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Journal of Pure and Applied Algebra 217 (2013) 1700-1701

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Maximal subalgebras of octonions

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ARTICLE INFO

ABSTRACT

Article history: Received 9 October 2012 Received in revised form 1 December 2012 Available online 7 March 2013 Communicated by C.A. Weibel

MSC: Primary: 17C35 Secondary: 12F15

1. Introduction

Gagola [1] has recently closed a small gap in Racine's classification [3, Thm. 5] of maximal subalgebras of an arbitrary octonion algebra *C* over a field *F*: the algebras missing in Racine's list are precisely the subfields of *C* that are purely inseparable of characteristic 2, exponent 1 and degree 4 over *F*. What [1] fails to address, however, is the question of why these subalgebras are indeed maximal.

Inseparable degree 4 subfields of octonion algebras are shown to be maximal subalgebras.

There are surely many ways of proving this. For example, I have been informed by Gagola himself that, in an unpublished preprint dating back more than five years ago, he derived the necessary conclusion from a general theory of maximal subloops sitting inside the Moufang loop of unit octonions. In what follows I will argue in a different manner by appealing to the non-orthogonal Cayley–Dickson construction due to Garibaldi–Petersson [2, Sec. 4].

2. Proposition

Suppose F has characteristic 2 and $K \subseteq C$ is an inseparable subfield of degree 4 over F. Then K is a maximal subalgebra of C.

Proof. Being an *F*-algebra of degree 2 and an inseparable field extension at the same time, K/F is in fact purely inseparable of exponent 1. From [2, Prop. 4.5] we therefore conclude that, for some linear form $s : K \to F$ taking 1 into 1 and for some $\mu \in F$, the natural embedding $K \hookrightarrow C$ extends to an isomorphism from *C* to the non-orthogonal Cayley–Dickson construction $C' := \text{Cay}(K; \mu, s)$ which lives on the vector space direct sum $K \oplus Kj$ of two copies of *K* under a multiplication having the following properties: identifying $K \subseteq K \oplus Kj$ through the first summand makes *K* a unital subalgebra of *C'*, and we have

$$u(vj) \equiv (uv)j \mod K \quad (u, v \in K);$$

for a full description of the algebra structure of C', which will not be needed here, see [2, 4.3]. Now suppose $A \subseteq C'$ is a subalgebra properly containing K. Since $A = K \oplus (A \cap Kj)$, we have $A \cap Kj \neq \{0\}$, so let $v \in K$ be non-zero satisfying $vj \in A$. Then (1) implies $Kj \equiv (Kv)j \equiv K(vj) \mod K$, forcing $Kj \subseteq A$, hence A = C', and the proposition is proved. \Box





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3. Remark

Examples of octonion algebras over a field of characteristic 2 containing purely inseparable subfields of exponent 1 and degree 4 can be easily constructed. In fact, if C is a division algebra, we refer to [2, 1.13] for details, while if C is split, we may generalize [1, Example 2] by using [2, Thm. 4.6 (b), Prop. 5.2] to conclude that every purely inseparable field extension of F having exponent 1 and degree 4 is isomorphic to a subalgebra of C.

Acknowledgment

I am grateful to S. Pumplün for having brought [1] to my attention.

References

- S.M. Gagola, Maximal subalgebras of the octonions, J. Pure Appl. Algebra 217 (1) (2013) 20–21. MR 2965899.
 S. Garibaldi, H.P. Petersson. Wild Pfister forms over Hencelize Fields. K. K. S. Garibaldi, H.P. Petersson, Wild Pfister forms over Henselian fields, K-theory, and conic division algebras, J. Algebra 327 (2011) 386-465. MR 2746044
- (2012e:11065).
- M.L. Racine, On maximal subalgebras, J. Algebra 30 (1974) 155-180. MR 50 #2264. [3]