Intervall decompositions on vector spaces over arbitrary fields

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1. Introduction. Given a vector space X over a field, Hitzemann and Hochstättler [1] have recently set up a(n almost) bijective correspondence between interval decompositions of the subspace lattice of X on the one hand and what they call families of point-wise reflexive and anti-symmetric linear forms on the other. In an effort to gain a better understanding of this correspondence, it will be recast here in a slightly different form. Examples of interval decompositions that seem to be new will also be presented.

2. The subspace lattice of X. Let X be a vector space, possibly infinite-dimensional, over an arbitrary field k. We denote by $\mathcal{L}(X)$ the lattice of all sub-vector spaces of X. Given $U, V \in \mathcal{L}(X)$, we denote by

$$[U, V] := \{W \in \mathcal{L}(X) \mid U \subseteq W \subseteq V\}$$

the (closed) interval determined by U, V in the lattice $\mathcal{L}(X)$. If V is finite-dimensional, we call

$$l([U, V]) := \dim_k(V) - \dim_k(U)$$

the *length* of $[U, V]$. Clearly, $[U, V]$ is not empty iff $U \subseteq V$ iff $U \in [U, V]$ iff $V \in [U, V]$. Moreover, for another pair of subspaces $U', V' \in \mathcal{L}(X)$,

$$[U, V] \cap [U', V'] = [U + U', V \cap V'],$$

and we conclude that that the intervals $[U, V]$, $[U', V']$ have a non-empty intersection iff $U + U' \subseteq V \cap V'$ iff U and U' are both subspaces of V and of V'.

3. Interval decompositions. By an *interval decomposition* of $\mathcal{L}(X)$ we mean a triple

$$\mathcal{Z} := (U_0, H_0, m)$$

satisfying the following conditions.

(i) $U_0 \in \mathcal{L}(X)$ has dimension 1.

(ii) $H_0 \in \mathcal{L}(X)$ is a hyperplane, i.e., a subspace of co-dimension 1 in X.

(iii) $m\colon \mathcal{P}(U_0, H_0) \to \mathcal{P}^*(U_0, H_0)$, where

$$\mathcal{P}(U_0, H_0) := \{U \in \mathcal{L}(X) \mid \dim(U) = 1,\ U_0 \neq U \not\subseteq H_0\},$$
$$\mathcal{P}^*(U_0, H_0) := \{H \in \mathcal{L}(X) \mid \mathrm{codim}_X(H) = 1,\ U_0 \not\subseteq H \neq H_0\},$$

is a map satisfying the following conditions:

(a) $U \subseteq m(U)$ for all $U \in \mathcal{P}(U_0, H_0)$.

(b) The intervals $[U, m(U)] \subseteq \mathcal{L}(X)$, $U \in \mathcal{P}(U_0, H_0)$, are mutually disjoint.

Here the map m is necessarily injective. Indeed, suppose $U, U' \in \mathcal{P}(U_0, H_0)$ satisfy $m(U) = m(U')$. Then (iii)(a) implies

$$m(U) = m(U') \in [U, m(U)] \cap [U', m(U'],$$

forcing $U = U'$ by (iii)(b).

It follows from **2.** that, in the presence of conditions (i)$-$(iii)(a), condition (iii)(b) is equivalent to the following:

(b$'$) If $U, U' \in \mathcal{P}(U_0, H_0)$ are distinct, then $U \nsubseteq m(U')$ or $U' \nsubseteq m(U)$.

We speak of a *proper* interval decomposition if the injective map m is surjective as well, hence bijective. This means that the intervals $[U, m(U)]$, $U \in \mathcal{P}(U_0, H_0)$, together with $[U_0, X]$ and $[\{0\}, H_0]$ form an interval partition of $\mathcal{L}(X)$.

4. Base points of interval decompositions. Let $\mathcal{Z} := (U_0, H_0, m)$ be an interval decomposition of $\mathcal{L}(X)$. Then that we have the splitting

$$X = U_0 \oplus H_0. \tag{1}$$

By a *base point* of $\mathcal{Z}$, we mean a non-zero element of U_0, i.e., a basis of the one-dimensional vector space U_0. A base point of $\mathcal{Z}$ is unique up to a non-zero scalar factor. By a *pointed interval decomposition* of $\mathcal{L}(X)$ we mean a pair $(\mathcal{Z}, p_0)$, where $\mathcal{Z}$ is an interval decomposition of $\mathcal{L}(X)$ as above and p_0 is a base point for $\mathcal{Z}$. We then claim *that the assignment*

$$p \longmapsto U_p := k(p_0 + p) \tag{2}$$

gives a bijection from $H_0 \setminus \{0\}$ onto $\mathcal{P}(U_0, H_0)$. Indeed, for $0 \neq p \in H_0$, the one-dimensional space U_p is clearly distinct from $U_0 = kp_0$ and not contained in H_0, hence belongs to $\mathcal{P}(U_0, H_0)$. The map in question is clearly injective and, given any $U \in \mathcal{P}(U_0, H_0)$, we may combine the definition of $\mathcal{P}(U_0, H_0)$ with (1) to find a scalar $\alpha \in k^\times$ and a vector $p' \in H_0$ such that $U_0 \neq U = k(\alpha p_0 + p') \nsubseteq H_0$. But then $U = U_p$ with $p = \alpha^{-1}p' \in H_0 \setminus \{0\}$, and the assertion follows.

Remark.. The preceding observation matches canonically with the the standard fact that the k-rational points of $\mathbb{P}^n_k$ whose $(n+1)$-th co-ordinate (say) is not zero are basically the same as the k-rational points of $\mathbb{A}^n_k$.

5. Irreflexive and anti-symmetric linear forms. A triple

$$\Sigma := \big(p_0, H_0, (\sigma_p)_{p \in H_0 \setminus \{0\}}\big)$$

is said to be a *point-wise irreflexive and anti-symmetric family of linear forms on X* if it satisfies the following conditions:

(i) $p_0 \in X$ is not zero.

(ii) $H_0 \in \mathcal{L}(X)$ is a hyperplane in X not containing p_0.

(iii) $(\sigma_p)_{p \in H_0 \setminus \{0\}}$ is a family of linear forms on X such that the following conditions are fulfilled, for all $p, q \in H_0 \setminus \{0\}$.

  (a) $\sigma_p(p_0) = -1$.

  (b) $\sigma_p(p) = 1$.

  (c) If $p \neq q$ and $\sigma_p(q) = 1$, then $\sigma_q(p) \neq 1$.

From now on, the term "point-wise" will always be suppressed in the preceding definition. Note that, thanks to conditions (i),(ii) above, we have the analogue of decomposition (1), i.e.,

$$X = U_0 \oplus H_0, \quad U_0 := kp_0. \tag{3}$$

Remark. By (iii)(a) and (3), the linear forms σ_p, $p \in H_0 \setminus \{0\}$, on X are completely determined by their action on H_0. Thus an irreflexive and anti-symmetric family of linear forms may be defined intrinsically on an arbitrary non-zero vector space Y over k as a family $(\sigma_y)_{y \in Y \setminus \{0\}}$ of linear forms on Y satisfying the condition

$$\forall y, z \in Y \setminus \{0\}: \ \sigma_y(z) = \sigma_z(y) = 1 \Longleftrightarrow y = z.$$

6. From interval decompositions to linear forms. Let $(\mathcal{Z}, p_0)$ with

$$\mathcal{Z} = (U_0, H_0, m)$$

be a pointed interval decomposition of $\mathcal{L}(X)$. For $0 \neq p \in H_0$, $U_0 = kp_0$ is not contained in $m(U_p)$, so we have the decomposition

$$X = U_0 \oplus m(U_p), \tag{4}$$

and find a unique linear form $\sigma_p \colon X \to k$ such that

$$\sigma_p(p_0) = -1, \quad \mathrm{Ker}(\sigma_p) = m(U_p). \tag{5}$$

We claim *that*

$$\mathbf{\Sigma}(\mathcal{Z}, p_0) := \big(p_0, H_0, (\sigma_p)_{p \in H_0 \setminus \{0\}}\big) \tag{6}$$

is an irreflexive and anti-symmetric family of linear forms on X. Indeed, conditions (i),(ii) in **5.** are clearly equivalent to the corresponding ones in **3.**, so we only have to worry about conditions (iii)(a)$-$(c). Here (a) is the first relation of (5). For (b),(c), let $p, q \in H_0 \setminus \{0\}$. Again by (5),

$$\sigma_p(q) = 1 \Longleftrightarrow \sigma_p(p_0 + q) = 0 \Longleftrightarrow p_0 + q \in \mathrm{Ker}(\sigma_p) \Longleftrightarrow U_q \subseteq m(U_p).$$

Therefore (iii)(b) (resp. (iii)(c)) follows from condition (iii)(a) (resp. (iii)(b$'$)) in **3.**.

What happens if we change the base point? To see this, let $\alpha \in k^\times$ and put

$$p_0' := \alpha^{-1} p_0, \quad \mathbf{\Sigma}(\mathcal{Z}, p_0') =: \big(U_0, H_0, (\sigma_p')_{p \in H_0 \setminus \{0\}}\big).$$

For $0 \neq p \in H_0$, we consult (2) and obtain

$$U_p' := k(p_0' + p) = k(p_0 + \alpha p) = U_{\alpha p}.$$

Combining this with (5), we obtain $\sigma_p' = \alpha \sigma_{\alpha p}$ for $p \in H_0 \setminus \{0\}$. Summing up we conclude

$$\mathbf{\Sigma}(\mathcal{Z}, \alpha^{-1} p_0) = \big(\alpha^{-1} p_0, H_0, (\alpha \sigma_{\alpha p})_{p \in H_0 \setminus \{0\}}\big). \tag{7}$$

7. From linear forms to interval decompositions. It is easy to reverse the preceding construction. Let $\Sigma = (p_0, H_0, (\sigma_p)_{p \in H_0 \setminus \{0\}})$ be an irreflexive and anti-symmetric family of linear forms on X. We put

$$\mathbf{Z}(\Sigma) := (\mathcal{Z}, p_0), \quad \mathcal{Z} := (U_0, H_0, m), \quad U_0 := kp_0, \tag{8}$$

where we observe **4.**, particularly (2), to define

$$m \colon \mathcal{P}(U_0, H_0) \longrightarrow \mathcal{P}^*(U_0, H_0), \quad m(U_p) := \mathrm{Ker}(\sigma_p) \qquad (p \in H_0 \setminus \{0\}). \tag{9}$$

We claim *that $\mathcal{Z}$ is an interval decomposition of $\mathcal{L}(X)$.* While conditions (i),(ii) of **3.** are obvious, condition (iii) follows from (iii) in **5.** and the following chain of equivalent conditions, for all $p, q \in H_0 \setminus \{0\}$.

$$U_q \subseteq m(U_p) \Longleftrightarrow p_0 + q \in \mathrm{Ker}(\sigma_p) \Longleftrightarrow \sigma_p(p_0 + q) = 0 \Longleftrightarrow \sigma_p(q) = 1.$$

Combining the two preceding constructions, we arrive at the following theorem.

8. Theorem. *The assignments*

$$(\mathcal{Z}, p_0) \longmapsto \mathbf{\Sigma}(\mathcal{Z}, p_0), \quad \Sigma \longmapsto \mathbf{Z}(\Sigma),$$

define inverse bijections between the set of pointed interval decompositions of $\mathcal{L}(X)$ and the set of irreflexive anti-symmetric families of linear forms on X. □

We now turn to examples of irreflexive anti-symmetric families of linear forms. In agreement with the remark of **5.**, we will construct such families on appropriate vector spaces Y over k. If Y has finite dimension n, this construction will give rise, via Thm. **8.**, to an interval decomposition in dimension $n + 1$.

We begin by generalizing [1, Example 2].

9. Example: Anisotropic bilinear forms. Let Y be a vector space over k and

$$\delta \colon Y \times Y \longrightarrow k$$

be a (possibly non-symmetric) bilinear form that is *anisotropic* in the sense that $\delta(y, y) \neq 0$ for all non-zero elements $y \in Y$. For $y \in Y \setminus \{0\}$ we define

$$\sigma_y \colon Y \longrightarrow Y, \quad z \longmapsto \sigma_y(z) := \delta(y, y)^{-1}\delta(y, z). \tag{10}$$

Clearly, σ_y is a linear form satisfying $\sigma_y(y) = 1$. Now suppose $y, z \in Y \setminus \{0\}$ are distinct with $\sigma_y(z) = \sigma_z(y) = 1$. Then y and z are linearly independent since, otherwise, $z = \alpha y$ for some $\alpha \in k$, forcing $\alpha = \sigma_y(\alpha y) = \sigma_y(z) = 1$, a contradiction. Now (10) gives $\delta(y, z) = \delta(y, y)$, $\delta(z, y) = \delta(z, z)$, hence

$$\det \begin{pmatrix} \delta(y, y) & \delta(y, z) \\ \delta(z, y) & \delta(z, z) \end{pmatrix} = \delta(y, y)\delta(z, z) - \delta(y, z)\delta(z, y) = 0.$$

Writing $Y' = ky + kz$ for the subspace of Y spanned by y, z, we conclude that there exists a non-zero vector $w \in Y'$ satisfying $\delta(Y', w) = \{0\}$. On the other hand, δ being anisotropic implies $\delta(w, w) \neq 0$, a contradiction. Thus $(\sigma_y)_{y \in Y \setminus \{0\}}$ is an irreflexive anti-symmetric family of linear forms on Y.

Remark. 1. It is a standard fact from the algebraic theory of quadratic forms that every quadratic form $q \colon Y \to k$ allows a bilinear form $\delta \colon Y \times Y \to k$, in general not symmetric, such that $q(y) = \delta(y, y)$ for all $y \in Y$. In particular, if q is anisotropic, so is δ, and conversely.

Remark. 2. Replacing δ by $\delta + \alpha$ for some *alternating* bilinear form $\alpha \colon Y \times Y \to k$ does not change the quadratic form corresponding to δ. Hence we obtain a whole family of irreflexive anti-symmetric families of linear forms on Y, parametrized by the alternating bilinear forms on Y.

Remark. 3. Let k be finite. Anisotropic quadratic forms of dimension n over k exist iff $n \leq 2$. We thus obtain examples of interval decompositions of $\mathcal{L}(X)$ if X has dimension ≤ 3 over k, in agreement with the first row the final table in [1].

10. Example: Anisotropic cubic forms. Again we let Y be a vector space over k but now assume

$$N \colon Y \longrightarrow k$$

is an anisotropic cubic form, so N is a polynomial law in the sense of Roby [3], homogeneous of degree 3, and representing zero only trivially: $N(y) = 0$, $y \in Y$, implies $y = 0$. We denote by

$$DN \colon Y \times Y \longrightarrow k, \quad (y, z) \longmapsto (DN)(y, z)$$

the total differential of N, which is quadratic in the first variable, linear in the second, and matches with N itself through the expansion

$$N(y+z) = N(y) + (DN)(y,z) + (DN)(z,y) + N(z), \tag{11}$$

valid in all scalar extensions. For $y \in Y \setminus \{0\}$, we define

$$\sigma_y \colon Y \longrightarrow Y, \quad z \longmapsto \sigma_y(z) := N(y)^{-1}(DN)(y,z) \tag{12}$$

and claim: *If k has characteristic 2, then $(\sigma_y)_{y \in Y \setminus \{0\}}$ is an irreflexive anti-symmetric family of linear forms on Y.* Since we are in characteristic 2, the relations $\sigma_y(y) = 1$ for $0 \neq y \in Y$ follow immediately from Euler's differential equation:

$$\sigma_y(y) = N(y)^{-1}(DN)(y,y) = 3N(y)^{-1}N(y) = 1.$$

Hence it remains to show for $y, z \in Y \setminus \{0\}$ distinct that the relations $\sigma_y(z) = \sigma_z(y) = 1$ lead to a contradiction. From (12) we conclude $(DN)(y,z) = N(y)$, $(DN)(z,y) = N(z)$, and (11) implies

$$N(y+z) = N(y) + N(y) + N(z) + N(z) = 0,$$

a contradiction since N was assumed to be anisotropic.

Remark. Let k be finite of characteristic 2, hence of the form $\mathbb{F}_{2^r}$ for some integer $r > 0$. By Chevalley's theorem [2, Chap. IV, Ex. 7], anisotropic cubic forms of dimension n over k exist iff $n \leq 3$. Thus we find interval decompositions over k in all dimensions ≤ 4, allowing us to replace the question mark in the second row of the final table in [1] by a "yes" provided q is a power of 2.

References

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