Moufang sets and the problem of commuting U-operators

Holger P. Petersson
Fakultät für Mathematik und Informatik
FernUniversität in Hagen
D-58084 Hagen
Germany
Email: Holger.Petersson@FernUni-Hagen.de

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Well, I guess everybody in this audience will agree with me that the theory of Jordan systems is important not only because of its intrinsic beauty but also because of its profound and extremely versatile connections with other branches of mathematics. Today I would like to focus on a connection that was brought up only a few years ago, in a paper by Tom deMedts and Richard Weiss [2] dating back to 2006. In this paper, the authors use a classical device from Jordan theory, namely, the Hua identity, to set up a correspondence between what are usually called Moufang sets on the one hand and honest-to-goodness (quadratic) Jordan division rings on the other. Let me begin by saying a few words about

1. Moufang sets.

Moufang sets are the brainchild of Jacques Tits [6]. Technically speaking, they are basically the same as Tits buildings of rank 1 carrying the additional structure of a split BN-pair. But I don't think you want to share with me my appallingly limited knowledge of the theory of buildings, so let me go straight to the formal definition of a Moufang set. In doing so, I follow the treatment of deMedts-Segev [1].

1.1. The concept of a Moufang set. A *Moufang set* is a pair $\mathbb{M} = (X, \mathfrak{W})$ satisfying the following conditions.

(i) X is a set of cardinality at least 3: $|X| \geq 3$.

(ii) $\mathfrak{W} = (W_x)_{x \in X}$ is a family of subgroups of $\mathrm{Sym}(X)$, the (full) permutation group of X.

(iii) Writing
$$G := G_{\mathbb{M}} := \langle W_x \mid x \in X \rangle$$
for the subgroup of $\mathrm{Sym}(X)$ generated by the members of $\mathfrak{W}$, the following conditions hold.

  (a) For all $x \in X$, $W_x \lhd G_x = \{g \in G \mid gx = x\}$ is a normal subgroup of the stabilizer, G_x, of x in G.

  (b) The subgroups W_x, $x \in X$, form a full conjugacy class of subgroups in G.

(c) For all $x \in X$, W_x is *regular*, i.e., simply transitive on $X \setminus \{x\}$: given $y, z \in X \setminus \{x\}$, there is a *unique* $g \in W_x$ sending y to z.

If $\mathbb{M}$ as above is a Moufang set, then $G_{\mathbb{M}}$ as defined in (iii), for reasons that will be explained below, is called the *little projective group* of $\mathbb{M}$, while the W_x, $x \in X$, are called its *root groups.*

Given two Moufang sets, $\mathbb{M} = (X, (W_x)_{x\in X})$ and $\mathbb{M}' = (X', (W'_{x'})_{x'\in X'})$, a *morphism* $\varphi\colon \mathbb{M} \to \mathbb{M}'$ is defined as a bijection $\varphi\colon X \to X'$ such that $W'_{\varphi(x)} = \varphi W_x \varphi^{-1}$ for all $x \in X$. In this way we obtain the groupoid of Moufang sets, a *groupoid* being a category all of whose morphisms are isomorphisms.

The preceding definition looks pretty intractable, and examples of Moufang sets seem difficult to construct. Nevertheless we will be able to do so. The following elementary observations will be crucial.

1.2. Lemma. *Let $\mathbb{M} = (X, (W_x)_{x\in X})$ be a Moufang set. Then*

(a) *The little projective group $G_{\mathbb{M}}$ is 2-transitive on X: for all $x, y, x', y' \in X$, $x \neq x'$, $y \neq y'$, there exists a $g \in G_{\mathbb{M}}$ such that $gx = y$, $gx' = y'$.*

(b) *$gW_xg^{-1} = W_{gx}$ for all $g \in \mathbb{M}$, $x \in X$, i.e., every $g \in G_{\mathbb{M}}$ is an automorphism of $\mathbb{M}$.* □

1.3. Towards Moufang sets via groups. Given any group W, written additively with neutral element 0 and inversion $a \mapsto -a$, *even though it may not be abelian*, we add a new symbol $\infty \notin W$ to W by forming the set

$$X := W \cup \{\infty\}$$

and consider any permutation $\tau \in \mathrm{Sym}(X)$ interchanging 0 and ∞: $\tau 0 = \infty$, $\tau\infty = 0$. We then define

$$b + \infty := \infty =: \infty + b \qquad (b \in W)$$

and, for any $a \in W$, the permutation $\alpha_a \in \mathrm{Sym}(X)$ by

$$\alpha_a x := a + x \qquad (x \in X),$$

so α_a fixes ∞ and agrees with the left translation by a on the group W. By means of

$$W_\infty := \{\alpha_a \mid a \in W\}, \tag{1}$$
$$W_0 := \tau W_\infty \tau^{-1}, \tag{2}$$
$$W_a := \alpha_a W_0 \alpha_a^{-1} = \alpha_a W_0 \alpha_{-a} \qquad (a \in W \setminus \{0\}), \tag{3}$$

we obtain

$$\mathbb{M}(W, \tau) := \big(X, (W_x)_{x\in X}\big), \tag{4}$$

which is just a set X together with a family of subgroups of the permutation group of X, one for each $x \in X$. Though $\mathbb{M}(W, \tau)$ will not be a Moufang set in general, it still makes sense to consider $G := \langle W_x \mid x \in X\rangle$, the subgroup of the permutation group $\mathrm{Sym}(X)$ generated by all W_x, $x \in X$. Of particular importance later on are

- the *Hua subgroup* of G defined by

$$H := \{g \in G \mid g0 = 0,\ g\infty = \infty\},$$

  whose elements may canonically be regarded as permutations of W fixing 0,

- and the *Hua maps* $h_a \in \mathrm{Sym}(X)$, $a \in W \setminus \{0\}$, defined by

  $$h_a := \alpha_{-\tau(-(\tau^{-1}a))}\tau\alpha_{-(\tau^{-1}a)}\tau^{-1}\alpha_a\tau,$$

  so $h_a\colon X \to X$ is given by

  $$h_a x = -\tau\big(-(\tau^{-1}a)\big) + \tau\big(-(\tau^{-1}a) + \tau^{-1}(a + \tau x)\big) \tag{5}$$

  for all $x \in X$. One checks easily that h_a fixes 0 and ∞, hence belongs to the Hua subgroup H of G.

As a converse of the procedure described in 1.3, we have

1.4. Proposition. *Let* $\mathbb{M} = (X, (W_x)_{x\in X})$ *be a Moufang set and pick two distinct elements* $0, \infty \in X$*. Put*

$$W := X \setminus \{\infty\}$$

and assume we are given a permutation $\tau \in \mathrm{Sym}(X)$ *interchanging* 0 *and* ∞ *as well as conjugating* W_∞ *to* W_0*:*

$$\tau 0 = \infty, \quad \tau\infty = 0, \quad \tau W_\infty \tau^{-1} = W_0. \tag{6}$$

Then the following statements hold.

(a) *For* $a \in W$*, there is a unique* $\alpha_a \in W_\infty$ *such that* $\alpha_a 0 = a$*.*

(b) *The binary operation* $+$ *defined on* W *by*

$$a + b := \alpha_a b \qquad\qquad (a, b \in W)$$

makes W *a (possibly non-abelian) group and the assignment* $a \mapsto \alpha_a$ *determines an isomorphism* $W \xrightarrow{\sim} W_\infty$*.*

(c) $\mathbb{M} = \mathbb{M}(W, \tau)$. □

Remark. By Lemma 1.2 (a), permutations τ of X satisfying the first two relations of (6) and even belonging to $G_{\mathbb{M}}$ always exist; the final condition of (6) is then automatic.

The big question that presents itself now reads: given W, τ as in 1.3, when is $\mathbb{M}(W, \tau)$ a Moufang set? Here is the answer.

1.5. Theorem. (deMedts-Weiss [2]) *If* W, τ *are as in 1.3, then* $\mathbb{M}(W, \tau)$ *is a Moufang set if and only if, for all* $a \in W \setminus \{0\}$*, the Hua map* $h_a \in \mathrm{Sym}(W)$ *is an automorphism of the group* W*.* □

This fundamental result, whose proof, though elementary, is not at all trivial, brings us to

2. The connection with Jordan division rings.

Using Theorem 1.5, we can now present

2.1. Examples of Moufang sets. Let J be a *(quadratic) Jordan division ring*, so J is a unital quadratic Jordan algebra over the ring $\mathbb{Z}$ of rational integers such that $J \neq \{0\}$ and $J^\times = J \setminus \{0\}$, i.e., all non-zero elements of J are invertible. We define

$$W := \mathrm{Add}(J)$$

as the additive group of J and, with $X := W \cup \{\infty\}$, let $\tau\colon\ X \to X$ be the map given by

$$\tau x := -x^{-1} \qquad\qquad (x \in X),$$

where we put $-\infty := \infty$, $\infty^{-1} := 0$, $0^{-1} := \infty$. Then $\tau^2 = \mathrm{Id}_X$ and we claim *that*

$$\mathbb{M}(J) := \mathbb{M}(W, \tau) = \mathbb{M}\big(\mathrm{Add}(J), x \longmapsto -x^{-1}\big)$$

is a Moufang set. By Theorem 1.5, it suffices to show that the Hua maps $h_a\colon\ J \to J$, $a \in J^\times$, are additive. But (1.5) shows

$$h_a x = a - \big(a^{-1} - (a - x^{-1})^{-1}\big)^{-1}$$

for $x \in J$, which agrees with $U_a x$ by the Hua identity [4, Prop. 1.7.10], even for the critical values $x = 0, a^{-1}$. Thus $h_a = U_a$ is an automorphism of the additive group of J and the assertion is proved.

Remark. For $J = F^+$, the Jordan division ring corresponding to an ordinary field F, one can show $G_{\mathbb{M}} = \mathrm{PSL}_2(F)$. This justifies the term "little projective group of $\mathbb{M}$".

Consulting (1.1)−(1.3), we see that since $\mathrm{Add}(J)$ is abelian, so are the root groups of $\mathbb{M}(J)$. The big question is the converse.

2.2. Big question. Is every Moufang set with abelian root groups isomorphic to the Moufang set of a Jordan division ring?

The answer to this question is not known. Yoav Segev, in a personal communication that I received a few months ago, described the big question as difficult, bordering on the intractable. He therefore suggested to specialize the situation by looking at

2.3. The Zassenhaus condition. A Moufang set $\mathbb{M} = (X, (W_x)_{x\in X})$ is said to satisfy the *Zassenhaus condition* if

(i) $\mathbb{M}$ is *proper*, i.e., its little projective group is *not* sharply 2-transitive on X, equivalently, realizing $\mathbb{M} = \mathbb{M}(W, \tau)$ as in Prop. 1.4, the Hua subgroup of $G_{\mathbb{M}}$ is *not* trivial.

(ii) The pointwise stabilizer in $G_{\mathbb{M}}$ of three distinct points in X is trivial.

The Zassenhaus condition leads to the following remarkable observation, which, for want of a better name, I call

2.4. The Segev alternative. (Segev, unpublished) *Let J be a Jordan division ring of characteristic not* 2 *and suppose $\mathbb{M}(J)$, the Moufang set corresponding to J, satisfies the Zassenhaus condition. Then one of the following holds.*

(a) *There exist elements $x, y \in J$ such that $x \circ y = 0$ and the U-operators U_x, U_y* do not *commute: $U_x U_y \neq U_y U_x$.*

(b) *J is a field.*

Proof. Let $x, y \in J^\times$. Standard identities from Jordan theory imply

$$U_x(x^{-1} \circ y) = x \circ y.$$

Applying this repeatedly, we conclude that the (multiplicative) commutator $U_xU_yU_x^{-1}U_y^{-1}$ fixes $x \circ y$:

$$U_xU_yU_x^{-1}U_y^{-1}(x \circ y) = x \circ y. \tag{1}$$

Now suppose (a) does not hold. Then $x \circ y = 0$ implies that U_x, U_y commute. Otherwise, by (1), the commutator $U_xU_yU_x^{-1}U_y^{-1}$ fixes three distinct elements in $\mathbb{M}(J)$, namely, $0, \infty, x \circ y$, which again implies, this times by the Zassenhaus condition,, that U_x, U_y commute. Thus the U-operators of J commute by pairs, forcing J to be a field under the bilinear multiplication $xy = \frac{1}{2}x \circ y$:

$$\begin{aligned}(xy)z &= \frac{1}{4}z \circ (x \circ y) = \frac{1}{4}V_zV_xy = \frac{1}{4}U_{z,1}U_{x,1}y \\ &= \frac{1}{4}U_{x,1}U_{z,1}y = \frac{1}{4}V_xV_zy = \frac{1}{4}x \circ (z \circ y) = x(yz).\end{aligned}$$

□

The Segev alternative brings us immediately to

3. The problem of commuting U-operators.

More specifically, the natural question to be raised here is

3.1. Tent's question. (Tent, unpublished) Given elements x, y in an arbitrary Jordan division ring J, does $x \circ y = 0$ imply that the U-operators U_x, U_y commute: $U_xU_y = U_yU_x$?

My answer to this question is provided by

3.2. Theorem. *Yes.*

Sketch of Proof. By the Zelmanov-McCrimmon structure theory [5, 15.7], there are two cases.

(a) J is special. Then, as a moment's reflection shows, the answer is trivially yes, *even without assuming that J be division.*

(b) J is an Albert division algebra over some field F. Then the answer is yes because elements $x, y \in J$ with $x \circ y = 0$ are extremely rare. More specifically, assuming $x \neq 0 \neq y$ (as we may), $x \circ y = 0$ if and only if F has characteristic 2 and x is a multiple of 1 or y belongs to the subalgebra of J generated by x:

$$x \circ y = 0 \Longleftrightarrow \text{char}(F) = 2 \text{ and } (x \in F1 \text{ or } y \in F[x].)$$

In any event, the conclusion that U_x and U_y commute provided $x \circ y = 0$ is obvious. □

While the preceding result is precisely what the Moufang people want, its proof is not. Instead, what these people want is a proof that can be mimicked in *arbitrary* Moufang sets, hopefully giving them a handle on the Zassenhaus condition in the general set-up. In other words, they want a proof by clever manipulations of identities valid in arbitrary Jordan algebras (not just division), equivalently, they want an affirmative answer to

3.3. The problem of commuting U-operators. Let J be a unital Jordan algebra over k, an arbitrary commutative associative ring of scalars, and suppose $x, y \in J$ satisfy $x \circ y = 0$. Does this imply that the U-operators U_x, U_y commute: $U_x U_y = U_y U_x$?

This problem is wide open. Let me use the remaining minutes of my talk to tell you what I know.

3.4. The answer is yes if J is special. This trivial observation was already noted in the proof (or, rather, its sketch) of Theorem 3.2.

3.5. Some time ago I thought I had a proof for the implication

$$x \circ y = 0 \Longrightarrow 2U_x U_y = 2U_y U_x$$

but, fortunately, Teresa and José were kind enough of pointing out to me a stupid mistake in my argument.

3.6. Pointed quadratic forms. By what we have seen before (3.4) the problem of commuting U-operators has a trivial (affirmative) answer unless J is exceptional. Here it were Ottmar and, again, Teresa and José who reminded me that the Jordan algebra $J = J(M, q, e)$ of a pointed quadratic form over k, where M is a k-module, $q\colon M \to k$ is a quadratic form and $e \in M$ has $q(e) = 1$, need not be special [4, p. 2.6], so the problem of commuting U-operators is open even in this case. Therefore I was asking myself whether it is possible, assuming $x \circ y = 0$, to derive a formula for $U_x U_y z$, in terms of q and its bilinearization, that is symmetric in x and y. I was able to do so *but only in the absence of* 2*-torsion, not in general*:

$$x \circ y = 0 \Longrightarrow U_x U_y z = q(x)q(y)z - q(x)q(y, z)y - q(y)q(x, z)x.$$

3.7. Cubic Jordan algebras. It is a natural question to ask wether the obvious analogue of the preceding game can be played on the level of cubic Jordan algebras. I failed miserably, even when confining myself to base *fields* (rather than rings) and excluding low (positive) characteristics. But at least I can record the following positive result.

3.8. Theorem. (Anquela-Cortes, unpublished) *Let J be a unital Jordan algebra over a field F of characteristic not* 2 *and suppose $x, y \in J$ satisfy $x \circ y = 0$. If x is algebraic of degree at most* 3 *over F but not nilpotent, then the U-operators U_x, U_y commute: $U_x U_y = U_y U_x$.* □

3.9. Strongly prime Jordan algebras. Once the problem of commuting U-operators in its most general form has been accepted as being fairly intractable, it is natural to impose regularity conditions on J, e.g., by insisting that it be strongly prime. In this case, again invoking the Zelmanov-McCrimmon structure theory, J is either special or an Albert form [5, 15.2], so it makes sense to look at arbitrary Albert algebras, division or not. Here I have

3.10. Theorem. *Let J be an arbitrary Albert algebra over an arbitrary field F and suppose $x, y \in J$ satisfy $x \circ y = 0$. Then the U-operators U_x, U_y commute: $U_x U_y = U_y U_x$.*

Sketch of Proof. 1^0. Changing scalars to the algebraic closure of F, we may assume that J is split.

2^0. By 1^0 and the Jacobson embedding theorem [3, Theorem IX.11], which also holds in characteristic 2 (not obvious!), there is a unital subalgebra $J' \subseteq J$ with $x \in J' \cong \mathrm{Mat}_3(F)^+$.

3^0. By 2^0, we may assume that

$$J = J(A,1) = A \oplus Aj_1 \oplus Aj_2, \quad A := \mathrm{Mat}_3(F),$$

is a first Tits construction and $x, y \in J$ have the form

$$x = x_0 := x_0 + 0j_1 + 0j_2, \quad y = y_0 + y_1 j_1 + y_2 j_2 \qquad (x_0, y_0, y_1, y_2 \in A).$$

4^0. Since there is an automorphism of J that stabilizes A (in its capacity as the initial summand of J) and interchanges Aj_1, Aj_2, we only have to show

$$U_x U_y z = U_y U_x z, \quad U_x U_y (zj_1) = U_y U_x (zj_1) \qquad (z \in A).$$

5^0. Working this out by means of the explicit formula for the U-operator in cubic Jordan algebras, it follows that Theorem 3.10 is equivalent to the validity of no less than eleven identities in ordinary $3 \times 3-$matrices x_0, y_0, y_1, y_2, z under the assumption $x \circ y = 0$, i.e.,

$$x_0 \circ y_0 = 0, \quad x_0 y_1 = \mathrm{tr}(x_0) y_1, \quad y_2 x_0 = \mathrm{tr}(x_0) y_2. \tag{1}$$

6^0. Surprisingly and annoyingly, the verification of these identities under the assumption (1), though elementary, is by no means straightforward. Here is a particularly notorious example:

$$\big((y_0 y_1) \times z\big) x_0^\sharp = (y_0 y_1) \times (x_0^\sharp z), \tag{2}$$

where $\sharp$ is the usual adjoint and $\times$ its bilinearization. If you could give a comparatively easy proof for the implication (1) $\Rightarrow$ (2), I would be happy.

4. Comparisons.

In this section, which is not part of the lecture, we compare the various notions of Hua maps floating around in [1, 2] and (1.5).

4.1. Groups with infinity switch. By a *group with infinity switch* we mean a pair (W, τ), where W is an additive group, possibly non-abelian, and $\tau \in \mathrm{Sym}(W \cup \{\infty\})$ is an *infinity switch*, i.e., a permutation of $W \cup \{\infty\}$ that interchanges $0 \in W$ with $\infty \notin W$. Given another group with infinity switch, (W', τ'), a *morphism* from (W, τ) to (W', τ') is a group *iso*morphism $\varphi\colon W \to W'$ that, when extended to a map $\varphi\colon W \cup \{\infty\} \to W' \cup \{\infty\}$ via $\varphi(\infty) := \infty$, satisfies the relation

$$\tau' = \varphi \tau \varphi^{-1}. \tag{1}$$

In this way we obtain the category $\mathbf{GrouSwi}_\infty$ of groups with infinity switch, which is, in fact, a groupoid.

4.2. Sets with families of permutation groups. By a *set with a family of permutation groups* we mean a pair $(X, \mathfrak{W})$ consisting of a set X and a family $\mathfrak{W} = (W_x)_{x \in X}$ of subgroups $W_x \subseteq \mathrm{Sym}(X)$, one for each $x \in X$. Given another set with a family of permutation groups, $(X', \mathfrak{W}')$, $\mathfrak{W}' = (W'_{x'})_{x' \in X'}$, a *morphism* from $(X, \mathfrak{W})$ to $(X', \mathfrak{W}')$ is a *bijective* map $\varphi\colon X \to X'$ such that

$$W'_{\varphi(x)} = \varphi W_x \varphi^{-1} \qquad (x \in X). \tag{2}$$

In this way we obtain the category **SeFaPer** of sets with families of permutation groups, which again is a groupoid.

4.3. Towards functoriality. For the remainder of this section, we adopt the conventions of 1.3 and fix the following situation. W, W' are additive groups, possibly non-abelian, and $\infty \notin W \cup W'$ is a new symbol. Setting

$$X := W \cup \{\infty\}, \quad X' := W' \cup \{\infty\}$$

and considering arbitrary permutations $\tau \in \mathrm{Sym}(X)$, $\tau' \in \mathrm{Sym}(X')$ interchanging 0 and ∞, we may then follow (1.3.1)−1.3.4) and form

$$\mathbb{M} := \mathbb{M}(W, \tau) = \big(X, (W_x)_{x \in X}\big), \quad \mathbb{M}' := \mathbb{M}(W', \tau') = \big(X', (W'_{x'})_{x' \in X'}\big) \tag{3}$$

as sets with families of permutations groups in the sense of 4.2.

To stress dependence on W, the left translation by $a \in W$ on X will be denoted by α_a^W, so

$$\alpha_a^W \colon X \longrightarrow X, \quad x \longmapsto \alpha_a^W(x) = a + x. \tag{4}$$

Ditto for the right translation,

$$\beta_a^W \colon X \longrightarrow X, \quad x \longmapsto \beta_a^W(x) = x + a. \tag{5}$$

4.4. Proposition. *Let $\varphi\colon (W, \tau) \to (W', \tau')$ be a morphism of groups with infinity switch. Then*

$$\alpha^{W'}_{\varphi(a)} = \varphi \alpha_a^W \varphi^{-1}, \quad \beta^{W'}_{\varphi(a)} = \varphi \beta_a^W \varphi^{-1}, \tag{6}$$

and $\mathbb{M}(\varphi) := \varphi\colon \mathbb{M}(W, \tau) \to \mathbb{M}(W', \tau')$ is a morphism of sets with families of permutation groups. In this way,

$$\mathbb{M}\colon \mathbf{GrouSwi}_\infty \longrightarrow \mathbf{SeFaPer}$$

becomes a functor.

Proof. We begin by verifying the first equation of (6); the second one will follow analogously. Since both sides fix ∞, is suffices to show that they take the same value at any $b \in W$. But $\varphi\colon W \to W'$ is a group isomorphism, so we have

$$\varphi \alpha_a^W \varphi^{-1} b = \varphi\big(a + \varphi^{-1}(b)\big) = \varphi(a) + b = \alpha^{W'}_{\varphi(a)} b,$$

as desired. To establish the remainder of the proposition, we only need to show that $\varphi\colon \mathbb{M}(W, \tau) \to \mathbb{M}(W', \tau')$ is a morphism of sets with families of permutation groups, equivalently, that (2) holds. Combining (1.3.1) with (6), we obtain

$$\begin{aligned} W'_{\varphi\infty} &= W'_\infty = \{\alpha^{W'}_{a'} \mid a' \in W'\} = \{\alpha^{W'}_{\varphi(a)} \mid a \in W\} \\ &= \{\varphi \alpha_a^W \varphi^{-1} \mid a \in W\} = \varphi\{\alpha_a^W \mid a \in W\}\varphi^{-1} = \varphi W_\infty \varphi^{-1}, \end{aligned}$$

hence (2) for $x = \infty$. Combining (1.3.2) with (1) and (2) for $x = \infty$, we obtain

$$\begin{aligned} W'_{\varphi 0} &= W'_0 = \tau' W'_\infty \tau'^{-1} = \varphi \tau \varphi^{-1} W'_\infty \varphi \tau^{-1} \varphi^{-1} \\ &= \varphi \tau W_\infty \tau^{-1} \varphi^{-1} = \varphi W_0 \varphi^{-1}, \end{aligned}$$

hence (2) for $x = 0$. Similarly, given $a \in W \setminus \{0\}$, we have $\varphi a \in W' \setminus \{0\}$ and then

$$\begin{aligned} W'_{\varphi a} &= \alpha^{W'}_{\varphi a} W'_0 (\alpha^{W'}_{\varphi a})^{-1} = \varphi \alpha_a^W \varphi^{-1} W'_0 \varphi (\alpha_a^W)^{-1} \varphi^{-1} \\ &= \varphi \alpha_a^W W_0 (\alpha_a^W)^{-1} \varphi^{-1} = \varphi W_a \varphi^{-1}, \end{aligned}$$

giving (2) for $x = a$. □

4.5. Opposites. The inversion $a \mapsto -a$ may be viewed as a group isomorphism

$$\iota\colon\ W \xrightarrow{\sim} W^{\mathrm{op}}, \quad a \longmapsto \iota a := -a \tag{7}$$

that satisfies $\iota^2 = \mathrm{Id}_W$. In fact, calling

$$(W, \tau)^{\mathrm{op}} := (W^{\mathrm{op}}, \tau^{\mathrm{op}}), \quad \tau^{\mathrm{op}} := \iota\tau\iota = \iota\tau\iota^{-1} \tag{8}$$

the *opposite* of (W, τ), we deduce from 4.1, particularly (1), that

$$\iota\colon\ (W, \tau) \xrightarrow{\sim} (W, \tau)^{\mathrm{op}} \tag{9}$$

is an isomorphism of groups with infinity switch, giving rise by Prop. 4.4 to an isomorphism

$$\iota = \mathbb{M}(\iota)\colon\ \mathbb{M}(W, \tau) \xrightarrow{\sim} \mathbb{M}(W, \tau)^{\mathrm{op}} := \mathbb{M}\big((W, \tau)^{\mathrm{op}}\big) \tag{10}$$

of sets with families of permutation groups. Hence (6) implies

$$\beta^W_{-a} = \alpha^{W^{\mathrm{op}}}_{\iota a} = \iota\alpha^W_a\iota, \tag{11}$$

while (2) yields

$$(W^{\mathrm{op}})_{\iota a} = \iota W_a \iota \tag{12}$$

for all $a \in W$.

By the first relation of (11), $\mathbb{M}(W^{\mathrm{op}}, \tau)$ agrees with what deMedts-Segev in [1, 1.3] and deMedts-Weiss in [2, §3] refer to as $\mathbb{M}(W, \tau)$.

4.6. Various Hua maps. To emphasize dependence on (W, τ), we write $h^{(W,\tau)}_a$ for the Hua map determined by $a \in W \setminus \{0\}$. Thus

$$h^{(W,\tau)}_a = \alpha^W_{-\tau(-(\tau^{-1}a))}\tau\alpha^W_{-(\tau^{-1}a)}\tau^{-1}\alpha^W_a\tau \tag{13}$$

by (1.5). On the other hand, we denote by $h'^{(W,\tau)}_a$ (resp. $h''^{(W,\tau)}_a$) the Hua map determined by a as defined in deMedts-Segev [1, 1.3.1.12] (resp. deMedts-Weiss [2, Def. 3.2]). Thus

$$h'^{(W,\tau)}_a = \beta^W_{-\tau(-(\tau^{-1}a))}\tau\beta^W_{-(\tau^{-1}a)}\tau^{-1}\beta^W_a\tau, \tag{14}$$

$$h''^{(W,\tau)}_a = \beta^W_{-\tau(-(\tau^{-1}a))}\tau\beta^W_{-(\tau^{-1}a)}\tau^{-1}\beta^W_a\tau, \tag{15}$$

the latter by [2, (14)] where, appearances to the contrary, the maps in question are being composed from left to right; otherwise, the authors' equation would not be consistent with [2, Def. 3.2].

4.7. Proposition. *With the notations of* 4.6, *the Hua maps of* (W, τ) *in the sense of deMedts-Segev agree with the ones in the sense of deMedts-Weiss:*

$$h'^{(W,\tau)}_a = h''^{(W,\tau)}_a \qquad (a \in W \setminus \{0\}).$$

Moreover, for all $a \in W \setminus \{0\}$*, we have a commutative diagram*

$$\begin{array}{ccc} W & \xrightarrow{\ \iota\ } & W^{\mathrm{op}} \\ {\scriptstyle h'^{(W,\tau)}_a}\downarrow & & \downarrow{\scriptstyle h^{(W,\tau)^{\mathrm{op}}}_{\iota a}} \\ W & \xrightarrow{\ \iota\ } & W^{\mathrm{op}}. \end{array}$$

Proof. The first part is clear by (14),(15), so it remains to verify that the diagram commutes. In order to do so, we apply (14),(11),(8) and obtain

$$\begin{aligned}\iota h_a'^{(W,\tau)} &= \iota\beta^W_{-\tau(-(\tau^{-1}a))}\tau\beta^W_{-(\tau^{-1}a)}\tau^{-1}\beta^W_a\tau\\ &= \alpha^W_{\tau(-(\tau^{-1}a))}\iota\tau\iota\alpha^W_{\tau^{-1}a}\iota\tau^{-1}\iota\alpha_{-a}\iota\tau\\ &= \alpha^W_{\tau(-(\tau^{-1}a))}\tau^{\mathrm{op}}\alpha^W_{\tau^{-1}a}(\tau^{\mathrm{op}})^{-1}\alpha^W_{\iota a}\tau^{\mathrm{op}}\iota,\end{aligned}$$

where

$$\begin{aligned}-\tau^{\mathrm{op}}\Big(-\big((\tau^{\mathrm{op}})^{-1}\iota a\big)\Big) &= \iota\tau^{\mathrm{op}}\iota\iota\tau^{-1}\iota\iota a = \tau\iota\tau^{-1}a = \tau\big(-(\tau^{-1}a)\big),\\ -(\tau^{\mathrm{op}})^{-1}\iota a &= \iota(\tau^{\mathrm{op}})^{-1}\iota a = \tau^{-1}a.\end{aligned}$$

Hence

$$\iota h_a'^{(W,\tau)} = \alpha^W_{-\tau^{\mathrm{op}}(-((\tau^{\mathrm{op}})^{-1}\iota a))}\tau^{\mathrm{op}}\alpha^W_{-(\tau^{\mathrm{op}})^{-1}\iota a}(\tau^{\mathrm{op}})^{-1}\alpha^W_{\iota a}\tau^{\mathrm{op}}\iota,$$

and by (13), this is the same as $h_{\iota a}^{(W,\tau)^{\mathrm{op}}}\iota$. □

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