Experiments in cubic systems

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The expermients I would like to discuss in this lecture have been designed in order to overcome certain difficulties that arise when dealing with the two Tits constructions of cubic Jordan algebras over arbitrary commutative rings. The key notion underlying both constructions has been introduced by Kevin McCrimmon [3] under the name "cubic forms with adjoint and base point". Later on, Michel Racine and I investigated the two Tits constructions more closely [6, 7] and, in the process of doing so, the name "cubic forms with adjoint and base point" was changed to "cubic norm structures". It should be added, however, that this shift in terminology was due neither to Michel nor to me but instead to the referee, who in actual fact was none other than Kevin McCrimmon himself. I therefore feel vindicated to use this latter terminology from now on, So let us begin by talking about

1. Cubic norm structures.

In order to do so, I start out from an arbitrary commutative associative ring of scalars, denoted by k, and write k-**mod** for the category of k-modules, k-**alg** for the category of unital commutative associative k-algebras, morphisms being k-algebra homomorphisms taking 1 into 1. For $M \in k$-**mod** and $R \in k$-**alg**, I always write $M_R := M \otimes R$, viewed as an R-module.

By a *cubic norm structure* over k I mean a quadruple

$$X = (X, 1 = 1_X, \sharp = \sharp_X, N = N_X)$$

consisting of the following components.

- The first component, also denoted by X, is an ordinary k-module.
- The second component, denoted by $1 = 1_X$ and called the *base point* of X, is a distinguished element of X which I assume to be unimodular: there exists a linear form λ on X taking the base point to the unit element of the base ring.
- The third component, denoted by $\sharp = \sharp_X$ and called the *adjoint* of X, is a quadratic map $X \to X$, $x \mapsto x^\sharp$, with bilinearization

$$x \times y = (x + y)^\sharp - x^\sharp - y^\sharp.$$

- The fourth component, denoted by $N = N_X$ and called the *norm* of X, is a cubic form in the sense of Roby [8], i.e., it is a family of set maps $N_R : X_R \to R$, one for each $R \in k$-**alg**, that are homogeneous of degree 3 and vary functorially with R.

For such a quadruple to be a cubic norm structure over k it is necessary and sufficient that a bunch of identities hold in all scalar extensions. Rather than writing them all down, I confine myself to the most important one, namely, the *adjoint identity*

$$x^{\sharp\sharp} = N(x)x,$$

which, as I just said, is assumed to hold in all scalar extensions of X.

Every cubic norm structure X as above allows a natural symmetric bilinear form

$$T := T_X : X \times X \longrightarrow k,$$

called its *bilinear trace*. An element $x \in X$ is said to be *invertible* if $N(x) \in k^\times$ is a unit in the base ring. The set of invertible elements will be denoted by

$$X^\times := \{x \in X \mid x \text{ is invertible}\} = \{x \in X \mid N(x) \in k^\times\}.$$

Homomorphisms of cubic norm structures are linear maps preserving base points, adjoints and norms in the obvious sense. Examples of cubic norm structures arise naturally from generically algebraic Jordan algebras of degree 3 over fields or, more generally, from generically algebraic Jordan algebras of degree 3 in the sense of Loos [2] over arbitrary commutative rings.

The two Tits constructions are about building up big cubic norm structures that are sufficiently regular out of smaller ones. Therefore we will have to talk about non-singularity and cubic sub-norm structures.

A cubic norm structure X over k is said to be *non-singular* if it is finitely generated projective as a k-module and its bilinear trace is *non-singular* in the sense that it induces an isomorphism from the k-module X onto its dual in the usual way. The condition on the module structure of X ensures that the property of being non-singular is stable under arbitrary base change.

We say $X_0 \subseteq X$ is a *cubic sub-norm structure* if

(i) X_0 is a cubic norm structure,

(ii) $X_0 \subseteq X$ is a submodule,

(iii) The inclusion $X_0 \hookrightarrow X$ is a homomorphism of cubic norm structures.

We now turn to

2. The two Tits constructions in their classical habitat.

This is provided by the assumption that the base ring k is, in fact, a field.

2.1. The first Tits construction. Its *input* consists of an associative k-algebra A of degree 3, with generic norm N and generic trace T, and a scalar $\mu \in k^\times$. Its *output* is a cubic norm structure $X := \mathcal{T}_1(A, \mu)$, where

$$\begin{aligned} X &= A \oplus Aj_1 \oplus Aj_2 \in k\text{-}\mathbf{mod}, && (1)\\ 1_X &= 1_A + 0j_1 + 0j_2, && (2)\\ u^\sharp &= (x_0^\sharp - \mu x_1 x_2) + (\mu x_2^\sharp - x_0 x_1)j_1 + (x_1^\sharp - x_2 x_0)j_2, && (3)\\ N_X(u) &= N(x_0) + \mu N(x_1) + \mu^2 N(x_2) - \mu T(x_0 x_1 x_2) && (4) \end{aligned}$$

for $u = x_0 + x_1 j_1 + x_2 j_2 \in X_R$, $R \in k$-**alg**.

2.2. The second Tits construction. Here the input is more complicated. It consists of

(i) an *associative k-algebra of degree* 3 *with involution of the second kind*, i.e., of a triple (K, B, N), where

  (a) $K \in k$-**alg** is *quadratic étale*, i.e., a composition algebra of dimension 2, with conjugation ι_K, $a \mapsto \bar{a}$,

  (b) B is an associative K-algebra of degree 3, with generic norm N and generic trace T,

  (c) $\tau : B \to B$ is an ι_K-linear algebra involution,

(ii) an *admissible pair* (p, μ), where $p \in H(B, \tau)$, $\mu \in K$ are invertible and satisfy $N(p) = \mu\bar{\mu}$.

Then the *output* is a cubic norm structure

$$X := \mathcal{T}_2(K, B, \tau, p, \mu),$$

where

$$X = H(B, \tau) \oplus Bj \in k\text{-}\mathbf{mod}, \tag{5}$$

$$1_X = 1_B + 0j, \tag{6}$$

$$u^\sharp = \big(x_0^\sharp - xp\tau(x)\big) + \big(\bar{\mu}\tau(x)^\sharp p^{-1} - x_0 x\big)j, \tag{7}$$

$$N_X(u) = N(x_0) + \mu N(x) + \bar{\mu}\,\overline{N(x)} - T\big(x_0, xp\tau(x)\big) \tag{8}$$

for $u = x_0 + xj \in X_R$, $R \in k$-**alg**.

In the 1980's, Michel and I were quite enthusiastic about the analogy between the two Tits constructions of cubic norm structures on the one hand and the Cayley Dickson construction of composition algebras on the other. Today I am distinctly less so, and this primarily for the following reason: composition algebras are well known and easily seen to satisfy the

2.3. Embedding property. *Suppose we are given*

- *a composition algebra C over k (any commutative ring), with norm n_C,*
- *a composition subalgebra $B \subseteq C$, for simplicity assumed to satisfy the condition*
$$\mathrm{rk}(B) = \frac{1}{2}\mathrm{rk}(C),$$
- *an invertible element $l \in C$ that is perpendicular to B at the same time.*

Then the inclusion $B \hookrightarrow C$ has a unique extension to an isomorphism

$$\mathrm{Cay}\big(B, n_C(l)\big) = B \oplus Bj \xrightarrow{\sim} C$$

sending j to l.

But the analogue of this result for cubic norm structures is known only in a special form and in special cases. Moreover, the proof relies heavily either on excessive computations [5] or a substantial amount of Jordan theory [3, 6] that is not readily available over arbitrary commutative rings.

In what follows, I would like to sketch an approach to the two Tits constructions that I hope will eventually lead to a natural proof of the embedding property. It will also work more generally for alternative rather than associative algebras of degree 3. While this turns out to be straightforward for the first Tits construction, it is distinctly less so for the second. There is, of course, an easy way out in the alternative setting by stipulating that the element p in 2.2 (ii) belong to the nucleus of B. But the nucleus of the standard examples of alternative algebras of degree 3 that are not associative is very small, so this kind of generalization doesn't look very interesting.

3. Cubic alternative algebras and pure elements.

We now return to our arbitrary commutative associative ring k of scalars and keep this amount of generality until the very end. By a *cubic alternative k-algebra* we mean a pair (A, N) such that

(i) A is a unital alternative algebra over k and $1_A \in A$ is unimodular,

(ii) $N : A \to k$ is a cubic form (again in the sense of Roby [8]), called the *norm* of (A, N), that satisfies the conditions

$$N(1_A) = 1,$$
$$N(xy) = N(x)N(y),$$
$$\text{the cubic polynomial } N(\mathbf{t}1_A - x) \text{ kills } x$$

strictly, i.e., in all scalar extensions.

Every cubic alternative algebra (A, N) over k is easily seen to carry

- a natural linear trace $T : A \to k$, and
- a natural quadratic adloint $\sharp : A \to A$

in such a way that

$$(A, N)^+ := (A \in k\text{-}\mathbf{mod}, 1_A, \sharp, N)$$

is a cubic norm structure over k, with bilinear trace given by $T(xy)$.

3.1. Theorem. (Petersson-Racine [7], Faulkner [1]) *Given a cubic alternative k-algebra (A, N) and a scalar $\mu \in k$, the relations* (1)−(4) *define a cubic norm structure $X = \mathcal{T}_1(A, N, \mu)$ over k with $(A, N)^+ \subseteq \mathcal{T}_1(A, N, \mu)$ through the initial summand.*

We now wish, conversely, to recover the first Tits construction as in Thm. 3.1 from an abstract setting consisting of a cubic sub-norm structure X_0 sitting inside a bigger cubic norm structure X.

3.2. Orthogonal complements. Suppose X is a cubic norm structure over k and $X_0 \subseteq X$ is a *non-singular* cubic sub-norm structure. Then the orthogonal splitting

$$X = X_0 \oplus V, \quad V := X_0^{\perp},$$

with respect to the bilinear trace of X comes quipped with two additional structural ingredients: there is a canonical bilinear action

$$X_0 \times V \longrightarrow V, \quad (x_0, v) \longmapsto x_0 \,.\, v := -x_0 \times v,$$

and there are quadratic maps $Q : V \to X_0$, $H : V \to V$ given by

$$v^\sharp = -Q(v) + H(v) \qquad (v \in V).$$

With these ingredients, an element $l \in X$ is said to be *pure* relative X_0 if

(i) $l \in V$ is invertible in X,

(ii) $l^\sharp \in V$ (equivalently, $Q(v) = 0$),

(iii) $X_0 \,.\, (X_0 \,.\, l) \subseteq X_0 \,.\, l$.

If this is so, we can give the k-module X_0 the structure of a well defined non-associative k-algebra

$$A_X(X_0, l) := \big(X_0 \in k\text{-}\mathbf{mod},\ (x_0 y_0) \,.\, l := x_0 \,.\, (y_0 \,.\, l)\big).$$

3.3. Theorem. *With the notations and assumptions of* 3.2,

(a) $((A_X(X_0, l), N_{X_0})$ *is a cubic alternative algebra over* k *with*

$$\big(A_X(X_0, l), N_{X_0}\big)^+ = X_0.$$

(b) *The inclusion* $X_0 \hookrightarrow X$ *has a unique extension to a homomorphism*

$$\mathcal{T}_1\big(A_X(X_0, l), N_{X_0}, N_X(l)\big) \longrightarrow X$$

of cubic norm structures that sends j_1 *to* l.

Among the conditions defining the notion of a pure element in 3.2, (iii) is the most delicate. It is therefore important to note that, under some additional "global" hypotheses involving X and X_0 but not l, it holds automatically once the other two are valid.

3.4. Theorem. *With the notations and assumptions of* 3.2, *suppose* X_0 *has rank* n *and* X *is finitely generated projective as a* k*-module of rank at most* $3n$. *Then, if* $l \in V$ *is invertible in* X *and has* $l^\sharp \in V$ *as well,* l *is pure relative to* X_0 *and the homomorphism of Thm.* 3.3 *is an isomorphism.*

We now turn the second Tits construction in the setting of cubic alternative algebras, which requires considerably more effort.

4. Isotopy involutions and étale elements.

The key notion that we will have to utilize here is again due to McCrimmon [4], but wil be needed here only in a special case.

4.1. Unital isotopes of alternative algebras. Let A be a unital alternative algebra over k and $p, q \in A^\times$. Then the k-algebra

$$A^p := \big(A \in k\text{-}\mathbf{mod}, (x, y) \longmapsto (xp^{-1})(py)\big)$$

is again unital alternative with

- $1_{A^p} = 1_A$,
- $\mathrm{Jord}(A^p) = \mathrm{Jord}(A)$,
- $(A^p)^q = A^{pq}$,
- $A^p = A$ if A is associative.

4.2. Isotopy involutions of the second kind. Let

$$\mathcal{B} := (K, B, N, p, \tau)$$

be a *cubic alternative k-algebra with isotopy involution of the second kind,* so

(i) $K \in k\text{-}\mathbf{alg}$ is quadratic étale, with conjugation ι_K, $a \mapsto \bar{a}$,

(ii) (B, N) is a cubic alternative algebra over K,

(iii) $p \in B^\times$,

(iv) $\tau : B \to (B^p)^{\mathrm{op}}$ is an ι_K-linear isomorphism satisfying

$$\tau(p) = p, \quad \tau^2 = \mathbf{1}_B, \quad N \circ \tau = \iota_K \circ N.$$

A scalar $\mu \in K$ is said to be *admissible* relative to $\mathcal{B}$ if $N(p) = \mu\bar{\mu}$. Condition (iv) implies in particular first $\tau(xy) = (\tau(y)p^{-1})(p\tau(x))$ and then

$$xp\tau(x) := x\big(p\tau(x)\big) \in H(\mathcal{B}) := H(B, \tau)$$

but NOT $(xp)\tau(x) \in H(\mathcal{B})$.

4.3. Theorem. *With $\mathcal{B}$ as in* 4.2, *suppose $\mu \in K$ is admissible relative to $\mathcal{B}$. Then the relations* (5)$-$(8) *define a cubic norm structure $\mathcal{T}_2(\mathcal{B}, \mu)$ over k with $H(\mathcal{B}) \subseteq \mathcal{T}_2(\mathcal{B}, \mu)$ through the initial summand.*

The proof of this result consists of some delicate computations.

4.4. Étale elements. Returning now to the set-up of 3.2, an element $v \in X$ is said to be *étale* relative to X_0 if it satisfies the conditions

$$v \in V, \quad Q(v) \in X_0^\times, \quad N_X(v)^2 - 4N_{X_0}\big(Q(v)\big) \in k^\times.$$

This implies that

$$K_v := k[\mathbf{t}]/\Big(\mathbf{t}^2 - N_X(v)\mathbf{t} + N_{X_0}\big(Q(v)\big)\Big)$$

is a quadratic étale k-algebra (hence the name) that is generated by an invertible element.

4.5. Theorem. *With the notations and assumptions of* 4.4, *suppose X_0 has rank n and $X \in k\text{-}\mathbf{mod}$ is finitely generated projective of rank at most $3n$. Then there are a cubic alternative k-algebra $\mathcal{B}$ with isotopy involution of the second kind as in* 4.2 *with $K = K_v$, an admissible scalar $\mu \in K$ relative to $\mathcal{B}$ and an isomorphism*

$$\mathcal{T}_2(\mathcal{B}, \mu) \xrightarrow{\sim} X$$

sending $H(\mathcal{B})$ to X_0 and j to v.

The proof of this theorem consists of

- extending scalars from k to $K = K_v$,
- exhibiting a pure element of X_K relative X_{0K},
- applying Thm. 3.3, and
- descending from K back to k.

5. The existence of étale elements.

The value of Thm. 4.5 hinges on existence criteria for étale elements. Here our results are even less complete than the previous ones. We begin with an important class of examples.

5.1. Hermitian matrices. Let C be a octonion algebra over k and $K \subseteq C$ a quadratic étale subalgebra. Then

$$C = K \oplus W, \quad W := K^{\perp},$$

and W is canonically a finitely generated projective (right) K-module of rank 3. Now suppose

$$\Gamma \in \mathrm{Diag}_3(k) \cap \mathrm{GL}_3(k)$$

is a 3×3 invertible diagonal matrix over k and put

$$X_0 := H_3(K, \Gamma) \subseteq H_3(C, \Gamma) =: X,$$

(where $H_3(?, \Gamma)$ refers to 3×3 Γ-hermitian matrices with scalars (in k) down the diagonal), first as generically algebraic Jordan algebras of degree 3 and then as cubic norm structures.

5.2. Theorem. *With the notations and assumptions of* 5.1, *the following conditions are equivalent.*

(i) *X contains étale elements relative to X_0.*

(ii) *$W \in K$-**mod** is free (of rank 3) and $K = k[a]$ for some invertible element $a \in K$.*

Combining Thm. 5.2 with a Zariski density argument and the fact that finite-dimensional absolutely simple Jordan algebras over a finite field are reduced, we obtain

5.3. Corollary. *Let k be a field, X the cubic norm structure corresponding to an Albert algebra J over k and $X_0 \subseteq X$ the cubic sub-norm structure corresponding to an absolutely simple nine-dimensional subalgebra $J' \subseteq J$. Then precisely one of the following holds.*

(a) *X contains étale elements relative to X_0.*

(b) *$J' \cong \mathrm{Jord}(\mathrm{Mat}_3(k))$ and $k = \mathbb{F}_2$.*

Moreover, if $J' \cong \mathrm{Jord}(\mathrm{Mat}_3(k))$, then X contains pure elements relative to X_0.

References

[1] J.R. Faulkner, *Finding octonion algebras in associative algebras*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1027–1030. MR 931729 (89g:17003)

[2] O. Loos, *Generically algebraic Jordan algebras over commutative rings*, J. of Algebra **297** (2006), 474–529.

[3] K. McCrimmon, *The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras*, Trans. Amer. Math. Soc. **139** (1969), 495–510. MR 0238916 (39 #276)

[4] ———, *Homotopes of alternative algebras*, Math. Ann. **191** (1971), 253–262. MR 0313344 (47 #1899)

[5] Holger P. Petersson and Maneesh Thakur, *The étale Tits process of Jordan algebras revisited*, J. Algebra **273** (2004), no. 1, 88–107. MR MR2032452 (2004m:17047)

[6] H.P. Petersson and M.L. Racine, *Classification of algebras arising from the Tits process*, J. Algebra **98** (1986), no. 1, 244–279. MR 87h:17038b

[7] ———, *Jordan algebras of degree* 3 *and the Tits process*, J. Algebra **98** (1986), no. 1, 211–243. MR 87h:17038a

[8] Norbert Roby, *Lois polynomes et lois formelles en théorie des modules*, Ann. Sci. École Norm. Sup. (3) **80** (1963), 213–348. MR MR0161887 (28 #5091)