A Computable Version of the Daniell-Stone Theorem on Integration and Linear Functionals

Yongcheng Wu
Mathematics Department, Nanking University, Nanking, China
Klaus Weihrauch
Fernuniversität, 58084 Hagen, Germany

29 June 04

Abstract
For every measure \( \mu \), the integral \( I : f \mapsto \int f \, d\mu \) is a linear functional on the set of real measurable functions. By the Daniell-Stone theorem, for every abstract integral \( \Lambda : F \to \mathbb{R} \) on a stone vector lattice \( F \) of real functions \( f : \Omega \to \mathbb{R} \) there is a measure \( \mu \) such that \( \int f \, d\mu = \Lambda(f) \) for all \( f \in F \). In this paper we prove a computable version of this theorem.

1 Introduction and Mathematical Preliminaries

In this section we summarize some notations, definitions and facts from measure theory and computable analysis.

As a reference to measure theory we use the book [1]. A \textit{ring} in a set \( \Omega \) is a set \( \mathcal{R} \) of subsets of \( \Omega \) such that \( \emptyset \in \mathcal{R} \) and \( A \cup B \in \mathcal{R} \) and \( A \setminus B \in \mathcal{R} \) if \( A, B \in \mathcal{R} \). A \( \sigma \)-\textit{algebra} in \( \Omega \) is a set \( \mathcal{A} \) of subsets of \( \Omega \) such that \( \Omega \in \mathcal{A} \), \( \emptyset \in \mathcal{A} \) if \( A \in \mathcal{A} \) and \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{A} \) if \( A_1, A_2, \ldots \in \mathcal{A} \). For any system \( \mathcal{E} \) of subsets of \( \Omega \) let \( \mathcal{A}(\mathcal{E}) \) be the smallest \( \sigma \)-algebra in \( \Omega \) containing \( \mathcal{E} \).

A \textit{premeasure} on a ring \( \mathcal{R} \) is a function \( \mu : \mathcal{R} \to \mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\} \) such that \( \mu(\emptyset) = 0, \mu(A) \geq 0 \) for \( A \in \mathcal{R} \) and

\[
\mu\left(\bigcup_{i=1}^{\infty} A_n\right) = \sum_{i=1}^{\infty} \mu(A_n)
\]

if \( A_1, A_2, \ldots \in \mathcal{A} \) are pairwise disjoint and \( \bigcup_{i=1}^{\infty} A_n \in \mathcal{A} \). A premeasure on an algebra is called a \textit{measure}. A premeasure \( \mu \) on a ring \( \mathcal{R} \) is called \( \sigma \)-\textit{finite}, if there is a sequence \( A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \) in \( \mathcal{R} \) such that \( A_1 \cup A_2 \cup \ldots = \Omega \) and \( \mu(A_i) < \infty \) for all \( i \).
Theorem 1.1 ([1]) Every σ-finite premeasure μ on a ring \( \mathcal{R} \) in \( \Omega \) has a unique extension to a measure on \( \mathcal{A}(\mathcal{R}) \) which (for convenience) we also denote by \( \mu \).

Let \((\Omega, \mathcal{A}, \mu)\) be a measure space. A function \( f : \Omega \to \mathbb{R} \) is called measurable, if \( \{ x \mid f(x) > a \} \in \mathcal{A} \) for all \( a \in \mathbb{R} \). The following condition is equivalent:

\[
(\forall a \in D) \{ x \mid f(x) > a \} \in \mathcal{A} \text{ for some set } D \text{ dense in } \mathbb{R}. \tag{1.1}
\]

As usual we will abbreviate \( \{ f > a \} := \{ x \in \Omega \mid f(x) > a \} \). In (1.1) the relation “>” can be replaced by “\( \leq \)”, “\( \geq \)” or “\( < \)”. A function \( f : \Omega \to \mathbb{R} \) is simple, if there are non-negative real numbers \( a_1, \ldots, a_n \) and pairwise disjoint sets \( A_1, \ldots, A_n \in \mathcal{A} \) of finite measure such that \( f(x) = \sum_{i=1}^{n} a_i \chi_{A_i} \), where \( \chi_{A_i} \) is the characteristic function of \( A_i \). For a simple function the integral is defined by

\[
\int \sum_{i=1}^{n} a_i \chi_{A_i} := \sum_{i=1}^{n} a_i \mu(A_i). \tag{1.2}
\]

For functions \( u, u_0, u_1, \ldots : \Omega \to \mathbb{R} \), \( u_i \nearrow u \) means: For all \( x \in \omega \), \( u_0(x) \leq u_1(x) \leq \ldots \) and \( \sup_i u_i(x) = u(x) \). For a non-negative measurable real

\[
\text{function } f : \Omega \to \mathbb{R} \text{ and } b \in \mathbb{R}, f \, d\mu = b, \text{ if there is some increasing sequence } (u_i)_{i \in \mathbb{N}} \text{ of simple functions such that}
\]

\[
u_i \nearrow f \text{ and } \sup_i \int u_i \, d\mu = b \tag{1.3}
\]

[1]. In particular, \( \int f \, d\mu \) does not exist (in \( \mathbb{R} \)), if the sequence \( (\int u_i \, d\mu)_i \) is unbounded. For an arbitrary real function \( f : \Omega \to \mathbb{R} \) let \( f^+ := \sup(0, f) \) (the positive part of \( f \)) and \( f^- := \sup(0, -f) \) (the negative part of \( f \)). By definition, a measurable function \( f \) is integrable, if \( \int f^+ \, d\mu \) and \( \int f^- \, d\mu \) exist and its integral is defined by

\[
\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu. \tag{1.4}
\]

For the following concepts from computable analysis see [4]. Let \( \mathbb{N} := \{0, 1, 2, \ldots\} \) be the set of natural numbers. A partial function from \( X \) to \( Y \) is denoted by \( f : \subseteq X \to Y \), a multifunction by \( f : \subseteq X \Rightarrow Y \). Let \( \Sigma \) be a sufficiently large finite alphabet such that \( \{0, 1\} \subseteq \Sigma \). The set of finite words over \( \Sigma \) is denoted by \( \Sigma^* \), the set of infinite sequences by \( \Sigma^w \). Computability of functions on \( \Sigma^* \) and \( \Sigma^w \) is defined by Turing machines which can read and write finite and infinite sequences, respectively. Standard pairing functions on \( \Sigma^* \) are denoted by \( \langle \cdot, \cdot \rangle \). For \( w \in \Sigma^* \) let \( \xi_w : \subseteq \Sigma^* \to \Sigma^* \) be the word function computed by the Turing machine with canonical code \( w \in \Sigma^* \). Like the
“effective Gödel numbering” \( \phi : \mathbb{N} \to P^{(1)} \) of the partial recursive functions the notation \( \xi \) satisfies the utm-theorem and the smn-theorem.

Computability on other sets is introduced by using finite or infinite sequences of symbols as “names”. For the natural numbers let \( \nu_i : \subseteq \Sigma^* \to \mathbb{N} \) be the notation by binary numbers and let \( b_{n_i} \) be the binary name of \( i \in \mathbb{N} \). Let \( \nu_k : \subseteq \Sigma^* \to \mathbb{Q} \) be some standard notation of the rational numbers. For the real numbers we use the standard Cauchy representation \( \rho : \subseteq \Sigma^\omega \to \mathbb{R} \), where \( \rho(p) = x \), iff \( p \) encodes a sequence \( (a_i)_i \) of rational numbers such that \( |a_i - x| \leq 2^{-i} \). For naming systems \( \delta_i : \subseteq Y_i \to M_i \), \( Y_i \subseteq \{ \Sigma^*, \Sigma^\omega \} \) for \( i = 1, 2 \), a multfunction \( f : \subseteq M_1 \Rightarrow M_2 \) is \( (\delta_1, \delta_2) \)-computable, iff there is a computable function \( h : \subseteq Y_1 \to Y_2 \) such that \( \delta_2 \circ h(p) \in f(\delta_1(p)) \) for all \( p \in \text{dom}(\delta_1) \) such that \( f(\delta_1(p)) \neq \emptyset \).

In this article we will consider computability on factorizations of several pseudometric spaces [2]. We generalize the definition of a computable metric space with Cauchy representation from [4] straightforwardly as follows: A computable pseudometric space is a quadruple \( \mathcal{M} = (M, d, A, \alpha) \) such that \((M, d)\) is a pseudometric space, \( A \subseteq M \) is dense and \( \alpha : \subseteq \Sigma^* \to A \) is a notation of \( A \) such that \( \text{dom}(\alpha) \) is recursive and the restriction of the pseudometric \( d \) to \( A \) is \( (\alpha, \alpha, \rho) \)-computable. (In \[4\], \( \text{dom}(\alpha) \) is assumed to be r.e. Notice that for every notation with r.e. domain there is an equivalent one with recursive domain.) In our applications, \( \mathcal{M} \) is a linear space and the pseudometric is derived from a seminorm \( ||| \cdot ||| \), \( d(x, y) = ||x - y|| \).

The factorization \((\mathcal{M}, \overline{d})\) of the pseudometric space \((M, d)\) is a metric space defined canonically as follows: \( \mathcal{F} := \{ y \in M \mid d(x, y) = 0 \}, \overline{M} := \{ \overline{x} \mid x \in M \}, \overline{d}(\overline{x}, \overline{y}) := d(x, y) \). We define the Cauchy representation \( \delta_{\mathcal{M}} \) of the factorization of a computable pseudometric space as follows: \( \delta_{\mathcal{M}}(p) = \overline{x} \), if \( p \in \Sigma^\omega \) encodes a sequence \( (a_i)_i \) (of \( \alpha \)-names) of elements of \( A \) such that \( d(a_i, x) \leq 2^{-i} \) for all \( i \). If \( \mathcal{M} \) is a linear space with seminorm \( ||| \cdot ||| \), by \( a\mathcal{F} := a\overline{x} \) and \( \mathcal{F} + \mathcal{F} := \overline{x} + \overline{y} \) the factor space becomes a linear space with norm \( ||| \cdot ||| \). In this case, \( \overline{d}(\overline{x}, \overline{y}) = ||x - y|| \).

### 2 Computable Measure Spaces

In this section let \( (\Omega, A, \mu) \) be a measure space. For any \( \mathcal{D} \subseteq A \) let \( \mathcal{D}^I := \{ A \in \mathcal{D} \mid \mu(A) < \infty \} \). In computable measure theory we want to identify two sets \( A, B \in A \), if their symmetric difference \( A \Delta B := (A \setminus B) \cup (B \setminus A) \) has measure 0 and distinguish them otherwise. Since \( A \Delta B \subseteq A \Delta C \cup C \Delta B \), on the set \( \mathcal{D}^I \) the mapping \( d : (A, B) \mapsto \mu(A \Delta B) \) is a pseudometric.

**Lemma 2.1** Let \( R \) be a ring such that \( A(R) = A \) and \( \mu \) is a \( \sigma \)-finite premeasure on \( R \). Then \((\mathcal{A}^I, \overline{d})\), \( d : (A, B) \mapsto \mu(A \Delta B) \), is a complete pseudometric space with \( R^I \) as a dense subset.

**Proof:** Straightforward. \( \square \)
For including sets with infinite measure consider the mapping $d_\infty : (A, B) \mapsto \mu(\Delta AB)/(1+\mu(\Delta AB))$ which is a pseudometric on $A$ (notice: $\infty/(1+\infty) = 1$). Its restriction to $A^f$ is equivalent to $d$. For introducing computability on a pseudometric space we need a countable dense subset $[4, 3]$. Unfortunately, there are important measure spaces such that the pseudometric space $(A, d_\infty)$ is not separable.

**Example:** Consider the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the set of Borel subsets of the real numbers and $\lambda$ is the Lebesgue-Borel measure. Let $(E_i)_{i \in \mathbb{N}}$ be any countable sequence in $\mathcal{B}$. Define $B := \bigcup (i; i+1) \setminus E_i$. Then for all $i$, $\lambda(B \Delta E_i) \geq 1$ and hence $d_\infty(B, E_i) \geq 1/2$. Therefore, the set of all $E_i$ cannot be dense. Since this is true for every sequence $(E_i)_{i \in \mathbb{N}}$, the pseudometric space $(B, d_\infty)$ is not separable.

We will consider measures which are completions of $\sigma$-finite premeasures on countable rings consisting of sets with finite measure. We assume that the operations on the ring and the premeasure are computable.

**Definition 2.2** A computable measure space is a quintuple $\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ such that

1. $\mathcal{A}$ is a $\sigma$-algebra in $\Omega$ and $\mu$ is a measure on it,
2. $\mathcal{R}$ is a countable ring such that $A = A(\mathcal{R})$,
3. $\mu(A) < \infty$ for all $A \in \mathcal{R}$,
4. the restriction of $\mu$ to $\mathcal{R}$ is $\sigma$-finite,
5. $\alpha : \subseteq \Sigma^* \rightarrow \mathcal{R}$ is a notation of $\mathcal{R}$ with recursive domain,
6. $(A, B) \mapsto A \cup B$ and $(A, B) \mapsto A \setminus B$ are $(\alpha, \alpha, \alpha)$-computable,
7. $\mu$ is $(\alpha, \rho)$-computable on $\mathcal{R}$.

By (4), $\Omega = \bigcup \mathcal{R}$. If $\bigcup \mathcal{R}$ is a proper subset of $\Omega$, then for obtaining a $\sigma$-finite measure, either restrict $\Omega$ to $\bigcup \mathcal{R}$ or add the set $\Omega \setminus \bigcup \mathcal{R}$ to $\mathcal{R}$ and define $\mu(\Omega \setminus \bigcup \mathcal{R}) = 0$.

**Theorem 2.3** Let $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ be a computable measure space. Then the quadruple $(A^f, d, \mathcal{R}, \alpha)$ is a computable complete pseudometric space, where $A^f = \{ A \in \mathcal{A} | \mu(A) < \infty \}$ and $d(A, B) = \mu(\Delta AB)$.

**Proof:** By Lemma 2.1, $(A^f, d)$ is a complete pseudometric space with $\mathcal{R}$ as a dense subset. By Def. 2.2(5)-(7) the notation $\alpha$ has recursive domain and the distance $d$ is $(\alpha, \alpha, \rho)$-computable.

Computability on the computable measure space can be defined via the Cauchy representation of the joined pseudometric space.

**Example 2.4** (Lebesgue-Borel measure on $\mathbb{R}$) Let $\Omega = \mathbb{R}$, let $D \subseteq \mathbb{R}$ be dense in $\mathbb{R}$ and let $\nu_D : \subseteq \Sigma^* \rightarrow D$ be a notation such that $\text{dom}(\nu_D)$ is recursive and $\nu \leq \rho$. Let $I_D$ be the set of all intervals $[a, b) \subseteq \mathbb{R}$ such that
\(a, b \in D\) and \(a < b\). Let \(\mathcal{R}_D\) be the set of all finite unions of intervals from \(\mathcal{I}_D\) and let \(\alpha_D\) be some notation of \(\mathcal{R}_D\) canonically derived from \(\nu_D\). Then \(B \coloneqq A(\mathcal{R}_D)\) is the set of Borel-subsets of \(\mathbb{R}\). The Lebesgue-Borel measure \(\lambda\) on \(B\) is defined uniquely by setting \(\lambda([a; b)) := b - a\) for all \(a, b \in D, a < b\) \cite{1}. \(\mathcal{M}_D \coloneqq (\mathbb{R}, B, \lambda, \mathcal{R}_D, \alpha_D)\) is a computable measure space.

## 3 Computability on the Integrable Functions

In this section we assume that \(\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)\) is a computable measure space. We introduce a computable pseudometric space for the integrable functions. On the set \(\mathcal{I}(\mathcal{M})\) of \(\mu\)-integrable functions \(f : \Omega \to \mathbb{R}\) a seminorm and a pseudometric are defined by

\[
\|f\|_\mathcal{M} \coloneqq \int |f| \, d\mu, \quad d_\mathcal{M}(f, g) \coloneqq \|f - g\|_\mathcal{M}.
\] (3.5)

(see \cite{1}). For introducing computability on \(\mathcal{I}(\mathcal{M})\) we consider a countable dense set.

**Definition 3.1**

1. A function \(u : \Omega \to \mathbb{R}\) is a rational step function, iff there are rational numbers \(a_1, \ldots, a_n\) and pairwise disjoint sets \(A_1, \ldots, A_n \in \mathcal{R}\) such that \(u = \sum_{i=1}^n a_i \cdot \chi_{A_i}\).
2. Let \(\beta : \subseteq \Sigma^* \to \text{RSF}\) be a canonical notation of the set RSF of rational step functions derived from the notation \(\alpha\) such that \(\text{dom}(\beta)\) is recursive.

In contrast to a simple function (see Sec. 1), for a rational step function \(f = \sum_{i=1}^n a_i \cdot \chi_{A_i}\) the sets \(A_i\) must be in \(\mathcal{R}\) and the coefficients must be rational, but may be negative. For a rational step function \(u = \sum_{i=1}^n a_i \cdot \chi_{A_i}\), \(f \cdot u \, d\mu = \sum_{i=1}^n a_i \cdot \mu(A_i)\) and \(\|u\|_\mathcal{M} = \sum_{i=1}^n |a_i| \cdot \mu(A_i)\).

**Lemma 3.2** For rational step functions \(u, v\) and \(a \in \mathbb{Q}\) the functions

1. \((a, u) \mapsto a \cdot u, (u, v) \mapsto u + v, u \mapsto |u|, u \mapsto \inf(u, 1), u \mapsto \int f \, du\)\mu,
2. \((u, v) \mapsto \sup(u, v), (u, v) \mapsto \inf(u, v), u \mapsto u_+, u \mapsto u_-, (u, a) \mapsto \inf(u, a), u \mapsto \|u\|_\mathcal{M}\)

are computable w.r.t. the notations \(\beta, \nu_2\) and \(\rho\).

**Proof:** Straightforward. \(\Box\)

In Def. 3.1(1) the condition “\(A_1, \ldots, A_n\) are pairwise disjoint” is not restrictive.

**Lemma 3.3** Let \(\beta\) be a canonical notation of all \(u = \sum_{i=1}^n a_i \cdot \chi_{A_i}\) such that \(a_i \in \mathbb{Q}\) and \(A_i \in \mathcal{R}\) (but the \(A_i\) are not necessarily disjoint). Then \(\beta' \equiv \beta\).
Proof: “≤”: From the sets $A_i$ by determining intersections and differences a finite set $B_1, \ldots, B_n$ of pairwise disjoint sets can be computed such that each $A_i$ is a finite union of $B_j$. Then coefficients $b_j \in \mathbb{Q}$ can be computed such that $\sum_{i=1}^{n} a_i \cdot \chi_{A_i} = \sum_{j=1}^{m} b_j \cdot \chi_{B_j}$. This procedure is computable w.r.t. the representations $\beta, \beta', \alpha_i \nu Q$ and $\nu_n$.

“≥”: Obvious. \hfill \square

Theorem 3.4 $(\mathcal{I}(\mathcal{M}), d_M, \text{RSF}, \beta)$ is a computable complete pseudometric space.

Proof: By Th. 15.5 in [1], $(\mathcal{I}(\mathcal{M}), d_M)$ is complete.

Consider $f \in \mathcal{I}(\mathcal{M})$ and $\varepsilon > 0$. Then $\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$. By (1.3) there is a simple function $u \leq f^+$ such that $0 \leq \int f^+ \, d\mu - \int u \, d\mu < \varepsilon/4$, hence $d_M(f^+, u) = \int (f^+ - u) \, d\mu = \int f^+ \, d\mu - \int u \, d\mu < \varepsilon/4$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathcal{R}$ is dense in $\mathcal{A}$ by Thm. 2.3, there is a rational step function $v$ such that $d_M(u,v) < \varepsilon/4$. We obtain $d_M(f^+, v) \leq d_M(f^+, u) + d_M(u,v) < \varepsilon/2$. Correspondingly, there is a rational step function $w$ such that $d_M(f^-, w) \leq \varepsilon/2$. We obtain $d_M(f,v-w) = ||f^+ - f^- - (v-w)|| \leq ||f^+ - v|| + ||f^- - w|| < \varepsilon$. Therefore, $v - w$ is a rational step function which is $\varepsilon$-close to $f$.

On RSF the distance $d_M$ is $(\beta, \beta, \rho)$-computable. This follows from Lemma 3.2. \hfill \square

Let $\delta_M : \subseteq \Sigma^\omega \rightarrow \mathcal{I}(\mathcal{M})/\equiv$ be the Cauchy representation of the set of equivalence classes of integrable functions (see Sec. 1).

4 The Computable Daniell-Stone Theorem

For two real-valued functions let $(f \wedge g)(x) := \inf(f(x), g(x))$. A Stone vector lattice of real functions is a vector space $\mathcal{F}$ of functions $f : \Omega \rightarrow \mathbb{R}$ such that the functions $x \mapsto |f(x)|$ and $x \mapsto \inf(f(x), 1)$ (denoted by $|f|$ and $f \wedge 1$, resp.) are in $\mathcal{F}$ if $f \in \mathcal{F}$.

Let $\mathcal{F}_+$ be the set of non-negative functions in $\mathcal{F}$. Let us call $\mathcal{F}$ complete, if $f \in \mathcal{F}$ whenever $u_i \nearrow f$ for $u_i \in \mathcal{F}_+$ and $f : \Omega \rightarrow \mathbb{R}$.

An abstract integral on a Stone vector lattice $\mathcal{F}$ of real functions is a linear functional $I : \mathcal{F} \rightarrow \mathbb{R}$ such that for all $f, f_0, f_1, \ldots \in \mathcal{F}_+$,

$$I(f) \geq 0 \quad \text{and} \quad I(f) = I(\sup f_n) = \sup_{n} I(f_n) \quad \text{if} \quad f_i \nearrow f. \quad (4.6)$$

Let $\mathcal{A}(\mathcal{F})$ be the smallest $\sigma$-algebra in $\Omega$ such that every function $f \in \mathcal{F}$ is measurable.

Theorem 4.1 (Daniell-Stone [1]) Let $\mathcal{F}$ be a Stone vector lattice with abstract integral $I$. Then there is a measure $\mu$ on $\mathcal{A}(\mathcal{F})$ such that $f$ is $\mu$-integrable and $I(f) = \int f \, d\mu$ for all $f \in \mathcal{F}$. Furthermore, if there is a
sequence \((f_i)_i\) in \(\mathcal{F}\) such that \((\forall x \in \Omega)(\exists i)f_i(x) > 0\), then the measure \(\mu\) is uniquely defined.

For a proof see Thms. 39.4 and Cor. 39.6 in [1]. On a Stone vector lattice with abstract integral a seminorm \(|||\|_S\) and a pseudometric \(d_S\) can be defined by

\[
|\langle f, g \rangle| := I(\langle f, g \rangle) \quad \text{and} \quad d_S(f, g) := \|f - g\|_S = I(|f - g|).
\]

(4.7)

For an effective version of Thm. 4.1 we consider a notation \(\gamma\) of a dense subset \(\mathcal{D}\) such that \((\mathcal{F}, d_S, \mathcal{D}, \gamma)\) is a computable pseudometric space. Furthermore, we assume that \([f], f \wedge 1 \in \mathcal{D}\) if \(f \in \mathcal{D}\) and that \(\mathcal{D}\) is closed under rational linear combination.

**Definition 4.2** A computable Stone vector lattice with abstract integral is a tuple \(\mathcal{S} = (\Omega, \mathcal{F}, I, \mathcal{D}, \gamma)\) such that

1. \(\mathcal{F}\) is a Stone vector lattice with abstract integral \(I\),
2. \(\mathcal{D} \subseteq \mathcal{F}\) is dense w.r.t the pseudometric \(d_S : (f, g) \mapsto I(|f - g|)\),
3. \(\gamma\) is a notation of \(\mathcal{D}\) with recursive domain,
4. if \(a \in \Omega\) and \(f, g \in \mathcal{D}\), then \(\{af, f + g, |f|, f \wedge 1\} \subseteq \mathcal{D}\),
5. for \(a \in \Omega\) and \(f, g \in \mathcal{D}\), the functions \(a, f) \mapsto af, (f, g) \mapsto f + g, f \mapsto |f|\) and \(f \mapsto f \wedge 1\) are computable w.r.t. \(\nu_\Omega, \gamma\) and \(\rho\).
6. the restriction of \(I\) to \(\mathcal{D}\) is \((\gamma, \rho)\)-computable.

Let \(\delta_S : \subseteq \Sigma^\omega \rightarrow \mathcal{F}/\mathcal{D}\) be the canonical Cauchy representation of the factorization of the computable pseudometric space \((\mathcal{F}, d_S, \mathcal{D}, \gamma)\).

It can be shown easily that \((\mathcal{F}, d_S, \mathcal{D}, \gamma)\) is a computable pseudometric space. For a computable measure space, the integrable functions with the integral as linear operator form a computable Stone vector lattice with abstract integral.

**Proposition 4.3** Let \(\mathcal{M} = (\Omega, A, \mu, R, \alpha)\) be a computable measure space. Then \((\Omega, I(\mathcal{M}), f \mapsto \int f d\mu, R_S, \beta)\) (see Def. 3.1(2)) is a computable complete Stone vector lattice with abstract integral.

**Proof:** Straightforward. \(\square\)

For two metric spaces \((M_i, d_i)\) \((i = 0, 1)\) call \(\psi : M_0 \rightarrow M_1\) a metric embedding, iff \(d_0(\psi(x), \psi(y)) = d_1(x, y)\) for all \(x, y \in M_0\). Obviously, a metric embedding \(\psi\) is injective, i.e., \((M_0, d_0)\) is, up to renaming, a subspace of \((M_1, d_1)\). For computable metric spaces \((M_i, d_i, A_i, \alpha_i)\) \((i = 0, 1)\) with Cauchy representations \(\delta_i\) \((i = 0, 1)\), if \(\psi : M_0 \rightarrow M_1\) is a \((\delta_0, \delta_1)\)-computable embedding, then its inverse \(\psi^{-1} : \subseteq M_1 \rightarrow M_0\) is \((\delta_1, \delta_0)\)-computable. In this case, the first space is, up to renaming, a very well behaved subspace of the second one.

We can now formulate and prove our computational version of the Daniell-Stone theorem. (We use the Cauchy representation \(\delta_M\) of a factorized pseudometric space of the integrable functions, see Thm. 3.4 and the end of Sec. 3.)
Theorem 4.4 (computable Daniell-Stone) Let $S = (\Omega, \mathcal{F}, I, D, \gamma)$ be a computable Stone vector lattice with abstract integral such that $(\forall x \in \Omega)(\exists f \in D)f(x) > 0$. Then there exist a computable measure space $M = (\Omega, A, \mu, R, \alpha)$ and a function $\psi$ such that

1. $\psi$ is a $(\delta_S, \delta_M)$ computable metric embedding $\psi : \mathcal{F}/= \to I(M)/=;$

2. $I(f) = \int g \, d\mu$ for all $f \in \mathcal{F}$ and $g \in \psi(f/=)$;

where $\delta_S$ is the Cauchy representation of the factorized pseudometric space derived from $S$ (Def. 4.2) and $\delta_M$ is the Cauchy representation of the factorized pseudometric space derived from $M$ (Thm. 3.4).

Proof: Our first goal is to define a ring $R$ on $\Omega$ with a notation $\alpha$. Consider $f \in D$. Since $f$ must be $\mu$-integrable by (1) and hence $A$-measurable, we must have $\{f > a\} \in A = A(\mathcal{R})$ for all $a \in R$. Since $\{f > a\} = \bigcup_{b \leq c} \{f > b\}$, it would suffice to require $\{f > b\} \in R$ for all $f \in D$ and $b \in \mathbb{Q}$. Unfortunately, some of the values $\mu(\{f > c\})$ (which will be defined canonically) might become non-computable. To avoid this problem we construct a new countable dense set $C$ of computable real numbers (see (1.1)) such that $\mu(\{f > c\})$ becomes computable. Moreover, we define a notation $\alpha : \subseteq \Sigma^* \to R$ such that (5) - (7) from Def. 2.2 are satisfied. The remainder of the proof will be split into several auxiliary propositions.

Define a notation $\gamma_+ = D_+ := D \cap \mathcal{F}_+$ by $\gamma_+(v) := |\gamma(v)|$. From Def. 4.2 we can conclude that $\gamma_+ = \text{reducible to } \gamma_+ (\gamma_+ \leq \gamma)$. Define a notation $\nu_\to$ of the computable sequences in $D_+$ by

$$\nu_\to(s) = (f_0, f_1, \ldots) \iff (\forall w \in \text{dom}(\nu_\to)) f_{\nu_\to(w)} = \gamma_+ \circ \xi_\to(w), \tag{4.8}$$

that is, if $\xi_\to$ is a $(\nu_\to, \gamma_+)$-realization of $i \mapsto f_i$ (see Section 1).

As a first step, for each $f = \gamma_+(v) \in D_+$ we compute some dense set $D_+ \subseteq \mathbb{R}$ such that $\mu(\{f > a\})$ is a computable real number for all $a \in D_+$ (and show how to compute these values).

Proposition 4.5 For every $f \in D_+$ and every $a_0, b_0 \in \mathbb{Q}$, $0 < a_0 < b_0$, a real number $c$ and two sequences $(g_n)_n$ and $(h_n)_n$ in $D_+$ can be computed w.r.t. the notations $\gamma_+, \nu_\to, \nu_\to$ such that

$$a_0 < c < b_0 \tag{4.9}$$

$$0 \leq h_0 \leq h_1 \leq \cdots \leq \chi(f > c) \leq \chi(f \geq c) \leq \cdots \leq g_n \leq g_0 \tag{4.10}$$

$$\sup I(h_n) = \inf I(g_n). \tag{4.11}$$

Proof: Consider $f \in D_+$ and $a \in R$, $a > 0$. For each $n > 0$ define

$$g_n^a := 2^n(f \wedge a - f \wedge (1 - 2^{-n}))/a, \tag{4.12}$$

$$h_n^a := 2^n(f \wedge (a + 2^{-n}) - f \wedge a)/a. \tag{4.13}$$

Since $f \wedge c = c \cdot (f/c \wedge 1)$ for any $c > 0$, $g_n, h_n \in \mathcal{F}_+$.  

8
If \( f(x) \geq a \) then \( g_n^a(x) = 1 = g_{n+1}^a(x) \). If \( f(x) \leq a(1 - 2^{-n-1}) \) then 
\[ g_{n+1}^a(x) = 0. \]
If \( a(1 - 2^{-n-1}) < f(x) < a \), then 
\[ g_{n+1}^a(x) = 2^{n+1}(f(x) - a + a2^{-n-1})/a < 2^n(f(x) - a + a2^{-n})/a = g_n^a(x). \]

Therefore, \( (\forall n) g_{n+1}^a \leq g_n^a. \)

If \( f(x) \geq a \) then \( g_n^a(x) = 1 \) for all \( n \) (see above). If \( f(x) < a \), then 
\( f(x) < a(1 - 2^{-k}) \) for some \( k \), hence \( g_n^a(x) = 2^n(f(x) - f(x)) = 0 \) for \( n \geq k \).

We obtain \( \chi(f > a) = \inf_n g_n^a \). Similarly, we can prove \( (\forall n) h_n^a \leq h_{n+1}^a \) and 
\[ \sup_n h_n^a = \chi(f > a). \]

Therefore, 
\[ h_1^a \leq h_2^a \leq \cdots \leq \sup_n h_n^a = \chi(f > a) \leq \inf_n g_n^a = \leq \cdots \leq g_2^a \leq g_1^a. \] (14)

Consider \( 0 < a < b. \) Then \( \chi(f \geq b) \leq \chi(f > a) \), therefore,
\[ \sup_n h_n^b \leq \inf_n g_n^a \leq \sup_n h_n^a \leq \inf_n g_n^a. \] (15)

It follows from (16), that the function \( I \) is monotone on \( F \), hence
\[ \sup_n I(h_n^b) \leq \inf_n I(g_n^a) \leq \sup_n I(h_n^a) \leq \inf_n I(g_n^b). \] (16)

The proof of Prop. 4.5 will be continued after the proof of the following proposition.

**Proposition 4.6** Let \( 0 < a < b \), \( a' := a + (b - a)/5, b' := a + 2(b - a)/5, \)
\[ a'' := a + 3(b - a)/5, b'' := a + 4(b - a)/5 \] and \( \varepsilon > 0 \). If
\[ \inf_n I(g_n^a) - \sup_n I(h_n^a) < \varepsilon, \] (17)

then for some \( k \)
\[ I(g_k^{a'}) - I(h_k^{b''}) < \varepsilon/2 \quad \text{or} \quad I(g_k^{a''}) - I(h_k^{b''}) < \varepsilon/2. \] (18)

**Proof:** Suppose (18) is false. Then \( \inf_n I(g_n^{a'}) - \sup_n I(h_n^{b''}) \geq \varepsilon/2 \) and 
\( \inf_n I(g_n^{a''}) - \sup_n I(h_n^{b''}) \geq \varepsilon/2 \). For convenience abbreviate \( a' := \sup_n I(h_n^a) \) and 
\( b' := \inf_n I(g_n^a). \) Thus \( a' - b' \geq \varepsilon/2 \) and \( a'' - b' \geq \varepsilon/2 \). Applying (16), we obtain
\[ a - b' = (a - a') + (a' - b') + (b' - a'') + (a'' - b'') + (b' - b) \]
\[ \geq 0 + \varepsilon/2 + 0 + \varepsilon/2 + 0 \]
\[ = \varepsilon \]

(contradiction). Thus, (18) is proved. \( \square \) (Prop. 4.6)

Now, let \( f \in D_k^a \) and \( a_0, b_0 \in \mathbb{Q}, \ 0 < a_0 < b_0 \). For each \( n \in \mathbb{N} \) we determine 
\( k_n \in \mathbb{N}, \ a_n, b_n \in \mathbb{Q} \) and \( h_n, g_n \in D_k \) as follows.

By (15), \( h_1^{a_0} \leq g_1^{a_0} \). Determine some \( m \in \mathbb{N} \) such that 
\[ I(g_1^{a_0}) - I(h_1^{a_0}) < 2^m. \] Let \( k_0 := 1 \).
Suppose $a_n, b_n \in \mathbb{Q}$ and $k_n \subseteq \mathbb{N}$ are defined such that $a_n < b_n$ and

$$I(g_{k_n}^{a_n}) - I(h_{k_n}^{b_n}) < 2^{m-n}. \quad (4.19)$$

Then $0 \leq \inf_i I(g_i^{a_n}) - \sup_j I(h_j^{b_n}) < 2^{m-n}$. Using this condition as (4.17) in Prop. 4.6, some $k_{n+1}$ and some $a_{n+1}, b_{n+1}$ can be computed such that $(a_{n+1}; b_{n+1})$ is the $2n$th fifth or the $4n$th fifth of the interval $(a_n; b_n)$ and $I(g_{k_{n+1}}^{a_{n+1}}) - I(h_{k_{n+1}}^{b_{n+1}}) < 2^{m-(n+1)}$. For $i \in \mathbb{N}$ define

$$h_i := \sup_{n \leq i} h_{k_n}^{b_n}, \quad g_i := \inf_{n \leq i} g_{k_n}^{a_n}. \quad (4.15)$$

Let $c \in \mathbb{R}$ be the single point such that $a_n < c < b_n$ for all $n$. Then by (4.15) for all $i \in \mathbb{N}$,

$$\sup_{n \leq i} h_{k_n}^{b_n} \leq \sup_{n \leq i} h_{k_n}^{b_n} \leq \sup_{j \leq i+1} h_j^{b_n} \leq \sup_{j} h_j^{b_n} \quad (4.16)$$

and therefore, $h_i \leq h_{i+1} \leq \sup_j h_j$, and correspondingly, $\inf_j g_j^{b_n} \leq g_{i+1} \leq g_i^{b_n}$.

Applying (4.14) we obtain (4.10). Since $0 \leq I(g_i) - I(h_i) < 2^{m-n}$ by (4.19), we obtain (4.11).

We consider computability. By the construction and Def. 4.2 there are computable word functions $H_0, T_\leq, T_\geq$ such that for $a_0 = \nu_Q(u_i) < \nu_Q(u_r) = b_0$, $f = \gamma_+(v) \in \mathcal{D}_+, \quad n = \nu_N(w)$,

$$c = \rho \circ H_0(v, u_i, u_r), \quad (4.20)$$

$h_n = \gamma_+ \circ T_{\leq}(v, u_i, u_r, w)$ and $g_n = \gamma_+ \circ T_{\geq}(v, u_i, u_r, w)$. By the smn-theorem for $\xi$, there are computable word functions $H_{\leq}, H_{\geq}$ such that

$$h_n = \gamma_+ \circ \xi_{H_{\leq}(v, u_i, u_r)}(w), \quad g_n = \gamma_+ \circ \xi_{H_{\geq}(v, u_i, u_r)}(w), \quad (4.21)$$

and therefore,

$$(h_n)_n = \nu \circ H_{\leq}(v, u_i, u_r), \quad (g_n)_n = \nu \circ H_{\geq}(v, u_i, u_r). \quad (4.22)$$

By the way, we mention that these functions from $(f, a_0, b_0)$ are multi-valued, since the choice in applying Prop. 4.6 cannot be made single-valued in general.

Notice that for every fixed $v \in \text{dom}(\gamma) = \text{dom}(\gamma_+)$, the set of constants $c$,

$$D_v := \{ \rho \circ H_0(v, u_i, u_r) \mid 0 < \nu_Q(u_i) < \nu_Q(u_r) \} \quad (4.23)$$

is dense in $\mathbb{R}$. We define the ring and the $\sigma$-algebra for the measure space $\mathcal{M}$.

**Definition 4.7**

$$\mathcal{R}_0 := \{ \{ \gamma_+(v) > \rho \circ H_0(v, u_i, u_r) \} \mid v \in \text{dom}(\gamma_+), 0 < \nu_Q(u_i) < \nu_Q(u_r) \}$$

$$\mathcal{R} := \text{the smallest ring containing } \mathcal{R}_0$$

$$\mathcal{A} := \mathcal{A}(\mathcal{R}) = \mathcal{A}(\mathcal{R}_0)$$

10
Notice that \( R_0 \) is not a ring in general. By Prop. 4.9 for every set \( A \in R_0 \) there are sequences \((h_i)\) and \((g_i)\) in \( D_+ \) such that
\[
0 \leq h_0 \leq h_1 \leq \ldots \leq \chi A \leq \ldots \leq g_1 \leq g_0 \quad \text{and} \quad \sup I(h_n) = \inf I(g_n). \quad (4.24)
\]
In the following we prove that this is true also for all \( A \in R \). Additionally we introduce a notation \( \alpha \) of \( R \) such that the sequences \((h_i)\) and \((g_i)\) can be computed from \( A \in R \).

**Proposition 4.8** For functions \( h_n, g_n, h'_n, g'_n \in D_+ \) and \( A, A' \subseteq \Omega \) let
\[
0 \leq h_0 \leq h_1 \leq \ldots \leq \chi A \leq \ldots \leq g_1 \leq g_0,
\]
\[
\sup I(h_n) = \inf I(g_n),
\]
\[
0 \leq h'_0 \leq h'_1 \leq \ldots \leq \chi A' \leq \ldots \leq g'_1 \leq g'_0,
\]
\[
\sup I(h'_n) = \inf I(g'_n).
\]
Then for \( h^+_n := \sup(h_n, h'_n), \quad g^+_n := \sup(g_n, g'_n), \quad h^-_n := (h_n - g'_n)^+ \) and \( g^-_n := (g_n - h'_n)^+ \),
\[
0 \leq h^+_0 \leq h^+_1 \leq \ldots \leq \chi A \cup A' \leq \ldots \leq g^+_1 \leq g^+_0,
\]
\[
\sup I(h^+_n) = \inf I(g^+_n),
\]
\[
0 \leq h^-_0 \leq h^-_1 \leq \ldots \leq \chi A \setminus A' \leq \ldots \leq g^-_1 \leq g^-_0,
\]
\[
\sup I(h^-_n) = \inf I(g^-_n).
\]

**Proof:**
Consider union. Since \( \sup(\chi A, \chi A') = \chi A \cup A' \),
\[
h^n_+ \leq h^{n+1}_+ \leq \chi A \cup A' \leq g^{n+1}_+ \leq g^n_+.
\]
for all \( n \in \mathbb{N} \). It remains to show \( \lim_n (I(g^n_+) - I(h^n_+)) = 0 \). Since \( \sup(f, g) = (f + g + |f - g|)/2 \),
\[
g^n_+ - h^n_+ = (g_n + g'_n + |g_n - g'_n| - h_n - h'_n - |h_n - h'_n|)/2.
\]
Since \( I(g_n - h_n) \to 0 \) and \( I(g'_n - h'_n) \to 0 \), it remains to show
\[
I(|h_n - h'_n| - |g_n - g'_n|) \to 0.
\]
But this follows from \( |h_n - h'_n| - |g_n - g'_n| \leq |h_n - g_n| + |g_n - g'_n| + |g'_n - h'_n| - |g_n - g'_n| = |h_n - g_n| + |g_n - h'_n| \) and correspondingly, \( |g_n - g'_n| - |h_n - h'_n| \leq |g_n - h_n| + |h_n - g'_n| \).
Consider difference. From the assumptions \( h_n - g_n \leq \chi A - \chi A' \leq g_n - h'_n \), since \( (\chi A - \chi A')^+ = \chi A \setminus A' \) and \( f^+ \leq g^+ \) if \( f \leq g \),
\[
h^n_+ \leq h^{n+1}_+ \leq \chi A \setminus A' \leq g^{n+1}_- \leq g^n_-
\]
for all \( n \in \mathbb{N} \). It remains to show \( \lim_n (I(g^n_-) - I(h^n_-)) = 0 \). Since \( f^+ = (f + |f|)/2 \),
\[
g^n_- - h^n_- = (g_n - h'_n + |g_n - h'_n| - h_n + g'_n - |h_n - g'_n|)/2
\]
Since $I(g_n - h_n) \to 0$ and $I(g'_n - h'_n) \to 0$, it remains to show
\[
I(\|g_n - h'_n\| - |h_n - g'_n|) \to 0.
\]
But this follows from $|g_n - h'_n| - |h_n - g'_n| \leq |g_n - h_n| + |h_n - g'_n| - |h_n - g_n| + |h'_n - g'_n|$ and correspondingly, $|h_n - g_n| - |g_n - h'_n| \leq |h_n - g_n| + |h'_n - g'_n|$. □ (Prop. 4.8)

By the next proposition the constructions in Prop. 4.8 are computable. Let us say that $t = (s_-, s_+)$ encloses a set $A \subseteq \Omega$, if (4.24) for the sequences $(h_0, h_1, \ldots) := \nu_+(s_-)$ and $(g_0, g_1, \ldots) := \nu_-(s_+)$.

**Proposition 4.9** There are computable functions $G_1$ and $G_2$ such that $G_1(t, t')$ encloses $A \cup A'$ and $G_2(t, t')$ encloses $A \setminus A'$, if $t$ encloses $A$ and $t'$ encloses $A'$.

**Proof:** Suppose, $t = (s_-, s_+)$ and $t' = (s'_-, s'_+)$. Let
\[
h_i := \gamma \circ \xi_{s_-}(bni), \quad g_i := \gamma \circ \xi_{s_+}(bni),
\]
\[
h'_i := \gamma \circ \xi'_{s_-}(bni), \quad g'_i := \gamma \circ \xi'_{s_+}(bni)
\]
(see (4.8), Prop. 4.8), where $bni$ is the binary notation of the natural number $i$. By Def. 4.2 there is a computable function $F : \subseteq \Sigma^* \times \Sigma^* \to \Sigma^*$ such that $\sup(\gamma(u), \gamma(v)) = \gamma \circ F(u, v)$. We obtain
\[
\sup(h_{\nu_n(w)}', h'_{\nu_n(w)}) = \sup(\gamma \circ \xi_{s_-}(w), \gamma \circ \xi'_{s_-}(w)) = \gamma \circ F(\xi_{s_+}(w), \xi'_{s_+}(w)) = \gamma \circ F_1(s_-, s'_-)(w)
\]
for some computable function $F_1 : \Sigma^* \times \Sigma^* \to \Sigma^*$ by the utm-theorem and the sum-theorem for the notation $\xi$. Correspondingly,
\[
\sup(g_{\nu_n(w)}', g'_{\nu_n(w)}) = \gamma \circ F_1(s_+, s'_+)(w).
\]
Define
\[
G_1((s_-, s_+), (s'_-, s'_+)) := \langle F_1(s_-, s'_-), F_1(s_+, s'_+) \rangle.
\]
The function $G_2$ can be defined accordingly. Then the claim follows from Prop. 4.8. □ (Prop. 4.9)

**Proposition 4.10** There is a computable function $L$ such that $\rho \circ L(\langle s_-, s_+ \rangle) = \sup I(h_n)$, if $\nu_+(s_-) = (h_i)_i$ and $\nu_-(s_+) = (g_i)_i$ such that (4.24).

**Proof:** This follows by standard arguments from Def. 4.2(6). □ (Prop. 4.10)
We define a notation $\alpha$ of $\mathcal{R}$ inductively as follows. Let $H_\rho$ be the function from (4.20). (For convenience we assume $\text{dom}(\gamma), \text{dom}(\nu_0) \subseteq (\Sigma \setminus \Sigma')^*$ and $\Sigma' \subseteq \Sigma \setminus \{0,1\}$ for $\Sigma := \{\{0\}, \cup, \\}$.)

$$\alpha(v, u, u_{r}) := \{ \gamma_{+}(v) \geq \rho \circ H_\rho(v, u, u_{r}) \} \in \mathcal{R}_0, \quad (4.25)$$

$$\alpha(w \cup w') := \alpha(w) \cup \alpha(w'), \quad (4.26)$$

$$\alpha((w \setminus w')) := \alpha(w) \setminus \alpha(w') \quad (4.27)$$

for $v \in \text{dom}(\gamma) = \text{dom}(\gamma_{+}), u, u_{r} \in \text{dom}(\nu_0)$ such that $0 < \nu_0(u) < \nu_0(u_{r})$ and $w, w' \in \text{dom}(\alpha)$. Let $\alpha(x)$ be undefined for all other $x \in \Sigma^*$. Then $\alpha$ is a notation of $\mathcal{R}$ such that $\text{dom}(\alpha)$ is recursive. Obviously, union and difference on $\mathcal{R}$ are $(\alpha, \alpha, \alpha)$-computable.

Thus we have proved (5) and (6) in Def. 2.2.

**Proposition 4.11** $\alpha : \Sigma^* \rightarrow \mathcal{R}$ is a notation of $\mathcal{R}$ with recursive domain and $(A, B) \mapsto A \cup B$ and $(A, B) \mapsto A \setminus B$ are $(\alpha, \alpha, \alpha)$-computable.

Next, we define the function $\mu$ on $A = A(\mathcal{R})$. For finding a $\sigma$-additive measure we apply the non-effective theorem 4.1 since $\mathcal{R}_0 \subseteq \mathcal{F}, A(\mathcal{R}) \subseteq A(\mathcal{F})$.

**Definition 4.11** Let $\mu'$ be the unique measure on $A(\mathcal{F})$ such that $f$ is $\mu'$-integrable and $I(f) = \int f \, d\mu'$ for all $f \in \mathcal{F}$ (Thm. 4.1). Let $\mu$ be the restriction of $\mu'$ to $A(\mathcal{R})$.

Since $A(\mathcal{R})$ is a $\sigma$-algebra, $\mu$ is a measure. Therefore, (1), (2), (5) and (6) from Def. 2.2 are true. It remains to prove (3) and (7). From Prop. 4.8 we obtain:

**Proposition 4.13** For every $A \in \mathcal{R}$ and sequences $(h_i)$ and $(g_i)$ in $D_+$ such that (4.24), $\int \chi_A \, d\mu = \mu(A) = \sup_i I(h_i) = \inf_i I(g_i)$. Furthermore, appropriate sequences $(h_i)$ and $(g_i)$ in $D_+$ can be computed from $A$ w.r.t. the notations $\alpha$ and $\nu_\alpha$.

**Proof:** For all $i$ we obtain: $I(h_i) = \int h_i \, d\mu' \leq \int \chi_A \, d\mu' \leq \int g_i \, d\mu' = I(g_i)$. Therefore, $\sup_i I(h_i) = \int \chi_A \, d\mu' = \mu'(A) = \mu(A)$. \hfill $\Box$ (Prop. 4.13)

Using the functions $G_1$ and $G_2$ from Prop. 4.9 and the function $L$ from Prop. 4.10 we prove that the measure $\mu$ is $(\alpha, \rho)$-computable on $\mathcal{R}$.

**Proposition 4.14** The measure $\mu$ is $(\alpha, \rho)$-computable on $\mathcal{R}$, in particular, $\mu(A) < \infty$ for all $A \in \mathcal{R}$.

**Proof:** By recursion we compute a function $H$ such that $H(y)$ encloses $\alpha(y)$ for all $y \in \text{dom}(\alpha)$.

$- \ y = \langle v, u_i, u_r \rangle \ (4.25)$: (see (4.21)) Define

$$H(y) := \langle H_<(v, u_i, u_r), H_>(v, u_i, u_r) \rangle .$$

Then $H(y)$ encloses the set $\alpha(y)$. 

13
\[ y = (t \cup t') \quad (4.26) \]: By induction, \( \langle s_-, s_+ \rangle := t \) encloses \( \alpha(t) \) and \( \langle s'_-, s'_+ \rangle := t' \) encloses \( \alpha(t') \). Define
\[ H(y) := G_1(t, t') . \]

By Prop. 4.9, \( H(y) \) encloses the set \( \alpha(y) \).

\[ y = (t \setminus t') \quad (4.27) \]: (Accordingly)

Therefore \( H \) is a computable function such that \( H(y) \) encloses the set \( \alpha(y) \) for all \( y \in \text{dom}(\alpha) \). This means that appropriate sequences \((h_k)\) and \((g_k)\) in \( \mathcal{D}_+ \) can be computed.

By Props. 4.10 and (4.13), \( \mu \circ \alpha(y) = \rho \circ L \circ H(y) \). Therefore, the measure \( \mu \) is \((\alpha, \rho)\)-computable on \( \mathcal{R} \).

Thus we have proved Def. 2.2(3) and (7). Finally we prove Def. 2.2(4).

**Proposition 4.15** The restriction of \( \mu \) to \( \mathcal{R} \) is \( \sigma \)-finite.

**Proof:** Since \( \mathcal{R} \) is countable and \( \mu(A) < \infty \) for all \( A \in \mathcal{R} \), it suffices to show \((\forall x \in \Omega)(\exists A \in \mathcal{R}) x \in A \). Consider \( x \in \Omega \). By assumption \( f(x) \neq 0 \) for some \( f \in \mathcal{D} \). Then \( |f| = \gamma_+ (v) \in \mathcal{D}_+ \) for some \( v \) and \( |f(x)| > 0 \). Therefore, there is some \( e \in D_v \) (see (4.23)) such that \( |f(x)| > c \). Therefore, \( x \in \{|f| > c\} \in \mathcal{R} \). \( \square \)(Prop. 4.15)

Altogether, we have defined a computable measure space \( \mathcal{M} = (\Omega, A, \mu, \mathcal{R}, \alpha) \).

Finally, we consider integration. First, we generalize Prop. 4.13 from characteristic functions \( \chi_A, A \subseteq \mathcal{R} \) to rational linear combinations of such functions, i.e., rational step functions. A notation \( \beta \) for the rational step functions is defined in Def. 3.1.

**Proposition 4.16** For every rational step function \( t \) with non-negative coefficients and every \( m \in \mathbb{N} \), functions \( H, G \in \mathcal{D}_+ \) can be computed (w.r.t. \( \beta \) and \( \gamma \)) such that \( H \leq t \leq G \) and
\[ \int t \, d\mu - 2^{-m} \leq I(H) \leq \int t \, d\mu \leq I(G) \leq \int t \, d\mu + 2^{-m} . \]

**Proof:** Straightforward from Prop. 4.13 . \( \square \)

Notice that a \( \mu' \)-integrable function \( f \in \mathcal{F} \) (see Def. 4.12) which is \( \mu \)-measurable may be not \( \mu \)-integrable. We prove the converse of Prop. 4.16.

**Proposition 4.17** For every function \( f \in \mathcal{D}_+ \) and every \( n \in \mathbb{N} \) a rational step function \( t \) in \( \mathcal{M} = (\Omega, A, \mu, \mathcal{R}, \alpha) \) with non-negative coefficients can be computed w.r.t. the notations \( \gamma, \nu_n \) and \( \beta \) (from Def. 3.1) such that
\[ t \leq f \quad \text{and} \quad 0 \leq I(f) - \int t \, d\mu \leq 2^{-n} . \]
\textbf{Proof:} First, we prove that for any \( f \in \mathcal{D}_+ \) and \( n \in \mathbb{N} \), rational step functions \( t \) and \( t' \) with non-negative coefficients can be computed such that for some \( \tilde{f} \in \mathcal{F}_+ \),
\begin{align}
0 &\leq t \leq f - \tilde{f} \leq t', \quad (4.28) \\
0 &\leq I(\tilde{f}) \leq 2^{-n-2}, \quad (4.29) \\
f(t' - t) \, \delta \mu &\leq 2^{-n-2}. \quad (4.30)
\end{align}
Let \( f = \gamma(v) \) (4.23). For any real numbers \( 0 < c_0 < c_1 < \ldots < c_k \),
\[ f = (f - f \wedge c_k) + \sum_{i=1}^{k} (f \wedge c_i - f \wedge c_{i-1}) + f \wedge c_0 =: S_1 + S + S_2. \quad (4.31) \]
By Props. 4.5, 4.10 and 4.13,
\begin{align}
\left\{ \begin{array}{l}
\text{from } f \text{ and rational numbers } a < b, \text{ constants } c, x \in \mathbb{R} \\
\text{can be computed such that } a < c < b \text{ and } \mu \{f > c\} = x.
\end{array} \right\} \quad (4.32)
\end{align}
Since \( \sup_n (f \wedge n) = f \) and \( I \) is computable on \( \mathcal{D}_+ \), some \( N \in \mathbb{N} \) can be computed from \( f \) and \( n \) such that \( I(f - f \wedge N) < 2^{-n-3} \).
Similarly, since \( \inf_n (f \wedge 1/n) = 0 \), some \( M \in \mathbb{N} \) can be computed from \( f \) and \( n \) such that \( I(f \wedge 1/M) < 2^{-n-3} \). By (4.32) constants \( c_0 > M \) and \( x_0 = \mu \{f > c_0\} > 0 \) can be computed from \( v \) such that
\[ I(f \wedge c_0) < 2^{-n-3}. \quad (4.33) \]
Let \( e := 2^{-n-4}/\mu \{f > c_0\} \) and choose \( k \in \mathbb{N} \) such that \( k \cdot e/2 \geq N \). Such a number \( k \) can be computed.
Inductively, for \( i = 1, \ldots, k \) find words \( u_i^t, u_i^\delta \in \text{dom}(\nu_0) \) such that
\[ c_{i-1} + \frac{e}{2} + \frac{(i - 1)e}{2k} \leq \nu_0(u_i^t) < \nu_0(u_i^\delta) \leq c_{i-1} + \frac{e}{2} + \frac{ie}{2k}. \quad (4.34) \]
and determine real numbers \( c_i \in [\nu_0(u_i^t); \nu_0(u_i^\delta)] \) and \( x_i = \mu \{f > c_i\} \) according to (4.32), for which, in particular,
\[ \alpha((v, (u_i^t, u_i^\delta)) = \{f > c_i\} \quad (4.35) \]
and
\[ c_{i-1} + \frac{e}{2} + \frac{(i - 1)e}{2k} < c_i < c_{i-1} + \frac{e}{2} + \frac{ie}{2k}. \quad (4.36) \]
Then for \( i = 1, \ldots, k, \) \( e/2 < c_i - c_{i-1} \) and therefore, \( c_k \geq c_0 + ke/2 > N \), hence
\[ I(f - f \wedge c_k) < 2^{-n-3}. \quad (4.37) \]
Also from (4.36), \( c_k - c_{k-1} \leq e \), therefore
\[ 2(c_k - c_{k-1})\mu \{f > c_0\} \leq 2e\mu \{f > c_0\} \leq 2^{-n-3}. \quad (4.38) \]
Finally, it follows from (4.36) that for all $i$,

$$c_i - c_{i-1} < \epsilon/2 + i\epsilon/(2k) < c_{i+1} - c_i. \quad (4.39)$$

Since $(c' - c) \chi_{(f > c')} \leq f \land c' - f \land c \leq (c' - c) \chi_{(f > c)}$ for any $c < c'$,

$$s := \sum_{i=1}^k (c_i - c_{i-1}) \chi_{(f > c_i)} \leq S \leq \sum_{i=1}^k (c_i - c_{i-1}) \chi_{(f > c_{i-1})} =: s'. \quad (4.40)$$

We estimate $s' - s$.

$$s' - s = \sum_{i=1}^k (c_i - c_{i-1}) \chi_{(f > c_{i-1})} - \sum_{i=1}^k (c_i - c_{i-1}) \chi_{(f > c_i)}$$

$$= (c_1 - c_0) \chi_{(f > c_0)} + \sum_{i=1}^{k-1} [(c_{i+1} - c_i) - (c_i - c_{i-1})] \chi_{(f > c_i)} + (c_k - c_{k-1}) \chi_{(f > c_k)}$$

$$\leq (c_1 - c_0) \chi_{(f > c_0)} + \sum_{i=1}^{k-1} [(c_{i+1} - c_i) - (c_i - c_{i-1})] \chi_{(f > c_i)} + (c_k - c_{k-1}) \chi_{(f > c_k)}$$

$$= [(c_1 - c_0) + \sum_{i=2}^k (c_i - c_{i-1}) - \sum_{i=1}^{k-1} (c_i - c_{i-1}) + (c_k - c_{k-1})] \chi_{(f > c_0)}$$

$$= 2(c_k - c_{k-1}) \chi_{(f > c_0)}$$

The “$\leq$” follows from (4.39). The step functions $s$ and $s'$ are $\mu$-integrable and by (4.38), $\int (s' - s) \, d\mu \leq 2(c_k - c_{k-1}) \mu{f > c_0} \leq 2^{-n-3}$.

By changing the coefficients of $s$ and $s'$ a little bit we obtain national step functions $t$ and $t'$ such that $0 < t < s$, $s' < t'$ and $\int (t' - t) \, d\mu \leq 2^{-n-2}$.

Now let $f := S_1 + S_2 = (f - f \land c_0) + f \land c_0$. Then $f \in \mathcal{F}_+$. By (4.33) and (4.37). Since $S = f - \tilde{f}$ we obtain $t' \leq f - \tilde{f} \leq t' \leq t$.

Thus we have proved (4.28) - (4.30).

For all $m \in \mathbb{N}$ by Prop. 4.16 and (4.28) there are functions $H, G \in \mathcal{D}_+$ such that $H \leq t \leq f - \tilde{f} \leq t' \leq G$ and

$$\int t \, d\mu - 2^{-m} \leq I(H) \leq I(f - \tilde{f}) \leq I(G) \leq \int t' \, d\mu + 2^{-m}.$$ 

Since this is true for all $m$, $\int t \, d\mu \leq I(f - \tilde{f}) \leq \int t' \, d\mu$. We obtain
\[ I(f) - \int t \, d\mu \leq I(f - \tilde{f}) - \int t \, d\mu + I(\tilde{f}) \]
\[ \leq \int (t' - t) \, d\mu + I(\tilde{f}) \]
\[ \leq 2^{-n-2} + 2^{-n-2} \]
\[ < 2^{-n}. \]

\( \Box \) (Prop. 4.17)

Let \( \mathcal{F}_+^n \) be the set of all \( f : \Omega \to \mathbb{R} \) such that \( f_i \nearrow f \) for some sequence of functions in \( \mathcal{F}_+^n \).

Define \( I^* : \mathcal{F}_+^n \to \mathbb{R} \) by
\[ I^*(f) := \sup_i I(u_i) \text{ if } u_i \nearrow f. \]

In [1] p. 189 it is proved that \( I^* \) is well-defined (i.e., \( \sup_i I(u_i) = \sup_i I(v_i) \) if \( u_i \nearrow f \) and \( v_i \nearrow f \)) and that \( I^* \) extends \( I \) on \( \mathcal{F}_+ \) such that \( I^*(af) = aI^*(f) \) \((a \geq 0)\), \( I^*(f + g) = I^*(f) + I^*(g) \) \((f, g \in \mathcal{F}_+^n)\) and \( I^*(\sup_i f_i) = \sup_i I^*(f_i) \) if \( f_i \nearrow f \) in \( \mathcal{F}_+^n \).

For every \( A \in \mathcal{R} \), there is a sequence \( (h_i)_i \) in \( \mathcal{D}_+ \) such that \( h_i \nearrow \chi_A \), hence by Prop. 4.13, \( \int \chi_A \, d\mu = \mu(A) = I^*(\chi_A) \), therefore
\[
\int t \, d\mu = I^*(t) \text{ for every non-negative rational step function } t. \quad (4.11)
\]

Now define the embedding \( \psi : \mathcal{F}_f/\equiv \to \mathcal{I}(\mathcal{M})/\equiv \). First, we define \( \psi(\tilde{f}) \) for \( f \in \mathcal{F}_+ \) by a \((\delta F, \delta \mathcal{M})\)-realization on names as follows.

Suppose \( \delta F(p) = \tilde{f} \). Then \( p \) encodes (\( \gamma \)-names of) elements \( f_i \in \mathcal{D}_+ \) such \( I(|f - f_i|) \leq 2^{-i} \). By Prop 4.17, for each \( i \) a rational step function \( s_i \) can be computed such that \( 0 \leq s_i \leq f_{i+2} \) and \( 0 \leq I(f_{i+2}) - \int s_i \, d\mu \leq 2^{-i-2} \), and hence
\[
0 \leq I^*(|f_{i+2} - s_i|) = I^*(f_{i+2}) - I^*(s_i) \leq 2^{-i-2}.
\]

Then for any \( k > i \),
\[
\int |s_i - s_k| \, d\mu = I^*(|s_i - s_k|) \text{ by (4.11)} \]
\[
\leq I^*(|s_i - f_{i+2}|) + I^*(|f_{i+2} - f|) + I^*(|f_{k+2} - s_k|) \]
\[
\leq 2^{-i-2} + 2^{-i-2} + 2^{-k-2} + 2^{-k-2} \]
\[
\leq 2^{-i}.
\]

17
By Thm 15.5 in [1], the sequence \((s_i)\) of rational step functions converges to some \(h \in \mathcal{I}(\mathcal{M})\) such that \(d_S(s_i, h) \leq 2^{-i}\).

Define \(\psi(J) := \overline{\eta} \bigg|_{J}\).

We show that \(\psi\) is well-defined on \(\mathcal{F}_+ \cap \mathcal{F}_-\). Suppose \(J = \mathcal{F}\) and \(\delta_S(q) = \mathcal{F}\).

The computation specified above gives a sequence \((g_i)\) of functions in \(D_+\) and a sequence \((t_i)\) of rational step functions such that

\[
I(|g - g_i|) \leq 2^{-i}, \quad 0 \leq t_i \leq g_{i+2} \quad \text{and} \quad 0 \leq I(g_{i+2}) - \int t_i \, du \leq 2^{-i-2}
\]

and \(d_S(t_i, h') \leq 2^{-i}\) for some \(h' \in \mathcal{I}(\mathcal{M})\). Therefore for all \(i\),

\[
d_S(h, h') \leq d_S(h, s_i) + d_S(s_i, t_i) + d_S(t_i, h') \leq 2^{-i} + \int |s_i - t_i| \, du + 2^{-i} = 2^{-i+1} + I^*(|s_i - t_i|) \leq 2^{-i+1} + I^*(|s_i - f_{i+2}| + |f_{i+2} - f| + |f - g| + |g - g_{i+2}| + |g_{i+2} - t_i|) \leq 2^{-i+1} + 2^{-i-2} + 2^{-i-2} + 0 + 2^{-i-2} + 2^{-i-2} \leq 2^{-i+2},
\]

and hence, \(\overline{h} = \overline{h}'\).

We extend \(\psi\) from \(\mathcal{F}_+ \cap \mathcal{F}_-\) to \(\mathcal{F}\). For \(\varepsilon = f_+ - f_-\), \((f_+, f_- \in \mathcal{F}_+\), define

\[
\psi(J) := \psi(J_+) - \psi(J_-).
\]

The definition is sound since \(f_+\) and \(f_-\) are uniquely defined.

We show that \(\psi\) is norm-preserving. Let \(f + f_- \in \mathcal{F}\). Let \(f^+_i, s^+_i, h^+_i\) and \(f^-_i, s^-_i, h^-\) be the functions used in the computation of \(\psi(J^+_i)\) and \(\psi(J^-_i)\), respectively. Then

\[
||\psi(J)|| = ||\psi(J^+_i) - \psi(J^-_i)|| = ||h^+ - h^-|| = I^*(|h^+ - h^-|)
\]

and for all \(i\),

\[
h^+ - h^- = (h^+ - s^+_i) + (s^+_i - f^+_i) + (f^+_i + f_-) + (f^+_i - f_-) + (f^-_i - f^+_i) + (f^-_i - f^+_i) + (f^-_i - s^-_i) + (s^-_i - h^-) =: (f^+_i - f^-_i) + \varepsilon_i.
\]

Then \(I^*(\varepsilon_i) \leq 2^{-i-2}\). Since in general \(|I^*(|g|) - I^*(|g + u|)| \leq I^*(|u|)\) we can conclude

\[
I^*(|h^+ - h^-|) - I^*(|f^+_i - f^-_i|) \leq 2^{-i-2}
\]
and therefore,

$$||\psi(\mathcal{F})|| = I^*(|h^+ - h^-|) = I^*(|f_+ - f_-|) = I^*(|f|) = ||f|| = ||\mathcal{F}||.$$ 

Similar considerations show that $\psi$ is a linear mapping and that $I(f) = \int g d\mu$ for all $f \in \mathcal{F}$ and $g \in \psi(\mathcal{F})$.

This ends the proof of the computable Daniell-Stone Theorem.

The authors want to thank the unknown referee for careful proofreading and valuable comments.

References


