Anisotropic quadratic forms over local fields

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In this lecture I would like to report on joint work with Bernhard Mühlherr from the University of Giessen in Germany and Richard Weiss from Tufts University in the United States. Let me begin by making a confession. I am now in the business of doing mathematical research for almost fifty years, and yet, in spite of this very long time, it is a completely new experience for me to be engaged in a research project I know practically nothing about. As a matter of fact, there are basically just two things I really know about this project: its name *The local structure of Bruhat-Tits buildings*, and the fact that it is intimately tied up with the algebraic theory of quadratic forms. More specifically, my own involvement with the project started approximately three years ago (incidentally, while I was staying near Malaga) when I received an Email from Richard Weiss asking me a number of very specific questions about some rather outlandish properties of anisotropic quadratic forms over local fields. After I had been able to answer these questions affirmatively, the collaboration with him intensified, culminating in yet another Email from Richard shortly before Christmas last year in which he stated two propositions and asked me to prove them, adding that it shouldn't be too difficult for me to do so.

Well, I wouldn't say that but still, eventually I succeeded. Moreover, in the process of doing so, I arrived at a much better understanding of anisotropic quadratic forms over local fields than I had before, and it is the purpose of my lecture today to share this better understanding with you, although it may not be really new for the experts.

In order to get the terminology straight, let me begin by reminding you of a few standard facts about

1. Quadratic forms over arbitrary fields.

Let K be an arbitrary field.

1.1. The concept of a quadratic form. Recall that a quadratic form over K is a map $Q\colon W \to K$, where

- W is a finite-dimensional vector space over K,
- The map Q is homogeneous of degree 2 and canonically induces a symmetric

bilinear form ∂Q on W defined by the formula

$$(\partial Q)(x,y) = Q(x+y) - Q(x) - Q(y). \tag{1}$$

Note that (1) implies

$$(\partial Q)(x,x) = 2Q(x). \tag{2}$$

Morphisms of quadratic forms are linear maps of the underlying vector spaces preserving the forms in the obvious sense. In this way, we obtain the category of quadratic forms over K. Isomorphic quadratic forms are mostly called *isometric*.

1.2. Non-singular versus anisotropic quadratic forms. (a) A quadratic form Q as in 1.1 is said to be *non-singular* if its induced symmetric bilinear form ∂Q is non-degenerate in the usual sense of linear algebra. For example, *hyperbolic quadratic forms are always non-singular*, where Q is said to be *hyperbolic* if, for some integer $r \geq 0$, it is isometric to the quadratic form

$$K^r \oplus K^r \longrightarrow K, \quad x \oplus y \longmapsto x^t y. \tag{3}$$

(b) On the other hand, we say our quadratic forms Q is *anisotropic* if $Q(x) \neq 0$ for all non-zero elements $x \in W$. From (2) we deduce that anisotropic quadratic forms are automatically non-singular provided the base field has characteristic not 2. *In characteristic* 2*, however, this conclusion does no longer hold.* This may be seen by looking at

1.3. Examples. (a) The non-zero one-dimensional quadratic forms over K have the form $\langle\alpha\rangle\colon K \to K$, $x \mapsto \alpha x^2$, for some $\alpha \in K^\times$. They are always anisotropic and satisfy $\partial\langle\alpha\rangle(x,y) = 2\alpha xy$, hence are non-singular if and only if K has characteristic not 2.

(b) Let E be a quadratic étale K-algebra, so either E/K is a separable quadratic field extension or $E \cong K \oplus K$ is split. Then n_E, the norm of E in its capacity as a two-dimensional composition algebra, is a non-singular binary quadratic form which is anisotropic if and only if E/K is a field.

(c) The quadratic forms over K that are non-singular are precisely the ones that are isometric to

$$\alpha_1 n_{E_1} \oplus \cdots \alpha_m n_{E_m} \oplus \varphi, \tag{4}$$

with $\alpha_i \in K^\times$, E_i quadratic étale K-algebras for $1 \leq i \leq m$ and φ a non-singular anisotropic quadratic form of dimension at most 1. Here by (a) the case $\dim(\varphi) = 1$ can only occur if K has characteristic not 2.

2. Residue forms.

The whole point of my lecture will be to specify the base field as follows.

2.1. The concept of a local field. From now on, we will always assume that K is a *local field*, so in addition to being a field, K is equipped with a *complete discrete valuation*, i.e., a map $\nu\colon K \to \mathbb{Z}_\infty$ that

- is surjective, definite, logarithmically multiplicative, and satisfies the additive version of the non-archimedean triangle inequality,
- makes K complete with respect to the valuation topology.

Then

$$\mathfrak{o} := \{\alpha \in K \mid \nu(\alpha) \geq 0\} \subseteq K, \tag{5}$$

is a subring, called the *valuation ring* of K. More precisely, $\mathfrak{o}$ is a *discrete valuation ring*, i.e., a principal ideal domain containing a unique non-zero prime ideal, furnished by the *valuation ideal*

$$\mathfrak{p} := \{\alpha \in K \mid \nu(\alpha) > 0\}. \tag{6}$$

The quotient $\bar{K} := \mathfrak{o}/\mathfrak{p}$ is a field, called the *residue field* of K. We always write $\alpha \mapsto \bar{\alpha}$ to indicate the natural map from $\mathfrak{o}$ to $\bar{K}$. We also fix a *uniformaizer* of K, i.e., an element of $\pi \in \mathfrak{o}$ that generates the valuation ideal $\mathfrak{p} = \mathfrak{o}\pi$, equivalently, that has value $\nu(\pi) = 1$. There are many useful elementary properties enjoyed by the discrete valuation of a local field, for example,

2.2. The principle of domination. This pompous name refers to the simple observation that, *given $\alpha_1, \dots, \alpha_m \in K$, the non-archimedian inequality can be converted into an equality,*

$$\nu(\sum_{i=1}^{m} \alpha_i) = \min\{\nu(\alpha_i) \mid 1 \leq i \leq m\}, \tag{7}$$

provided the minimum on the right of (7) *is attained at a* unique *index $j = 1, \dots, m$.*

2.3. Examples of local fields. Examples of local fields are provided by

- the p-adics $\mathbb{Q}_p$, p a prime, with residue field $\bar{\mathbb{Q}}_p = \mathbb{F}_p$, the field with p elements,
- the formal Laurent series field $F((\mathbf{t}))$, with resdue field F, for any field F.

More generally, a classical theorem of Teichmüller implies that, given any field F of characteristic $p > 0$ and any positive integer r, there exists a local field K of characteristic zero with residue field $\bar{K} = F$ having *absolute ramification index* $e_K := \nu(p \cdot 1_K) = r$. When trying to reduce properties of quadratic forms over a local field K to its residue field (cf. 2.1), this means that you cannot avoid difficulties in characteristic $p = 2$ even if you want to, e.g., by assuming that K has characteristic zero.

2.4. The concept of residue forms. When dealing with algebraic properties of a local field K, the idea is to reduce matters to the (presumably much simpler) residue field $\bar{K}$. The theory of quadratic forms is a case in point: let $Q\colon W \to K$ be an anisotropic quadratic form. Then Hensel's Lemma implies that

$$\nu\big(Q(x+y)\big) \geq \min\big\{\nu\big(Q(x)\big), \nu\big(Q(y)\big)\big\}, \tag{8}$$

$$\nu\big((\partial Q)(x,y)\big) \geq \frac{1}{2}\Big(\nu\big(Q(x)\big) + \nu\big(Q(y)\big)\Big). \tag{9}$$

We therefore obtain a filtration

$$\{0\} \subseteq \cdots \subseteq W_{i+1} \subseteq W_i \subseteq \cdots \subseteq W, \qquad (i \in \mathbb{Z}) \tag{10}$$

consisting of *full $\mathfrak{o}$-lattices*

$$W_i := \big\{x \in W \mid \nu\big(Q(x)\big) \geq i)\big\} \subseteq W \qquad (i \in \mathbb{Z}), \tag{11}$$

i.e., of finitely generated $\mathfrak{o}$-submodules containing a vector space basis of W. In particular, each W_i is torsion-free as an $\mathfrak{o}$-module. But since $\mathfrak{o}$ is a PID, we conclude that W_i is in fact a free $\mathfrak{o}$-module of finite rank equal to the dimension of Q, forcing $\bar{W}_i := W_i/W_{i+1}$ to be a finite-dimensional vector space over $\bar{K}$, endowed with the natural map $x \mapsto \bar{x}$ from W_i to $\bar{W}_i$ such that the relation

$$\bar{Q}_i(\bar{x}) := \overline{\pi^{-i}Q(x)} \tag{12}$$

determines a well defined anisotropic quadratic form $\bar{Q}_i$ on $\bar{W}_i$ over $\bar{K}$, called the *i-th residue form* of Q. Since Q is homegenous of degree 2, the assignment $x \mapsto \pi x$ gives an isometry from $\bar{Q}_i$ onto $\bar{Q}_{i+2}$, so it suffices to consider the residue forms $\bar{Q}_0, \bar{Q}_1$. Note by (12) that $\bar{Q}_1$ depends on the choice of the uniformizer π but $\bar{Q}_0$ doesn't.

2.5. Unramified and tame quadratic forms. Let Q be an anisotropic quadratic form as in 2.4 and put $W^\times := W \setminus \{0\}$.

(a) One checks easily that Q satisfies the dimension formula

$$\dim_K(Q) = \dim_{\bar{K}}(\bar{Q}_0) + \dim_{\bar{K}}(\bar{Q}_1). \tag{13}$$

Moreover, the following conditions are easily seen to be equivalent.

(i) $\bar{Q}_1 = 0$ (equivalently, $\bar{W}_1 = \{0\}$).

(ii) $W_1 = pW_0$.

(iii) $\nu(Q(W^\times)) = 2\mathbb{Z}$.

In this case, Q is said to be *unramified*. For example, If E/K is a separable quadratic field extension, then its norm $n_E \colon E \to K$, which by 1.3 (b) is a non-singular anisotropic quadratic form over K, is unramified if and only if the field extension E/K has ramification index 1 in the sense of classical valuation theory.

(b) Q is said to be *tame* if both of its residue forms $\bar{Q}_0, \bar{Q}_1$ are non-singular. By 1.2 (b), this holds automatically if $\bar{K}$ has characteristic not 2 but fails to do so in general. In fact, most of the anisotropic quadratic forms over local fields arising in the theory of Bruhat-Tits buildings are *wild*, i.e., not tame. On the positive side, it is easily seen that *tame anisotropic quadratic forms over K are always non-singular.*

3. Residue forms under scalar extensions.

Let $Q \colon W \to K$ be an anisotropic quadratic form over the local field K and suppose F/K is a finite algebraic field extension, making F a local field in its own right. One could ask naively what happens to the residue forms of Q when extending scalars from K to F, bearing in mind that $\bar{F}/\bar{K}$ may naturally be viewed as a finite algebraic extension as well? Unfortunately, this question doesn't make too much sense since the base change of Q from K to F may loose the property of being an *anisotropic* quadratic form (over F). Therefore our naive question should be phrased somewhat more carefully as follows: *what happens to an anisotropic quadratic form over a local field when extending scalars to a finite algebraic extension and prescribing the behavior of its residue forms under base change to the corresponding residue field?*

The two propositions stated by Richard Weiss a couple of months ago provide partial answers to this very general question and may be condensed into a single statement as follows.

3.1. Theorem. *Let $Q\colon W \to K$ be an anisotropic quadratic form over K and suppose F/K is a finite algebraic field extension such that the extended residue forms $(\bar{Q}_i)_{\bar{F}}$, $i = 0, 1$, are both hyperbolic* (resp. *anisotropic*). *Then the base change Q_F of Q from K to F is hyperbolic* (resp. *anisotropic with residue forms $\overline{Q_F}_i \cong (\bar{Q}_i)_{\bar{F}}$*).

I will spend the remaining time of my lecture to sketch a proof of this theorem. One of its key technical ingredients is the possibility of

4. Gluing unramified forms.

4.1. The gluing process. (a) The simplest way of gluing unramified anisotropic quadratic forms $q\colon V \to K$, $q'\colon V' \to K$ over our local field K consists in looking at the (orthogonal) direct sum $Q := q \oplus \pi q'$, which, by the principle of domination 2.2, is easily seen to be anisotropic with residue forms $\bar{Q}_0 \cong \bar{q}_0$, $\bar{Q}_1 \cong \bar{q}'_0$. However, this procedure is not the most general and, in particular, turns out to be inadequate when dealing with wild anisotropic quadratic forms.
(b) Suppose that, in addition to q, q' as in (a), we are given a bilinear form $\sigma\colon V \times V' \to K$. Then we may consider the map

$$Q := q \oplus_\sigma \pi q'\colon V \oplus V' \to K, \quad x \oplus x' \longmapsto q(x) + \sigma(x, x') + \pi q'(x'), \tag{14}$$

which is clearly a quadratic form over K. But is it anisotropic?

4.2. Proposition. *With the notation and assumptions of* 4.1, *the following conditions are equivalent.*

(i) *Q is anisotropic.*

(ii) *For all $v \in V$, $v' \in V'$ we have $\nu(\sigma(v, v')) \geq 1$.*

(iii) *If $(e_1, \dots, e_m)$* (resp. *$(e_{m+1}, \dots e_r)$*) *is an $\mathfrak{o}$-basis of V_0* (resp. *V'_0*), *then $\nu(\sigma(e_i, e_j)) \geq 1$ for all $1 \leq i \leq m < j \leq r$.*

In this case, we have $\bar{Q}_0 \cong \bar{q}_0$, $\bar{Q}_1 \cong \bar{q}'_0$.

Again this result follows from the principle of domination, but not quite so easily as before. Its significance derives from the fact that *every* anisotropic quadratic form over K allows a decomposition of type (14).

4.3. Proposition. *Let $Q\colon W \to K$ be an anisotropic quadratic forms over K. Then there are unramified anisotropic quadratic forms q, q' as in* 4.1 (a) *and a bilinear form σ as in* 4.1 (b) *such that $Q \cong q \oplus_\sigma \pi q'$.*

Sketch of proof. By the Elementary Divisor Theorem, there exist an $\mathfrak{o}$-basis $(e_1, \dots, e_r)$ of W_0 and integers $s_1 \geq \dots \geq s_r \geq 0$ making $(s_1 e_1, \dots, s_r e_r)$ an $\mathfrak{o}$-basis of $W_1 \subseteq W_0$. Here $\pi W_0 \subseteq W_1$ forces $1 \geq s_1$, and we find an integer m, $0 \leq m \leq r$, such that $s_i = 1$ for $1 \leq i \leq m$ and $s_j = 0$ for $m < j \leq r$. Now put $V := \sum_{i=1}^m K e_i$, $V' := \sum_{j=m+1}^r K e_j$ and define q (resp. q', resp. σ) as the restriction of Q to V (resp. of $\pi^{-1} Q$ to V', resp. of ∂Q to $V \times V'$). $\square$

4.4. Corollary. *If Q as in* 4.3 *has $\bar{Q}_0$ non-singular, then $Q \cong q \oplus \pi q'$ as a direct orthogonal sum, for some unramified anisotropic quadratic forms q, q' over K.*

Sketch of proof. By Prop. 4.3, there exists an unramified subform $q \subseteq Q$ such that $\bar{q}_0 \cong \bar{Q}_0$. In particular, q is tame, hence non-singular (2.5 (b)). We therefore obtain a direct orthogonal decomposition $Q = q \oplus \varphi$, for some anisotropic quadratic form φ over K. Viewing $\bar{q}_0$ and $\bar{\varphi}_0$ canonically as subforms of $\bar{Q}_0$, we conclude that $\bar{\varphi}_0$ becomes orthogonal to $\bar{q}_0 = \bar{Q}_0$. But $\bar{Q}_0$ is non-singular by hypothesis, forcing $\bar{\varphi}_0 = 0$. Applying Props. 4.3, 4.2 to φ in place of Q, we therefore deduce $\varphi = \pi q'$, for some unramified anisotropic quadratic form q' over K. □

Another application of Prop. 4.3 is addressed to the

4.5. Springer-Tietze Theorem. ([1, 2]) *Suppose Q'_0, Q'_1 are non-singular anisotropic quadratic forms over $\bar{K}$. Then there exists an anisotropic quadratic form Q over K uniquely determined up to isometry by the condition $\bar{Q}_i \cong Q'_i$ for $i = 0, 1$.*

Sketch of proof. (a) *Springer's* proof works only if $\bar{K}$ has characteristic not 2 and makes use of the fact that, in this case, the (orthogonally) indecomposable non-singular quadratic forms over $\bar{K}$ are one-dimensional.
(b) *Tietze's* proof works only if $\bar{K}$ has characteristic equal to 2 and, in this case, makes use of his own presentation [3] of the Witt group of $\bar{K}$ by generators and relations.
(c) *My own* proof works uniformly in all characteristics (of $\bar{K}$) and makes use of 4.1 (a) combined with Cor. 4.4 and 1.3 (c). □

5. Towards the proof of Theorem 3.1.

In order to tackle the proof of Theorem 3.1, the following technical result will be crucial.

5.1. Proposition. *Let X be a free $\mathfrak{o}$-module of finite rank and consider the following conditions on a quadratic form $\varphi\colon X \to \mathfrak{o}$.*

(i) *the quadratic form $\bar{\varphi} := \varphi \otimes_{\mathfrak{o}} \bar{K}\colon \bar{X} := X \otimes_{\mathfrak{o}} \bar{K} \to \bar{K}$ is anisotropic.*

(ii) *$\varphi(x) \neq 0$ for all $x \in X \setminus \pi X$.*

(iii) *φ is anisotropic: $\varphi(x) \neq 0$ for all non-zero elements $x \in X$.*

(iv) *The quadratic form $Q := \varphi \otimes_{\mathfrak{o}} K\colon W := X \otimes_{\mathfrak{o}} K \to K$ is anisotropic.*

Then the following statements hold.
(a) *We have the implications* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv).
(b) *If* (i) *holds, then Q is unramified anisotropic with $W_0 = X$ and $\bar{Q}_0 = \bar{\varphi}$.*
(c) *If $\bar{\varphi}$ is non-singular, then conditions* (i)$-$(iv) *are all equivalent, and Q is tame.* □

5.2. Corollary. *If $\bar{\varphi}$ is hyperbolic, so is φ, hence Q.*

Sketch of proof. By Prop. 5.1 (ii), some $x \in X \setminus \pi X$ has $\varphi(x) = 0$. Hence $\bar{x} := x \otimes_{\mathfrak{o}} 1_{\bar{K}} \in \bar{X}$ can be extended to a hyperbolic pair of $\bar{X}$ relative to $\bar{\varphi}$, which by standard arguments lifts to an extension of x to a hyperbolic pair of X relative to φ. We can thus split off a hyperbolic plane from X, and the result follows by induction on the rank of X. □

5.3. Sketching the proof of Theorem 3.1. (a) Suppose first that $(\bar{Q}_i)_{\bar{F}}$ are both hyperbolic for $i = 0, 1$. Then Q is tame, and by Cor. 4.4 we are reduced to the case that $Q\colon W \to K$ is unramified. Put $X := W_0$, write $\varphi\colon X \to \mathfrak{o}$ for the restriction of Q to X, and $\psi := \varphi \otimes_{\mathfrak{o}} \mathfrak{o}_F : Y := X \otimes_{\mathfrak{o}} \mathfrak{o}_F \to \mathfrak{o}_F$ for the base change of φ to the valuation ring of F. Then $\bar{\psi} \cong (\bar{Q}_0)_{\bar{F}}$ is hyperbolic by hypothesis,

forcing ψ and $\psi \otimes_{\mathfrak{o}_F} F \cong Q_F$ to be hyperbolic by Cor. 5.2.
(b) Now assume that $(\bar{Q}_i)_{\bar{F}}$ are both anisotropic for $i = 0, 1$. Using Prop. 5.1, the unramified case can be treated in pretty much the same way as before. But the reduction of the general case to the unramified one is different. By Prop. 4.3, we can write $Q = q \oplus_\sigma \pi q'$ for some unramified anisotropic quadratic forms q, q' and some bilinear form σ satisfying the conditions of Prop. 4.2. The unramified case now guarantees that q_F, q'_F are both anisotropic over F, and one checks, using Prop. 5.1 again, that condition (iii) of Prop. 4.2 extends from σ to the base change σ_F. Thus Q_F is anisotropic with $\overline{Q_F}_i \cong (\bar{Q}_i)_{\bar{F}}$ for $i = 0, 1$. $\square$

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