An algebraic formalism for the octonionic structure of the E_8-lattice

Holger P. Petersson
Fakultät für Mathematik und Informatik
FernUniversität in Hagen
D-58084 Hagen
Germany
Email: holger.petersson@fernuni-hagen.de

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It is a great honor for me, and a great pleasure at the same time, having the opportunity to give a talk at a conference dedicated to Richard and, in a wider sense, to Jacques Tits as well. My personal acquaintance with Richard goes back almost twenty years, and during this very long period, he has been, and continues to be, a constant source of personal friendship and mathematical inspiration. For both I am very grateful.

The background of my lecture today is dominated by the E_8-lattice, this famous indecomposable unimodular positive definite quadratic lattice of rank 8 over the integers that has come to the fore again a few weeks ago when Maryna Viazovska [9] proved that the densest sphere packing of eight-dimensional euclidean space is the E_8-lattice sphere packing having density $\pi^4/384$. My aim today is much more modest, focusing instead on a discovery made by Coxeter [1] in 1946, to the effect that the E_8-lattice carries the structure of an octonion algebra over the integers whose generic fiber is the unique octonion division algebra over the rationals. To put the historical record straight, I should add that Coxeter's discovery had its predecessors, most notably a similar construction due to Dickson [2] dating back to 1923, and a result on the ideal structure of the Dickson octonions due to Mahler [5] from 1942.

My principal objective in the present talk will be to provide you with an elementary purely algebraic formalism that collapses to Coxeter's and Dickson's constructions once the appropriate specifications have been made. This formalism is based upon one of the most fruitful among the classical constructions in non-associative algebra, namely, the Cayley-Dickson construction. The natural habitat of the Cayley-Dickson construction are what I call

1. Conic algebras.

Working over an arbitrary commutative ring k remaining fixed throughout, conic algebras, more commonly known under the name *algebras of degree* 2 (McCrimmon [6]) or *quadratic algebras* (Osborn [7]), are defined as follows.

1.1. The notion of a conic algebra. By a *conic algebra over k* I mean a non-associative k-algebra C satisfying the following four conditions.

(i) C is *unital*, i.e., it contains an identity element.

(ii) C is finitely generated projective as a k-module.

(iii) The identity element of C is *unimodular*, so there exists a linear form $\lambda\colon C \to k$ such that $\lambda(1_C) = 1$.

(iv) There exists a quadratic form $n_C\colon C \to k$, necessarily unique and called the *norm* of C, such that $n_C(1_C) = 1$ and

$$x^2 - t_C(x)x + n_C(x)1_C = 0$$

for all $x \in C$.

Here $t_C\colon C \to k$ is the *trace* of C defined as the linear form $x \mapsto (Dn_C)(1_C, x)$, where Dn_C stands for the *bilinearized norm* given by

$$(Dn_C)(x, y) := n_C(x + y) - n_C(x) - n_C(y)$$

for all $x, y \in C$.

We then define the *conjugation* of C as the map

$$\iota_C\colon C \longrightarrow C, \quad x \longmapsto \bar{x} := t_C(x)1_C - x,$$

which is linear of period 2 but will fail in general to be an (algebra) involution.

Before presenting some examples, it will be convenient to discuss two

1.2. Regularity conditions. A conic algebra C over k is said to be *non-singular* if the natural map from the k-module C to its dual given by the bilinearized norm, i.e.,

$$C \longrightarrow C^*, \quad x \longmapsto n_C(x, -),$$

is a (linear) bijection. By contrast, C is said to be *weakly non-singular* if the aforementioned map is (only) injective. Weak non-singularity is not nearly as useful a notion as non-singularity since, for example, non-singularity is stable under arbitrary base change while weak non-singularity is not.

1.3. Examples. (a) The most trivial example of a conic k-algebra is the base ring itself, with norm, trace and conjugation respectively given by $n_k(\alpha) = \alpha^2$, $t_k(\alpha) = 2\alpha$ and $\bar{\alpha} = \alpha$. This conic algebra is non-singular if and only if $\frac{1}{2} \in k$.

(b) Let R be a *quadratic* algebra over k, so R is unital and finitely generated projective of rank 2 as a k-module. Then R is a conic k-algebra, with norm, trace given by $n_R(x) = \det(L_x)$, $t_R(x) = \mathrm{trace}(L_x)$, where $L_x\colon R \to R$, $y \mapsto xy$ is the left multiplication affected by $x \in R$. Recall that (i) R is commutative associative, (ii) R is non-singular if and only if it is quadratic étale, and (iii) its conjugation is an automorphism.

(c) *Quaternion algebras* over k are non-singular associative conic k-algebras that have rank 4 as finitely generated projective k-modules.

(d) *Octonion algebras* C over k are non-singular alternative conic k-algebras that have rank 8 as finitely generated projective k-modules. Their conjugation is an involution, and their norm *permits composition*: $n_C(xy) = n_C(x)n_C(y)$. Here *alternativity* of C means that the *associator*, i.e., the map

$$C \times C \times C \longrightarrow C, \quad (x, y, z) \longmapsto [x, y, z] := (xy)z - x(yz)$$

is an alternating (trilinear) function of its arguments.

2. The Cayley-Dickson construction.

After this short digression into conic algebras, we will now be able to introduce the Cayley-Dickson construction.

2.1. Defining the Cayley-Dickson construction. Its input consists of a conic k-algebra B and a scalar $\mu \in k$. Its output is a k-algebra $C := \mathrm{Cay}(B, \mu)$ living on the direct sum $B \oplus Bj$ of two copies of B as a k-module under a bilinear multiplication uniquely determined by the condition that

(i) B (identified in C through the initial summand) is a subalgebra, and

(ii) the multiplication rules

$$u(vj) = (vu)j, \quad (vj)u = (v\bar{u})j, \quad (vj)(wj) = \mu\bar{w}v$$

hold for all $u, v, w \in B$.

It is then straightforward to check that C is again a conic k-algebra, with identity element, norm, trace and conjugation respectively given by

$$\begin{aligned} 1_C &= 1_B, \\ n_C(u + vj) &= n_B(u) - \mu n_B(v), \\ t_C(u + vj) &= t_B(u), \\ \overline{u + vj} &= \bar{u} - vj \end{aligned}$$

for all $u, v \in B$.

Properties of conic algebras preserved by the Cayley-Dickson construction are in short supply. On the positive side, we can tell rather precisely to what extent certain properties get lost when performing the Cayley-Dickson construction. Calling an algebra *flexible* if it satisfies the identity $(xy)x = x(yx) =: xyx$, the principal result in this direction, basically well known, may be summarized as follows.

2.2. Theorem. *Let B be a conic k-algebra, $\mu \in k$ and $C := \mathrm{Cay}(B, \mu)$.*

(i) *ι_C is an involution iff ι_B is an involution.*

(ii) *C is flexible iff B is flexible.*

(iii) *C is commutative iff B is commutative and has trivial conjugation.*

(iv) *C is associative iff B is commutative associative.*

(v) *C is alternative iff B is associative.*

(vi) *C is weakly non-singular iff B is weakly non-singular and μ is not a zero divisor in k.*

(vii) *C is non-singular iff B is non-singular and μ is invertible in k.*

By Thm. 2.2 (vii), it is impossible to realize octonion algebras that are indecomposable as quadratic spaces, like the Coxeter (or Dixon) octonions or the examples constructed by Knus-Parimala-Sridharan [4] and Thakur [8], by means of the Cayley Dickson construction.

In order to find a way out of this impasse, we will therefore introduce a non-orthogonal version of the Cayley-Dickson construction. This is best motivated by first looking at a

2.3. Motivation of the orthogonal Cayley-Dickson construction. The best motivation for the orthogonal Cayley-Dickson construction is arguably the following. Let C be an *alternative* conic algebra over k and suppose we are given a unital subalgebra $B \subseteq C$ as well as an element $l \in C$ that is perpendicular to B relative to the bilinearized norm. Then $B + Bl \subseteq C$ is the subalgebra generated by B and l, and the multiplication rules 2.1 (ii) hold with j replaced by l, and μ replaced by $-n_C(l)$.

3. The non-orthogonal Cayley-Dickson construction.

The idea behind the non-orthogonal Cayley-Dickson construction consists in looking at the motivation 2.3 but dropping the assumption that l be orthogonal to B relative to the bilinearized norm of C. Then $u \mapsto (Dn_C)(u, l)$ defines a linear form s on B that measures the deviation of l from being orthogonal to B, and arguing as before, it follows that $B + Bl \subseteq C$ continues to be the subalgebra generated by B and l whose algebra structure can be described by explicit formulas involving only B, s and $\mu := -n_C(l)$. Ignoring the background of these formulas, we arrive at the following definition.

3.1. Defining the non-orthogonal Cayley-Dickson construction. The input of the non-orthogonal Cayley-Dickson construction consists of a conic k-algebra B, a scalar $\mu \in k$ and a linear form $s\colon B \to k$. Its output is a k-algebra $C := \mathrm{Cay}(B; \mu, s)$ living on the direct sum $B \oplus Bj$ of two copies of B as a k-module under the unique bilinear multiplication that

(i) makes B (identified in C through the initial summand) a subalgebra, and

(ii) with $\lambda := s(1_B)$, obeys the multiplication rules

$$u(vj) = \big[-s(\bar{v}u)1_B + s(u)v + s(\bar{v})u - \lambda vu\big] + \big[vu\big]j, \tag{1}$$

$$(vj)u = \big[-s(u)v + \lambda vu\big] + \big[v\bar{u}\big]j, \tag{2}$$

$$\begin{aligned}(vj)(wj) = {} & \big[-\lambda s(\bar{w}v)1_B + \lambda s(v)w + \lambda s(\bar{w})v - \lambda^2 wv + \mu\bar{w}v\big] \\ & + \big[s(\bar{w}v)1_B - s(v)w + \lambda wv\big]j\end{aligned} \tag{3}$$

for all $u, v, w \in B$.

It follows that C is again a conic k-algebra whose identity element, norm, trace and conjugation are respectively given by

$$\begin{aligned}1_C &= 1_B,\\ n_C(u + vj) &= n_B(u) + s(\bar{v}u) - \mu n_B(v),\\ t_C(u + vj) &= t_B(u) + s(\bar{v}),\\ \overline{u + vj} &= \bar{u} + s(\bar{v})1_B - vj\end{aligned}$$

for all $u, v \in B$.

Our next aim is to derive an analogue of Thm. 2.2. This will be achieved by including conditions on the linear form s and by a horrendous amount of computations.

3.2. Theorem. *Let B be a conic k-algebra, $\mu \in k$, $s\colon B \to k$ a linear form, $\lambda := s(1_B)$ and $C := \mathrm{Cay}(B; \mu, s)$.*

(i) *ι_C is an involution iff ι_B is an involution.*

(ii) *C is flexible iff B is flexible and s is* alternative, *i.e., the expression $s([u, v, w]) \in k$ is alternating (trilinear) in $u, v, w \in B$.*

(iii) *C is commutative iff B is commutative, has trivial conjugation and $s(u)v = s(v)u$ for all $u, v \in B$.*

(iv) *C is associative iff B is commutative associative and*

$$s(uv)1_B = s(u)v + s(v)\bar{u} - \lambda\bar{u}v$$

for all $u, v \in B$.

(v) *C is alternative iff B is associative and $H_{B,s} = 0$, where $H_{B,s} \colon B \times B \times B \to B$ is the trilinear map defined by*

$$\begin{aligned} H_{B,s}(u, v, w) := & -s(uvw)1_B + s(uv)\bar{w} + s(vw)\bar{u} - s(u\bar{w})v \\ & + s(u)vw - s(v)\bar{u}\bar{w} - s(w)\bar{u}v + \lambda\bar{u}v\bar{w} \end{aligned}$$

for all $u, v, w \in B$.

The condition on s in part (iv) of the theorem is easily verified if B is a quadratic k-algebra. Hence we obtain

3.3. Corollary. *If B is a quadratic k-algebra, then C is associative.*

The condition on s in part (v) of the theorem looks particularly frightening but, actually, it isn't because a straightforward computation shows that, for any associative conic algebra B,

- $H := H_{B,s}$ is alternating,
- $H(u, v, 1_B) = H(u, v, uv) = 0$ for all $u, v \in B$.

This is easily seen to imply $H = 0$ if B is locally generated by two elements, e.g., a quaternion algebra. Hence we obtain

3.4. Corollary. *If B as in Thm.* 3.2 *is associative and locally generated by two elements, then C is alternative.*

4. Towards non-singularity.

We now come to a property of the non-orthogonal Cayley-Dickson construction that has no analogue in the orthogonal case. More specifically, we present two instances of a rather general situation where the input algebra of the non-orthogonal Cayley-Dickson construction is singular but the output algebra is not. Thus no obvious analogue of Thm. 2.2 (vii) seems to exist in the non-orthogonal case.

4.1. Fields of characteristic 2 (Garibaldi-Petersson [3])**.** Let k be a field of characteristic 2 and K/k a purely inseparable field extension of finite degree and exponent at most 1. Then K is a conic k-algebra whose bilinearized norm is identically zero; in particular, K is not even weakly non-singular. Let $\mu \in k$ and $s \colon K \to k$ be a linear form normalized by the condition $s(1_K) = 1$. Then $C := \mathrm{Cay}(K; \mu, s)$ is a non-singular conic k-algebra whose norm is a Pfister (quadratic) form over k; in fact, every *anisotropic* Pfister form over k can be written in this manner. Moreover, if K has degree 4 over k, it is generated by two elements, so Cor. 3.4 shows that C is alternative, hence an octonion algebra over k.

4.2. Integral domains. We now assume that k is an integral domain and write $F := \mathrm{Quot}(k)$ for its quotient field. We let B be an alternative conic k-algebra, $s\colon B \to k$ a linear form and $\mu \in k$ a scalar.

Let us assume that B is weakly non-singular, so the natural map from the k-module B to its dual determined by the bilinearized norm is injective. Since F is a flat k-algebra, the scalar extension B_F is weakly non-singular over F. But it is also finite-dimensional, so in actual fact B_F is an honest-to-goodness non-singular composition algebra. Now consider the scalar extension $s_F\colon B_F \to F$ of s from k to F. Since B_F is non-singular, there is a unique $a \in B_F$ such that $s_F = n_{B_F}(a, -)$, and a moment's reflection shows that a belongs to

$$B^\sharp := \{u \in B_F \mid (Dn_{B_F})(u, v) \in k \text{ for all } v \in B\},$$

the dual module of B in B_F. But $B^\sharp \supseteq B$ is is a finitely generated projective k-module, so there exists a non-zero element $\delta \in k$ satisfying $B^\sharp \subseteq \delta^{-1}B$. Hence $a = \delta^{-1}a_0$ for some $a_0 \in B$, which implies $\varepsilon := n_B(a_0) \in k$.

4.3. Theorem. *With the notation and assumptions of* 4.2, *for the conic algebra* $C := \mathrm{Cay}(B; \mu, s)$ *to be non-singular it is sufficient that* $\varepsilon + \delta^2\mu$ *be invertible in* k.

However, this condition is not necessary, even if we assume that $\delta^{-1}B$ is minimal among the principal "fractional" ideals of B containing $B^\sharp$.

4.4. Application. Let B_0 be the algebra of Hamiltonian quaternions over the reals and $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ the standard "Hamiltonian" basis of B_0. Then

$$B := \mathbb{Z}1 \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k}$$

is a weakly non-singular conic algebra over $k = \mathbb{Z}$ satisfying $B^\sharp = \frac{1}{2}B \subseteq B_\mathbb{Q}$. Thus we may choose $\delta = 2$ in Thm. 4.3. We put $\mu := -1$ and $s := n_{B_\mathbb{Q}}(a, -)|_B$ where

$$a := \frac{1}{2}a_0 \in B^\sharp, \quad a_0 := 1 + \mathbf{i} + \mathbf{j} \in B.$$

Then $\varepsilon := n_B(a_0) = 3$, hence $\varepsilon + \delta^2\mu = 3 - 4 = -1 \in \mathbb{Z}^\times$. Thus Thm. 4.3 implies that $C := \mathrm{Cay}(B; -1, s)$ is an octonion algebra over $\mathbb{Z}$. Moreover, one checks easily that its generic fiber $C_\mathbb{Q}$ is isomorphic to $\mathrm{Cay}(B_\mathbb{Q}, -1)$, i.e., to the unique octonion division algebra over the rationals. Thus the quadratic lattice underlying C is the E_8-lattice, forcing C to be the algebra of Coxeter octonions. In fact, our construction mimics almost verbatim Coxeter's original description presented in [1].

On the other hand, put

$$\mathbf{h} := \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Then

$$B' := \mathbb{Z}1 \oplus \mathbb{Z}\mathbf{i} + \mathbb{Z}\mathbf{j} + \mathbb{Z}\mathbf{h},$$

the algebra of *Hurwitz quaternions*, is weakly non-singular over $\mathbb{Z}$ such that $B \subseteq B'$, hence

$$B'^\sharp \subseteq B^\sharp = \frac{1}{2}B \subseteq \frac{1}{2}B'.$$

Thus again we may assume $\delta = 2$. Again we put $\mu := -1$; but now we deviate from the preceding choices by setting $s := n_{B_\mathbb{Q}}(a, -)|_{B'}$ where

$$a := \frac{1}{2}a_0 \in B'^\sharp, \quad a_0 := 1 + \mathbf{i} \in B \subseteq B'.$$

Then $\varepsilon = n_{B'}(a_0) = 2$, hence $\varepsilon + \delta^2\mu = 2 - 4 = -2 \notin \mathbb{Z}^\times$. And yet the non-orthogonla Cayley-Dickson construction $C' := \mathrm{Cay}(B'; -1, s)$ turns out to be exactly the octonion algebra over the integers exhibited by Dickson in [2].

4.5. Open questions. (a) By 4.2, Thm. 4.3 also applies when $B_0 = \mathbb{O}$ is the algebra of Graves-Cayley octonions over the reals. Arguing as in 4.4, we find unimodular positive definite quadratic lattices of rank 16 over $\mathbb{Z}$ carrying the structure of a *sedenion* algebra over $\mathbb{Z}$ with generic fiber $\mathrm{Cay}(B_{\mathbb{Q}}, -1)$ over $\mathbb{Q}$. These sedenion algebras have zero divisors. Thanks to the work of Witt [10], their underlying quadratic lattices are either indecomposable or the direct sum of two copies of the E_8-lattice. It is easy to see that the decomposable case of the direct sum of two E_8-lattices can be obtained from our construction. But is this true also for the indecomposable case?

(b) Is it possible to realize the examples of Knus-Parimala-Sridharan [4] by means of the non-orthogonal Cayley-Dickson construction? I suspect the answer is yes.

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