On a class of exotic cubic polynomial laws

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In this short note we will be concerned with non-zero cubic polynomial laws $g\colon V \to W$ between vector spaces V, W over a field k such that the set maps $g_k\colon V \to W$ are (identically) zero. A complete characterization of these objects may be found in Thm. 7 below. Along the way towards proving this result, a few standard properties of cubic polynomial laws over arbitrary commutative rings will be derived in an ad-hoc manner.

1. Expansion formulas for cubic maps. Let k be a commutative ring and $f\colon M \to N$ with k-modules M, N a cubic polynomial law over k, so f is a polynomial law in the sense of Roby [2] (or Petersson-Racine [1, 3.1]) and homogeneous of degree 3 at the same time. For $R \in k$-**alg** and $x, y \in M_R$ we put

$$f(x, y) = (Df)(x, y), \tag{1}$$

which is bi-homogeneous of degree $(2, 1)$ since $Df = \Pi^{(2,1)}f$ (by [1, 3.1.9]) has this property in view of [1, 3.1.5]. Moreover, [1, (3.1.13)] yields

$$(D^2 f)(x, y) = f(y, x), \tag{2}$$

and, with a variable $\mathbf{t}$, the Taylor expansion [1, (3.1.10)] attains the form

$$f(x + \mathbf{t}y) = f(x) + \mathbf{t}f(x, y) + \mathbf{t}^2 f(y, x) + \mathbf{t}^3 f(y). \tag{3}$$

Finally, the evaluation of the total linearization of f at x, y, z will be abbreviated as

$$f(x, y, z) = \big(\Pi^{(1,1,1)}f\big)(x, y, z), \tag{4}$$

which is trilinear and totally symmetric in its arguments.

2. Lemma. *With the assumptions and notations of* **1.**, *we have*

$$f(x + y, z) = f(x, z) + f(x, y, z) + f(y, z) \tag{5}$$

for all $x, y, z \in M_R$, $R \in k$-**alg**.

Proof. Combining [1, Lemma 3.1.2] for $n = 2$, $p = 1$ under the specialization $\mathbf{t}_j \mapsto 1$ $(1 \le j \le n)$ with [1, 3.1.6] and (1), we obtain

$$\begin{aligned} f(x + y, z) = (Df)(x + y, z) &= \sum_{\nu \in \mathbb{N}_0^2} \big(\Pi^{(\nu,1)}f\big)(x, y, z) = \sum_{\nu \in \mathbb{N}_0^2, |\nu| = 2} \big(\Pi^{(\nu,1)}f\big)(x, y, z) \\ &= \big(\Pi^{(2,0,1)}f\big)(x, y, z) + \big(\Pi^{(1,1,1)}f\big)(x, y, z) + \big(\Pi^{(0,2,1)}f\big)(x, y, z). \end{aligned}$$

Here $(\Pi^{(1,1,1)}f)(x, y, z) = f(x, y, z)$ by (4) gives the middle term on the right-hand side of (5). On the other hand, [1, (3.1.5)] for $n = 3$ under the specialization $\mathbf{t}_1 \mapsto \mathbf{s}$, $\mathbf{t}_2 \mapsto 0$, $\mathbf{t}_3 \mapsto \mathbf{t}$ shows that $(\Pi^{(2,0,1)}f)(x, y, z)$ is the coefficient of $\mathbf{s}^2\mathbf{t}$ in the expansion of $f(\mathbf{s}x + \mathbf{t}z)$, hence by (3) agrees with $f(x, z)$. Similarly, specializing $\mathbf{t}_1 \mapsto 0$, $\mathbf{t}_2 \mapsto \mathbf{s}$, $\mathbf{t}_3 \mapsto \mathbf{t}$ in [1, (3.1.5)] for $n = 3$ identifies $(\Pi^{(0,2,1)}f)(x, y, z)$ with the coefficient of $\mathbf{s}^2\mathbf{t}$ in the expansion of $f(\mathbf{s}y + \mathbf{t}z)$, hence with $f(y, z)$. The lemma follows. ∎

3. Corollary. *For $n \in \mathbb{N}$, $v_1, \ldots, v_n, v \in M_R$, $R \in k$-**alg**, we have*

$$f\big(\sum_{i=1}^{n} v_i, v\big) = \sum_{i=1}^{n} f(v_i, v) + \sum_{1 \le i < j \le n} f(v_i, v_j, v).$$

Proof. By induction on n. For $n = 1$, there is nothing to prove. For $n > 1$, Lemma 2 and the induction hypothesis yield

$$\begin{aligned} f\big(\sum_{i=1}^{n} v_i, v\big) &= f\big(\sum_{i=1}^{n-1} v_i + v_n, v\big) = f\big(\sum_{i=1}^{n-1} v_i, v\big) + f\big(\sum_{i=1}^{n-1} v_i, v_n, v\big) + f(v_n, v) \\ &= \sum_{i=1}^{n-1} f(v_i, v) + f(v_n, v) + \sum_{1 \le i < j < n} f(v_i, v_j, v) + \sum_{i=1}^{n-1} f(v_i, v_n, v) \\ &= \sum_{i=1}^{n} f(v_i, v) + \sum_{1 \le i < j \le n} f(v_i, v_j, v), \end{aligned}$$

as claimed □

4. Corollary. *For all $x, y, z \in M_R$, $R \in k$-**alg**, we have*

$$f(x, y, z) = f(x + y + z) - f(x + y) - f(y + z) - f(z + x) + f(x) + f(y) + f(z).$$

Proof. Expanding the right-hand side by using (3) and Lemma 2, we obtain

$$\begin{aligned} f(x+y) + f(x+y, z) + f(z, x+y) + f(z) - f(x+y) - f(y) - f(y,z) - \\ f(z,y) - f(z) - f(z) - f(z,x) - f(x,z) - f(x) + f(x) + f(y) + f(z) &= \\ f(x,z) + f(x,y,z) + f(y,z) + f(z,x) + f(z,y) - \\ f(y,z) - f(z,y) - f(z,x) - f(x,z) &= f(x,y,z). \end{aligned}$$

□

5. Proposition. *With the assumptions and notations of **1.**, we have*

$$f\big(\sum_{i=1}^{n} r_i v_i\big) = \sum_{i=1}^{n} r_i^3 f(v_i) + \sum_{1 \le i,j \le n, i \ne j} r_i^2 r_j f(v_i, v_j) + \sum_{1 \le i < j < l \le n} r_i r_j r_l f(v_i, v_j, v_l)$$

*for all $n \in \mathbb{N}$, $r_1, \ldots, r_n \in R$, $v_1, \ldots, v_n \in M_R$, $R \in k$-**alg**.*

Proof. Again by induction on n, the case $n = 1$ again being obvious. For $n > 1$, we combine the induction hypothesis with the Taylor expansion (3) and Cor. 3 to obtain

$$\begin{aligned} f\big(\sum_{i=1}^{n} r_i v_i\big) &= f\big(\sum_{i=1}^{n-1} r_i v_i + r_n v_n\big) = f\big(\sum_{i=1}^{n-1} r_i v_i\big) + r_n f\big(\sum_{i=1}^{n-1} r_i v_i, v_n\big) + r_n^2 f\big(v_n, \sum_{i=1}^{n-1} r_i v_i\big) + r_n^3 f(v_n) \\ &= \sum_{i=1}^{n-1} r_i^3 f(v_i) + r_n^3 f(v_n) + \sum_{1 \le i,j < n, i \ne j} r_i^2 r_j f(v_i, v_j) + \sum_{1 \le i < j < l < n} r_i r_j r_l f(v_i, v_j, v_l) \\ &\quad + \sum_{i=1}^{n-1} r_i^2 r_n f(v_i, v_n) + \sum_{1 \le i < j < n} r_i r_j r_n f(v_i, v_j, v_n) + \sum_{i=1}^{n-1} r_i r_n^2 f(v_n, v_i) \\ &= \sum_{i=1}^{n} r_i^3 f(v_i) + \sum_{1 \le i,j \le n, i \ne j} r_i^2 r_j f(v_i, v_j) + \sum_{1 \le i < j < l \le n} r_i r_j r_l f(v_i, v_j, v_l), \end{aligned}$$

again as claimed. □

6. Notations and conventions. We now assume that we are given a free k-module V of finite rank $n > 0$, with basis $(e_i)_{1\leq i\leq n}$. We use this basis to identify V with n-dimensional column space k^n, which in turn will be viewed as the split étale k-algebra of rank n under the componentwise multiplication. Given another k-module W and a matrix $S = (s_{ij}) \in \mathrm{Mat}_n(W)$, we obtain an induced bilinear map

$$\langle S\rangle\colon\ V \times V \longrightarrow W, \quad (x, y) \longmapsto \langle S\rangle(x, y) := x^t S y,$$

where in explicit "co-ordinate" terms

$$x^t S y = \sum_{i,j=1}^{n} \alpha_i s_{ij} \beta_j = \sum_{i,j=1}^{n} \alpha_i \beta_j s_{ij} \in W \qquad (x = \begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix},\ y = \begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix} \in V = k^n).$$

The usual formalism of matrix multiplication obviously prevails also in this more general set-up. In particular, the matrix S is *alternating*, i.e., skew-symmetric with zeroes down the diagonal, if and only if $\langle S\rangle\colon\ V \times V \to W$ is an alternating bilinear map.

In dealing with polynomial laws $f\colon\ M \to N$ over k, we have allowed ourselves so far the notational laxity of writing the induced set maps $M_R \to N_R$, $R \in k$-**alg**, simply as f. For greater clarity, this laxity will not be tolerated anymore, so from now on we will consistently use the elaborate notation $f_R\colon\ M_R \to N_R$.

7. Theorem. *With the notations and conventions of* **6.**, *assume k is a field and let $g\colon\ V \to W$ be a cubic polynomial law over k. Then the following conditions are equivalent.*

(i) *The set map $g_k\colon\ V \to W$ is zero, but g itself is not.*

(ii) *We have $k = \mathbb{F}_2$,*

$$g_k(e_i) = 0, \quad g_k(x, x) = 0, \quad g_k(x, y, z) = 0 \qquad (1 \leq i \leq n, x, y, z \in V) \tag{6}$$

and there are $x_0, y_0 \in V$ such that $g_k(x_0, y_0) \neq 0$.

(iii) *We have $k = \mathbb{F}_2$ and the exists a non-zero alternating matrix $S \in \mathrm{Mat}_n(W)$ such that*

$$g_R(x) = x^t S_R x^2 \tag{7}$$

*for all $x \in R^n = \mathbb{F}_2^n \otimes R = V_R$, $R \in \mathbb{F}_2$-**alg**.*

Proof. (i) $\Rightarrow$ (ii). The first relation in (6) is obvious, as is the last one, by Cor. 4. For the middle one, we use Euler's differential equation and obtain $g(x, x) = (Dg)(x, x) = 3g(x) = 0$ for all $x \in V$, as claimed. It remains to show $k = \mathbb{F}_2$ and the final statement of (ii). Given $x, y, z \in V$, we obtain $g_k(x) = g_k(y) = g_k(x + y) = 0$ by (i), and (3) for $\mathbf{t} \mapsto \alpha \in k^\times$ yields

$$g_k(x, y) + \alpha g_k(y, x) = 0 \qquad (\alpha \in k^\times). \tag{8}$$

Assuming k contains more than two elements, (8) implies $g_(x, y) = 0$ for all $x, y \in V$, which in turn implies $g_R(\sum r_i e_i) = 0$ for all $r_1, \dots, r_n \in R$, $R \in k$-**alg** by Prop. 5, and we conclude that the cubic polynomial law g is zero. This contradiction shows not only $k = \mathbb{F}_2$ but also $g_k(x_0, y_0) \neq 0$ for some $x_0, y_0 \in V$.

(ii) $\Rightarrow$ (iii). By (6) and Lemma 2, the map $V \times V \to W$, $(x, y) \mapsto g(x, y)$, is $\mathbb{F}_2$-bilinear and alternating, so we obtain in

$$S := \big(g(e_i, e_j)\big)_{1\leq i,j\leq n} \in \mathrm{Mat}_n(W)$$

an alternating matrix, which by (ii) is non-zero. Combining Prop. 5 with (6), we therefore conclude, for

$$x = \begin{pmatrix}r_1\\ \vdots\\ r_n\end{pmatrix} = \sum_{i=1}^{n} r_i e_i \in R^n = \mathbb{F}_2^n \otimes R = V_R, \qquad (r_1, \dots, r_n \in R),$$

that

$$
\begin{aligned}
g_R(x) = g_R\big(\sum_{i=1}^{n} r_i e_i\big) &= \sum_{i=1}^{n} r_i^2 g(e_i)_R + \sum_{1\le i,j\le n, i\ne j} r_i^2 r_j g(e_i,e_j)_R + \sum_{1\le i<j<l\le n} r_i r_j r_l g(e_i,e_j,e_l)_R \\
&= \sum_{1\le i,j\le n} r_i g(e_i,e_j)_R r_j^2 = x^t S_R x^2,
\end{aligned}
$$

as desired.

(iii) ⇒ (i). The elements of the $\mathbb{F}_2$-algebra $\mathbb{F}_2^n$ are idempotents, which implies $g_k(x) = g_{\mathbb{F}_2}(x) = x^t Sx = 0$ for all $x \in V$ since S is alternating. Thus the set map $g_k\colon V \to W$ is zero. On the other hand, there are $x_0, y_0 \in V$ such that $x_0^t S y_0 \ne 0$, and passing from $k = \mathbb{F}_2$ to $K = \mathbb{F}_4 = \mathbb{F}_2(\theta)$, $\theta^2 = \theta + 1$, we deduce

$$
\begin{aligned}
g_K(x_0+\theta y_0) &= (x_0+\theta y_0)^t S(x_0+\theta y_0)^2 = (x_0+\theta y_0)^t S(x_0^2+\theta^2 y_0^2) = (x_0+\theta y_0)^t S(x_0+\theta^2 y_0) \\
&= (x_0+\theta y_0)^t S(x_0+\theta y_0) + (x_0+\theta y_0)^t S y_0 \\
&= x_0^t S y_0 \ne 0.
\end{aligned}
$$

Thus the set map $g_K\colon V_K \to W_K$ is not zero, forcing g to be non-zero as well, and we have (i). □

8. Remark. In the course of establishing the implication (iii) ⇒ (i) above, we have shown that a polynomial law g satisfying the conditions of the theorem does not vanish as a set map from V_K to W_K, $K = \mathbb{F}_4$. Actually, this is part of the result: for any *proper* extension field L of $k = \mathbb{F}_2$, the extension $g \otimes L\colon V_L \to W_L$ is a non-zero cubic polynomial law over L which, thanks to Thm. 7, cannot induce the zero set map from V_L to W_L since the base field L is *not* $\mathbb{F}_2$.

References

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