R}) \), the set of infinitely differentiable functions \( f : \mathbb{R} \to \mathbb{C} \) by \( C^\infty(\mathbb{R}) \), and, for a compact set \( K \subseteq \mathbb{R} \), the set of test functions with support in \( K \) by \( C^\infty(\mathbb{R}; K) \). Together with the usual addition and the usual scalar product on functions, \( \mathcal{D}(\mathbb{R}), C(\mathbb{R}), C^\infty(\mathbb{R}), \) and \( C^\infty(\mathbb{R}; K) \) are known to form vector spaces over the complex field \( \mathbb{C} \) (cf. [BN73]).

By using the construction methods in the previous sections, we equip these vector spaces with representations such that the resulting represented spaces are computable vector spaces. A computable vector space is a pair \( (\mathcal{V}, \delta) \) such that \( \mathcal{V} \) is a vector space
over \( \mathbb{C} \), \( \delta \) is a representation of the underlying set of \( \mathcal{V} \), addition is \((\delta, \delta, \delta)\)-computable, scalar multiplication is \((\delta, \delta, \delta)\)-computable, where \( \delta_{\mathbb{C}} := \delta_{\mathbb{R}} \otimes \delta_{\mathbb{R}} \), and the origin \( \bar{0} \) is a \( \delta \)-computable element (cf. [Bra01]). It is well-known that \( \delta_{\mathbb{R}} \) and thus \( \delta_{\mathbb{C}} \) admit computability of addition and multiplication (cf. Example 4.3.17 and [Wei00, Theorem 4.3.2]). Lemma 4.2.8 implies that addition and scalar multiplication in \( C(\mathbb{R}) \) is computable w.r.t.

\[
\delta_{C(\mathbb{R})} := [\delta_{\mathbb{R}} \rightarrow \delta_{\mathbb{C}}]
\]

and \( \delta_{\mathbb{C}} \) (see also [Bra01, Proposition 3.6]). For \( m \in \mathbb{N} \), let \( \partial^{(m)} : C^\infty(\mathbb{R}) \rightarrow C(\mathbb{R}) \) denote the operator that maps an infinitely differentiable function to its \( m \)-th derivative. We define the representation \( \delta_{C^\infty(\mathbb{R})} : \mathbb{N}^\omega \rightarrow C^\infty(\mathbb{R}) \) by

\[
\delta_{C^\infty(\mathbb{R})}(p) = f := (\forall m \in \mathbb{N}) \partial^{(m)}(f) = \delta_{C(\mathbb{R})}(\pi_{\infty, m}(p)).
\]

With the help of Proposition 4.1.3 one can easily verify that \( (C^\infty(\mathbb{R}), \delta_{C^\infty(\mathbb{R})}) \) is a computable vector space, too. Finally we define \( \delta_{D(\mathbb{R})} : \mathbb{N}^\omega \rightarrow D(\mathbb{R}) \) by

\[
\delta_{D(\mathbb{R})}(p) = f := (\text{supp}(f) \subseteq [-p(0):p(0)] \land g_{C^\infty(\mathbb{R})}(p^\geq 0) = f).
\]

Since for functions \( f, g \in D(\mathbb{R}) \) and \( \alpha \in \mathbb{C} \) we have \( \text{supp}(f + g) \subseteq \text{supp}(f) \cup \text{supp}(g) \) and \( \text{supp}(\alpha \cdot f) \subseteq \text{supp}(f) \), the pair \( (D(\mathbb{R}), \delta_{D(\mathbb{R})}) \) inherits from \( (C^\infty(\mathbb{R}), \delta_{C^\infty(\mathbb{R})}) \) the property of being a computable vector space.

By Propositions 4.2.5 and 4.1.8, \( \delta_{C(\mathbb{R})} \) is admissible w.r.t. the compact open topology \( \tau_{\text{co}} \) on \( C(\mathbb{R}) \). Let \( \tau_{C^\infty(\mathbb{R})} \) be the topology generated from \( \mathcal{C}(\mathbb{R}) := (C(\mathbb{R}), \tau_{\text{co}}) \) by the sequence \((\partial^{(m)})_m \), cf. Subsection 4.2.5. By [BN73, Section 2.1], \( \tau_{C^\infty(\mathbb{R})} \) is a compatible\(^{19} \) and locally convex\(^{20} \) topology on the vector space \( C^\infty(\mathbb{R}) \); moreover, \( \tau_{C^\infty(\mathbb{R})} \) is induced by some complete metric \( d_{C^\infty(\mathbb{R})} \). Proposition 4.1.4 implies that \( \delta_{C^\infty(\mathbb{R})} \) is an admissible representation of the space \( \mathcal{C}^\infty(\mathbb{R}) := (C^\infty(\mathbb{R}), \tau_{C^\infty(\mathbb{R})}) \). For every \( k \in \mathbb{N} \), let \( \mathcal{C}^\infty_k(\mathbb{R}) \) be the topological subspace of \( \mathcal{C}^\infty(\mathbb{R}) \) with underlying set \( C^\infty(\mathbb{R}; [-k; k]) \). Then \( (\mathcal{C}^\infty_k(\mathbb{R}))_k \) is an increasing sequence of metrizable and thus sequential topological spaces. The natural topology \( \tau_{D(\mathbb{R})} \) on \( D(\mathbb{R}) \) is defined to be the inductive limit topology \( \tau_{\text{Lim}} \mathcal{E}^\infty_k(\mathbb{R}) \).

By Proposition 4.5.19, \( \delta_{D(\mathbb{R})} \) is an admissible representation of the inductive limit space \( \mathcal{D}(\mathbb{R}) := (D(\mathbb{R}), \tau_{\text{Lim}} \mathcal{E}^\infty_k(\mathbb{R})) \). Moreover, since \( \mathcal{D}(\mathbb{R}) \) is sequential by Lemma 4.5.18, the final topology of \( \delta_{D(\mathbb{R})} \) is equal to \( \tau_{\text{Lim}} \mathcal{E}^\infty_k(\mathbb{R}) \) by virtue of Proposition 2.4.18.

By \( \mathcal{D}'(\mathbb{R}) \) we denote the set of distributions, i.e. the set of linear functions from the vector space \( D(\mathbb{R}) \) to \( \mathbb{C} \). From Lemma 4.5.18 we obtain that the convergence relation \(-\rightarrow_{\mathcal{D}'(\mathbb{R})} \) of \( \mathcal{D}(\mathbb{R}) \) is given by

\[
(f_n)_n \rightarrow_{\mathcal{D}'(\mathbb{R})} f_\infty \iff \lim_{n \rightarrow \infty} d_{C^\infty(\mathbb{R})}(f_n, f_\infty) = 0 \land (\exists k \in \mathbb{N})(\forall n \in \mathbb{N}) \text{supp}(f_n) \subseteq [-k; k].
\]

\(^{19}\)A topology on a vector space is called compatible iff addition and scalar multiplication is topologically continuous w.r.t. that topology (cf. [BN73]).

\(^{20}\)A topology on a vector space is called locally convex iff it is compatible and the origin \( \bar{0} \) has a neighbourhood base of convex sets (cf. [BN73]).
Lemma 2.2.3 implies that addition and scalar multiplication in $D(\mathbb{R})$ are sequentially continuous (w.r.t. $\rightarrow_D$ and $\rightarrow_C$). Thus $\varrho_D(\mathbb{R}) := (\{\varrho_D(\mathbb{R}) \rightarrow \varrho_C\})|D(\mathbb{R})$ is a representation of the set $D'(\mathbb{R})$ of distributions.

By local compactness of $(\mathbb{C}, \tau_C)$ and by [Eng89, Ex. 3.3.J], the product topology $\tau_C \otimes \tau_{\lim\mathcal{C}_k}(\mathbb{R})$ is sequential, hence scalar multiplication in $D(\mathbb{R})$ is even topologically continuous (cf. [Eng89, Proposition 1.6.15]). On the other hand, Equivalence (4.27) implies that $\rightarrow_D$ is the convergence relation on $D(\mathbb{R})$ considered in [Dud64, Sections 5, 9]. Therefore $\tau_{\lim\mathcal{C}_k}(\mathbb{R})$, being equal to $\text{seq}(\rightarrow_D)$, is not a compatible topology on the vector space $D(\mathbb{R})$ by a remark in [Dud64, Section 9]. We conclude that addition, although sequentially continuous, is not topologically continuous w.r.t. $\tau_{\lim\mathcal{C}_k}(\mathbb{R})$. Hence, $D(\mathbb{R})$ is an example for a natural sequential topological space $(X, \tau)$ with admissible representation such that the product space $(X \times X, \tau \otimes \tau)$ is not sequential.

The usually considered topology $\tau_{\mathcal{C}}$ on $D(\mathbb{R})$ is defined to be the finest locally convex topology for which the injections $\iota_k : C^\infty(\mathbb{R}; \mathbb{k}; \mathbb{k}) \rightarrow D(\mathbb{R})$ are topologically continuous (cf. [BN73, Section 2.2]). From [BN73, Theorem 2.9] it follows that a sequence $(f_n)_{n \leq \infty}$ is a convergent sequence of $\mathcal{D}_{\mathcal{C}}(\mathbb{R}) := (D(\mathbb{R}), \tau_{\mathcal{C}})$ if and only if $(f_n)_{n \leq \infty}$ satisfies the right hand side of Equivalence (4.27). Hence $\mathcal{D}_{\mathcal{C}}(\mathbb{R})$ and $D(\mathbb{R})$ have the same convergent sequences. By Lemma 2.4.30, $\varrho_D(\mathbb{R})$ is an admissible representation of $\mathcal{D}_{\mathcal{C}}(\mathbb{R})$ as well. Since $\tau_{\mathcal{C}}$, being locally convex, is compatible in contrast to $\tau_{\lim\mathcal{C}_k}(\mathbb{R})$, these topologies are not equal. Therefore, $\mathcal{D}_{\mathcal{C}}(\mathbb{R})$ is a natural non sequential topological space with admissible representation.
Chapter 5

Conclusion

We have introduced an extended notion of admissibility that allows to handle a much larger class of spaces than countably based spaces. Admissible (multi-) representations are those which yield topologically the best encoding. In Section 4.3 we have shown that arbitrary multirepresentations can be transformed into admissible ones of the same spaces. The resulting multirepresentations are well-behaved with regard to computability aspects. The admissibly representable weak limit spaces as well as the admissibly representable topological spaces contain many important spaces used in analysis. Both classes enjoy excellent closure properties, for instance, they form Cartesian closed categories.

One open problem is to characterize weakly continuously realizable correspondences, i.e., to find a Main Theorem suitable for weakly continuous realizability similar to Theorem 2.4.25. This notion of realizability defined in Equation (2.11) seems to be more useful than the stricter one considered in Theorem 2.4.25.

Another future task is to study multirepresentations with regard to complexity aspects. This requires to characterize and to investigate the class of admissibly representable spaces for which a reasonable complexity theory is possible. There is some evidence that these spaces form a subclass of the uniform weak limit spaces: we have seen in Subsection 4.5.6 that at least all weak limit spaces which have an admissible fiber-compact multirepresentation are uniform weak limit spaces. A notion of efficiently admissible representations seems to be desirable.
Symbols

Sets

\( \mathbb{N} \) the set of natural numbers \( \{0, 1, 2, \ldots \} \)
\( \mathbb{N}^* = \mathbb{N} \cup \{\infty\} \), where \( \infty \) denotes infinity
\( \mathbb{Z} \) the set of integers
\( \mathbb{Q} \) the set of rational numbers
\( \mathbb{R} \) the set of real numbers
\( \mathbb{C} \) the set of complex numbers
\( \Sigma \) some finite or countably infinite alphabet containing 0, 1 and equipped with a standard numbering \( \nu : \Sigma \to \mathbb{N} \), cf. Subsection 1.1.1
\( \Sigma^* \) the set of finite words over \( \Sigma \)
\( \mathbb{N}^* \) the set of finite words over \( \mathbb{N} \)
\( \Sigma^\omega \) the set \( \Sigma^\mathbb{N} \) of sequences over \( \Sigma \) considered as \( \omega \)-words
\( \mathbb{N}^\omega \) the set \( \mathbb{N}^\mathbb{N} \) of sequences over \( \mathbb{N} \) considered as \( \omega \)-words
\( (a; b) \) the open interval \( \{x \in \mathbb{R} \mid a < x < b\} \)
\( [a; b] \) the closed interval \( \{x \in \mathbb{R} \mid a \leq x \leq b\} \)
\( 2^X \) the power set \( \{M \mid M \subseteq X\} \) of the set \( X \)
\( X \times Y \) Cartesian product of the sets \( X \) and \( Y \)
\( \prod_{i \in I} X_i \) Cartesian product of the sets \( X_i \)
\( \bigcup_{i \in I} X_i \) union of the sets \( X_i \)
\( \bigcap_{i \in I} X_i \) intersection of the sets \( X_i \)

Correspondences and functions

\( F : \subseteq X \implies Y \) a (partial) correspondence between sets \( X \) and \( Y \), cf. Section 1.1
\( G(F) \) the graph of \( F : \subseteq X \Rightarrow Y \), being a subset of \( X \times Y \)
\( \text{dom}(F) \) the domain of \( F \), i.e. \( \text{dom}(F) = \{x \in X \mid (\exists y \in Y)(x, y) \in G(F)\} \)
\( \text{range}(F) \) the range of \( F \), i.e. \( \text{range}(F) = \{y \in Y \mid (\exists x \in X)(x, y) \in G(F)\} \)
\( F^{-1} \) the inverse of \( F \), i.e. \( G(F^{-1}) = \{(y, x) \in Y \times X \mid (x, y) \in G(F)\} \)
\( F[x] \) the image of \( x \) in \( X \), i.e. \( F[x] = \{y \in Y \mid (x, y) \in G(F)\} \)
\( F^{-1}[y] \) the preimage of \( y \) in \( Y \), i.e. \( F^{-1}[y] = \{x \in X \mid (x, y) \in G(F)\} \)
\( F[A] \) the image of \( A \subseteq X \), i.e. \( F[A] = \{y \in Y \mid (\exists x \in A)(x, y) \in G(F)\} \)
\( F^{-1}[B] \) the preimage of \( B \subseteq Y \), i.e. \( \{x \in X \mid (\exists y \in B)(x, y) \in G(F)\} \)
\( F : X \Rightarrow Y \) a total correspondence between \( X \) and \( Y \), i.e. \( \text{dom}(F) = X \)
\( f : \subseteq X \to Y \) a (partial) function between \( X \) and \( Y \), i.e. a correspondence \( f : \subseteq X \Rightarrow Y \) satisfying \( (y_1 \in f[x] \land y_2 \in f[x]) \implies y_1 = y_2 \)
\( f : X \to Y \) a total function between \( X \) and \( Y \), i.e. \( \text{dom}(f) = X \)
Symbols

\( f(x) \)  
the only element in \( f[x] \), if \( x \) is in the domain of the function \( f \)

\( f(x) = \text{div} \)  
abbreviation for \( x \notin \text{dom}(f) \)

\( f^t \)  
the transpose \( f^t : \mathbb{N} \times X \to Y \) of a function sequence \( (f_n)_{n \leq \infty} \), cf. Subsection 4.2.1

\( \text{eval} \)  
evaluation function, cf. Proposition 4.2.3, Lemma 4.2.9

\( e_{x, \mathbb{N}} \)  
transpose of the evaluation function, cf. Subsection 4.3.1

\( \text{pr}_j \)  
the \( j \)-th projection of a finite or an infinite tuple, i.e., respectively, \( \text{pr}_j(x_1, \ldots, x_k) := x_j \) and \( \text{pr}_j(x_0, x_1, \ldots) := x_j \)

Some weak limit spaces and topological spaces

\( (\mathbb{N}, \tau_{\mathbb{N}}) \)  
natural numbers with the discrete topology

\( \rightarrow_{\mathbb{N}} \)  
the convergence relation of \( (\mathbb{N}, \tau_{\mathbb{N}}) \)

\( (\overline{\mathbb{N}}, \tau_{\overline{\mathbb{N}}}) \)  
one point compactification of \( (\mathbb{N}, \tau_{\mathbb{N}}) \), cf. Subsection 2.2.2

\( \rightarrow_{\overline{\mathbb{N}}} \)  
the convergence relation of \( (\overline{\mathbb{N}}, \tau_{\overline{\mathbb{N}}}) \)

\( (\Sigma^\omega, \tau_{\Sigma^\omega}) \)  
\( \Sigma^\omega \) with the Cantor topology \( \tau_{\Sigma^\omega} \), cf. Subsection 2.2.2

\( \rightarrow_{\Sigma^\omega} \)  
the convergence relation of \( (\Sigma^\omega, \tau_{\Sigma^\omega}) \)

\( (\mathbb{N}^\omega, \tau_{\mathbb{N}^\omega}) \)  
the Baire space, cf. Subsection 2.2.2

\( \rightarrow_{\mathbb{N}^\omega} \)  
the convergence relation of \( (\mathbb{N}^\omega, \tau_{\mathbb{N}^\omega}) \)

\( (\mathbb{R}, \tau_{\mathbb{R}}) \)  
the Euclidean space of real numbers, cf. Subsection 2.2.2

\( \rightarrow_{\mathbb{R}} \)  
the convergence relation of \( (\mathbb{R}, \tau_{\mathbb{R}}) \)

\( (\mathbb{C}, \tau_{\mathbb{C}}) \)  
the complex numbers with the product topology \( \tau_{\mathbb{C}} := \tau_{\mathbb{R}} \otimes \tau_{\mathbb{R}} \)

\( \rightarrow_{\mathbb{C}} \)  
the convergence relation of \( (\mathbb{C}, \tau_{\mathbb{C}}) \)

\( \mathbb{W} = (\mathbb{W}e, \rightarrow_{\mathbb{W}e}) \)  
universal codomain for \( \text{AdmWeakLim} \), cf. Subsection 4.3.1

\( \mathbb{L} = (\mathbb{L}i, \rightarrow_{\mathbb{L}i}) \)  
universal codomain for \( \text{Lim} \), cf. Subsection 4.3.1

\( \mathbb{L} = (\mathbb{L}e, \rightarrow_{\mathbb{L}e}) \)  
another universal codomain for \( \text{Lim} \), cf. Subsection 4.3.1

\( \mathbb{S} = (\mathbb{S}i, \tau_{\mathbb{S}i}) \)  
the Sierpiński space, cf. Example 2.3.7 and Subsection 4.3.1

\( \rightarrow_{\mathbb{S}i} \)  
the convergence relation of \( \mathbb{S} \)

Sequences, convergence relations, topologies

\( (x_n)_n \)  
a sequence \( x : \mathbb{N} \to X \)

\( (x_n)_{n \leq \infty} \)  
a (generalized) sequence \( x : \overline{\mathbb{N}} \to X \)

\( \rightarrow_{\mathbb{x}} \)  
the convergence relation of the sequential convergence space \( \mathfrak{X} = (X, \rightarrow_{\mathfrak{x}}) \), cf. Subsection 2.2.1

\( (x_n)_n \rightarrow_{\mathfrak{x}} x_{\infty} \)  
\( (x_n)_n \) converges to \( x_{\infty} \) with respect to \( \rightarrow_{\mathfrak{x}} \), cf. Subsection 2.2.1

\( \rightarrow_{\mathfrak{t}} \)  
the convergence relation induced by a topology \( \tau \), cf. Subsection 2.2.2

\( \rightarrow_{\mathfrak{t}, Q}, \leftrightarrow_{\mathfrak{t}, Q} \)  
final convergence relations induced by a surjective correspondence \( Q \), cf. Subsections 2.4.5, 2.2.4

\( \rightarrow_{\delta}, \leftrightarrow_{\delta} \)  
final convergence relations induced by a multirepresentation \( \delta \), cf. Subsections 2.4.5, 2.2.4

\( \text{seq}(\rightarrow_{\mathfrak{x}}) \)  
the associated topology of sequentially open subsets of a weak limit space \( \mathfrak{X} = (X, \rightarrow_{\mathfrak{x}}) \), cf. Subsection 2.2.5

\( \Rightarrow_{\mathcal{X}, \mathcal{Y}}, \Rightarrow_{\mathcal{P}} \)  
continuous convergence of function sequences, cf. Subsection 4.2.1

\( \Rightarrow_{\mathcal{P}} \)  
continuous convergence for partial function sequences, cf. Subsection 4.2.5
Operators on weak limit spaces and topological spaces

\(\mathcal{W}(\mathfrak{A})\) the limit space \((A, \rightarrow_\mathfrak{A})\), where \(\mathfrak{A} = (A, \tau_\mathfrak{A})\), cf. Subsection 2.2.2

\(\mathcal{T}(\mathfrak{X})\) the sequentialization of a topological space \(\mathfrak{X}\), cf. Subsection 2.2.6

\(\mathcal{T}(\mathfrak{X})\) the associated topological space of \(\mathfrak{X}\), cf. Subsection 2.2.6

\(\mathcal{L}(\mathfrak{X})\) the associated limit space of \(\mathfrak{X}\), cf. Subsection 2.2.6

\(\mathfrak{X} \otimes \mathfrak{Y}, \bigotimes_{i \in \mathbb{N}} \mathfrak{X}_i\) product of weak limit spaces, cf. Subsections 2.2.1, 4.1.4

\(\mathfrak{X} \wedge \mathfrak{Y}, \bigwedge_{i \in \mathbb{N}} \mathfrak{X}_i\) conjunction of weak limit spaces, cf. Subsection 4.1.4

\(\mathfrak{X} \vee \mathfrak{Y}, \bigvee_{i \in \mathbb{N}} \mathfrak{X}_i\) disjunction of weak limit spaces, cf. Subsection 4.5.3

\(\mathcal{C}(\mathfrak{X}, \mathfrak{Y})\) the underlying set of \(\mathcal{C}(\mathfrak{X}, \mathfrak{Y})\): total sequentially continuous functions between \(\mathfrak{X}\) and \(\mathfrak{Y}\)

\(\Rightarrow_{\mathfrak{X}, \mathfrak{Y}}\) the convergence relation of \(\mathcal{C}(\mathfrak{X}, \mathfrak{Y})\): continuous convergence

\(\mathcal{F}(A, B)\) the set \(\{f \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y}) \mid f[A] \subseteq B\}\), cf. Subsection 4.2.1

\(\mathfrak{P}(\mathfrak{X}, \mathfrak{Y})\) the function space \((\mathcal{P}(\mathfrak{X}, \mathfrak{Y})), \Rightarrow_p\), cf. Subsection 4.2.5

\(\text{seq}(\mathfrak{X}), \text{seq}(\rightarrow_\mathfrak{X})\) the associated topology of sequentially open subsets of a weak limit space \(\mathfrak{X} = (X, \rightarrow_\mathfrak{X})\), cf. Subsection 2.2.5

\(\text{sclo}(\mathfrak{X})\) the family of all sequentially closed sets of \(\mathfrak{X}\), cf. Subsection 2.2.5

\(\mathcal{SC}(\mathfrak{X})\) the family of all sequentially compact subsets of \(\mathfrak{X}\), cf. Subsection 4.4.3

\(\mathcal{K}(\mathfrak{X})\) the family of all compact subsets of \(\mathcal{T}(\mathfrak{X})\), cf. Subsection 4.4.3

\(e_{\mathfrak{X}, \mathfrak{M}}\) transpose of the evaluation function, cf. Subsection 4.3.1

\(e_{\mathfrak{X}, \mathfrak{M}}^\perp\) extension of \(e_{\mathfrak{X}, \mathfrak{M}}\) to arbitrary subsets, cf. Subsection 4.4.2
Symbols

\( e_{X,\delta} \) extension of \( e_{X,\delta} \) to compact sets, cf. Subsection 4.4.3
\( e_{X,\delta} \) extension of \( e_{X,\delta} \) to sequentially compact sets, cf. Subsection 4.4.3

Categories

\textbf{Top} the category of topological spaces and of total topologically continuous functions, cf. Subsection 2.2.2
\textbf{Seq} the category of sequential topological spaces and of total topologically continuous functions, cf. Subsection 2.2.6
\textbf{Lim} the category of limit spaces and of total sequentially continuous functions, cf. Subsection 2.2.2
\textbf{WeakLim} the category of weak limit spaces and of total sequentially continuous functions, cf. Subsection 2.2.3
\textbf{AdmWeakLim} the category of weak limit spaces with admissible multirepresentations and of total sequentially continuous functions, cf. Subsection 2.4.6
\textbf{AdmLim} analogous category of limit spaces, cf. Subsection 2.4.6
\textbf{AdmSeq} analogous category of sequential topological spaces, cf. Subsection 2.4.6
\textbf{EffWeakLim} the category of weak limit spaces with effectively admissible multirepresentations and of relatively computable functions, cf. Subsection 4.3.5
\textbf{EffLim} analogous category of limit spaces, cf. Subsection 4.3.5
\textbf{EffSeq} analogous category of sequential topological spaces, cf. Subsection 4.3.5
\textbf{PQ} the category of \( \omega \)–projecting topological quotients of countably–based spaces and of continuous functions, cf. Subsection 3.4.2
\textbf{PQL} analogous category of limit spaces, cf. Subsection 3.4.2
\textbf{PQW} analogous category of weak limit spaces, cf. Subsection 3.4.2
\textbf{Equ} the category of equilogical spaces and of equivariant maps, cf. Subsection 3.4.3
\textbf{\( \omega \)Equ} the category of countably based equilogical spaces and of equivariant maps, cf. Subsection 3.4.3

Standard numberings and multirepresentations

\( \nu_\Sigma \) standard numbering of an alphabet \( \Sigma \), cf. Subsection 1.1.1
\( \nu_{\Sigma^*}, \nu_{\mathbb{N}^*} \) standard numberings of \( \Sigma^* \) and of \( \mathbb{N}^* \) cf. Subsection 1.1.3
\( \nu_{\mathbb{N}}, \nu_{\mathbb{Z}} \) standard numberings of \( \mathbb{N} \) and of \( \mathbb{Z} \), cf. Subsection 1.1.1
\( \nu_{\mathbb{Q}} \) standard numbering of \( \mathbb{Q} \), cf. Subsection 1.1.3
\( \nu_{\mathcal{F}\mathbb{N}} \) bijective numbering of \( \mathcal{F}\mathbb{N} = \{ M \subseteq \mathbb{N} | M \text{ finite} \} \), cf. Subsection 4.4.1
\( \rho_{(\Sigma^\omega)^k}, \rho_{(\Sigma^\omega)^{\mathbb{N}}} \) inverses of \( \langle \cdot, \ldots, \cdot \rangle \) and \( \langle \cdot, \cdot, \cdot \rangle \)
\( \langle \cdot, \ldots, \cdot \rangle \) tupling function from \( \mathbb{N}^k \) to \( \mathbb{N} \), cf. Subsection 1.1.3, or from \( (\Sigma^\omega)^k \) to \( \Sigma^\omega \), or from \( (\mathbb{N}^\omega)^k \) to \( \mathbb{N}^\omega \), cf. Subsection 2.1.7
\( \langle \cdot, \cdot, \ldots \rangle \) tupling function from \( (\Sigma^\omega)^{\mathbb{N}} \) to \( \Sigma^\omega \) or from \( (\mathbb{N}^\omega)^k \) to \( \mathbb{N}^\omega \), cf. Subsection 2.1.7
\( \langle \cdot, \cdot \rangle^X_{\mathbb{N}} \) tupling of functions in a category, cf. Subsection 4.1.4
\( \pi_{k,i}, \pi_{\infty,i} \) computable projections of the inverses of \( \langle \cdot, \ldots, \cdot \rangle \) and \( \langle \cdot, \cdot, \cdot \rangle \) to one component, cf. Subsections 1.1.3, 2.1.7
\( \varrho_{\Sigma^\omega} \) injective representation \( \varrho_{\Sigma^\omega} : \subseteq \mathbb{N}^\omega \rightarrow \Sigma^\omega \), cf. Subsection 2.1.7
Symbols

$\vartheta$ injective representation $\vartheta: \Sigma^\omega \rightarrow N^\omega$, cf. Subsection 2.1.7

$\eta$ effective representation of the set $F^\omega$ of continuous functions $g: \subseteq N^\omega$ with a $G_\delta$-domain, cf. Subsection 4.2.3

$\eta_p$ abbreviation for $\eta(p)$

$En$ representation of $2^N$ defined by $En(p) = N \cap \{p(i) - 1 \mid i \in N\}$ for $p \in N^\omega$

$\varrho_{\omega}, \varrho_{\bar{\omega}}, \varrho_{\omega^k}$ multirepresentations of $\mathfrak{G}, \mathfrak{L}, \mathfrak{M}$, cf. Example 2.3.7, Subsection 4.3.1

$\varrho_N, \varrho_{\bar{N}}$ representations of $N$ and $\bar{N}$, cf. Example 2.3.6

$\varrho_R, \rho_R$ signed-digit-representations of $\mathbb{R}$, cf. Example 2.3.8

$\varrho_C$ representation of $C$ defined by $\varrho_C := \varrho_R \boxtimes \varrho_R$

$\rho_X$ standard representation of a computable topological space $X$, cf. Subsection 4.3.6

$\varrho_{<}, \varrho_{<^*}$ lower representations of $\mathbb{R}$, cf. Example 2.4.15

$\rho_{10}$ decimal representation, cf. Examples 2.1.2, 4.3.19

$\rho_{	ext{Cn}}$ naive Cauchy representation of $\mathbb{R}$, cf. Example 2.3.11

$\gamma_{\text{ub}}$ upper-bound multirepresentation of $\mathbb{R}$, cf. Example 2.4.22

Operations on multirepresentation

$\delta \leq_t \gamma$ $\delta$ is continuously translatable to $\gamma$, cf. Subsections 2.1.6, 2.4.3

$\delta \leq_{cp} \gamma$ $\delta$ is computably translatable to $\gamma$, cf. Subsections 2.1.6, 2.4.3

$\delta \equiv_t \gamma$ $\delta$ is continuously equivalent to $\gamma$, cf. Subsections 2.1.6, 2.4.3

$\delta \equiv_{cp} \gamma$ $\delta$ is computably equivalent to $\gamma$, cf. Subsections 2.1.6, 2.4.3

$\delta \boxtimes \gamma$, $(\boxtimes_{i \in N} \delta_i)$ product of multirepresentations, cf. Subsection 4.1.4

$\delta \land \gamma$, $(\land_{i \in N} \delta_i)$ conjunction of multirepresentations, cf. Subsection 4.1.4

$\delta \lor \gamma$, $(\lor_{i \in N} \delta_i)$ disjunction of multirepresentations, cf. Subsection 4.5.2

$C(\delta, \gamma)$ the set of all total $(\delta, \gamma)$-continuous functions, cf. Subsection 4.2.4

$C_{\text{cp}}(\delta, \gamma)$ the set of all total $(\delta, \gamma)$-computable functions, cf. Subsection 4.2.4

$[\delta \rightarrow \gamma]$ multirepresentation of $C(\delta, \gamma)$, cf. Subsection 4.2.4

$P(\delta, \gamma)$ the set of all partial $(\delta, \gamma)$-continuous functions, cf. Subsection 4.2.5

$P_{\text{cp}}(\delta, \gamma)$ the set of all partial $(\delta, \gamma)$-computable functions, cf. Subsection 4.2.5

$[\delta \rightarrow \gamma]$ multirepresentation of $P(\delta, \gamma)$, cf. Subsection 4.2.5

$\rightarrow_t, \rightarrow_{cp}$ final convergence relations induced by $\delta$, cf. Subsections 2.4.5, 2.2.4

$\tau_\delta$ the final topology of $\delta$, cf. Subsection 2.4.5 and Equation (2.3)

$\delta^\text{mo}, \delta^\text{e}, \delta^\text{ul}$ transformation of $\delta$ into admissible multirepresentations, cf. Subsection 4.3.2

$\delta^\text{so}, \theta^{\text{so}}_X$ positive representations of sequentially open sets, cf. Subsection 4.4.1

$\delta^\text{nc}, \psi^\text{en}_X$ negative representations of sequentially closed sets, cf. Subsection 4.4.1

$\delta^\text{sc}, \psi^\text{sc}_X$ positive representations of sequentially closed sets, cf. Subsection 4.4.2

$\delta^\text{hit}$ positive multirepresentation of arbitrary subsets, cf. Subsection 4.4.2

$\delta^\text{miss}, \kappa_c$ multirepresentations of compact subsets, cf. Subsection 4.4.3

$\delta^\text{Sc}$ multirepresentation of sequentially compact subsets, cf. Subsection 4.4.3

$\psi^{\text{reg}}_X, \xi^{\text{reg}}_X$ representations of regularly closed sets, cf. Subsection 4.4.4
Symbols

\( \gamma \) \( \text{chaotic multirepresentation, cf. Example 2.4.24} \)

**Words and \( \omega \)-words**

- \( \lg(u) \) the length of a word \( u \)
- \( u(n) \) the \((n + 1)\)-st symbol of a word \( u \), thus \( u = u(0) \ldots u(n-1) \)
- \( a^n \) the word consisting of \( n \) symbols \( a \)
- \( a^\omega \) the \( \omega \)-word \( aaa \ldots \)
- \( p<^n \) the prefix of length \( n \) of a \( \omega \)-word \( p \), i.e. the word \( p(0)p(1) \ldots p(n-1) \)
- \( p<^\omega \) the \( \omega \)-word \( p \) itself
- \( p<^n \) the word \( p(0)p(1) \ldots p(n) \)
- \( p>n \) the \( \omega \)-word \( p(n+1), p(n+2), \ldots \)
- \( u \subseteq v \) the word \( u \) is prefix of the word \( v \)
- \( u \subseteq p \) the word \( u \) is prefix of the \( \omega \)-word \( p \)
- \( up, u::p \) the \( \omega \)-word with prefix \( u \) followed by \( p \)
- \( u\Sigma^\omega \) the set of \( \omega \)-words in \( \Sigma^\omega \) with prefix \( u \)
- \( W\Sigma^\omega \) the set of \( \omega \)-words in \( \Sigma^\omega \) with a prefix in the set \( W \)

**Quantifier**

- \((\exists x \in X) P(x)\) there is some \( x \in X \) satisfying property \( P \)
- \((\forall x \in X) P(x)\) all \( x \in X \) satisfy property \( P \)
- \((\exists \infty n \in \mathbb{N}) P(n)\) infinitely many natural numbers satisfy property \( P \)
- \((\forall \infty n \in \mathbb{N}) P(n)\) almost all natural numbers satisfy property \( P \)
- \((\forall \infty n \in \mathbb{N}) P(n)\) almost all natural numbers and \( \infty \) satisfy property \( P \)

**Miscellaneous**

- \( k \) sometimes the arity of a multivariate function
- \( \text{cf}_M \) characteristic function of the set \( M \), cf. Subsection 4.4.1
- \( \max(M) \) maximum of the set \( M \)
- \( \min(M) \) minimum of the set \( M \)
- \( \sup(M) \) supremum of the set \( M \)
- \( \inf(M) \) infimum of the set \( M \)
- \( \mathcal{B}^\cap \) closure of the family \( \mathcal{B} \) under finite intersections, cf. Subsection 4.4.1
- \( \mathcal{B}^\cup \) closure of the family \( \mathcal{B} \) under finite unions, cf. Subsection 4.4.1
- \( \beta^\cap \) numbering of \( \mathcal{B}^\cap \), if \( \beta \) is a numbering of \( \mathcal{B} \), cf. Subsection 4.4.1
- \( \beta^\cup \) numbering of \( \mathcal{B}^\cup \), if \( \beta \) is a numbering of \( \mathcal{B} \), cf. Subsection 4.4.1
- \( \beta^\cap \cup \) numbering of \( (\mathcal{B}^\cap) \cup \), if \( \beta \) is a numbering of \( \mathcal{B} \), cf. Subsection 4.4.3
- \( |x| \) absolute value of a real number \( x \)
- \( [x] \) the integer \( \max\{z \in \mathbb{Z} \mid z \leq x\} \)
- \( \lfloor x \rfloor \) the integer \( \min\{z \in \mathbb{Z} \mid z \geq x\} \)
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