Errata and addenda
for the book
Albert algebras over commutative rings

Version of May 3, 2026

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Contents

Errata for the published edition of *Albert algebras over commutative rings*

Page numbers refer to the published/paper edition of the book, published by Cambridge in November 2024. These typos have been corrected in the online/"draft" version you can find on the web. The online/draft versions have a date on their front cover, and the "fixed" date says that the typo has been corrected in all online/draft versions starting with that date.

This errata sheet was compiled on May 3, 2026.

Thank you to Alberto Elduque, Darij Grinberg, and Eric Rains for identifying some of the following.

If you find other typos, please do tell us! You can send us an email or use the contact form on Skip's website at this link.

Notation and conventions

Page xxii, line -17 (fixed 20 Feb 2025): The R in $M \times M \to R$ should be replaced by k.

Chapter I: Prologue: the ancient protagonists

Page 38, item 6.4, line 4 (fixed 16 Dec 2024): The U-operator maps $U\colon J \to \mathrm{End}(J)$. That is, the codomain is $\mathrm{End}(J)$ not J.

Chapter II: Foundations

Page 94, item 12.16, line -2 (fixed 9 Nov 2025): The (Df) on this line should be $(Df)_R$.

Page 95, item 12.17, line 1 after (8) (fixed 9 Nov 2025): Insert the following sentence before "Note": To verify each of (1)–(8), one can use (12.15.4) or (12.15.5) to expand both sides and equate coefficients of ε.

Page 103, item 12.36, line -1 (fixed 26 Nov 2025): The lower bound on p should read "$1 \leq p$" instead of "$0 \leq p$".

Page 104, item 12.39, line 2 (fixed 2 Jan 2025): Insert after the first sentence: (See for example [27[1], §VI.3.6] for background on discrete valuations on fields.)

Chapter IV: Composition algebras

Page 157, item 19.24, line 5 (fixed 16 Dec 2024): Replace $\mathrm{Her}_r(\mathbb{D})$ with $\mathrm{Her}_r(\mathbb{O})$.

Chapter V: Jordan algebras

Page 275, item 29.11, line 3 (fixed 20 Feb 2025): Replace "that that" with a single "that".

Chapter VI: Cubic Jordan algebras

Page 344, item 34.25, line -2 (fixed 19 Jan 2026): Replace "in (c)" with "in (b)".

Chapter VII: The two Tits constructions

Page 458, item 42.18, line 1 (fixed 16 Jan 2025): The first sentence should read: Let $J = J(D, \mu)$ be an Albert algebra arising from a first Tits construction over a field F of characteristic $\neq 3$.

Page 495, item 45.1, line -2 (fixed 31 Dec 2025): Delete the phrase "of the second kind".

Page 509, item 46.6 proof, line 3 (fixed 16 Dec 2024): Should refer to Lemma 46.5, not Proposition.

[1] For the convenience of the reader: [27] in the printed book = [Bou72].

Chapter VIII: Lie algebras

Page 527, line 1 (fixed 20 Feb 2025): The first sentence should read: Let V be the subspace of $E = \mathbb{R}^8$ whose points have coordinates (ξ_i) such that $\xi_6 = \xi_7 = -\xi_8$.

(That is, replace every ϵ in the printed version with a ξ, to align with the notation in [Bou02].)

Page 529, item 47.17, line 4 of the remark (fixed 20 Feb 2025): Replace "very" with "every".

Page 565, 51.29 (fixed 20 Feb 2025): The statement of part (a) should also include the hypothesis $2 \in k^\times$, in order for the proof provided to be sufficient. The proof needs that the map $\mathfrak{g}_0 \to \mathfrak{o}(n_C)$ is bijective from 51.26(b), which relies on $2 \in k^\times$.

The claim in part (a), that $\mathrm{Der}(J)$ is finitely generated projective of rank 52, does hold in greater generality. Here is a proof using the material from the next chapter, Chapter IX.

First suppose that k is a field, in which case we only need to prove that $\dim(\mathrm{Der}(J)) = 52$. By 52.1(b), $\mathrm{Lie}(\mathbf{Aut}(J)) = \mathrm{Der}(J)$. By Theorem 53.4, $\mathbf{Aut}(J)$ is semisimple of type F_4, so $\dim \mathrm{Lie}(\mathbf{Aut}(J)) = \dim \mathbf{Aut}(J)$ because $\mathbf{Aut}(J)$ is smooth and $\dim \mathbf{Aut}(J) = 52$ because type F_4.

Next suppose that J is the split Albert algebra $\mathrm{Her}_3(\mathrm{Zor}(k))$ over a principal ideal domain k. Since $\mathrm{Zor}(k)$ is a finitely generated free module, so is $\mathrm{End}(\mathrm{Zor}(k))$; since additionally k is a principal ideal domain we conclude that the submodule $\mathrm{Der}(J)$ is also finitely generated and free [Sta18, Tag 0AUW]. The field of fractions F of k is a flat extension of k, so $\mathrm{Der}(J)_F \cong \mathrm{Der}(J_F)$ by Prop. 50.4, and this has dimension 52 by the previous paragraph, hence $\mathrm{Der}(J)$ has rank 52 at the prime 0. Since k is connected, $\mathrm{Der}(J)$ has constant rank 52.

If J is any Albert algebra over a principal ideal domain k, then there is some faithfully flat $K \in k\text{-}\mathbf{alg}$ such that J_K is split. Then $\mathrm{Der}(J)_K \cong \mathrm{Der}(J_K) \cong \mathrm{Der}(\mathrm{Her}_3(\mathrm{Zor}(K))) \cong \mathrm{Der}(\mathrm{Her}_3(\mathrm{Zor}(k))) \otimes K$ is finitely generated projective of rank 52 by the previous paragraph, and it follows that $\mathrm{Der}(J)$ is finitely generated projective of rank 52.

Page 557, item (51.10.2), line 1 (fixed 16 Dec 2024): The right side of the equation should read $\{(a_1, a_1^\top) \mid a_1 \in \mathrm{Mat}_3(k)\}$.

Page 566, alternative proof of 51.29(b), line 2 (fixed 20 Feb 2025): Replace $\mathrm{Zor}(k)$ with $\mathrm{Her}_3(\mathrm{Zor}(k))$.

Chapter IX: Group schemes

Page 567, item 52.1, paragraph 2, line -2 (fixed 22 Feb 2025): Replace G with $\mathbf{G}$.

Page 588, item proof of 54.6, line 3 (fixed 20 Feb 2025): Replace "(1 3 2)" by "(1 2)".

Page 593, item 54.11, line 2 (fixed 20 Feb 2025): Replace "structure" by "system".

Page 593, item 54.12, line 2 (fixed 20 Feb 2025): Replace "structure" by "system".

Page 603, 55.12 (fixed 29 Mar 2025): Replace the statement of the exercise with the following, clearer, version:

Let F be a field. Show that the maps $(A, \tau) \mapsto H(A, \tau)$ and $J \mapsto \mathbf{Aut}(J)$ define bijections between the isomorphism classes of

(i) Azumaya algebras (A, τ) of degree 3 with unitary involution over F, as in 44.23,
(ii) rank 9 Freudenthal F-algebras J, and
(iii) adjoint semi-simple F-group schemes of type A_2.

Page 619, bottom of page (fixed 20 Feb 2025): Insert new paragraph: Results for groups of type E_7 — analogous to the results in this section for type E_6 — can be found in sections 16 and 17 of [95[2]].

Subject Index

Page 648, middle of left column (fixed 1 Dec 2025): Insert line: balanced pair, 470

[2] For the convenience of the reader: [95] in the printed book = [GPR23].

Addenda to the published edition of *Albert algebras over commutative rings*

This chapter contains material that could have been in the book and does not require substantial extra background.

A1 Albert algebras are exceptional

In his first paper on the subject of Jordan algebras, Albert [Alb34] showed that (1) the euclidean Albert algebra $\mathrm{Her}_3(\mathbb{O})$ over the reals is a (linear) Jordan algebra and (2) that it is exceptional. We have shown (1) in Theorem 5.10. More generally, for every octonion algebra C over every ring k, we have shown that $\mathrm{Her}_3(C)$ is a Jordan algebra in Theorem 36.5. (Note that Albert algebras are defined in the book to be cubic Jordan algebras, hence Jordan algebras.) In this addendum we would like to extend (2), the exceptionality of Albert algebras, to an arbitrary ring k. We prove the following slightly stronger result.

A1.1 Theorem. *Every Albert algebra over every non-zero ring is i-exceptional.*

We will define the term i-exceptional in a moment; it is stronger than the notion of exceptional defined in 29.9. When the ring is a field of characteristic different from 2, Theorem A1.1 was first proved in [AP59]. A proof of Theorem A1.1 for every field can be found in Jacobson's Arkansas notes [Jac81, §2.5]. That proof considers separately the cases of fields of characteristic 2 and different from 2. We provide a proof that does not contain special considerations involving 2.

In the excluded case of the zero ring, all algebras are zero and so the notion of being exceptional or i-exceptional do not make sense.

Here is a beautiful application of the theorem.

A1.2 Theorem. *A central simple Jordan algebra over a field is exceptional if and only if it is an Albert algebra.*

Proof The central simple exceptional Jordan algebras over a field have been classified into types in §15 of [MZ88]. One finds that each type is either special or an Albert algebra. Albert algebras are exceptional by Theorem A1.1. □

We now provide the promised definition.

A1.3 Definition. A Jordan algebra J is *i-special* if J is a homomorphic image of a special algebra and *i-exceptional* if it is not.

Clearly any special Jordan algebra is i-special. However P.M. Cohn has given an example of an i-special Jordan algebra that is exceptional [Jac68, §I.3, Thm. 2]. Therefore being i-exceptional is stronger than being exceptional. We also have the following:

A1.4 Lemma. *Let J be a Jordan k-algebra. If J_R is i-exceptional for some flat $R \in k$-**alg**, then J is i-exceptional.*

Proof We prove the contrapositive. Suppose that J is not i-exceptional, i.e., there is a special Jordan algebra J' and a (surjective) homomorphism $f\colon J' \to J$. Hence there exists a unital associative algebra A and an injective Jordan homomorphism $j\colon J' \to A^{(+)}$. It follows that $f_R\colon J'_R \to J_R$ is still surjective and, by flatness, $j_R\colon J'_R \to (A_R)^{(+)}$ is still injective, i.e., J_R is i-special. □

As a first step in the proof of Theorem A1.1, we note the following, which could have appeared in 37.7.

A1.5 Lemma. *For $J = \mathrm{Her}_3(C)$, C a multiplicative conic alternative algebra, and $a[ij]$, $b[ji]$, $c[il] \in J$ with $\{i, j, k\} = \{1, 2, 3\}$,*

$$\{a[ij]\, b[ji]\, c[il]\} = a(bc)[il]. \tag{a1}$$

We remark that if C is associative, then (a1) follows immediately from matrix multiplication.

Proof Using (33.6.2), (36.4.6), (36.4.7),

$$\begin{aligned}\{a[ij]\, b[ji]\, c[il]\} &= T_J(a[ij], \bar{b}[ij])c[il] - (c[il] \times a[ij]) \times b[ji] \\ &= n_C(a, \bar{b})c[il] - \bar{a}c[jl] \times b[ji] \\ &= (n_C(a, \bar{b})c - \bar{b}(\bar{a}c))[il].\end{aligned}$$

Replace $\bar{a}$ and $\bar{b}$ using (16.5.4) and expand. By (16.5.5), Proposition 16.10 and alternativity, we obtain (a1). □

The following polynomial map was introduced by Glennie [Gle66]:

$$g_9(x,y,z) := U_x z \circ \{x\, U_z y^2\, y\} - U_y z \circ \{x\, U_z x^2\, y\} \\ - U_x U_z\{x\, U_y z\, y\} + U_y U_z\{x\, U_x z\, y\}.$$

It is homogeneous of degree 9 and skew-symmetric in x and y.

A1.6 Lemma. *Let C be a multiplicative conic alternative algebra. If C is not associative, then g_9 is not identically zero on* $\mathrm{Her}_3(C)$.

Proof Consider the following substitution,

$$x = 1[12], \quad y = 1[23], \quad z = a[12] + b[23] + c[31], \quad x,y,z \in \mathrm{Her}_3(C),$$

for a, b, $c \in C$. Using (16.12.2), (17.4.2), the Peirce decomposition rules (32.15), the identities of 37.7 and Lemma A1.5, one computes

$$x^2 = e_{11} + e_{22}, \quad y^2 = e_{22} + e_{33}, \quad U_x z = \bar{a}[12], \quad U_y z = \bar{b}[23].$$

Futhermore

$$\begin{aligned}
U_z x^2 &= n_C(a)e_{11} + n_C(a)e_{22} + (n_C(b) + n_C(c))e_{33} + \bar{a}\,\bar{c}[23] + \bar{b}\,\bar{a}[31].\\
U_z y^2 &= n_C(b)e_{22} + n_C(b)e_{33} + (n_C(a) + n_C(c))e_{11} + \bar{c}\,\bar{b}[12] + \bar{b}\,\bar{a}[31].\\
\{x\, U_z y^2\, y\} &= t_C(ab)e_{22} + \bar{c}\,\bar{b}[23] + n_C(b)1[31].\\
\{x\, U_z x^2\, y\} &= t_C(ab)e_{22} + \bar{a}\,\bar{c}[12] + n_C(a)1[31].\\
U_x z \circ \{x\, U_z y^2\, y\} &= t_C(ab)\bar{a}[12] + n_C(b)a[23] + (bc)a[31].\\
U_y z \circ \{x\, U_z x^2\, y\} &= n_C(a)b[12] + t_C(ab)\bar{b}[23] + b(ca)[31].\\
\{x\, U_y z\, y\} &= \bar{b}[12],\\
U_z\{x\, U_y z\, y\} &= n_C(b)t_C(c)e_{33} + aba[12] + n_C(b)\bar{a}[23] + (c\bar{b})\bar{a}[31].\\
U_x U_z\{x\, U_y z\, y\} &= \overline{aba}[12].\\
\{x\, U_x z\, y\} &= \bar{a}[23].\\
U_z\{x\, U_x z\, y\} &= n_C(a)t_C(c)e_{11} + n_C(a)\bar{b}[12] + bab[23] + \bar{b}(\bar{a}c)[31].\\
U_y U_z\{x\, U_x z\, y\} &= \overline{bab}[23].
\end{aligned}$$

Therefore

$$g_3(x,y,z) = (t_C(ab)\bar{a} - n_C(a)b - \overline{aba})[12] + (n_C(b)a - t_C(ab)\bar{b} + \overline{bab})[23] \\ + [b,c,a][31].$$

The [12] and [23] entries are 0 by (17.4.2). If C is not associative, one can choose $a, b, c \in C$ such that the associator $[a,b,c] \neq 0$, in which case the [31] entry is not 0. ◻

A1.7 Lemma. *The Glennie polynomial $g_9(x, y, z)$ vanishes on every i-special Jordan algebra.*

Proof A special Jordan algebra is a subalgebra of $A^{(+)}$ for some associative algebra A. Let u, v, $w \in A$. Substituting $x = u$, $y = v$, $z = w$ in $g_9(x, y, z)$, we obtain

$$\begin{aligned} &uwuuwv^2wv + uwuvwv^2wu + uwv^2wvuwu + vwv^2wuuwu \\ &\quad - vwvuwu^2wv - vwvvwu^2wu - uwu^2wvvwv - vwu^2wuvwv \\ &\quad - uwuvwvvwu - uwvvwvuwu + vwuuwuvwv + vwvuwuuwv = 0. \end{aligned}$$

If J is an i-special Jordan algebra, then by definition there is a surjection $f: J' \to J$ such that J' is special. Then g_9 on J can be computed as the composition $g_9 f$ on J', which is identically zero by the preceding paragraph. □

In fancier language, we have shown that g_9 is an *s-identity*, i.e., it is not identically zero for every Jordan algebra (Lemma A1.6) and it vanishes on every special Jordan algebra (Lemma A1.7).

Proof of Theorem A1.1 If J is an Albert algebra, by Corollary 39.32, there exists a faithfully flat extension $R \in k$-**alg** such that $J_R \cong \mathrm{Her}_3(\mathrm{Zor}(R))$. Since $\mathrm{Zor}(R)$ is not associative, Lemmas A1.6 and A1.7 show that J_R is not i-special, i.e., is i-exceptional, and it follows (Lemma A1.4) that J is i-exceptional. □

A2 Descent of algebras to Dedekind domains

Chapter I of the book, especially section 6, addressed the question of what properties of a $\mathbb{Z}$-submodule of a real Jordan or Albert algebra correspond to it being a Jordan or Albert algebra over $\mathbb{Z}$. That part of the text, however, takes a naive view of the objects involved and therefore does not obviously give statements about the more sophisticated notions of cubic Jordan and Freudenthal algebras defined later in the text. We rectify that here.

The general setting we describe is some integral domain k contained in a field F and we wish to give properties of a k-submodule M of some algebraic object over F to guarantee that M is also of the same type of object. We need the following basic tool.

A2.1 Lemma. *Let M, N be k-modules for some integral domain k, and let $f: M \to N$ be a polynomial law. If there exists a field F containing k such that $f \otimes F = 0$, then $f = 0$ as a polynomial law.*

Proof If k is itself a field, then F is a faithfully flat k-algebra. Since $0 \otimes F = f \otimes F$, $f = 0$ by Exercise 25.35(b). If k is not a field, then it is infinite, so F is infinite. The Principle of Permanence [Bou81, §IV.2.3, Scholium] then says that $f_R = 0$ for every k-algebra R and therefore $f = 0$. □

Jordan algebras

Recall from Definition 29.1 that a Jordan algebra is a para-quadratic algebra such that two identities concerning the U-operator hold strictly, meaning as polynomial laws.

A2.2 Proposition (Jordan algebras, first version)**.** *Let A be a para-quadratic k-algebra where k is an integral domain. If there is a field F containing k such that $A \otimes_k F$ is a Jordan F-algebra, then A is a Jordan k-algebra.*

Proof We must verify that the polynomial laws $A \times A \times A \to A$ given by sending (x, y, z) to

$$U_{U_x y} z - U_x U_y U_x z \quad \text{and} \quad U_x V_{y,x} z - V_{x,y} U_x z$$

are zero as polynomial laws. This follows from Lemma A2.1. □

We can rephrase this in terms of lattices. Let M be a vector space over a field F. For an integral domain k contained in F, a *k-lattice* Λ in M is a finitely generated k-submodule of M such that the natural F-linear map $\Lambda \otimes_k F \to M$ is an isomorphism of F-modules.

A2.3 Proposition (Jordan algebras, second version)**.** *Suppose J is a Jordan F-algebra for a field F, k is an integral domain contained in F, and Λ is a k-lattice J that contains 1_J. If*

(1) Λ is closed under the U-operator for J, or
(2) $2 \in k^\times$ and $x^2 \in \Lambda$ for every $x \in \Lambda$,

then the restriction of the U-operator of J turns Λ into a Jordan k-algebra.

Proof Assume (1), i.e., the restriction of U to Λ defines a function $\Lambda \to \mathrm{End}_k(\Lambda)$. Since U on J is a quadratic map in the sense of 11.1, so is the restriction of U to Λ. Therefore, Λ together with 1_J and the restriction of U is a para-quadratic k-algebra. Now apply Proposition A2.2.

Now assume (2). Since $2 \in k^\times$, we view J as a linear Jordan algebra and write xy for the product of $x, y \in J$. Again because $2 \in k^\times$, for $x, y \in \Lambda$, $xy := \frac{1}{2}((x + y)^2 - x^2 - y^2)$ belongs to Λ. It follows that Λ is closed under the U-operator (27.10.1), and we are done by (1). □

Jordan algebras of degree 3

Next we wish to treat Jordan algebras of degree 3, for which we will restrict to the case where k is a Dedekind domain. In that case, because a k-lattice Λ is necessarily torsion-free (and by definition finitely generated), it follows that Λ is projective [Sta18, Tags 00NX, 0AUW].

Recall that a cubic Jordan algebra can be defined as a Jordan algebra J together with a cubic form N on J as in 34.1, or as a cubic norm structure meaning a module J together with a base point 1_J, quadratic map $\sharp$, and cubic form N as in 33.1 and 33.4; the two notions are interchangeable by Corollary 34.6. Certain identities regarding these maps are required to hold as polynomial laws.

A Jordan algebra of degree 3 is defined in 38.1(b) as a cubic Jordan algebra J such that the set map $J_K \to \wedge^3 J_K$ given by $x \mapsto 1_{J_K} \wedge x \wedge x^2$ is not the zero map for every algebraically closed field $K \in k$-**alg**.

A2.4 Proposition (Compare Theorem 6.11)**.** *Let F be a field of characteristic $\neq 2$, and suppose J is a Jordan algebra of degree 3 over F. Let k be an infinite Dedekind domain contained in F. For a k-lattice Λ in J containing 1_J, the following are equivalent:*

- *(i) The restriction of the U-operator and N turn Λ into a cubic Jordan algebra.*
- *(ii) $x^2 \in \Lambda$ for all $x \in \Lambda$.*
- *(iii) $x^\sharp \in \Lambda$ for all $x \in \Lambda$.*

Proof It is trivial that (i) implies (ii) and (iii), so the work is in proving the opposite implications. The idea is to imitate the proof of Theorem 6.11 in the book, substituting items according to the following table, where K denotes the fraction field of k:

book	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathrm{Her}_3(\mathbb{D})$
here	k	K	F	J

We also replace various identities regarding $\mathrm{Her}_3(\mathbb{D})$ from Chapter I with the corresponding properties for cubic Jordan algebras from Chapter VI, for example: (5a.8) is replaced by (34.1.4); Exercise 6.16 is replaced by (33a.6); and Exercise 6.12(b) is replaced by (33.6.1) and (33.24). We now step through the various arguments leading up to Theorem 6.11, and comment on what further changes are required.

The proof of Proposition 3.5 goes through, using that a Dedekind domain is integrally closed. Lemma 3.7, Proposition 3.8, and Exercises 3.19 and 6.15 go through.

Lemma 6.7 shows that (ii) implies that the trace T, quadratic trace S, and norm N of J restrict to maps $\Lambda \otimes K \to K$. In the proof, we use that K has characteristic $\neq 2$ so that we can work with linear Jordan algebras, and so that (ii) gives that $\Lambda \otimes K$ is a K-algebra. For the Zariski-density argument following equation (1) in the proof of Lemma 6.7, we require J to be a Jordan algebra of degree 3 and not merely a cubic Jordan algebra in order that the set of elements in k with minimal polynomial of degree 3 is not empty.[3] We require k to be infinite so that the Zariski-open set constructed there has a k-point.

Remark 6.8 observes that every K-subspace B of J that is closed under squaring is a K-subalgebra, because char $K \neq 2$.

Proposition 6.9 shows that, if Λ is closed under taking powers $x \mapsto x^n$ for $n \in \mathbb{N}$, then every $x \in \Lambda$ is integral over k and has $T(x), S(x), N(x) \in k$. In the proof, when Cor. 5.14 is invoked to produce elements $c_1, c_2 \in \mathbb{R}[x]$, we instead produce elements in $\bar{F}[x]$, where $\bar{F}$ denotes the algebraic closure of F.

Assuming (ii), the argument in the proof of Theorem 6.11 gives (iii), where Lemma 6.10 is replaced by Lemma A2.5 below.

Assuming (iii), the argument in the proof of Theorem 6.11 shows that the adjoint $\sharp$ and norm N on J restrict to Λ to give Λ the structure of a cubic array as defined in 33.1. To be a cubic array requires that the element 1_J be unimodular, but that is automatic by Exercise 12.44 since Λ is a projective k-module.

For this cubic array to be a cubic norm structure, certain identities must hold strictly, i.e., as polynomial laws. For each such polynomial law f that we wish to prove is zero, we use Lemma A2.1 to reduce to checking it over F, where it holds by hypothesis. The equivalence between cubic norm structures and cubic Jordan algebras (Cor. 34.6) completes the proof of (i). ◻

Here is the promised generalization of Lemma 6.10 about $\mathbb{Z}$ to the case of a Dedekind domain.

A2.5 Lemma (Compare Lemma 6.10)**.** *Let k be a Dedekind domain with quotient field F and suppose $X \subseteq F$ is closed under squaring (i.e., $\xi \in X \Rightarrow \xi^2 \in X$). If the k-submodule of F generated by X is finitely generated, then $X \subseteq k$.*

The proof is only cosmetically different from the one in the book.

Proof For any $\xi \in F$, we denote by

$$\mathfrak{d}_\xi := \{b \in k \mid b\xi \in k\} \subseteq k$$

the *denominator ideal* of ξ. These ideals satisfy the following obvious relations

[3] Note that this property is lacking for the example in Exercise 34.27 that shows that cubic Jordan algebras are not stable under faithfully flat descent.

for all $\xi, \eta \in F$ and $a \in k$:

$$\mathfrak{d}_\xi = k \Longleftrightarrow \xi \in k, \quad \mathfrak{d}_\xi \mathfrak{d}_\eta \subseteq \mathfrak{d}_\xi \cap \mathfrak{d}_\eta \subseteq \mathfrak{d}_{\xi+\eta}, \quad \mathfrak{d}_\xi \subseteq \mathfrak{d}_{a\xi}. \tag{a1}$$

Slightly less obvious is

$$\mathfrak{d}_\xi = \prod_{\mathfrak{p}, v_\mathfrak{p}(\xi)<0} \mathfrak{p}^{-v_\mathfrak{p}(\xi)}. \tag{a2}$$

In order to see this, let $b \in k$. Then $b \in \mathfrak{d}_\xi$ if and only if $v_\mathfrak{p}(b) \geq -v_\mathfrak{p}(\xi)$ for all $\mathfrak{p}$, which holds trivially for $v_\mathfrak{p}(\xi) \geq 0$ and thus is equivalent to its validity for all $\mathfrak{p}$ having $v_\mathfrak{p}(\xi) < 0$, that is, to b belonging to the right-hand side of (a2). This verifies (a2). As an immediate consequence, we conclude

$$\mathfrak{d}_{\xi^2} = \mathfrak{d}_\xi^2. \tag{a3}$$

Now let $\xi_1, \ldots, \xi_n$ be a finite sequence in X generating the k-submodule of F generated by X. Then any $\xi \in X$ may be written as $\xi = \sum_{i=1}^n a_i \xi_i$, for some $a_1, \ldots, a_n \in k$, which by (a1) implies

$$\mathfrak{d}_\xi \supseteq \prod_{i=1}^n \mathfrak{d}_{a_i \xi_i} \supseteq \mathfrak{d} := \prod_{i=1}^n \mathfrak{d}_{\xi_i},$$

the latter being a fractional ideal of F depending only on X. Thus $\mathfrak{d}_\xi$ divides $\mathfrak{d}$ for all $\xi \in X$. But $\mathfrak{d}$ has only finitely many divisors, so $\{\mathfrak{d}_\xi \mid \xi \in X\}$ is a finite set. On the other hand, the fractional ideals of F form a free, hence torsion-free, abelian group, and for any $\xi \in X$ having $\mathfrak{d}_\xi \neq (1) = k$, the hypothesis of the lemma would yield a sequence $(\xi^{2^n})_{n \geq 0}$ of elements in X such that the $\mathfrak{d}_{\xi^{2^n}}$, $n = 0, 1, 2, \ldots$, by (a3) would be mutually distinct, a contradiction. Thus $\mathfrak{d}_\xi = k$ for all $\xi \in X$, and (a1) implies $X \subseteq k$. □

Albert algebras

Next we wish to give a result for Albert algebras that is similar to Propositions A2.2, which we will do in Theorem A2.7. Recall that an Albert k-algebra J is a cubic Jordan algebra whose underlying module is finitely generated projective[4] and such that J_K is simple of rank 27 for every field $K \in k$-**alg**.

The heavy lifting for this is done by the following lemma, which encapsulates arguments by Racine and van der Blij–Springer.

[4] The definition in 39.8 has that the module is projective and its rank is locally constant, which is a priori a weaker condition, yet Corollary 39.11 proves that the module is finitely generated. Conversely, a finitely generated projective module has locally constant rank [Sta18, Tag 00NX].

A2.6 Lemma. *Let F be a field that is complete with respect to a discrete valuation and let J be a cubic Jordan algebra over the valuation ring k of F. Then J is the split Albert k-algebra if and only if*

(i) $J \otimes_k F$ *is the split Albert F-algebra and*
(ii) *the trace bilinear form on J is regular.*

Proof Since the trace bilinear form on an Albert algebra is regular (39.19(b)) and split Albert algebras remain so after base change, the two conditions are clearly necessary.

Conversely, given the two conditions and since k is a Dedekind domain, [Rac73, p. 115, §IV.4, Prop. 5] says that J is isomorphic to $\mathrm{Her}_3(C)$ for C a maximal order in the split octonions over F. (We note that while the isomorphism statements for J in [Rac73] are as quadratic Jordan algebras, such isomorphisms are isomorphisms also as cubic Jordan algebras by Cor. 38.18.) Since k is a complete discrete valuation ring, all maximal orders in the split octonions are isomorphic [VdBS59, (3.4)], so we may assume that C is the split octonions over k, ergo J is itself the split Albert algebra. □

A2.7 Theorem. *Let k be a Dedekind domain and suppose J is a cubic Jordan algebra over k that is regular. If there is a field $F \supseteq k$ such that $J \otimes_k F$ is an Albert algebra, then J is an Albert algebra.*

Recall from 33.3 that J is defined to be regular if its underlying k-module is finitely generated projective and the bilinear trace form is regular as a symmetric bilinear form.

Proof Because the rank of J is locally constant and k is an integral domain, the rank of J at every prime equals $\dim_F(J_F) = 27$. Consequently, it suffices to prove that J_K is an Albert algebra for every field $K \in k$-**alg**. So suppose that $K = k(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of k.

If $\mathfrak{p} = 0$, then K is the fraction field of k. Since F is faithfully flat over K, the base change J_K is an Albert algebra by Cor. 39.32.

Assume $\mathfrak{p} \neq 0$, and write E for the completion of F with respect to the discrete valuation defined by $\mathfrak{p}$. Now J_E is an Albert algebra so there is a finite[5] extension E' of E that splits J_E by Cor. 54.19 and Prop. 55.5. The discrete valuation on E extends to one on E' and we take R to be the valuation ring of E' with respect to that extension. Since E' is finite over E, it is complete with respect to the valuation. Now J_R is regular because J is, and $J_{E'}$ is the split Albert algebra, so J_R is the split Albert algebra by Lemma A2.6. For ℓ the

[5] In fact, separable of dimension dividing 6, by Proposition A3.3.

residue field of R, then, J_ℓ is the split Albert algebra. But ℓ is a finite extension of K, so we deduce that J_K is also an Albert algebra.

For any $K \in k$-**alg**, there is some prime ideal $\mathfrak{p}$ (possibly zero) such that $K \supseteq k(\mathfrak{p})$. Since $J_{k(\mathfrak{p})}$ is an Albert algebra, so is J_K. □

A2.8 Application. Pick k to be an infinite Dedekind domain of characteristic $\neq 2$. Let F be any field containing k, and let A be an Albert algebra over F. For example, you could take k to be the ring of integers in a number field with a real embedding, $F = \mathbb{R}$, and A to be the euclidean Albert algebra.

If you pick any k-lattice J in A that contains 1_J and is closed under the $\sharp$ operator in A, then by Theorem A2.4 J is a cubic Jordan algebra. If furthermore the bilinear trace form on J (the restriction of the bilinear form on A) is regular, then Theorem A2.7 gives that J is an Albert algebra over k as defined in 39.18.

A3 Splitting and reducing fields for Freudenthal algebras

In this section, we consider a Freudenthal algebra J over a field F and prove the existence of separable extensions $K \supseteq L \supseteq F$ of small dimension such that J_K is split (as defined in 39.20) and J_L is reduced (as defined in 41.1).

Dimensions 1 and 3

If $\dim_F J = 1$, then $J \cong k^{(+)}$, so J is split and we may take $K = F$.

If $\dim_F J = 3$, then $J \cong E^{(+)}$ for a cubic étale F-algebra E that is uniquely determined up to isomorphism, see Cor. 55.2 or Exercise 39.42(b). For such a J, being reduced is equivalent to being split — i.e., $J \cong (F \times F \times F)^{(+)}$ — which is equivalent to $E \cong F \times F \times F$. If $E \cong F \times K$ for a field K, then $E \otimes_F K$ is split. Otherwise, E is a field, and the normal closure K of E has dimension 3 or 6 and $E \otimes_F K$ is split. In summary, one can take $L = K$ and K of dimension dividing 6.

Dimensions 9, 15, and 27

A3.1 Lemma. *Suppose J is a Freudenthal F-algebra of rank ≥ 9. If J is reduced, then there is a separable extension K of J of degree dividing 2 such that J_K is split.*

Proof The hypothesis that J is reduced says that $J \cong \mathrm{Her}_3(C, \Gamma)$ for some composition algebra C (Prop. 39.17) of rank ≥ 2. If C is split, then J is split

(Prop. 40.6), and we take $K = F$. Otherwise, there is a separable quadratic extension K of F contained in C and $C \otimes_F K$ is split (Prop. 22.20, Cor. 22.16). □

Because every Freudenthal algebra of dimension 15 is reduced (Th. 46.8), we obtain:

A3.2 Corollary. *If J is a Freduenthal F-algebra of dimension 15, then there is a separable extension K of F of dimension dividing 2 such that J_K is split.* □

A3.3 Proposition. *Let J be a Freudenthal F-algebra of dimension 9 or 27. There exist separable extensions $K \supseteq L \supseteq F$ such that $[K : L]$ divides 2, $[L : F]$ divides 3, J_K is split, and J_L is reduced.*

Proof #1 If J is reduced, take $L = F$. Otherwise, J is a division algebra (Prop. 39.17), so it contains a a separable cubic subfield L (Cor. 46.7). Since $L \otimes_F L$ is not a field, $(L \otimes_F L)^{(+)}$ contains a nonzero element of norm zero, ergo so does J_L, and we conclude that J_L is reduced (Prop. 39.17).

Applying Lemma A3.1 to J_L provides K. □

Proof #2 (sketch) To produce K, use Exercise 55.12 or Prop. 55.5 to translate the problem into producing a field K that splits an absolutely simple algebraic group of type A_2 or F_4. Then apply the main result of [Tit92]. □

A3.4 Example. We sketch an example of a field F and an Albert F-algebra J that shows that the dimensions in Prop. A3.3 are best possible.

Pick a field k. There exists a field $F \supseteq k$ and an Albert F-algebra J such that the Rost invariant $r(J)$ of J as described in 55.10 has order 6 as an element of the abelian group $H^3(F, \mathbb{Z}/6\mathbb{Z}(2))$. Here are two ways to produce F and J:

(1) Do it explicitly, using the second Tits construction and the formulas for $r(J)$ provided in [PR96], [PR95], [PR97], and [KMRT98, §40].

(2) Take $\mathbf{G}$ to be the automorphism group of the split Albert algebra, which is a split semi-simple group scheme of type F_4. Find a versal $\mathbf{G}$-torsor $\mathbf{X}$, which exists by [GMS03, p. 12] and is defined over some extension field F of k. Take J to be the Albert F-algebra corresponding to $\mathbf{X}$ by Prop. 55.5. The fact that $r(J)$ has order 6 follows from [GMS03, pp. 31, 135].

Suppose now that K is a finite separable extension of F such that J_K is split. Then the restriction $\mathrm{res}_{K/F}(r(J)) \in H^3(K, \mathbb{Z}/6\mathbb{Z}(2))$ is zero, consequently the corestriction satisfies

$$[K : F]\, r(J) = \mathrm{cor}_{K/F}\, \mathrm{res}_{K/F}(r(J)) = \mathrm{cor}_{K/F}(0) = 0 \qquad \text{(a1)}$$

in $H^3(F, \mathbb{Z}/6\mathbb{Z}(2))$, ergo 6 divides $[K : F]$. A similar argument shows that every separable extension L of F such that J_L is reduced has dimension divisible by 3.

The argument in the example works with essentially no changes also for dimension 9.

The restriction/corestriction argument around (a1) gives a converse to Proposition A3.3: *If J is an Albert F-algebra that is not reduced and L is a separable extension of F such that J_L is reduced, then 3 divides* $[L : F]$. (Compare Exercise 12.41.)

Using the same restriction/corestriction argument or combining Th. 41.26 and Springer's theorem on quadratic forms under odd-degree extensions [EKM08, Cor. 18.5, 18.6] completes the converse: *If J is an Albert F-algebra in which every nilpotent element is zero and K is a separable extension of F such that J_K does contain nonzero nilpotents, then 2 divides* $[K : F]$.

Dimension 6

It remains to treat the case of a Freudenthal algebra J of rank 6. We will assume that J is regular, equivalently (by Cor. 39.15) that char $F \neq 2$. By Th. 46.8, any such J is reduced, i.e., of the form $\mathrm{Her}_3(F, \Gamma)$ for some $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3) \in \mathbf{GL}_3(F)$. The algebra J that is defined to be split in 39.20 has $\Gamma = \mathrm{diag}(1, 1, 1)$, whereas the J that has $\mathbf{Aut}(J)$ split has $\Gamma = \mathrm{diag}(1, -1, -1)$ (Prop. 55.6). Example 39.35 provides an example of a J that is not split by any separable extension of F. Consequently, rather than splitting J, we instead split $\mathbf{Aut}(J)$.

A3.5 Lemma. *Let J be a Freudenthal F-algebra of dimension 6 and suppose that* char $F \neq 2$. *Then there exists a separable extension K of F of dimension dividing 2 such that* $J_K \cong \mathrm{Her}_3(F, \mathrm{diag}(1, -1, -1))$.

Proof There is a 3-dimensional regular quadratic form Q_J defined in (41.5.1). Adjoining a square root (which would be a separable extension) is sufficient to make Q_J isotropic, which implies the necessary isomorphism by Th. 41.26. □

A4 Describing Albert algebras by the Tits constructions

The classical treatment of the two Tits constructions as set forth in Jacobson [Jac68, IX.12] and McCrimmon [McC69, McC70], culminates in the following result ([Jac68, Thm. IX.22], [McC70, Thm. 8]): If J is an Albert algebra over a field F and A is a central simple associative F-algebra of degree 3 making $A^{(+)}$ a subalgebra of J, then there exists a scalar $\lambda \in F^\times$ such that the inclusion $A^{(+)} \hookrightarrow J$ can be extended to an isomorphism from the first Tits construction $J(A, \lambda)$ onto J. As an immediate corollary, basically the same conclusion can be drawn after replacing $A^{(+)}$ by the Jordan algebra of symmetric elements of a central simple associative F-algebra of degree 3 with unitary involution.

These results are quick to derive, given the approach to the Tits constructions as described in Chap. VII, so we give a proof here. The approach adopted here is different from that taken by Jacobson and McCrimmon. They relied on properties of the special universal envelope of $A^{(+)}$, where A is a central simple associative algebra of degree 3, whereas we combine results from the book with [JR50].

Throughout we let F be an arbitrary field.

A4.1 Proposition. *For a central simple associative F-algebra (B, τ) of degree 3 with unitary involution, the following conditions are equivalent.*

(i) *The center of B is not a field.*

(ii) *B is not simple.*

(iii) *There exists a central simple associative F-algebra A of degree 3 such that $H(B, \tau) \cong A^{(+)}$.*

Proof Put $K := \mathrm{Cent}(B)$, a quadratic étale F-algebra.

(i)⇒(ii). The center of a unital simple algebra is a field (Cor. 9.20).

(ii)⇒(iii). Since (B, τ) is simple but B is not, Prop. 10.5 yields a simple associative F-algebra A having $(B, \tau) \cong (A \times A^{\mathrm{op}}, \varepsilon_A)$ where ε_A is the exchange involution. We conclude $H(B, \tau) \cong A^{(+)}$ from (29.8.1) and with $L := \mathrm{Cent}(A)$ have $L \times L = K$, hence $L = F$. Thus A is central, while a dimension count shows that A has degree 3.

(iii)⇒(i). Suppose (iii) and not (i), i.e., that K is a field. From Exc. 44.29 (a) we deduce $A_K^{(+)} \cong H(B, \tau)_K = B^{(+)}$, where A_K and then B are both central simple over K (Cor. 9.23 (a)). By [JR50, Cor. of Thm. 21], therefore, either $A_K \cong B$ or $A_K^{\mathrm{op}} \cong B$. Replacing A by A^{op} if necessary, we may assume $A_K \cong B$. The action of τ on A_K induced by any such isomorphism fixes A since $A^{(+)} \cong H(B, \tau)$ and is the conjugation on K, hence an automorphism on all of A_K, a contradiction to τ being an anti-automorphism on B. □

A4.2 Corollary. *For $i = 1, 2$, let (B_i, τ_i) be central simple associative F-algebras of degree 3 with unitary involutions and $K_i := \mathrm{Cent}(B_i)$. If $H(B_1, \tau_1) \cong H(B_2, \tau_2)$, then $K_1 \cong K_2$ over F.*

Proof If K_1 is split, then Prop. A4.1 yields a central simple associative F-

algebra A_1 of degree 3 such that $H(B_2, \tau_2) \cong A_1^{(+)}$. Hence K_2 is split as well. By symmetry, we may therefore assume that K_1, K_2 are both fields. Then

$$H((B_1)_{K_2}, (\tau_1)_{K_2}) \cong B_2^{(+)},$$

so $(K_1)_{K_2}$ by Prop. A4.1 is a split quadratic étale K_2-algebra. By Exc. 23.40, this implies $K_1 \cong K_2$ over F. □

A4.3 Lemma. *Let $\mathcal{B} = (K, B, \tau, p)$ be an involutorial system of the second kind over F such that $H(\mathcal{B})$ is a Freudenthal F-algebra of dimension* 9*. Then (B, τ) is a central simple associative F-algebra of degree* 3 *with unitary involution and $K = \mathrm{Cent}(B)$.*

Proof $B^{(+)} = H(\mathcal{B})_K$ (by Exc. 44.29 (a)) is a Freudenthal K-algebra of constant rank 9 as a K-module. If K is a field, we therefore conclude that $B^{(+)}$ is simple. Hence so is B, and Exc. 42.21 implies that B is, in fact, a central simple associative K-algebra of degree 3 on which τ acts as a K/F-involution. On the other hand, if $K = F \times F$ is split, Exc. 44.31 (a) yields a pointed cubic alternative F-algebra (A, q) such that $B = A \times (A^q)^{\mathrm{op}}$, $\tau = \varepsilon_A$ is the switch and $p = (q, q)$. Letting K act on F through the first factor, we obtain $F \in K$-**alg** and, for the same reason as before, $A = B_F$ is a central simple associative F-algebra of degree 3. In both cases, therefore, (B, τ) is a central simple associative F-algebra of degree 3 with unitary involution having $K = \mathrm{Cent}(B)$. □

We can now prove the results announced at the beginning of this section, albeit in the reverse order.

A4.4 Theorem. *Let J be an Albert F-algebra and (B, τ) a central simple associative algebra of degree* 3 *with unitary involution over F making $J_0 := H(B, \tau)$ a Freudenthal subalgebra of J. Then there exists an admissible scalar (p, μ) for (B, τ) such that the inclusion $J_0 \hookrightarrow J$ can be extended to an isomorphism from the second Tits construction $J(B, \tau, p, \mu)$ onto J.*

Proof We first assume that F is finite. Then J is split (Exc. 40.17 (b)). But $J' := J(B, \tau, 1, 1)$ is a split Albert F-algebra as well, so there is an isomorphism $\Psi\colon J' \xrightarrow{\sim} J$, and this isomorphism maps the subalgebra $J_0 \subseteq J'$ onto a subalgebra $J_0' \subseteq J$. Since F is finite, both J_0 and J_0' are reduced simple Freudenthal subalgebras of J, and Ψ induces an isomorphism $\varphi\colon J_0 \xrightarrow{\sim} J_0'$. By the Skolem-Noether theorem (Exc. 41.34), φ can be extended to an automorphism Φ of J. Hence $\Phi^{-1} \circ \Psi\colon J' \xrightarrow{\sim} J$ is an isomorphism of the desired kind.

Next assume that F is infinite. Applying Thm. 45.10, we find an étale element of J relative to J_0. Hence Cor. 44.17 yields an involutorial system $\mathcal{A} =$

(K, A, σ, p) of the second kind over F as well as an admissible scalar $\mu \in K^\times$ such that

$$H(A, \sigma) = H(\mathcal{A}) = H(B, \tau) \tag{a1}$$

and the inclusion $J_0 \hookrightarrow J$ can be extended to an isomorphism from $J(\mathcal{A}, \mu)$ onto J. By Lemma A4.3, (A, σ) is a central simple associative F-algebra of degree 3 with unitary involution and $K = \mathrm{Cent}(A)$. From Cor. A4.2 we deduce $K = \mathrm{Cent}(B)$, while (a1) implies

$$A^{(+)} = H(A, \sigma) \otimes_F K = H(B, \tau) \otimes_F K = B^{(+)},$$

hence $A = B$ by [JR50, Cor. of Thm. 21], after replacing A by A^{op} if necessary. But σ acts on $H(A, \sigma) = H(B, \tau)$ as the identity and on K by conjugation. Hence $\sigma = \tau$, and the proof is complete. □

A4.5 Remark. The full force of the difficult Thm. 45.10 was not needed in the preceding proof. Instead, by a Zariski density argument, we are reduced to the case that F is algebraically closed. Then $J = \mathrm{Her}_3(C)$, $C := \mathrm{Zor}(F)$, is split, and, again by the Skolem-Noether theorem, we may assume $J_0 = \mathrm{Her}_3(D)$, D being the diagonal of $\mathrm{Zor}(F)$. Since $D^\perp \subseteq \mathrm{Zor}(F)$ is a free right D-module of rank 3, étale elements of J relative to J_0 exist, thanks to Exc. 45.17 (b).

A4.6 Corollary. *Let J be an Albert F-algebra and A a central sinple associative algebra of degree 3 over F making $A^{(+)}$ a subalgebra of J. Then there exists a $\lambda \in F^\times$ such that the inclusion $A^{(+)} \hookrightarrow J$ can be extended to an isomorphism from the first Tits construction $J(A, \lambda)$ onto J.*

Proof Put $(B, \tau) := (A \times A^{\mathrm{op}}, \varepsilon_A)$, ε_A being the exchange involution, and identify $A^{(+)} = H(B, \tau)$ canonically. By Thm. A4.4, there exists an admissible scalar (p, μ) for (B, τ) such that the inclusion $A^{(+)} \hookrightarrow J$ can be extended to an isomorphism $\Psi\colon J(B, \tau, p, \mu) \xrightarrow{\sim} J$. Write $p = (q, q)$ with $q \in A^\times$, and $\mu = (\lambda, \lambda')$ with $\lambda, \lambda' \in F^\times$. Applying Thm. 44.19, we find an isomorphism $\Phi\colon J(B, \tau, p, \mu) \xrightarrow{\sim} J(A, \lambda)$ extending the identity of $A^{(+)}$. Thus $\Psi \circ \Phi^{-1}\colon J(A, \lambda) \xrightarrow{\sim} J$ is an isomorphism of the desired kind. □

A counter example to the Skolem-Noether theorem

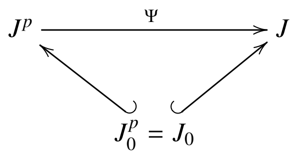
The Skolem-Noether theorem in its classical form may be phrased as follows [Jac89, Thm. 4.9]: *If A is a central simple associative algebra of finite dimension over a field F, then any isomorphism between simple subalgebras of A can be extended to an (inner) automorphism of A.* This result survives in many different settings. In the present one, we will be concerned with Albert algebras. Given an Albert F-algebra J, its most important class of subalgebras is provided by the Freudenthal subalgebras of dimension 9, and, indeed, the Skolem-Noether theorem does hold in this particular setting ([PST98, Thm. 2.7], [Pet04, Thm. 5.2]).

Replacing the base field F by an arbitrary commutative ring, and passing from central simple to Azumaya algebras in the process, a straightforward extension of the Skolem-Noether theorem in its classical form does not hold, even for principal ideal domains [CD67, p. 27]. A similar extension of its version for Albert algebras alluded to above seems to be even less likely. In the present note, we will confirm this suspicion. More precisely, we will establish the following result.

A4.7 Theorem. *Let k be a commutative ring whose split octonion algebra of Zorn vector matrices, $C := \mathrm{Zor}(k)$, contains an invertible element p such that C^p, the unital p-isotope of C, is not split. Then the split Albert k-algebra $J := \mathrm{Her}_3(C)$ admits two split Freudenthal subalgebras of rank 9 as k-modules as well as an isomorphism between them that cannot be extended to an automorphism of J.*

Commutative rings satisfying the hypothesis of the theorem do in fact exist, see Cor. 23.11.

Proof Write D for the diagonal of $C = \mathrm{Zor}(k)$, which is a split quadratic étale subalgebra. After the natural identifications of Prop. 36.9, therefore, $J_0 := \mathrm{Mat}_3(k)^{(+)} = \mathrm{Her}_3(D) \subseteq J$ is a split Freudenthal subalgebra of rank 9. Note that C^p is an octonion algebra which, though no longer split by hypothesis, has the same norm, trace and unit element as C (15.9 and Exc. 17.12). In particular, the diagonal idempotent $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Zor}(k)$ is elementary not only in C but also in C^p. It follows that D is a split quadratic étale subalgebra not only of C but also of C^p. Hence J_0 may be viewed not only as a split Freudenthal subalgebra of J but also as one of $J^p := \mathrm{Her}_3(C^p)$. In the latter capacity, it will be denoted by J_0^p, so we have $J_0^p \subseteq J^p$ as a split Freudenthal subalgebra of rank 9 and $J_0^p = J_0$ as cubic Jordan algebras. Since the Albert algebra J^p is split (Exc. 42.27 (b)), there exists an isomorphism $\Phi\colon J^p \to J$. Hence $J_0' := \Phi(J_0^p) \subseteq J$ is a split Freuidenthal subalgebra of rank 9, and the restriction of Φ to J_0^p may be regarded as an isomorphism $\varphi\colon J_0 \to J_0'$. We claim that φ cannot be extended to an automorphism of J. Arguing indirectly, assume that $\eta \in \mathrm{Aut}(J)$ is such an extension. Then $\Psi := \eta^{-1} \circ \Phi\colon J^p \to J$ is an isomorphism rendering the diagram



commutative. Now let $\mathfrak{S} := (e_{11}, e_{22}, e_{33}, u_{23}, u_{31})$ be the diagonal coordinate system of J_0, so $u_{jl} = 1_D[jl] = 1_C[jl] = 1_{C^p}[jl]$ for $i = 1, 2$, and write $\mathfrak{S}^p$ for $\mathfrak{S}$ regarded as a coordinate system of J_0^p. Then $\mathfrak{S}, \mathfrak{S}^p$ are coordinate systems for J, J^p, respectively, and Ψ, by the diagram above, maps $\mathfrak{S}^p$ to $\mathfrak{S}$. Since Ψ fixes u_{23} and u_{31}, it follows from (37.15.3), (37.15.6) that the restriction of Ψ to J_{12}^p gives an isomorphism $\Psi_{12} \colon C_{J^p,\mathfrak{S}^p} \to C_{J,\mathfrak{S}}$ of octonion algebras. On the other hand, since $\mathfrak{S}$ (resp., $\mathfrak{S}^p$) is the *diagonal* coordinate system of J (resp., of J^p), Ex. 37.16 implies $C_{J,\mathfrak{S}} \cong C$, $C_{J^p,\mathfrak{S}^p} \cong C^p$, and we arrive at the contradiction that C and C^p are isomorphic. □

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