Integral octonions

Holger P. Petersson
Fakultät für Mathematik und Informatik
FernUniversität in Hagen
D-58084 Hagen
Germany
Email: holger.petersson@fernuni-hagen.de

Malaga Workshop on Non-associative algebras 2018
September 4 – 6*, 2018*
Department of Mathematics
University of Malaga
September 6, 2018, 11:00–12:00

The purpose of my lecture today will be to share with you a number of casual observations on both the history and the mathematics of integral octonions, that is, of the unique octonion algebra over the integers which is not split. Let me begin with a little bit of history.

The standard reference to this topic is a paper by Coxeter [4], dating back to 1946 and published in the Duke Mathematical Journal under the title "Integral Cayley numbers". Upon close inspection, this paper reveals quite a few remarkable properties. First of all, at least for an algebraist, it is awfully hard to read, but this will be of no concern to my lecture. Secondly, it becomes quite clear from the very beginning that Coxeter, when he started his investigation, was mainly interested in understanding an earlier paper by Kirmse [8], Johannes Kirmse to be precise, dating back to 1924 and published by the Saxonian Academy of Sciences under a long and typically German title whose English translation reads roughly as follows: "On the representability of natural integers as sums of eight squares and a non-commutative, non-associative number system connected with this problem".

Who was this Johannes Kirmse, and what did he accomplish? His name will forever be associated either with *Kirmse's identities* or with *Kirmse's mistake*, depending on whether you are mainly concerned with non-associative algebras or with integral quadratic forms. The story behind Kirmse's identities is very simple. As you all know, they have the form $x(\bar{x}y) = n_C(x)y = (y\bar{x})x$ and are valid, for example, in an arbitrary composition algebra C over any commutative ring, where n_C is the norm of C and $x \mapsto \bar{x}$ is its conjugation. Also, they may be found in Kirmse's aforementioned paper, presumably for the first time, so it seems their name correctly reflects the historical development of the subject.

On the other hand, the story behind Kirmse's mistake is much more involved and, strictly speaking, goes back to the year 1867, when Henry John Stephen Smith [14] showed in a non-constructive manner that there exists what I call a positive definite unimodular integral quadratic lattice of rank 8. The precise definition of this concept will be given in a moment, but let me just add that, ten years later, in 1877, Korkine and Zolotarev [9] gave an explicit description of this lattice and connected it with the sphere packing problem.

Now what is a *positive definite unimodular integral quadratic lattice of rank* n, say? By this I mean a pair (M, Q) with the following properties:

(i) M is a free additive abelian group of rank n.

(ii) $Q\colon M \to \mathbb{Z}$ is a positive definite quadratic form which is also *non-singular* in the sense that its bilinearization $Q(x,y) := Q(x+y) - Q(x) - Q(y)$ induces an isomorphism from the $\mathbb{Z}$-module M onto its dual in the usual way, equivalently, that the matrix of $Q(x,y)$ relative to any $\mathbb{Z}$-basis of M, which is positive definite of size n with integral entries and even ones on the diagonal, has determinant 1.

In this case, extending scalars to the reals by passing from M to $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, it follows that M basically sits in $\mathbb{R}^n$ as an honest-to-goodness *lattice*, i.e., as an additive subgroup generated by a basis of the ambient space; moreover, $Q_{\mathbb{R}} := Q \otimes_{\mathbb{Z}} \mathbb{R}$ basically becomes a positive definite inner product on $\mathbb{R}^n$. We know today, e.g., from the theory of modular forms [3], that the rank of any positive definite unimodular integral quadratic lattice is divisible by 8, so 8 is the smallest positive number where such a phenomenon can occur. We also know that the Smith-Korkine-Zolotarev lattice is basically unique and, in modern terminology, is nothing else than the E_8*-lattice*, i.e., the lattice spanned by the roots of the root system E_8 in the ambient Euclidean space.

How do Johannes Kirmse and his work fit into this picture? Actually, not at all, at least not explicitly. But before I go into the details of this, let me briefly digress by saying a few words about Kirmse's early life. Johannes Kirmse was born on December 26, 1894, in the Thuringian town of Schmölln and began the study of mathematics and physics at the University of Leipzig in 1914. He was drafted into the German Army in 1915 and took part in the nightmare of World War I till its very end, to be discharged from the army only as late as 1919. He resumed his mathematical studies at Leipzig University and received his P.D. from there in 1923. The title of his thesis ("Beitrag zur Theorie endlicher Körper im Gebiete der Quaternionen", i.e., "Contribution to the theory of finite fields in the domain of the quaternions" in English translation) strongly suggests that it belonged to the field of algebra, which is all the more remarkable since his thesis advisor, Gustav Herglotz, was *not* an algebraist but, instead, had built up a reputation as an outstanding researcher by contributions to such diverse fields as analysis, mathematical physics, and astronomy. The paper [8] was published in 1924 when Kirmse had already taken up a position as a high school teacher in the Thuringian town of Apolda. As an amusing side remark I would like to add that Max Koecher was born in Apolda in exactly the same year, i.e., in 1924.

But let me now return to the subject of my lecture. What eventually would lead to Kirmse's mistake, originally started out as something completely different: as the bold attempt of a young mathematician having just completed his Ph.D. and entering into the career of a high school teacher to investigate the Graves-Cayley octonions, a weird and bizarre topic even by today's standards, let alone the ones of a hundred years ago, in an arithmetic setting. One cannot be certain but it could very well be that by doing so Kirmse inadvertently initiated the study of alternative algebras. Because Emil Artin, one of the towering figures of twentieth century mathematics, had received his Ph.D. also from Leipzig University two years prior to Kirmse, in 1921, and his thesis adviser was none other than Gustav Herglotz. So it is more than plausible that Artin and Kirmse knew each other personally, and it wouldn't be too far fetched to believe that Artin's lifelong interest in alternative algebras (just remember Artin's theorem and his second Ph.D. student, Max Zorn) was inspired by [8].

What, exactly, did Kirmse do in [8]? Inspired by earlier work of Hurwitz [7], Kirmse, as I just mentioned, considered the real algebra $\mathbb{O}$ of Graves-Cayley octonions, which he defined, as was customary at the time, by means of an explicit multiplication table relative to a fixed basis of eight-dimensional Euclidean space. By a judicious choice of a new set of basis vectors, he exhibited a lattice $M \subseteq \mathbb{O}$ containing $1_{\mathbb{O}}$ on which the norm $n_{\mathbb{O}}$ of $\mathbb{O}$ not only continues to be positive definite but also becomes non-singular. Thus $(M, n_{\mathbb{O}}|_M)$ is a positive definite unimodular integral quadratic lattice of rank 8, hence by what I said a few minutes ago, must be isomorphic to the one exhibited by Smith-Korkine-Zolotarev more than fifty years earlier. But Kirmse wasn't aware of this; in fact, it is safe to assume that he was not familiar with the papers [9, 14] at all. He then proceeded to claim without proof that, in addition to the aforementioned properties, M is closed under multiplication. Coxeter, when studying Kirmse's paper, tried to verify this assertion, and got stuck. Eventually, he was able to show that*Kirmse's assertion is false*, and Kirmse's mistake was born. Coxeter then turned to Bruck, one of the leading experts in non-associative systems at the time, who remedied the defect by a modification of Kirmse's original construction. Summing up, Coxeter had thus exhibited on M the structure of a unital algebra over $\mathbb{Z}$, endowed with a positive definite non-singular quadratic form permitting composition; details may be found in [4, §4].

This is an excellent moment for a brief interruption in order to introduce the concept of an octonion algebra. By an *octonion algebra* over a commutative ring k, I mean a unital algebra C over k which is finitely generated projective (e.g., free) of rank 8 as a k-module and carries a non-singular quadratic form $n_C\colon C \to k$, necessarily unique and called the *norm* of C, permitting composition; *quaternion algebras* are defined analogously, rank 8 being replaced by rank 4. Many among the standard properties of octonion algebras you are familiar with over fields continue to hold over arbitrary commutative rings, but there are exceptions. A particularly notorious one is the fact that octonion algebras cannot always be obtained from of the Cayley-Dickson construction. For example, I claim that this is so for the algebra constructed by Coxeter, which obviously is an octonion algebra over $\mathbb{Z}$, henceforth referred to as the *integral* (or *Coxeter*) *octonions*, and denoted by $\mathrm{Cox}(\mathbb{O})$. Indeed, since positive definite unimodular integral quadratic lattices, as I have mentioned before, exist only in ranks divisible by 8, there are no non-split quaternion algebras over $\mathbb{Z}$, let alone ones contained in $\mathrm{Cox}(\mathbb{O})$.

Just like Kirmse, Coxeter defines the Cayley-Graves octonions $\mathbb{O}$ by a multiplication table relative to an appropriate basis and then proceeds to do the same for the integral subalgebra of Coxeter octonions. I for one would find it much more natural, to use the algebra $\mathbb{H}$ of Hamiltonian quaternions, with its standard basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ subject to the relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$, as a starting point in order to reach $\mathbb{O}$ by means of the Cayley Dickson construction:

$$\mathbb{O} = \mathrm{Cay}(\mathbb{H}, -1) = \mathbb{H} \oplus \mathbb{H}\mathbf{l}, \quad \mathbf{l}^2 = -1.$$

This has the advantage of allowing a fairly intrinsic description of the Coxeter octonions as follows. Denoting by

$$\mathrm{Ga}(\mathbb{H}) := \mathbb{Z}1 \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k} \subseteq \mathbb{H}$$

what I call the *Gaussian integers* of $\mathbb{H}$, which is obviously a sub-lattice (but a singular one!) and an integral subalgebra, the Coxeter octonions may be described as

$$\mathrm{Cox}(\mathbb{O}) = \mathrm{Ga}(\mathbb{H}) \oplus \mathrm{Ga}(\mathbb{H})\mathbf{p}, \quad \mathbf{p} := \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{l}).$$

Using this description, which may be regarded as a non-orthogonal version of the Cayley-Dickson construction, it is rather straightforward to check that $\mathrm{Cox}(\mathbb{O})$ is indeed an integral octonion subalgebra of $\mathbb{O}$. For the most general form of the non-orthogonal Cayley-Dickson construction, see [12].

Coxeter's paper [4] contains yet another remarkable feature. Right in the middle of it, Coxeter refers to a paper by Mahler [10], dating back to 1942 and published in the Proceedings of the Royal Irish Academy under the title "On ideals in the Cayley-Dickson algebra". A particularly puzzling aspect of Mahler's paper is that it seems to be dealing with the integral octonions four years prior to Coxeter, who not only published but also submitted his paper [4] in 1946. And indeed, Mahler relied on a paper by Dickson [5], dating back to 1923 and published in the Journal de Mathématiques pure et appliquée under the title"A new simple theory of hypercomplex integers". In this paper, Dickson constructs and investigates, among other things, an algebra of integral octonions which is easily seen to be isomorphic to Coxeter's. Thus the Coxeter octonions are more than twenty years older than is often assumed. To be fair, in a postscript almost at the very end of the paper, Coxeter himself reports that, after having finished writing up his results, he was informed by Olga Taussky Todd about Mahler's paper [10], which in turn directed him to Dickson's paper [5]. Incidentally, it shouldn't come as too much of a surprise that this part of Dickson's paper is so little known, and so rarely quoted: though reviewed in zbMath by Emmy Noether, no mention is made in this review of the connection with octonions. Moreover, the paper is not even listed in Math. Reviews, even though they have presumably compiled all of Dickson's scientific publications, including the ones appearing before 1940, the year when Math. Reviews were established.

Let us return again for a moment to Mahler's paper [10]. In its main result, Mahler describes the one-sided ideals of the integral octonions in a very explicit but still incomplete manner, by showing that every such ideal is principal and, in fact, can be generated by a multiple of an integral octonion having norm 1, 2, or 4. But the trouble is that, conversely, not every such integral octonion will lead to a one-sided ideal in this way. Fortunately, one can do much better than that.

t.IDINOC

1. Theorem. *Let C be an octonion algebra over the commutative ring k and identify $k = k1_C \subseteq C$ canonically. Then the following statements hold.*

(a) (Petersson-Racine [13, Exc. 87]) *The assignments $\mathfrak{a} \mapsto \mathfrak{a}C$ and $I \mapsto I \cap k$ give inclusion preserving inverse bijections between the ideals of k and the two-sided ideals of C.*

(b) (Petersson-Racine [13, Exc. 107]) *If k is a Dedekind domain, then every one-sided ideal $I \subseteq C$ is two-sided, hence has the form $\mathfrak{a}C$ for some ideal $\mathfrak{a} \subseteq k$.*

Sketch of proof. (a) I confine myself to performing one of the key steps, which consists in showing $I = (I \cap k)C$ for any two-sided ideal $I \subseteq C$. For this to hold it is necessary that $I \cap k = \{0\}$ implies $I = 0$. In order to see this, I first assume $I = \bar{I}$, i.e., that I is stable under conjugation. Writing t_C for the trace of C, and letting $x \in I$, we have $t_C(x) = x + \bar{x} \in I \cap k = \{0\}$. But $I \subseteq C$ is an ideal, so we may conclude $t_C(xy) = 0$ for all $y \in C$, hence $x = 0$ by non-degeneracy. If I is arbitrary, then what we had just proved implies $I \cap \bar{I} = \{0\}$. Now let $0 \neq x \in I$. Then $\bar{x} = t_C(x) - x \notin I$, and we conclude $t_C(x) \neq 0$. Hence t_C is injective on I. But the trace of C is an associative linear form, so we have $t_C(xy) = t_C(yx)$, $t_C((xy)z)) = t_C(x(yz))$ for all $x \in I$, $y, z \in C$, which by the injectivity just proved implies that x belongs to the centre of C, which is k. Thus $I \subseteq I \cap k = \{0\}$, as desired. The general formula $I = (I \cap k)C$ now follows by applying the special case just treated after reduction modulo $I \cap k$, i.e., by passing to the base change from k to $k/(I \cap k)$.

(b) Passing from C to C^{op} if necessary, it will be enough to show that any right ideal $I \subseteq C$ has the form as indicated. To this end, we consider the following cases.

Case 1. $k = F$ is a field. Then a routine argument involving standard properties of octonion algebras, particularly the Zorn vector matrices, over fields shows $I = \{0\}$ or $I = C$.

Case 2. $k = \mathfrak{o}$ is a discrete valuation ring. We write $\mathfrak{p} \subseteq \mathfrak{o}$ for the corresponding valuation ideal and $\kappa = \mathfrak{o}/\mathfrak{p}$ for the corresponding residue field. We may assume $I \neq \{0\}$ and must show $I = \mathfrak{p}^r C$, for some integer $r \geq 0$. . Since $\bigcap_{i\geq 0} \mathfrak{p}^i = \{0\}$ and C is a free $\mathfrak{o}$-module of finite rank, we conclude $\bigcap_{i\geq 0} \mathfrak{p}^i C = \{0\}$ as well. Thus there exists a natural number r which is maximal with respect to the property $I \subseteq \mathfrak{p}^r C$. In particular, $I' := \mathfrak{p}^{-r} I \subseteq \mathfrak{o}$ is a right ideal, and the assumption $I' \subseteq \mathfrak{p}C$ would imply the contradiction $I = \mathfrak{p}^r I' \subseteq \mathfrak{p}^{r+1} C$. Hence $I' \nsubseteq \mathfrak{p}C$. Consequently, $(I' + \mathfrak{p}C)/\mathfrak{p}C$ is a non-zero right ideal in the octonion algebra $C(\mathfrak{p})$ over the field κ, which by Case 1 must be all of $C(\mathfrak{p})$. Thus $I' + \mathfrak{p}C = C$, and Nakayama's lemma implies $I' = C$, hence $I = \mathfrak{p}^r I' = \mathfrak{p}^r C$.

Case 3. The general case. By (a) it will be enough to show that $I \subseteq C$ is, in fact, a two-sided ideal, equivalently, that $I_\mathfrak{p} \subseteq C_\mathfrak{p}$ is a two-sided ideal for any prime ideal $\mathfrak{p} \subseteq k$. But this follows immediately from Case 2. □

r.IDINOC

2. Remark. Part (a) of Thm. 1 underscores the analogy of octonions over commutative rings with Azumaya algebras. Part (b) generalizes not only Mahler's aforementioned result but also a theorem of Van der Blij-Springer [15, p. 414], which reaches the conclusion of (b) for the Coxeter octonions, see also Allcock [1] for a different proof. It is an open question whether part (a) remains valid for one-sided rather than two-sided ideals. The answer to the same question for quaternion algebras is certainly negative, the split quaternions over a field serving as a particularly easy counter example.

The ideal structure of the integral octonions as described in Thm. 1 suggests the natural question of what happens when they are reduced mod m, for any positive integer m. The answer is not hard to guess but still requires a proof. In fact, we will be able to establish a much more general result.

p.RECOM

3. Proposition. *Let K be an algebraic number field, i.e., a finite algebraic extension of the rationals, $\mathfrak{o}$ its ring of integers, C an octonion algebra over $\mathfrak{o}$ and $\mathfrak{a}$ a non-zero ideal in $\mathfrak{o}$. Then the octonion algebra over $\mathfrak{o}/\mathfrak{a}$ obtained from C after reduction* mod $\mathfrak{a}$, *i.e.,*

$$C/\mathfrak{a}C = C \otimes_\mathfrak{o} \mathfrak{o}/\mathfrak{a},$$

is split.

Proof. Consider $C' := C/\mathfrak{a}C$ as an octonion algebra over $\mathfrak{o}/\mathfrak{a}$. By the Chinese Remainder Theorem, we are reduced to the case $\mathfrak{a} = \mathfrak{p}^r$, for some non-zero prime ideal $\mathfrak{p} \subseteq \mathfrak{o}$ and some positive integer r. If $r = 1$, then C' becomes an octonion algebra over the finite field $\mathfrak{o}/\mathfrak{p}$ and hence is known to be split. If r is arbitrary, we have to argue in a slightly different manner. Let $\hat{\mathfrak{o}}_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of $\mathfrak{o}_\mathfrak{p}$, the localization of $\mathfrak{o}$ at $\mathfrak{p}$. Then $\hat{C}_\mathfrak{p} := C \otimes_\mathfrak{o} \hat{\mathfrak{o}}_\mathfrak{p}$ is an octonion algebra over $\hat{\mathfrak{o}}_\mathfrak{p}$. On the other hand, $\hat{K}_\mathfrak{p} = \mathrm{Quot}(\hat{\mathfrak{o}}_\mathfrak{p})$ is a local field with finite residue field, and it is well known [13, 25.13] that any octonion algebra over $\hat{K}_\mathfrak{p}$, in particular $\hat{C}_\mathfrak{p} \otimes_{\hat{\mathfrak{o}}_\mathfrak{p}} \hat{K}_\mathfrak{p}$, is split. By a result of Van der Blij-Springer [15, (3.4)], therefore, so is $\hat{C}_\mathfrak{p}$, and we conclude that C' is split since a $\mathfrak{p}$-adic density argument shows $C' \cong \hat{C}_\mathfrak{p} \otimes_{\hat{\mathfrak{o}}_\mathfrak{p}} (\hat{\mathfrak{o}}_\mathfrak{p}/\mathfrak{p}^r\hat{\mathfrak{o}}_\mathfrak{p})$. □

Let me close my talk by discussing an open problem. Paradoxically, this problem starts out with a positive result related to one of the most important properties of composition algebras over fields: two composition algebras over a field are ismorphic if and only if their norms are isometric. In my survey lectures at Ottawa and Lens of 2012, I raised the question of whether this property extends to composition algebras over arbitrary commutative rings. This question has meanwhile been answered in the negative by Gille [6], while a more recent result of Alsaody-Gille

[2] yields a more detailed picture. In order to clarify their answers, I must remind you of the notion of isotopy, due to McCrimmon [11], for alternative algebras.

Let A be a unital alternative algebra over a commutative ring k and suppose $p, q \in A$ are invertible. Write $A^{(p,q)}$ for the non-associative k-algebra living on the k-module A under the multiplication $(xp)(qy)$. McCrimmon [11] has shown that $A^{(p,q)}$ is again a unital alternative algebra over k, called the (p,q)-*isotope* of A, with identity element $1^{(p,q)} = 1_{A^{(p,q)}} = (pq)^{-1}$.

For example, if C is an octonion algebra over k, then so is $C^{(p,q)}$, for all invertible elements $p, q \in C$; in fact, one checks immediately that $n_{C^{(p,q)}} := n_C(pq)n_C$ is a non-singular quadratic form acting on $C^{(p,q)}$ and permitting composition. It follows that the left multiplication operator $L_{pq} \colon C^{(p,q)} \to C$, $x \mapsto (pq)x$, is an isometry from $n_{C^{(p,q)}}$ to n_C preserving units but almost never an isomorphism. McCrimmon [11] also provides examples showing that isotopes of alternative algebras are in general not isomorphic. But these examples display pathologies in characteristic 3, for which alternative algebras are notorious, and it has long been suspected that examples avoiding such pathologies do not exist. This suspicion, however, is confounded by the following results.

t.NOREQ

4. Theorem. (a) (Gille [6]) *There exist non-isomorphic octonion algebras over appropriate commutative base rings whose norms are isometric.*

(b) (Alsaody-Gille [2]) *Two octonion algebras C and C' over k have isometric norms if and only if C' is isomorphic to an isotope of C.* □

With regard to (a), Gille actually proves much more: *there exist octonion algebras over appropriate base rings which are hyperbolic as quadratic spaces but not split as octonion algebras.*. By (b), therefore, they are isotopic, but *not* isomorphic, to the algebra of Zorn vector matrices. All these remarkable results are obtained with a heavy dose of group scheme theory attached. Possibly for this reason, the examples corroborating part (a) of Thm. 4 are not particularly explicit. On the other hand, it would be nice to have examples at your disposal you could really put your hands on.

Up to now, such examples have not been found. For instance, by a result of Van der Blij-Springer [15], octonion algebras over the integers are either split or isomorphic to the Coxeter octonions, so the isotopes of the latter are all isomorphic. In view of this, a natural question presents itself.

pr.OCNUFI

5. Problem. *Let K be an algebraic number field and let $\mathfrak{o} \subseteq K$ be its ring of integers. Classify the octonion algebras over $\mathfrak{o}$,* (i) *up to isomorphism,* (ii) *up to isotopy, and decide whether the classification problems* (i), (ii) *are the same.*

You are cordially invited to try your luck with this problem.

Acknowledgements. I am grateful to Winfried Hochstättler for having provided me with a short outline of Kirmse's early life through the appropriate link to the website of Deutsche Mathematiker-Vereinigung. Also, I would like to express my gratitude to Achim Schneider† for having gone out of his way to provide me with hard-copies of [8, 10].

References

MR1716786 [1] D. Allcock, *Ideals in the integral octaves*, J. Algebra **220** (1999), no. 2, 396–400. MR 1716786

AlGi17 [2] S. Alsaody and P. Gille, *Isotopes of octonion algebras and triality*, arXiv 1704.05229vl (2017).

MR1662447 [3] J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, Springer-Verlag, New York, 1999. MR 1662447

MR0019111 [4] H. S. M. Coxeter, *Integral Cayley numbers*, Duke Math. J. **13** (1946), 561–578. MR 0019111 (8,370b)

Di23 [5] L.E. Dickson, *A new simple theory of hypercomplex integers*, J. Math. Pures Appl. **9** (1923), 281–326.

MR3194176 [6] P. Gille, *Octonion algebras over rings are not determined by their norms*, Canad. Math. Bull. **57** (2014), no. 2, 303–309. MR 3194176

zbMATH02675822 [7] A. Hurwitz, *Ueber die Zahlentheorie der Quaternionen.*, Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. **1896** (1896), 314–340 (German).

Ki [8] J. Kirmse, *Über die Darstellbarkeit natürlicher ganzer Zahlen als Summen von acht Quadraten und über ein mit diesem Problem zusammenhängendes nichtkommutatives und nichtassoziatives Zahlensystem*, Sächs. Akad. Wissensch. Leipzig **76** (1924), 63–82.

KoZo [9] A. Korkine and G. Zolotarev, *Sur les formes quadratiques positives*, Math. Ann. **11** (1877), 242–292.

MR0007745 [10] K. Mahler, *On ideals in the Cayley-Dickson algebra*, Proc. Roy. Irish Acad. Sect. A. **48** (1942), 123–133. MR 0007745 (4,185e)

MR0313344 [11] K. McCrimmon, *Homotopes of alternative algebras*, Math. Ann. **191** (1971), 253–262. MR 0313344 (47 #1899)

MR3725090 [12] H.P. Petersson, *The non-orthogonal Cayley-Dickson construction and the octonionic structure of the E_8-lattice*, J. Algebra Appl. **16** (2017), no. 12, 1750230, 52. MR 3725090

P14 [13] H.P. Petersson and M.L. Racine, *Octonions and Albert algebras over commutative rings*, Monograph in progress.

Smi [14] H.J.S. Smith, *On the orders and genera of quadratic forms containing more than three indeterminates*, Proc. Royal Soc. **16** (1867), 197–208.

MR0152555 [15] F. Van der Blij and T. A. Springer, *The arithmetics of octaves and of the group G_2*, Nederl. Akad. Wetensch. Proc. Ser. A 62 = Indag. Math. **21** (1959), 406–418. MR 0152555 (27 #2533)